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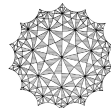
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# On maximally non-factorial nodal Fano threefolds

Ivan Cheltsov, Igor Krylov, Jesus Martinez-Garcia and Evgeny Shinder

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**Abstract.** We classify non-factorial nodal Fano threefolds with 1 node and class group of rank 2.

## 1. Introduction

Let  $X$  be a Fano threefold that has at worst isolated ordinary double points (nodes). Then both the Picard group  $\text{Pic}(X)$  and the class group  $\text{Cl}(X)$  are torsion-free of finite rank, and  $\text{rk Cl}(X) - \text{rk Pic}(X)$  is known as the *defect* of the threefold  $X$  [13, 18, 19, 31]. If the defect is zero, we say that  $X$  is *factorial* [6, 7]. Factoriality imposes significant constraints on the geometry of the Fano threefold [8, 9, 39, 50].

It is well known that the defect of  $X$  does not exceed the number of its singular points (see, e.g., Corollary 3.8 in [30]). If

$$\text{rk Cl}(X) - \text{rk Pic}(X) = |\text{Sing}(X)|,$$

then we say that  $X$  is *maximally non-factorial*. This property is also called  $\mathbb{Q}$ -maximal non-factoriality; see Proposition 6.13 in [36] and Proposition A.14 in [37] for various ways to define it for a nodal Fano threefold  $X$ .

By definition, if  $X$  has a single node, then  $X$  is maximally non-factorial if and only if it is non-factorial. Let us give the simplest example of a non-factorial threefold with one node.

**Example 1.1.** Let  $X$  be the quadric cone in  $\mathbb{P}^4$  with one node. Then  $X$  is a maximally non-factorial nodal Fano threefold. Let  $\eta: \tilde{X} \rightarrow X$  be the blow up at the singular point of the threefold  $X$ , and let  $E$  be the  $\eta$ -exceptional surface. Then

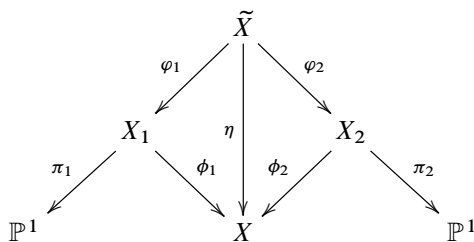
$$E \cong \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{and} \quad E|_E \cong \mathcal{O}_E(-1, -1),$$

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and there exists the following commutative diagram:



where  $\varphi_1$  and  $\varphi_2$  are contractions of the surface  $E$  to curves such that  $\varphi_2 \circ \varphi_1^{-1}$  is an Atiyah flop, both  $\phi_1$  and  $\phi_2$  are small projective resolutions, and both  $\pi_1$  and  $\pi_2$  are  $\mathbb{P}^2$ -bundles.

While maximally non-factorial nodal Fano threefolds are important in birational geometry because of their rich geometry, they also play a central role in the recent study of the derived categories of coherent sheaves for singular varieties (this in turn allows the study of the birational geometry with a very different toolset, such as stability conditions). Indeed, maximally non-factorial nodal Fano threefolds are very special from the perspective of derived categories of coherent sheaves; in particular, their derived categories can often be separated into a smooth proper part and a singular part [30, 36, 37, 43, 54].

Let us explain the connection to derived categories in some more detail. In [30, 43, 54], the authors, inspired by the work of Kawamata [33], introduced and studied maximal non-factoriality of del Pezzo threefolds (a subset consisting of 8 of the 105 families of Fano threefolds). They proved that a del Pezzo threefold is maximally non-factorial if and only if its derived category admits a Kawamata semiorthogonal decomposition, that is an admissible semiorthogonal decomposition into a perfect part and derived categories of singular finite-dimensional algebras. It is thus natural to ask whether being a maximally non-factorial Fano threefold is a sufficient condition for the existence of a Kawamata decomposition. The proof of [43] relied on the classification of del Pezzo threefolds which are maximally non-factorial. Thus, in order to study Kawamata decompositions, it is natural to have a classification of maximally non-factorial Fano threefolds.

A slightly weaker notion of categorical absorption of singularities was introduced in [36, 37]. By Corollary 6.17 in [36], every maximally non-factorial Fano threefold with one ordinary double point admits a categorical absorption of singularities; the converse is also true, and holds for any number of nodes (see Proposition 6.12 in [36]). A highlight of this theory in [37] is the deformation between the main components of the derived categories of one-nodal prime Fano threefolds of genus  $2d + 2$  and del Pezzo threefolds of rank one and degree  $d$ , for  $d \in \{1, 2, 3, 4, 5\}$ , which solves the so-called Fano threefold conjecture of Kuznetsov.

There is also a consequence of maximal non-factoriality to intermediate Jacobians. Namely, in some sense, a maximally non-factorial nodal Fano threefold  $X$  has a smooth projective intermediate Jacobian, so that the singularities of  $X$  can be ignored from the Hodge theory perspective. The precise statement, see Proposition A.16 in [37], is that a family of smooth Fano threefolds degenerating to a 1-nodal maximally non-factorial Fano threefold has a smooth projective family of intermediate Jacobians, i.e., no actual degeneration takes place in the middle degree cohomology.

On the other hand, maximally non-factorial Fano threefolds are rather rare among all nodal Fano threefolds. Motivated by the recent advances in derived categories of singular Fano threefolds, we pose the following problem.

**Problem 1.2.** *Classify all maximally non-factorial nodal Fano threefolds.*

The goal of this article is to partially solve this problem. Namely, we aim to classify maximally non-factorial nodal Fano threefolds of Picard rank one that have exactly one singular point (node). This case is particularly well behaved from the viewpoint of birational geometry, see the chain of equivalences in Proposition 6.13 of [36], that applies only when one singular point is present.

Now, we are ready to present the main result of this article. To do this, we suppose that

- the nodal Fano threefold  $X$  has one node,
- the rank of the Picard group  $\text{Pic}(X)$  is one,
- the rank of the class group  $\text{Cl}(X)$  is two.

Let  $\eta: \tilde{X} \rightarrow X$  be the blow up of the node of the threefold  $X$ , and let  $E$  be the  $\eta$ -exceptional surface. Then  $\tilde{X}$  is smooth,  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $E|_E \simeq \mathcal{O}_E(-1, -1)$ , and it follows from [15] that  $X$  uniquely determines the following Sarkisov link:

$$(1.1) \quad \begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow \varphi_1 & & \searrow \varphi_2 & \\ & X_1 & \eta & X_2 & \\ \swarrow \pi_1 & & \downarrow \phi_1 & \downarrow \phi_2 & \searrow \pi_2 \\ Z_1 & & X & & Z_2 \end{array}$$

where  $\varphi_1$  and  $\varphi_2$  are contractions of the surface  $E$  to curves such that  $\varphi_2 \circ \varphi_1^{-1}$  is an Atiyah flop, both  $\phi_1$  and  $\phi_2$  are small projective resolutions, and both  $\pi_1$  and  $\pi_2$  are extremal contractions [40]. Note that  $-K_{X_1} \sim \phi_1^*(-K_X)$  and  $-K_{X_2} \sim \phi_2^*(-K_X)$ , so that

$$-K_{X_1}^3 = -K_{X_2}^3 = -K_X^3.$$

It follows from [29, 42] that  $X$  admits a smoothing  $X \rightsquigarrow X_s$ , where  $X_s$  is a smooth Fano threefold,  $-K_X^3 = -K_{X_s}^3$ , and the rank of the Picard group  $\text{Pic}(X_s)$  is 1. We also know from [13] that

$$(1.2) \quad h^{1,2}(\tilde{X}) = h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(X_s),$$

which imposes a significant constraint on the link (1.1). We set

$$d = -K_X^3, \quad h^{1,2} = h^{1,2}(X_s),$$

and

$$I = \max\{n \in \mathbb{Z}_{>0} \text{ such that } -K_{X_s} \sim nH \text{ for } H \in \text{Pic}(X_s)\}.$$

Then  $I$  is the *index* of the Fano threefold  $X_s$ , which is also the index of the Fano threefold  $X$ , see [29].

In the remaining part of this article, we prove the following theorem.

**Theorem 1.3.** *There are exactly 17 types of non-factorial Fano threefolds of Picard rank one with one node. All possibilities for (1.1), up to swapping the left and right sides of the diagram, are described in Table 1.*

Each Sarkisov link in the table exists and can be described explicitly. For the reader’s convenience, we provide the relevant references in Table 1. For the particular case of  $-K_X^3 = 22$ , the result is proved in [44]. A. Kuznetsov and Y. Prokhorov have independently obtained the same classification of non-factorial Fano threefolds of Picard rank one, see [35].

**Remark 1.4.** It should be pointed out that it follows from our classification that one-nodal maximally non-factorial degenerations of smooth Fano threefolds of Picard rank one (if any) have the same rationality as their smoothing (in the cases 2 and 7 in the table, we need to assume that the smooth Fano threefolds are general). Indeed, this can be verified case by case, using the rationality results from [3, 14, 22, 45, 48, 53].

## 2. Proofs and classification

**Observation 2.1.** If  $X$  is a del Pezzo threefold ( $I = 2$ ) of Picard rank one with  $-K_X^3 \leq 32$ , then the nodal Fano threefold  $X$  is never maximally non-factorial. This follows from the defect computation [18, 19], see Corollary 2.5 in [43]. Therefore, the only options for  $X$  when  $I > 1$  are these two Fano threefolds:

- the nodal quadric threefold in  $\mathbb{P}^4$  ( $I = 3$ ,  $-K_X^3 = 54$ , the Sarkisov link 17);
- a quintic del Pezzo threefold ( $I = 2$ ,  $-K_X^3 = 40$ , the Sarkisov link 16).

We prove Theorem 1.3 by analyzing the possible links (1.1) in the following order:

- (1)  $\pi_1$  is a del Pezzo fibration, and  $\pi_2$  is arbitrary;
- (2) both  $\pi_1$  and  $\pi_2$  are birational;
- (3)  $\pi_1$  is a conic bundle and  $\pi_2$  is arbitrary.

These cover all possible Mori fiber spaces arising in (1.1), up to swapping  $\pi_1$  and  $\pi_2$ .

Note that all possibilities for the smooth Fano variety  $X_s$  are known, and can be found in [24]. Using this classification, we list the possible values of  $h^{1,2}$  as follows.

|           |        |        |        |        |         |         |         |
|-----------|--------|--------|--------|--------|---------|---------|---------|
| $(d, I)$  | (2, 1) | (4, 1) | (6, 1) | (8, 1) | (10, 1) | (12, 1) | (14, 1) |
| $h^{1,2}$ | 52     | 30     | 20     | 14     | 10      | 7       | 5       |

|           |         |         |         |
|-----------|---------|---------|---------|
| $(d, I)$  | (16, 1) | (18, 1) | (22, 1) |
| $h^{1,2}$ | 3       | 2       | 0       |

|           |        |         |         |         |         |         |         |
|-----------|--------|---------|---------|---------|---------|---------|---------|
| $(d, I)$  | (8, 2) | (16, 2) | (24, 2) | (32, 2) | (40, 2) | (54, 3) | (64, 4) |
| $h^{1,2}$ | 21     | 10      | 5       | 2       | 0       | 0       | 0       |

Possibilities for (1.1) are studied in [1, 4, 16, 17, 20, 21, 25–29, 32, 44, 46, 47, 51, 52, 55]. Using some of these results, we immediately obtain the following corollary.

**Corollary 2.2.** *Suppose that  $\pi_1$  is a fibration into del Pezzo surfaces. Then (1.1) is one of the links*

**1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 17**

in Table 1.

*Proof.* If  $\pi_1$  is a fibration into del Pezzo surfaces of degree 6, the assertion follows from [20, 21], in which case we get the link 15. In the remaining cases, the required assertion follows from [52]. ■

Therefore, we may assume that neither  $\pi_1$  nor  $\pi_2$  is a fibration into del Pezzo surfaces.

**Proposition 2.3.** *Suppose that  $\pi_1$  and  $\pi_2$  are birational. Then (1.1) is the link 13 in Table 1.*

*Proof.* Both  $Z_1$  and  $Z_2$  are (possibly singular) Fano threefolds, and we have  $\text{rk Pic}(Z_1) = \text{rk Pic}(Z_2) = 1$ .

Suppose  $Z_1$  is smooth, so that  $\pi_1$  is a contraction of type  $E1$  or  $E2$  in Theorem 1.32 of [34] and  $\pi_2$  is a contraction of type  $E1$ – $E5$ . Then possibilities for  $h^{1,2}(Z_1)$  are listed in the two tables presented above. Using [17], we obtain all possible values of  $h^{1,2}(X_1)$ . Now, using (1.2), in combination with the list of Sarkisov links in Tables 1–7 of [17], we see, carrying out a case-by-case analysis, that  $Z_1 \cong Z_2 \cong \mathbb{P}^3$ , and  $\pi_1$  and  $\pi_2$  are the blow ups along smooth rational curves of degree 5. Alternatively, one can run a short computer program exhausting all the possibilities for  $Z_1$  and  $Z_2$  and reach the same conclusion.

Therefore, to show that (1.1) is the link 13 in the table, it suffices to explain why the rational quintic curves are not contained in a quadric. Indeed, none of these curves are contained in a smooth quadric surface, because in that case one of the rulings of this quadric will be contracted in the anticanonical model, but birational morphisms  $\phi_1$  and  $\phi_2$  are small by construction. Furthermore, a degree 5 smooth rational curve in  $\mathbb{P}^3$  is never contained in a singular quadric.

Thus, we may assume that both  $Z_1$  and  $Z_2$  are singular. Now, using Tables 8–9 in [17], we get  $-K_X^3 \in \{2, 4\}$ . Hence, if  $|-K_X|$  does not have base points, then  $X$  is one of the following threefolds:

- (1) a sextic hypersurface in  $\mathbb{P}(1, 1, 1, 1, 3)$ ,
- (2) a quartic hypersurface in  $\mathbb{P}^4$ ,
- (3) a complete intersection of a quadric cone and a quartic hypersurface in the weighted projective space  $\mathbb{P}(1, 1, 1, 1, 1, 2)$ .

Indeed, by the Riemann–Roch theorem, see Corollary 2.1.14 in [24],  $|-K_X|$  defines a finite map  $\phi: X \rightarrow \mathbb{P}^N$  with  $N = 3$  (respectively  $N = 4$ ) when  $-K_X^3 = 2$  (respectively  $-K_X^3 = 4$ ). We have  $\text{deg}(\phi(X)) \cdot \text{deg}(\phi) = -K_X^3$ . If  $-K_X^3 = 2$ , then  $\phi$  is a double cover of  $\mathbb{P}^3$  ramified at a sextic hypersurface by Hurwitz’s formula, thus giving the first case. If  $-K_X^3 = 4$ , we either get that  $\text{deg}(\phi) = 1$  and  $\phi(X)$  is a quartic threefold, or  $\text{deg}(\phi) = \text{deg}(\phi(X)) = 2$  and we get the last case.

By studying the defect, in each of these cases, the threefold  $X$  is factorial as it follows from [5–8, 50], contradicting our assumption.

Therefore,  $|-K_X|$  has base points. Hence, using Theorem 1.1 (i) in [28], we see that  $-K_X^3 = 2$ , and  $X$  is the complete intersection of a quadric cone and a sextic hypersurface

in  $\mathbb{P}(1, 1, 1, 1, 2, 3)$  on variables  $x_0, \dots, x_5$ . We can assume that the quadric cone is given by  $x_0x_1 - x_2x_3 = 0$ . Then the projection on  $x_0, x_2$  coordinates gives (after a small resolution of the singularity) a fibration by del Pezzo surfaces of degree 1. Similarly, the projection on  $x_0, x_3$  coordinates gives another such fibration. This implies that (1.1) is the Sarkisov link **1** in Table 1, so that  $\pi_2$  is not birational, which contradicts our assumption. ■

Thus, we may assume that  $\pi_1$  is a conic bundle, and either  $\pi_2$  is birational, or  $\pi_2$  is a conic bundle. Then the surface  $Z_1$  is smooth (see (3.5.1) in [40]), which implies that  $Z_1 = \mathbb{P}^2$ , since  $X_1$  has Picard rank two. Let  $d_1$  be the degree of the discriminant curve of the conic bundle  $\pi_1$ . Then the Main Theorem in Section 1.6 of [49] implies  $0 \leq d_1 \leq 11$ ;  $d_1 = 0$  if  $\pi_1$  is a  $\mathbb{P}^1$ -bundle. By [3], we get

$$(2.1) \quad h^{1,2}(X_1) = \frac{d_1(d_1 - 3)}{2},$$

so  $d_1 \notin \{1, 2\}$ . Using (1.2) and the list of possible values of  $h^{1,2}$  presented in tables above, we get

$$d_1 \in \{0, 3, 4, 5, 7, 8\}.$$

Using Observation 2.1 above, for the rest of the proof we will assume that  $I = 1$ . Therefore we have

$$(2.2) \quad (d, h^{1,2}, d_1) \in \{(6, 20, 8), (8, 14, 7), (14, 5, 5), (18, 2, 4), (22, 0, 0), (22, 0, 3)\}.$$

Let  $D_2$  be a Cartier divisor on  $X_2$ , let  $D_1$  be its strict transform on  $X_1$ , and let  $H_1$  be a sufficiently general surface in  $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ . Then  $D_1 \sim_{\mathbb{Q}} a(-K_{X_1}) - bH_1$  for some rational numbers  $a$  and  $b$ . Moreover, if  $d_1 \neq 0$ , then  $a$  and  $b$  are integers, because the conic bundle has no sections and the Picard group of the generic fiber is generated by its canonical class. If  $d_1 = 0$ , then  $2a$  and  $2b$  are integers, because the Picard group of the generic fiber is generated by the class of a section. On the other hand we have (e.g., see Lemma A.3 in [12])

$$\begin{aligned} -K_{X_1} \cdot D_1^2 &= -K_{X_2} \cdot D_2^2, \\ (-K_{X_1})^2 \cdot D_1 &= (-K_{X_2})^2 \cdot D_2. \end{aligned}$$

Moreover, we have (see Proposition 6 in [11])

$$(2.3) \quad -K_{X_1}^3 = d, \quad (-K_{X_1})^2 \cdot H_1 = 12 - d_1, \quad -K_{X_1} \cdot H_1^2 = 2 \quad \text{and} \quad H_1^3 = 0.$$

This gives

$$(2.4) \quad \begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = -K_{X_2} \cdot D_2^2, \\ da - (12 - d_1)b = (-K_{X_2})^2 \cdot D_2. \end{cases}$$

**Lemma 2.4.** *Suppose that  $\pi_2$  is birational. Then (1.1) is either the link **11** or the link **14** in Table 1.*

*Proof.* Let  $D_2$  be the  $\pi_2$ -exceptional surface. Then  $a = D_1 \cdot H_1^2 \geq 0$ .

If  $\pi_2(D_2)$  is a point, then it follows from Theorem (3.3) in [40] that one of the following cases holds:

- (A)  $D_2 = \mathbb{P}^2$  with normal bundle  $\mathcal{O}(-1)$ ,
- (B)  $D_2 = \mathbb{P}^2$  with normal bundle  $\mathcal{O}(-2)$ ,
- (C)  $D_2$  is an irreducible quadric surface in  $\mathbb{P}^3$  with normal bundle  $\mathcal{O}(-1)$ .

A simple computation using the adjunction formula implies that  $-K_{X_2} \cdot D_2^2 = -2$  and

$$(-K_{X_2})^2 \cdot D_2 = \begin{cases} 4 & \text{in the case (A),} \\ 1 & \text{in the case (B),} \\ 2 & \text{in the case (C).} \end{cases}$$

Now, solving (2.4) for each triple  $(d, h^{1,2}, d_1)$  listed in (2.2), we see that  $2a$  is never a non-negative integer. This shows that  $\pi_2(D_2)$  is not a point.

We see that  $Z_2$  is a smooth Fano threefold of Picard rank 1, and  $\pi_2(D_2)$  is a smooth curve in  $Z_2$ . Then it follows from Theorem 7.14 in [27] and from (1.2) that (1.1) is one of the Sarkisov links 11 and 14, which would complete the proof of the lemma.

Note, however, that the paper [27] has gaps, see Remark 1.18 in [10]. For instance, the link in Construction 2.5 below contradicts Theorem 7.4 in [27], and a few examples constructed in [55] contradict Proposition 7.2 in [27]. Keeping this in mind, let us complete the proof of the lemma without using Theorem 7.14 in [27].

Set  $C_2 = \pi_2(D_2)$ . Let  $d_2 = -K_{Z_2} \cdot C_2$ , and let  $g_2$  be the genus of the curve  $C_2$ . Then as  $\pi_2$  is the blow up along a curve on a threefold, we have that

$$h^{1,2}(Z_2) + g_2 = h^{1,2} \in \{0, 2, 5, 14, 20\},$$

where the inclusion follows from (2.2). As a result, using the classification of smooth Fano threefolds (see Section 12.2 of [24]), we get  $h^{1,2}(Z_2) \in \{0, 2, 3, 5, 7, 10, 14, 20\}$ . In fact, we can say a bit more. Let  $e = -K_{Z_2}^3$  and let  $i$  be the index of the Fano threefold  $Z_2$ . Then

- $(e, i) = (64, 4) \iff Z_2 = \mathbb{P}^3$ ,
- $(e, i) = (54, 3) \iff Z_2$  is a smooth quadric threefold in  $\mathbb{P}^4$ .

Moreover, the possible values  $h^{1,2}(Z_2) \leq 20$  can be listed as follows.

|                |        |        |         |         |          |         |         |         |
|----------------|--------|--------|---------|---------|----------|---------|---------|---------|
| $(e, i)$       | (6, 1) | (8, 1) | (10, 1) | (12, 1) | (14, 1), | (16, 1) | (18, 1) | (22, 1) |
| $h^{1,2}(Z_2)$ | 20     | 14     | 10      | 7       | 5        | 3       | 2       | 0       |

|                |         |         |         |         |         |         |
|----------------|---------|---------|---------|---------|---------|---------|
| $(e, i)$       | (16, 2) | (24, 2) | (32, 2) | (40, 2) | (54, 3) | (64, 4) |
| $h^{1,2}(Z_2)$ | 10      | 5       | 2       | 0       | 0       | 0       |

This leaves not so many possibilities for the genus  $g_2 = h^{1,2} - h^{1,2}(Z_2)$ .

One the other hand, it follows from Lemma 4.1.2 in [24] that

$$\begin{aligned} -K_{X_2} \cdot D_2^2 &= 2g_2 - 2, \\ (-K_{X_2})^2 \cdot D_2 &= d_2 + 2 - 2g_2, \\ -K_{X_2}^3 &= e - 2 + 2g_2 - 2d_2, \end{aligned}$$



so that (2.4) gives

$$\begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = 2g_2 - 2, \\ da - (12 - d_1)b = d_2 + 2 - 2g_2, \\ d = e - 2 + 2g_2 - 2d_2. \end{cases}$$

Now, solving this system of equations for each triple  $(d, I, h^{1,2}, d_1)$  listed in (2.2), and each possible triple  $(e, i, g_2) = (e, i, h^{1,2} - h^{1,2}(Z_2))$ , we obtain the following two cases:

- (I)  $d = 18, I = 1, h^{1,2} = 2, d_1 = 4, Z_2 = \mathbb{P}^3, d_2 = 24, g_2 = 2, a = 3, b = 4;$
- (II)  $d = 22, I = 1, h^{1,2} = 0, d_1 = 3, Z_2$  is a smooth quadric in  $\mathbb{P}^4, d_2 = 15, g_2 = 0, a = 3, b = 4;$

In the case (I), (1.1) is the link **11** in the table. In the case (II), (1.1) is the link **14** in the table. ■

Therefore, we may assume that  $\pi_2$  is also a conic bundle and  $Z_2 = \mathbb{P}^2$ . Let  $d_2$  be the discriminant curve of the conic bundle  $\pi_2$ . Using (2.1) and  $h^{1,2}(X_1) = h^{1,2}(X_2)$ , we obtain that either  $d_1 = d_2$  or  $d_1, d_2 \in \{0, 3\}$ . Now, we let  $D_2$  be a general surface in  $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ . Then (2.4) simplifies as

$$\begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = 2, \\ da - (12 - d_1)b = 12 - d_2. \end{cases}$$

Solving these equations for each quadruple  $(d, h^{1,2}, d_1)$  listed in (2.2), we get the following cases:

- (1)  $a = 0, b = -1;$
- (2)  $d = 14, I = 1, h^{1,2} = 5, d_1 = d_2 = 5, a = 1, b = 1.$

In the case (1), the composition  $\varphi_2 \circ \varphi_1^{-1}$  is biregular. This contradicts our initial assumption. So, the case (2) holds. Then (1.1) is the link **7** in the table, which proves Theorem 1.3.

Let us conclude this article by showing that the Sarkisov link **7** in Table 1 is always obtained using the following.

**Construction 2.5** (Case 4° in Section 3.4 of [47]). Let  $\bar{E} = \{z_1 = z_2 = 0\} \subset \mathbb{P}_{x_1, y_1, z_1}^2 \times \mathbb{P}_{x_2, y_2, z_2}^2$ , and let

$$\bar{X} = \{z_1 f(x_1, y_1, z_1; x_2, y_2, z_2) = z_2 g(x_1, y_1, z_1; x_2, y_2, z_2)\},$$

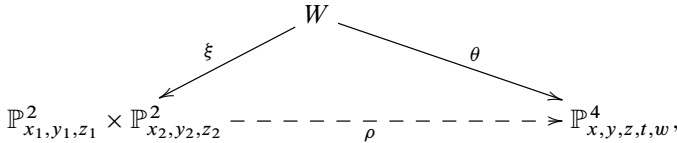
where  $f$  and  $g$  are sufficiently general polynomials of bi-degrees  $(1, 2)$  and  $(2, 1)$ , respectively. Then  $\bar{X}$  is a singular Verra threefold (a bidegree  $(2, 2)$  threefold in  $\mathbb{P}^2 \times \mathbb{P}^2$ ) with 5 nodes. Note that  $\bar{E} \cong \mathbb{P}^1 \times \mathbb{P}^1, \bar{E} \subset \bar{X}$  and

$$\text{Sing}(\bar{X}) = \{z_1 = z_2 = f = g = 0\} \subset \bar{E}.$$

Let  $\rho: \mathbb{P}_{x_1, y_1, z_1}^2 \times \mathbb{P}_{x_2, y_2, z_2}^2 \dashrightarrow \mathbb{P}_{x, y, z, t, w}^4$  be the rational map given by

$$([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) \mapsto [x_1 z_2 : y_1 z_2 : x_2 z_1 : y_2 z_1 : z_1 z_2].$$

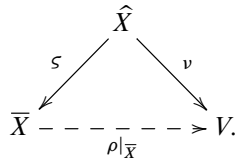
Then  $\rho$  is birational, and the inverse map  $\rho^{-1}$  is given by  $[x : y : z : t : w] \mapsto ([x : y : w], [z : t : w])$ . Let  $\xi: W \rightarrow \mathbb{P}^2_{x_1, y_1, z_1} \times \mathbb{P}^2_{x_2, y_2, z_2}$  be the blow up along the surface  $\bar{E}$  and let  $\mathcal{E}$  be its exceptional divisor. Let  $\bar{G}_1 = \{z_1 = 0\}$  and  $\bar{G}_2 = \{z_2 = 0\}$ , and let  $G_1$  and  $G_2$  be the proper transforms on  $W$  of  $\bar{G}_1$  and  $\bar{G}_2$ . Then we have the following commutative diagram:



where  $\theta$  blows down  $G_1$  and  $G_2$  to the lines  $\ell_1 = \{z = t = w = 0\}$  and  $\ell_2 = \{x = y = w = 0\}$ . Note that  $\theta(\mathcal{E})$  is the hyperplane  $\{w = 0\}$  – the unique hyperplane containing the lines  $\ell_1$  and  $\ell_2$ . Set  $V = \rho(\bar{X})$ . Then  $V$  is a smooth cubic threefold in  $\mathbb{P}^4_{x, y, z, t, w}$ . Moreover, we have

$$V = \{f(x, y, w; z, t, w) = g(x, y, w; z, t, w)\} \subset \mathbb{P}^4_{x, y, z, t, w}.$$

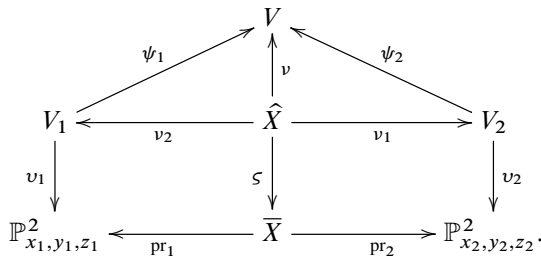
Now, let  $\hat{X}$  be the strict transform of the threefold  $\bar{X}$  on  $W$ , let  $\varsigma: \hat{X} \rightarrow \bar{X}$  be the morphism induced by  $\xi$ , and let  $\nu: \hat{X} \rightarrow V$  be the morphism induced by  $\theta$ . Then  $\hat{X}$  is smooth,  $\varsigma$  is a small projective resolution, and we have the following commutative diagram:



Note that  $\nu$  is the blow up of the cubic threefold  $V$  along the lines  $\ell_1$  and  $\ell_2$ . Let  $\hat{E} = \mathcal{E}|_{\hat{X}}$ . Then

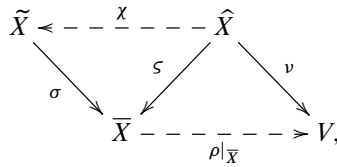
- the induced map  $\varsigma|_{\hat{E}}: \hat{E} \rightarrow \bar{E}$  is the blow up at the points in  $\text{Sing}(\bar{X})$ ,
- $\hat{E}$  is isomorphic to a smooth cubic surface,
- $\nu(\hat{E})$  is the hyperplane section  $\{w = 0\} \cap V$ .

Now, we extend the last commutative diagram to the following commutative diagram:



Here  $\psi_1$  and  $\psi_2$  are the blow ups along the lines  $\ell_1$  and  $\ell_2$ , respectively,  $\nu_1$  and  $\nu_2$  are the blow ups along the strict transforms of the lines  $\ell_1$  and  $\ell_2$ , respectively, both  $\nu_1$  and  $\nu_2$  are

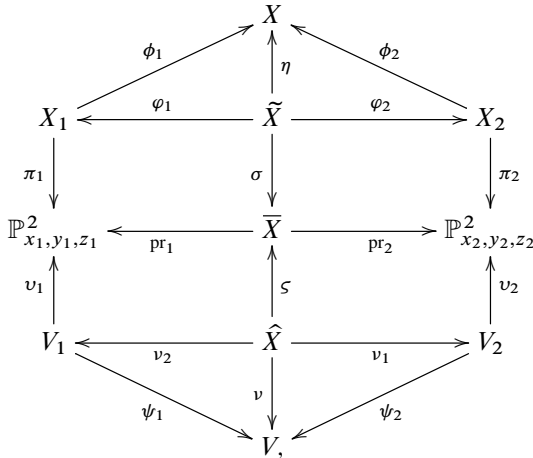
standard conic bundles [45], and  $\text{pr}_1$  and  $\text{pr}_2$  are the natural projections. Let  $\Delta_1$  and  $\Delta_2$  be the discriminant curves of the conic bundles  $\nu_1$  and  $\nu_2$ , respectively. Then  $\Delta_1$  and  $\Delta_2$  are quintic curves with at most nodal singularities. Since  $\zeta$  is a flopping contraction, there exists a composition of flops  $\chi: \widehat{X} \dashrightarrow \widetilde{X}$  of the 5 curves contracted by  $\zeta$  (this is the only projective flop which exists because the relative Picard number of  $\zeta$  equals 1). Then  $\widetilde{X}$  is smooth and projective, and we have another commutative diagram:



where  $\sigma$  is a small resolution. Let  $E = \chi(\widehat{E})$ . Then  $\chi$  induces a morphism  $\widehat{E} \rightarrow E$  that blows down all five curves contracted by  $\zeta$ , which implies that  $\sigma$  induces an isomorphism  $E \cong \overline{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Note that  $E|_E \sim \mathcal{O}_E(-1, -1)$ , and that there exists a birational morphism  $\eta: \widetilde{X} \rightarrow X$  that blows down the surface  $E$  to an ordinary double point of the threefold  $X$ . We have  $-K_X^3 = -K_{\widetilde{X}}^3 - 2 = 14$  and

$$1 = \text{rk Pic}(X) < \text{rk Cl}(X) = 1 + |\text{Sing}(X)| = 2.$$

Therefore, the threefold  $X$  is a non-factorial nodal Fano threefold that has one node. We complete the picture with the following commutative diagram:



where  $\phi_1$  and  $\phi_2$  are two small resolutions such that the composition  $\phi_1^{-1} \circ \phi_2$  is an Atiyah flop, both  $\varphi_1$  and  $\varphi_2$  are contractions of the surface  $E$  to curves,  $\pi_1$  and  $\pi_2$  are standard conic bundles whose discriminant curves are  $\Delta_1$  and  $\Delta_2$ , respectively. Note that  $X$  is irrational as it is birational to a smooth cubic threefold [14], and

$$h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(\widetilde{X}) = h^{1,2}(\widehat{X}) = h^{1,2}(V) = 5.$$

Instead of using the Verra threefold  $\overline{X}$  containing  $\overline{E}$ , we can construct the nodal threefold  $X$  using the birational map  $\rho^{-1}$ , and the smooth cubic threefold  $V$  containing the lines  $\ell_1$  and  $\ell_2$ .

We now explain that link 7 in Table 1 is obtained by Construction 2.5. We have  $Z_1 = Z_2 = \mathbb{P}^2$ , and both  $\pi_1$  and  $\pi_2$  are conic bundles with discriminant curves of degree 5. Let  $C_1$  and  $C_2$  be the curves contracted by  $\phi_1$  and  $\phi_2$ , respectively.

Recall that we denote by  $H_1$  (respectively,  $D_2$ ) the pullback of the ample generator by  $\pi_1$  from  $Z_1$  (respectively, by  $\pi_2$  from  $Z_2$ ), and  $D_1$  is the divisor corresponding to  $D_2$  on  $X_1$  under flop. Then it follows from the calculations above (see case (2) before Construction 2.5) that  $D_1 \sim -K_{X_1} - H_1$ . We have

$$-1 = (-K_{X_1} - H_1)^3 = D_1^3 = D_2^3 - (D_2 \cdot C_2)^3 = -(D_2 \cdot C_2)^3,$$

where we used (2.3) in the first equality, and Lemma A.3 in [12] in the third one. It follows that  $D_2 \cdot C_2 = 1$ . Similarly, we get  $H_1 \cdot C_1 = 1$ .

Let  $h_1 = \varphi_1^*(H_1)$  and  $h_2 = \varphi_2^*(D_2)$ . A simple computation using  $D_2 \cdot C_2 = 1$  implies that

$$\varphi_1^*(D_1) \sim h_2 + E.$$

Thus we can express the canonical class  $-K_{\tilde{X}}$  in terms of  $h_1$  and  $h_2$  as follows:

$$-K_{\tilde{X}} \sim -\varphi_1^*(K_{X_1}) - E \sim \varphi_1^*(H_1 + D_1) - E \sim h_1 + h_2.$$

Note that  $-K_{\tilde{X}}^3 = 12$ ,  $h^{1,2}(\tilde{X}) = 5$  and  $\text{rk Pic}(\tilde{X}) = 3$ , which implies that  $-K_{\tilde{X}}$  is not ample, because smooth Fano threefolds with these invariants do not exist, see Table 3 in [41].

Combining  $\pi_1 \circ \phi_1$  and  $\pi_2 \circ \phi_2$ , we obtain a morphism  $\tilde{X} \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ . Let  $\bar{X}$  be its image, and let  $\sigma: \tilde{X} \rightarrow \bar{X}$  be the induced morphism.

**Claim 2.6.** *The threefold  $\bar{X} \subset \mathbb{P}^2 \times \mathbb{P}^2$  is a divisor of bidegree (2, 2) with terminal singularities, containing a linearly embedded surface  $\mathbb{P}^1 \times \mathbb{P}^1$ , and  $\sigma$  is a small resolution.*

Therefore,  $X$  is obtained by taking a small resolution of the singular Verra threefold  $\bar{X}$  containing a divisor  $\mathbb{P}^1 \times \mathbb{P}^1$  as in Construction 2.5 above.

*Proof.* The threefold  $\bar{X}$  is a divisor of bidegree  $(e_1, e_2)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ , with  $e_1, e_2 > 0$ , because  $\bar{X}$  dominates both factors. We have

$$12 = (h_1 + h_2)^3 = \text{deg}(\sigma) \text{deg}(\bar{X}) = 3 \text{deg}(\sigma)(e_1 + e_2).$$

This implies that either  $\text{deg}(\sigma) = 1$  and  $e_1 + e_2 = 4$ , in which case  $e_1 = e_2 = 2$  because the two projections give rise to conic bundle structures on  $\tilde{X}$ , or  $\text{deg}(\sigma) = 2$  and  $e_1 + e_2 = 2$ , so that  $e_1 = e_2 = 1$  because  $e_1, e_2 > 0$ . In other words,

- either  $\bar{X}$  is a divisor of degree (2, 2), and  $\sigma$  is birational,
- or  $\bar{X}$  is a divisor of degree (1, 1), and  $\sigma$  is generically two-to-one.

In the former case,  $\sigma$  is crepant, and it follows from the subadjunction formula that the threefold  $\bar{X}$  is normal. In the latter case, the threefold  $\bar{X}$  is also normal, because there are only two isomorphism classes of irreducible (1, 1) divisors in  $\mathbb{P}^2 \times \mathbb{P}^2$ : one is smooth and the other has one node.

Set  $\bar{E} = \sigma(E)$ . Let  $\text{pr}_1: \bar{X} \rightarrow \mathbb{P}^2$  and  $\text{pr}_2: \bar{X} \rightarrow \mathbb{P}^2$  be the projections to the first and the second factors of the fourfold  $\mathbb{P}^2 \times \mathbb{P}^2$ , respectively. Then  $\text{pr}_1(\bar{E})$  and  $\text{pr}_2(\bar{E})$  are lines

**Table 1.** Description of all possibilities for the Sarkisov link (1.1).

| Nº | $d$ | $I$ | $h^{1,2}$ | $\pi_1: X_1 \rightarrow Z_1$   | $\pi_2: X_2 \rightarrow Z_2$   | References  |
|----|-----|-----|-----------|--|--|---|
| 1  | 2   | 1   | 52        | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>del Pezzo surfaces of degree 1. | $Z_2 = \mathbb{P}^1$ ,<br>$\pi_2$ is a fibration into<br>del Pezzo surfaces of degree 1.   | [22, 23, 28],<br>[52, (2.5.2)].   |
| 2  | 6   | 1   | 20        | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>del Pezzo surfaces of degree 2. | $Z_2$ is a del Pezzo threefold of degree 1<br>that has one singular double point,<br>$\pi_2$ is the blow up at the singular point.             | [48, Proposition 5.6],<br>[22, 23],<br>[46, Example 4.3],<br>[52, (2.7.3)]. |
| 3  | 8   | 1   | 14        | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into cubic surfaces.                    | $Z_2 \cong \mathbb{P}^2$ ,<br>$\pi_2$ is a conic bundle<br>with septic discriminant curve.   | [48, Proposition 5.9],<br>[46, Example 4.6],<br>[52, (2.9.4)].              |
| 4  | 10  | 1   | 10        | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into cubic surfaces.                    | $Z_2$ is a smooth del Pezzo threefold of degree 2,<br>$\pi_2$ is the blow up along a smooth rational curve<br>that has anticanonical degree 2. | [48, Example 1.11],<br>[47, § 3.12 Case 11°],<br>[52, (2.9.3)].             |
| 5  | 12  | 1   | 7         | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>quartic del Pezzo surfaces.     | $Z_2 \cong \mathbb{P}^3$ ,<br>$\pi_2$ is the blow up along a smooth<br>curve of degree 8 and genus 7.  | [27, Proposition 6.5],<br>[52, (2.11.5)].                                   |
| 6  | 14  | 1   | 5         | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>quartic del Pezzo surfaces.     | $Z_2$ is a smooth cubic threefold,<br>$\pi_2$ is the blow up at a smooth conic.  | [27, Proposition 6.5],<br>[47, § 3.13 Case 12°],<br>[52, (2.11.4)].         |
| 7  | 14  | 1   | 5         | $Z_1 = \mathbb{P}^2$ ,<br>$\pi_1$ is a conic bundle<br>with quintic discriminant curve.  | $Z_2 = \mathbb{P}^2$ ,<br>$\pi_1$ is a conic bundle<br>with quintic discriminant curve.  | [47, § 3.4 Case 4°],<br>Construction 2.5 and<br>Claim 2.6.                  |
| 8  | 16  | 1   | 3         | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>quintic del Pezzo surfaces.     | $Z_2$ is a smooth quadric in $\mathbb{P}^4$ ,<br>$\pi_2$ is the blow up along a smooth<br>curve of degree 7 and genus 3.                       | [27, Proposition 6.5],<br>[52, (2.13.4)].                                   |

|           |    |   |   |  |  |  |
|-----------|----|---|---|--|--|--|
| <b>9</b>  | 16 | 1 | 3 | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a quadric bundle  | $Z_2 = \mathbb{P}^1$ ,<br>$\pi_2$ is a fibration into<br>quartic del Pezzo surfaces.   | [2, Example 4.9],<br>[52, (2.3.8)],<br>[52, (2.11.2)]. |
| <b>10</b> | 18 | 1 | 2 | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>quintic del Pezzo surfaces.   | $Z_2$ is a smooth complete<br>intersection of two quadrics in $\mathbb{P}^5$ ,<br>$\pi_2$ is the blow up along a twisted cubic.      | [27, Proposition 6.5],<br>[52, (2.13.3)].              |
| <b>11</b> | 18 | 1 | 2 | $Z_1 \cong \mathbb{P}^2$ ,<br>$\pi_1$ is a conic bundle<br>with quartic discriminant curve.  | $Z_2 = \mathbb{P}^3$ ,<br>$\pi_2$ is the blow up along a smooth<br>curve of degree 6 and genus 2.                                    | [4, Example 4.8],<br>[27, Theorem 7.14],<br>Lemma 2.4. |
| <b>12</b> | 22 | 1 | 0 | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>quintic del Pezzo surfaces.   | $Z_2 \cong \mathbb{P}^2$ ,<br>$\pi_2$ is a $\mathbb{P}^1$ -bundle.   | [44, (IV)],<br>[52, (2.13.1)].                         |
| <b>13</b> | 22 | 1 | 0 | $Z_1 = \mathbb{P}^3$ ,<br>$\pi_1$ is the blow up along a smooth<br>rational curve of degree 5<br>that is not contained in a quadric. | $Z_2 = \mathbb{P}^3$ ,<br>$\pi_1$ is the blow up along a smooth<br>rational curve of degree 5<br>that is not contained in a quadric. | [17, Proposition 2.11],<br>[44, (I)].                  |
| <b>14</b> | 22 | 1 | 0 | $Z_1 \cong \mathbb{P}^2$ ,<br>$\pi_1$ is a conic bundle<br>with cubic discriminant curve.  | $Z_2$ is a smooth quadric threefold,<br>$\pi_2$ is the blow up along a smooth<br>rational quintic curve.                             | [27, Theorem 7.14],<br>[44, (II)],<br>Lemma 2.4.       |
| <b>15</b> | 22 | 1 | 0 | $Z_1 \cong \mathbb{P}^1$ ,<br>$\pi_1$ is a fibration into<br>sextic del Pezzo surfaces.  | $Z_2 \cong V_5$ ,<br>$\pi_2$ is the blow up along<br>a rational quartic curve.   | [27, Proposition 6.5],<br>[44, (III)].                 |
| <b>16</b> | 40 | 2 | 0 | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a quadric bundle.   | $Z_2 = \mathbb{P}^2$ ,<br>$\pi_2$ is a $\mathbb{P}^1$ -bundle.   | [25, Theorem 3.5],<br>[52, (2.3.2)].                   |
| <b>17</b> | 54 | 3 | 0 | $Z_1 = \mathbb{P}^1$ ,<br>$\pi_1$ is a $\mathbb{P}^2$ -bundle.   | $Z_2 = \mathbb{P}^1$ ,<br>$\pi_2$ is a $\mathbb{P}^2$ -bundle.   | Example 1.1.   |

by  $H_1 \cdot C_1 = D_2 \cdot C_2 = 1$ , so we can choose coordinates  $([x_1 : y_1 : z_1], [x_2 : y_2 : z_2])$  on  $\mathbb{P}^2 \times \mathbb{P}^2$  such that

$$\bar{E} = \{z_1 = z_2 = 0\}.$$

RMI-1495 Since  $\bar{E} \subset \bar{X}$ , we see that  $\bar{X}$  is singular. Note also that  $\sigma$  induces an isomorphism  $E \cong \bar{E} = \mathbb{P}^1 \times \mathbb{P}^1$ .

Divisor classes  $h_1, h_2$  and  $E$  generate the group  $\text{Pic}(\tilde{X})$ . We have

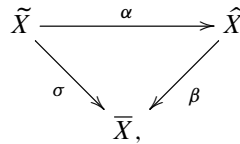
$$h_1^2 \cdot h_2 = h_1 \cdot h_2^2 = 2, \quad h_1 \cdot h_2 \cdot E = 1 \quad \text{and} \quad h_1^2 \cdot E = h_2^2 \cdot E = 0.$$

Assume that  $\sigma$  contracts a divisor  $F \sim a_1 h_1 + a_2 h_2 + a_3 E$ . Then we have

$$\begin{aligned} 2a_2 &= F \cdot h_1^2 = 0, \\ 2a_1 &= F \cdot h_2^2 = 0, \\ 2a_1 + 2a_2 + a_3 &= F \cdot h_1 \cdot h_2 = 0, \end{aligned}$$

which gives  $a_1 = 0, a_2 = 0, a_3 = 0$ . This shows that  $\sigma$  does not contract any divisors.

The Stein factorization of  $\sigma$  is the following commutative diagram:



where  $\alpha$  is a birational morphism, and  $\beta$  is either an isomorphism or a (ramified) double cover. Since  $\sigma$  does not contract divisors and  $-K_{\tilde{X}}$  is not ample, we see that  $\alpha$  is a flopping contraction, and  $\hat{X}$  has terminal Gorenstein singularities. We must show that  $\beta$  is an isomorphism.

Suppose  $\beta$  is a double cover. Its Galois involution induces a birational involution  $\tau \in \text{Bir}(\hat{X})$ . Then  $\tau$  induces an action  $\tau_*$  on  $\text{Pic}(\hat{X}) = \text{Cl}(\hat{X})$  such that  $\tau_* h_1 \sim h_1, \tau_* h_2 \sim h_2$ , and

$$\tau_*(E) \sim b_1 h_1 + b_2 h_2 + b_3 E$$

for some integers  $b_1, b_2$  and  $b_3$ . Then

$$\begin{aligned} 2b_2 &= \tau_*(E) \cdot h_1^2 = E \cdot h_1^2 = 0, \\ 2b_1 &= \tau_*(E) \cdot h_2^2 = E \cdot h_2^2 = 0, \\ 2b_1 + 2b_2 + b_3 &= \tau_*(E) \cdot h_1 \cdot h_2 = E \cdot h_1 \cdot h_2 = 1, \end{aligned}$$

which gives  $b_1 = 0, b_2 = 0, b_3 = 1$ , so  $\tau_*(E) \sim E$ , which gives  $\tau(E) = E$ , since  $E$  is  $\eta$ -exceptional.

Since  $\tau(E) = E$  and  $\sigma$  induces an isomorphism  $E \cong \bar{E}$ , we see that the surface  $\bar{E}$  is contained in the branch divisor of the double cover  $\beta$ . On the other hand,  $\bar{E}$  cannot be equal to this branch divisor by degree reasons, thus the branch divisor is reducible. This implies that  $\hat{X}$  has non-isolated singularities, which is impossible, since  $\hat{X}$  has terminal singularities. Thus, we see that  $\beta$  is an isomorphism.

We see that  $\bar{X}$  is a singular divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$  of degree  $(2, 2)$ , containing  $\bar{E}$  and  $\sigma$  is a small resolution. ■

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