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Hamiltonian Formulation and Aspects of Integrability of Generalised Hydrodynamics

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Abstract. Generalised hydrodynamics (GHD) describes the large-scale inhomogeneous dynamics of integrable (or close to integrable) systems in one dimension of space, based on a central equation for the fluid density or quasi-particle density: the GHD equation. We consider a new, general form of the GHD equation: we allow for spatially extended interaction kernels, generalising previous constructions. We show that the GHD equation, in our general form and hence also in its conventional form, is Hamiltonian. This holds also including force terms representing inhomogeneous external potentials coupled to conserved densities. To this end, we introduce a new Poisson bracket on functionals of the fluid density, which is seen as our dynamical field variable. The total energy is the Hamiltonian whose flow under this Poisson bracket generates the GHD equation. The fluid density depends on two (real and spectral) variables, and the GHD equation can be seen as a $2+1$-dimensional classical field theory. In its $1+1$-dimensional reduction corresponding to the case without external forces, we further show the system admits an infinite set of conserved quantities that are in involution for our Poisson bracket, hinting at integrability of this field theory.

Contents

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1. Introduction

Hydrodynamic equations of Euler type are hyperbolic equations that emerge at large scales of space and time in many-body systems [1]. They describe the propagation of local relaxation—the separation between a slow, emergent dynamics and fast projection of local observables onto conserved quantities.

In one dimension of space, the situation of interest here, they take the local "conservation form"

$$\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_i = \mathscr{F}_i\,, \tag{1.1}$$

where i parametrises the set of admitted local conservation laws, and $\mathscr{F}_i$ represents the contributions from external force fields (which naturally break the conservation laws). The fluxes $\mathtt{j}_i$ and force fields contributions $\mathscr{F}_i$ are functions of the conserved densities $\mathtt{q}_i$ only (the equations of state) and are determined by thermodynamic considerations, such as entropy maximisation. For instance, the standard Euler equations for Galilean fluid, and Relativistic hydrodynamics, are of this type.

In one dimension of space, many-body systems often display the property of integrability [2,3]. In such cases, there are infinitely many conservation laws, and the universal theory for their emergent Euler-scale hydrodynamics is Generalised Hydrodynamics (GHD) [4,5]. This recovers the Euler-scale hydrodynamic equations for hard rods [1,6] and soliton gases [7–9] obtained earlier, but applies more universally, including in classical and quantum systems of interacting particles, spin chains, and quantum field theory; see the reviews in [10].

GHD recasts the infinite set of (broken) conservation laws (1.1) into a family parametrised by a continuous "spectral" parameter θ instead of the discrete index i; we denote the conserved densities in real space, spectral space, and time as $\rho(x,\theta,t)$. The spectral parameter enumerates the possible asymptotic objects of the corresponding scattering theory (particles, solitons, etc.), including their momentum and possible internal states. The precise set in which θ lies depends on the model, but in many simple situations it is a subset of $\mathbb{R}$, or $\mathbb{R}$ itself, and represents the asymptotic momenta. In these cases, the coordinates x,θ span an effective "spectral phase space", on which ρ is a density.[1] The inclusion of external force fields coupled to conserved densities was achieved in [11], where it was shown that GHD takes the form

$$\partial_t\rho + \partial_x(v^{\rm eff}\rho) + \partial_\theta(a^{\rm eff}\rho) = 0. \tag{1.2}$$

Here $v^{\rm eff}$ and $a^{\rm eff}$ are appropriate functionals of $\rho(x,\cdot,t)$, and the last term on the left-hand side is the contribution from force fields. Other types of forces have been studied [12,13], which we will not consider here. GHD can therefore be understood, at this level of generality, as hydrodynamics for a *two-dimensional fluid*, with a conserved fluid density. It generically has extended interaction range on the spectral component θ of phase space, which represents scattering between asymptotic objects of different spectral parameters. In systems of non-interacting particles, the spectral phase space is the true phase space of classical mechanics, and the GHD equation is the (single-particle) Liouville equation, or collision-less Boltzmann equation, as arises from basic kinetic theory. GHD generalises this to integrable interactions.

[1] It may be defined physically by a time-of-flight thought experiment: $\rho(x,\theta){\rm d}x{\rm d}\theta$ is the number of particles with asymptotic spectral parameters in $[\theta,\theta+{\rm d}\theta]$, that are asymptotically observed if the fluid element on $[x,x+{\rm d}x]$ is extracted and let to expand in the vacuum.

What are the intrinsic properties of Eq. (1.2), as a dynamical system? In particular, a natural question in hydrodynamics is that of the Hamiltonian structure: is there a Poisson bracket for the conserved densities seen as the dynamical variables, such that the total fluid energy is a Hamiltonian generating the fluid equations? The standard Euler equations, in any dimension, do have a Hamiltonian structure, this is well documented [14–17] and has important consequences [17], see also the recent study [18] in two spatial dimensions. When it comes to GHD, it is known that, without external force fields, so-called polychromatic reductions have Hamiltonian structures, and even are integrable [19–22]. Regarding the non-reduced, force-less GHD equation, although integrability has not been addressed, solutions by quadrature to the initial value problem are known [23,24], and with external force fields, stationary solutions have been analysed [11,25,26]. On top of that, perhaps most notably, a bi-Hamiltonian structure was recently proposed for the special case of force-less hard rods hydrodynamics [22]. In fact, one may expect that an appropriate "continuous-spectrum limit," from the polychromatic reduction of GHD, of the construction of [22], would provide a Hamiltonian structure for the GHD equation more generally. But this appears to be a non-trivial task—in the hard rods case, the fact that the scattering shift is constant seemed to be play an important role. As far as we are aware, there are currently no results about Hamiltonian structure and integrability for the GHD equation in general, in the continuous case for $\rho(x, \theta, t)$, and especially with external force fields.

In this paper we establish the Hamiltonian structure of the GHD Eq. (1.2). The fluid density ρ is the dynamical variable, spanning the dynamical phase space $\mathcal{M}$ of this two-dimensional field theory. The set of observables on $\mathcal{M}$ are functionals (not necessarily linear) of ρ. The fundamental Poisson bracket can be expressed, using the derivative of the Dirac delta-function $\delta'(x)$, the standard dressing operation $^{\mathrm{dr}}$, and the occupation function $n(x, \theta)$ (see e.g. [4] for a review of the GHD formalism), as

$$\begin{aligned} &\{\rho(x_1,\theta_1),\rho(x_2,\theta_2)\} \\ &\quad= -\frac{1}{2\pi}\delta'(x_1-x_2)\Big(\big[\delta'(\cdot-\theta_2)\big]^{\mathrm{dr}}(x_2,\theta_1)\,n(x_2,\theta_1) \\ &\qquad+\big[\delta'(\cdot-\theta_1)\big]^{\mathrm{dr}}(x_1,\theta_2)\,n(x_1,\theta_2)\Big). \end{aligned} \tag{1.3}$$

In Sect. 3, Eq. (3.3), we provide the general expression in terms of the algebra of observables on $\mathcal{M}$; we show it satisfies the Leibniz property and the Jacobi identity. The Hamiltonian then takes the simple form of the total fluid energy,

$$H = \int \mathrm{d}x\mathrm{d}\theta\, E(x,\theta)\rho(x,\theta): \qquad \partial_t\rho = \{\rho, H\} \text{ is equivalent to (1.2)}, \tag{1.4}$$

where $E(x, \theta)$ is the energy associated to the quasi-particle of spectral parameter θ when it is at position x.

In fact, it is convenient to establish our results for an even more general form of the GHD equation that we introduce. This form allows for non-locality in space (more precisely, it is not "ultra-local" in space, but may still have finite interaction range), thus recovering a structural symmetry between how the spatial and spectral variables, x and θ, are treated. The dressing operation acts on the full spectral phase space; the Poisson bracket is

$$\begin{aligned}\{\rho(x_1,\theta_1),\rho(x_2,\theta_2)\}\\ &= \frac{1}{2\pi}\Big(\big[\delta'(\cdot - x_1)\delta'(\cdot-\theta_2)\big]^{\rm dr}(x_2,\theta_1)\,n(x_2,\theta_1)\\ &\quad - \big[\delta'(\cdot - x_2)\delta'(\cdot-\theta_1)\big]^{\rm dr}(x_1,\theta_2),n(x_1,\theta_2)\Big),\end{aligned} \tag{1.5}$$

and the Hamiltonian still takes the form (1.4).

The Poisson structure (1.5) and the statement in (1.4) are the main results of this work. The fact that the Hamiltonian is the total fluid energy is natural and also appears in [22][2] and most works on Hamiltonian structures of fluid equations more generally. However the Poisson brackets (1.3) and (1.5) are, we believe, new; in particular, it is a non-trivial problem, that we solve in this paper, to show that they indeed satisfy the appropriate requirements for a Poisson bracket (Leibniz and Jacobi). We show that the spatially extended form of the GHD equation arises from a family of classical particle systems that we construct. This is an adaptation of the semiclassical Bethe systems introduced in [28,29] to the spatially extended scaling of interactions and the presence of external force fields; it can be seen as an "atomic reduction" of the GHD equation.

We also propose that in the case without external force fields ($a^{\rm eff}=0$, $E(x,\theta)=E(\theta)$), and in the case without "kinetic term" ($v^{\rm eff}=0$, $E(x,\theta)=E(x)$), the resulting two-dimensional field theory may be integrable, by exhibiting a large family of conserved quantities in involution.

We do not require any particular form for the scattering shift or more generally the interaction kernel, besides a basic symmetry of the associated scattering phase (which is often satisfied and follows from the Kubo-Martin-Schwinger relation for the associated microscopic model [30]). Thus our results apply to the GHD equation for the KdV soliton gas, the Lieb-Liniger quantum gas, the Toda model of classical particles, the sinh-Gordon relativistic quantum field theory, etc. We show that one of the aforementioned Hamiltonian structures in [22] for the hard rod gas is a special case of our construction,[3] when expressed in terms of moments of the field $\rho(x,t)$ and specialised to constant scattering shift.

The paper is organised as follows. In Sect. 2 we introduce elements of the GHD formalism that will be necessary for our discussion. Section 3 contains the main result of our manuscript: it is where we construct the Hamiltonian structure of the GHD equation and show that there exists an infinite family

[2] The definition of what we mean by energy is somewhat ambiguous, see Section 4.5 of [8] and 3.2 of [27], and does not always coincide with that of [22].

[3] Preprint [22] appeared, while this paper was in preparation.

of linearly independent, conserved quantities, that are in involution with respect to the associated Poisson bracket. In Sect. 4, we go over some previously obtained results regarding the Hamiltonian structure of particular reductions of the GHD equation and introduce the theoretical framework in which they were obtained; we also show that our construction is compatible with that of [22]. Finally, in Sect. 5, we comment on some aspects of our construction, on some restrictions that may be lifted, and on some interesting perspectives.

2. GHD and Dressing Operations

2.1. Fluid Density, Dynamical, and Spectral Phase Spaces

The GHD equation describes the large-scale behaviour of integrable many-body systems, in terms of the propagation of local (generalised) "equilibria"—more precisely, local projection onto conserved quantities. At any given time, every expectation value of local observables is completely characterised by the fluid density $\rho(x,\theta)$, which is therefore the dynamical variable of interest. The fluid density can be interpreted as the density of quasi-particles in *spectral phase space* $\Lambda = \mathscr{L} \times \mathscr{P}$, such that $x \in \mathscr{L}$ and $\theta \in \mathscr{P}$. In this paper we concentrate on the simplest case $\mathscr{L} = \mathscr{P} = \mathbb{R}$ and comment on other cases in Sect. 5. We assume that the fluid density quickly vanishes in unbounded directions:

$$\Lambda = \mathbb{R}^2, \qquad \lim_{|x|+|\theta|\to\infty} \rho(x,\theta) = 0. \tag{2.1}$$

The *dynamical phase space* is the space on which the Poisson structure will be constructed. This is the space in which the dynamical variable ρ lies. There are two possible setups.

In the *abstract setup*, ρ is a *formal dynamical variable* out of which observables are constructed, see Sect. 3.1. In this case, vanishing at infinity (2.1) is to be understood in an abstract sense. By this, we mean that we consider densities ρ such that (i) integrals of ρ over the spectral phase-space Λ are bounded and (ii) that boundary terms, notably when performing integration by parts, cancel out. This includes, but is not limited to, the case of quickly vanishing densities as defined in (2.1).

In the *concrete setup*, ρ is in a specific function space. For simplicity we assume smoothness, and thus the dynamical phase space is some space of smooth functions on $\mathbb{R}^2$ that asymptotically vanish,

$$\mathcal{M} \subset \mathcal{C}_0^\infty(\mathbb{R}^2). \tag{2.2}$$

In many situations of physical interest, $\mathcal{M}$ is restricted to non-negative functions $\rho(x,y) \geq 0,\ \forall\,(x,y) \in \mathbb{R}^2$; however, this is not necessary; for instance, in the KdV soliton gas, see below, it is convenient to allow for negative fluid densities. We explain in "Appendix B", with further comments in "Appendix E.1", how taking

$$\mathcal{M} = \{|\rho| \leq \rho_* : \rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)\}\,, \tag{2.3}$$

the set of smooth compactly supported functions that are bounded by a given function $\rho_*(x,\theta)$, or any subset thereof, gives an explicit setup for our general construction (this requires some technical conditions on our GHD data, Eqs. (B.1), (B.2) and (B.12)). The use of $\mathcal{C}_c^\infty(\mathbb{R}^2)$ is convenient as it allows for a large set of GHD data and simple analysis techniques; however, by adding constraints on the GHD data or using different techniques, different families of functions may be allowed. This includes Schwartz functions or other space of rapidly decaying functions, and even bounded functions not required to be smooth, with relatively weak differentiability requirement, see [31]. The arguments we present in the main text and in "Appendices B and E.1" can be adapted to such situations.

We do not discuss under what conditions $\mathcal{M}$ may be invariant under the dynamics, Eq. (2.7) below, although in the abstract setup this is immediate (see [31] for rigorous results in a concrete setup.

2.2. Conventional Data of GHD

In its conventional, original formulation, the GHD equation is determined by a few data: the scattering shift $\varphi(\theta,\theta')$, the momentum function $P(\theta)$, and the energy function $E(\theta)$. These data encode the system in terms of the scattering properties of the set of asymptotic "objects", parametrised by θ: the energy function determines the dynamics, the scattering shift fixes the two-body displacements, and the momentum sets the relation to real space. The asymptotic objects may be particles, solitons, quantum excitations, etc.; the GHD equation takes the same form in all cases. As shown in [30], the Kubo-Martin-Schwinger relation imposes that there exist $\phi(\theta,\theta')$ such that $\phi(\theta,\theta') = -\phi(\theta',\theta)$ and $\varphi(\theta,\theta') = \partial_\theta\phi(\theta,\theta')$. But otherwise, there are no strong constraints on the data. GHD is therefore a universal framework for the hydrodynamic limit of many-body integrable systems.

As was explained in [32], the choice of a parametrisation θ for the asymptotic objects is largely arbitrary. In many (but not all) cases one may choose θ to correspond to the physical momentum, $P(\theta) = \theta$, or one may choose it to guarantee that the scattering shift is a function of differences of spectral parameters, $\varphi(\theta,\theta') = \varphi(\theta-\theta')$ (only in Galilean systems may it be chosen such that both hold). Moreover, as was explained in [8] and further discussed in [27], in classical systems, even given a parametrisation θ, the choice of a momentum function may be arbitrary, being relevant in quantum models only due to the discretisation of phase space. That is, there is a general theory for the transformation of the main objects in the GHD equation under change of parametrisation $\theta \to \tilde{\theta}$, and, in classical cases, under change of momentum function $P(\theta) \to \tilde{P}(\theta)$, which keeps the GHD equation invariant. In this paper, the former transformation theory will play an important role (we will recall it), but as we consider the general form of the GHD equation, we will not discuss the latter; and we keep $P(\theta)$ in its general form.

The emergent GHD equation receives modifications when external fields, which may be varying in space and time, influence the dynamics of the many-body systems; or when the dynamics itself, say the coupling parameters of the model, is inhomogeneous in space and/or non-autonomous. These modifications can be seen as "acceleration" terms. This was first obtained for space-varying external fields coupled to conserved densities (such as minimal coupling of electric or magnetic fields to charges), in [11]; this case corresponds to taking the energy function to be space-dependent, $E(\theta) \to E(x, \theta)$. It was then generalised to space-time varying external fields and coupling parameters in [12,13], in fact admitting all data to be space-time dependent, $E(\theta) \to E(x, \theta, t)$, $P(\theta) \to P(x, \theta, t)$ and $\phi(\theta - \theta') \to \phi(x, \theta - \theta', t)$.

2.3. Spatially Extended GHD

In this paper, we consider autonomous systems, hence all data are time independent, but we admit space-dependent energy and momentum functions.

Instead of the momentum function, the quantity that is most relevant in GHD is the spectral derivative of the momentum, $P_\theta(x, \theta) = \partial_\theta P(x, \theta)$. This, divided by 2π for conventional normalisation, is physically interpreted as a bare *spectral phase-space state density*: $\mathrm{d}x\mathrm{d}\theta\, |P_\theta|/(2\pi)$ is the number of physical states that are available within the infinitesimal spectral phase-space element $\mathrm{d}x\mathrm{d}\theta$, without accounting for interactions (the spectral phase-space state density that accounts for interaction is ρ_{s}, defined in (2.19) below). Geometrically, it is a volume density—a discussion of the associated metric structure is left for future works. For our discussion, an object that appears to be more fundamental is the *phase function* $\Omega(x, \theta)$, defined such that $P(x, \theta) = \partial_x \Omega(x, \theta)$.

Thus, parts of our data are phase-space-dependent energy and phase functions

$$E(x, \theta), \quad \Omega(x, \theta), \qquad \text{with} \qquad P(x, \theta) = \Omega_x(x, \theta)\,, \tag{2.4}$$

where here and below indices $x, \theta, \ldots$ mean partial derivatives with respect to $x, \theta, \ldots$. For definiteness we assume[4] $E,\ \Omega \in \mathcal{C}^\infty(\mathbb{R}^2)$.

More importantly, we admit an extended form of the GHD equation not considered until now, which we refer to as its *spatially extended form*. In this form, the datum of the scattering shift $\varphi(\theta, \theta')$ is replaced by an *interaction kernel* $\psi(x, \theta; x', \theta')$ on $\mathbb{R}^2 \times \mathbb{R}^2$, with the main requirement that it be symmetric

$$\psi(x, \theta; x', \theta') = \psi(x', \theta'; x, \theta). \tag{2.5}$$

ψ is a smooth-function-valued distribution in x', θ'—that is, it may be a generalised function and should be seen as the kernel of an integral operator. For definiteness, we require that ψ be integrable in x', θ' against smooth functions supported on compact subsets of $\mathbb{R}^2$ and that the result be smooth functions of x, θ:

$$\int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta'\, \psi(\cdot, \cdot; x', \theta') g(x', \theta') \ \in \mathcal{C}^\infty(\mathbb{R}^2) \tag{2.6}$$

for every $g \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. This includes ψ being any smooth function on $\mathbb{R}^2 \times \mathbb{R}^2$, as well as all examples considered below. This requirement on ψ is adapted to

[4]Smoothness can easily be relaxed to finite-order differentiability.

the specific context of Eq. (2.3); for different choices of the space $\mathcal{M}$, one may require similar properties for a larger family of g.

The use of the interaction kernel ψ, instead of the scattering shift φ, will allow us to treat the spatial and spectral coordinates on equal footing—restoring a structural symmetry in spectral phase space. The spatially extended GHD equation, Eq. (2.7) below, is more naturally a hydrodynamic equation on two "spatial" dimensions, with coordinates x, θ. It (generically) has extended interaction range not only in spectral space θ (as is the case for the conventional GHD equation), but also in real space x. This spatially extended form is convenient to consider, as it admits a Hamiltonian formulation in its full generality, which is the main object of this paper.

The spatially extended GHD equation is an equation for the dynamics of the fluid density $\rho(x, \theta, t)$. It takes the conservation form

$$\partial_t \rho + \partial_x (v^{\text{eff}} \rho) + \partial_\theta (a^{\text{eff}} \rho) = 0 \tag{2.7}$$

for effective velocity and acceleration $v^{\text{eff}}(x, \theta, t)$, $a^{\text{eff}}(x, \theta, t)$ which are nonlinear functionals of $\rho(\cdot, \cdot, t)$. These functionals are defined as solutions to the following linear integral equations (here taken on any time slice, keeping the time variable implicit):

$$E_\theta = v^{\text{eff}}(x, \theta)\Omega_{x\theta} + \int_{\mathbb{R}^2} \mathrm{d}x' \mathrm{d}\theta' \, \rho(x', \theta') \left[\psi_{x\theta} \, v^{\text{eff}}(x, \theta) + \psi_{x'\theta} \, v^{\text{eff}}(x', \theta')\right] \tag{2.8}$$

$$-E_x = a^{\text{eff}}(x, \theta)\Omega_{x\theta} + \int_{\mathbb{R}^2} \mathrm{d}x' \mathrm{d}\theta' \, \rho(x', \theta') \left[\psi_{x\theta} \, a^{\text{eff}}(x, \theta) + \psi_{x\theta'} \, a^{\text{eff}}(x', \theta')\right] \tag{2.9}$$

where $E = E(x, \theta)$, $\Omega_{x\theta} = \Omega_{x\theta}(x, \theta) = P_\theta(x, \theta)$, $\psi = \psi(x, \theta; x', \theta')$ as above. Equations (2.8) and (2.9) are inhomogeneous Fredholm equations of the second kind; we will see below that solutions are unique in the setting of Eq. (2.3) (c.f. Appendix B), adapting techniques from [31].

Effectively, we are considering space-dependent external fields as in [11], space-dependent momenta, and certain types of space-dependent, inhomogeneous and generically spatially extended scattering shifts. It is not clear if it is possible to specialise our theory to the type of local, space-dependent scattering shifts considered in [13]; the form of GHD equation we obtain appears to be new.

Our construction naturally reduces to conventional GHD by imposing that the interaction kernel ψ produces spatially homogeneous scattering shifts:

$$\psi(x, \theta; x', \theta') = \frac{1}{2} \operatorname{sgn}(x - x'), \phi(\theta, \theta')$$

$$\text{(conventional GHD with homogeneous scattering shift).} \tag{2.10}$$

Due to the symmetry condition (2.5), ϕ must be anti-symmetric, and we recover the scattering shift of conventional GHD as $\varphi(\theta, \theta') = \partial_\theta \phi(\theta, \theta')$. In this case, the spatially extended GHD equation becomes the conventional GHD equation with (in general) external inhomogeneous fields linearly coupled to the conserved densities as represented by the energy function $E(x, \theta)$; see [11]. In our notation, it takes the form (2.7) with

$$E_\theta = v^{\rm eff}(x,\theta)\Omega_{x\theta} + \int_{\mathbb{R}} {\rm d}\theta'\, \rho(x,\theta')\varphi(\theta,\theta')\left[v^{\rm eff}(x,\theta) - v^{\rm eff}(x',\theta')\right] \tag{2.11}$$

$$-E_x = a^{\rm eff}(x,\theta)\Omega_{x\theta} + \int_{\mathbb{R}} {\rm d}\theta'\, \rho(x,\theta')\big[\varphi(\theta,\theta')\, a^{\rm eff}(x,\theta) - \varphi(\theta',\theta), a^{\rm eff}(x,\theta')\big]$$
$$\text{(conventional GHD with homogeneous scattering shift).} \tag{2.12}$$

At the time of writing, this is probably the most relevant configuration.

Remark 2.1. We discussed above the "interaction range" of the GHD equation, both in real and spectral space. An extended range means that the instantaneous change of $\rho(x, \theta, t)$ depends on $\rho(x', \theta', t)$ for $(x', \theta') \neq (x, \theta)$. As in many-body systems, one may also talk about "finite-range", "short-range", or "long-range" interactions. As is clear from (2.8) and (2.9), the interaction distance may be taken as the minimal value of $|x - x'|$ or $|\theta - \theta'|$ above which $\psi_{\tilde{x}\tilde{\theta}}$ vanishes for every choice of $\tilde{x} = x,\ x'$ and $\tilde{\theta} = \theta,\ \theta'$; if it is finite, we have a finite-range interaction. We may say that we have a short-range interaction if $\psi_{\tilde{x}\tilde{\theta}}$ decays exponentially with $|x-x'|+|\theta-\theta'|$ (taking, say, the L^1 distance on $\mathbb{R}^2$), and long-range if the decay is algebraic. One may expect "local physics" to hold in two-dimensional spectral phase space in all cases, for the latter if the power of the algebraic decay is large enough; however, a full analysis is beyond the scope of this paper.

2.4. Examples

Important known examples of our construction occur in the case of conventional GHD (2.10), where typically one takes $P(x, \theta) = P(\theta)$. For instance, setting

$$E(x,\theta) = \tfrac{\theta^2}{2}, \quad \Omega(x,\theta) = x\theta,$$
$$\phi(\theta,\theta') = 2\,{\rm Arctan}\tfrac{\theta-\theta'}{c} \quad \text{(Lieb-Liniger)}\ , \tag{2.13}$$

reproduces the GHD equations for the Lieb-Liniger gas with coupling c [33]; setting

$$E(x,\theta) = \frac{\theta^2}{2}, \quad \Omega(x,\theta) = x\theta, \quad \phi(\theta,\theta') = a\,(\theta-\theta') \quad \text{(hard rods)} \tag{2.14}$$

gives the hard rod gas for rods of length a; and restricting $\mathcal{M}$ to odd functions under $\theta \to -\theta$ on $\mathbb{R}^2$, and setting

$$E(x,\theta) = 8\theta^4, \quad \Omega(x,\theta) = 4x\theta^2,$$
$$\phi(\theta,\theta') = 8\Big((\theta-\theta')(\log|\theta-\theta'| - 1)\Big) \quad \text{(KdV)}\ , \tag{2.15}$$

gives the KdV soliton gas. In the latter case, we have taken the "convenient" choice of momentum function (see [8, Sec 4.5]), and extended the physical domain of spectral parameters from $\mathbb{R}^+$ to $\mathbb{R}$, performing an odd continuation of ρ as in [34]. As E_θ is odd in θ, Eq. (2.8) reproduces the usual equation of state of the KdV soliton gas. This is the way in which the KdV soliton gas fits within our general framework.

Setting $\psi(x,\theta;x',\theta') = \psi(x-x',\theta-\theta')$ along with $E(x,\theta) = E(\theta)$, $\Omega(x,\theta) = x\theta$, the spatially extended GHD equation emerges as an appropriate hydrodynamic limit of the family of classical interacting, integrable Hamiltonian particle models referred to as Bethe semiclassical systems, introduced in [28, 29]. In fact, we show in Appendix A that, for any $P(x,\theta)$, $E(x,\theta)$, and $\psi(x,\theta;x',\theta')$, the spatially extended GHD equation arises as an appropriate hydrodynamic limit of a microscopic Hamiltonian model that generalises the semiclassical Bethe systems. The phase function $\Omega(x,\theta)$ and the interaction kernel $\psi(x,\theta;x',\theta')$ appear naturally in the canonical transformation that defines the models.

2.5. Spectral Phase-Space Reparametrisation

The theory of phase-space reparametrisation (generalising that introduced in [32]) will be crucial in order to understand the dressing operations that we now discuss. For our purposes, it is sufficient to consider factorised changes of coordinates $(x,\theta) \to (\tilde{x},\tilde{\theta}) = (f(x), g(\theta))$, where f and g are smooth functions with positive derivative. We note that Eqs. (2.7), (2.8), and (2.9) are invariant under such transformations if the objects involved transform in the appropriate way. In order to describe the transformation properties, we say that a spectral phase-space function $U(x,\theta)$ is of type (j,k) if, under a change of coordinates, it transforms as

$$U(x,\theta) \to \tilde{U}(\tilde{x},\tilde{\theta}) = (f_x)^{-j}(g_\theta)^{-k}U(x,\theta): \qquad U \text{ is of type } (j,k). \tag{2.16}$$

We will only consider $j,k \in \{-1,0,1\}$. Note that if $U(x,\theta)$ is of type $(0,0)$ (scalar), then $\partial_x U(x,\theta)$ is of type $(1,0)$ (spatial vector field), etc. Invariance is obtained by claiming[5]:

$$E, \Omega, \psi \quad \text{are of type } (0,0), \qquad \rho \quad \text{is of type } (1,1)\,, \tag{2.17}$$

where the statement for $\psi(x,\theta;x',\theta')$ holds independently for x,θ and x',θ'. As a consequence,

$$v^{\text{eff}} \quad \text{is of type } (-1,0), \qquad a^{\text{eff}} \quad \text{is of type } (0,-1)\,, \tag{2.18}$$

and this guarantees invariance of (2.7). Note how the momentum function P is of type $(1,0)$, and therefore P_θ is of type $(1,1)$ as it should for a density in spectral phase space (like ρ). The energy and momentum functions are not on the same footing; rather, it is the energy and phase functions that are.

A full theory of spectral phase-space reparametrisation would account for more general transformations $(\tilde{x},\tilde{\theta}) = (f(x,\theta), g(x,\theta))$, including those that do not preserve orientation. We will develop this in a separate work.

[5] Those transformation types are to be understood in terms of the invariance, covariance, and contravariance of multilinear algebra. To avoid confusion, in definition (2.16), it is important to note that the function U is changed to the new function $\tilde{U}$ under reparametrisation; for instance, even if $\Omega = x\theta$ in Lieb-Liniger (c.f. the data (2.13)), Ω is still of type $(0,0)$.

2.6. Dressing and Riemann Invariants

It is possible to rewrite the spatially extended GHD equation in terms of a "continuum of Riemann invariants"—the occupation function $n(x,\theta,t)$—using dressing operations, as in conventional GHD. For this purpose, we need to define dressing operations. As was first realised in [32], the dressing operation acts differently on objects that transform differently. We generalise these concepts to spatially extended GHD as follows. We define the occupation function as

$$n(x,\theta) = \frac{\rho(x,\theta)}{\rho_{\rm s}(x,\theta)},$$
$$2\pi\rho_{\rm s}(x,\theta) = \Omega_{x\theta} + \int_{\mathbb{R}^2} {\rm d}x'{\rm d}\theta'\,\rho(x',\theta')\psi_{x\theta}(x,\theta;x',\theta'). \tag{2.19}$$

It is clear that n is a scalar:

$$n \quad \text{is of type } (0,0). \tag{2.20}$$

A function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ is dressed, $f \mapsto f^{\rm dr}$, by solving the following linear integral equation (an inhomogeneous Fredholm equation of the second kind), which depends on the transformation type of the function f:

$$f^{\rm dr}(x,\theta) = f(x,\theta) + \int_{\mathbb{R}^2} \frac{{\rm d}x'{\rm d}\theta'}{2\pi}\,\psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')\,n(x',\theta')\,f^{\rm dr}(x',\theta') \tag{2.21}$$

where $\tilde{x} = x$ or $-x'$ and $\tilde{\theta} = \theta$ or $-\theta'$ as per the transformation type of f:

$$\psi_{\tilde{x}\tilde{\theta}} = \begin{cases} \psi_{x'\theta'} & (f \text{ is of type } (0,0)) \\ -\psi_{x'\theta} & (f \text{ is of type } (0,1)) \\ -\psi_{x\theta'} & (f \text{ is of type } (1,0)) \\ \psi_{x\theta} & (f \text{ is of type } (1,1)). \end{cases} \tag{2.22}$$

This dressing operation does not apply on functions with other transformation types. See "Appendix B" where we show that in the context of Eq. (2.3) the occupation function and the dressing operation are well-defined. With this definition, we see that the dressing preserves the transformation type:

$$f \text{ is of type } (j,k) \quad \Rightarrow \quad f^{\rm dr} \text{ is of type } (j,k). \tag{2.23}$$

By our general requirements Eq. (2.6) on the interaction kernel ψ, it also maps

$$^{\rm dr}: \quad \mathcal{C}^\infty(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^2), \tag{2.24}$$

(at least in the context of Eq. (2.3)—see "Appendix B)". Further, if $\psi(x,\theta;x',\theta') = \psi(x-x',\theta-\theta')$, then the dressing operation does not depend on the transformation type. Combining the definition of dressing with (2.19), we find

$$\rho_{\rm s}(x,\theta) = \frac{\Omega_{x\theta}^{\rm dr}}{2\pi}, \tag{2.25}$$

that is $\rho_{\rm s}$ is the interaction-dependent spectral phase-space state density. Defining

$$\Psi = \frac{\psi}{2\pi}, \tag{2.26}$$

the dressing can be written in integral-operator form as

$$f^{\rm dr} = (1 - \partial_{\tilde{x}}\partial_{\tilde{\theta}}\Psi\, n)^{-1} f. \tag{2.27}$$

It also satisfies the symmetry relation [4]

$$\int_{\mathbb{R}^2} {\rm d}x{\rm d}\theta\, n f g^{\rm dr} = \int_{\mathbb{R}^2} {\rm d}x{\rm d}\theta\, n f^{\rm dr} g\,, \tag{2.28}$$

for any f, g such that the sum of their types is $(1,1)$ (the resulting integral is a scalar). Note how, in this symmetry relation, the dressing may act differently on f and g, as they have different transformation types.

We observe that (2.8) simply expresses the fact that $2\pi\rho_{\rm s}v^{\rm eff}$ is the dressing of E_θ (both of type $(0,1)$ as it should), and (2.9) expresses the fact that $2\pi\rho_{\rm s}a^{\rm eff}$ is the dressing of $-E_x$ (both of type $(1,0)$):

$$v^{\rm eff} = \frac{E_\theta^{\ \rm dr}}{\Omega_{x\theta}^{\ \rm dr}}, \quad a^{\rm eff} = -\frac{E_x^{\ \rm dr}}{\Omega_{x\theta}^{\ \rm dr}}. \tag{2.29}$$

In the context of "Appendix B" we show that, under certain technical conditions, integral equations (2.8) and (2.9) indeed have unique solutions for $\rho \in \mathcal{M}$.

Much like in conventional GHD, we can show that $n(x,\theta)$ diagonalises the spatially extended GHD Eq. (2.7)

$$\partial_t n + v^{\rm eff}\partial_x n + a^{\rm eff}\partial_\theta n = 0. \tag{2.30}$$

The derivation of this important fact, which generalises the usual one to the spatially extended case and to the transformation-dependent dressing operation as defined above, is provided in "Appendix C". Hence, we can claim that $n(x,\theta)$ can be interpreted as a continuum of Riemann invariants (or normal modes). With these definitions, we are now ready to establish the Hamiltonian structure of spatially extended GHD.

Remark 2.2. In many situations, such as in the GHD of quantum models, ρ is non-negative, and $\rho_{\rm s}$ and $\Omega_{x\theta}$ are strictly positive. Then n, from Eq. (2.19), is non-negative and upper bounded. But in classical systems, as mentioned in the example of the KdV soliton gas description, ρ and $\Omega_{x\theta}$ may become negative, and further ρ_s might vanish and n diverge. Physical quantities and the present construction still make sense; see for example discussions regarding the so-called condensate limit of the KdV [34] and NLS [35] soliton gas (with dictionary [8]). Note also that condition (2.3), along with the assumptions in "Appendix B", are sufficient for the dressing to be well-defined, but not necessary; see for instance the analyses in [31,36]. Regarding classical systems in particular, it is easy to see that, for any smooth function f, if the interaction kernel is integrable against smooth functions, $f^{\rm dr}$ vanishes as $n \to \infty$. Then $\tilde{f} := f^{\rm dr}/n$ remains smooth even as $n \to \infty$. For instance, when it comes to the KdV soliton gas in the condensate limit mentioned earlier, $\tilde{f}$ simply is the inverse Hilbert transform of f_θ/π [34]. In particular, the Poisson bracket Eq. (3.3) below remains well-defined even as $n \to \infty$, since it involves the product of n with a dressed function.

Remark 2.3. As hinted in the Introduction, in particular in Eqs. (1.3) and (1.5), to define a Poisson bracket it will be convenient to introduce the dressing of distributions, which would appear to go against definition (2.24). However, dressed distributions are to be interpreted in terms of their effect when integrated against a compactly supported smooth test function g; we will further comment on this aspect in Appendix B.

3. Hamiltonian Formulation and Aspects of Integrability

In this section we establish a Hamiltonian formulation for the GHD equation; this is done for its most general (spatially extended) form, but immediately holds, by specialisation (2.10), to conventional GHD as well. We take a rather direct approach, inspired by Faddeev and Takhtajan's [3], and define a Poisson structure on the algebra of observables on the dynamical space $\mathcal{M}$.

3.1. Algebra of Observables

For definiteness, we define the algebra of observables $\mathfrak{U}$ on the dynamical phase space as the algebra of real-valued functionals on $\mathcal{M}$ of the form

$$F[\rho] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{2n}} c_n(x_1, \cdots, x_n; \theta_1, \cdots, \theta_n) \prod_{i=1}^{n} \rho(x_i, \theta_i) \mathrm{d}x_i \mathrm{d}\theta_i \ . \tag{3.1}$$

Here $c_0 \in \mathbb{R}$ and, given a particular dynamical space $\mathcal{M}$, c_n, $n \geq 1$ are elements of an appropriate space of distributions $\mathcal{D}_{\mathcal{M}}(\mathbb{R}^{2n})$ such that every integral in (3.1) exists. For instance, in the context of Eq. (2.3), $\mathcal{D}_{\mathcal{M}}(\mathbb{R}^{2n})$ is the space of distributions and notably includes smooth functions, while if $\mathcal{M}$ is a space of Schwartz functions, $\mathcal{D}_{\mathcal{M}}(\mathbb{R}^{2n})$ is instead the space of tempered distributions. We further require that the series (3.1) be convergent (see "Appendix E.1" for more information). Without loss of generality, we assume the c_n's to be symmetric under permutations:

$$c_n(x_1, \cdots, x_n; \theta_1, \cdots, \theta_n) = c_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; \theta_{\sigma(1)}, \cdots, \theta_{\sigma(n)})$$

for any permutation σ on n elements. For a functional $F[\rho]$, we denote by

$$F'[\rho](x, \theta) = \frac{\delta F[\rho]}{\delta \rho(x, \theta)} = \sum_{n=1}^{\infty} n \int_{\mathbb{R}^{2(n-1)}} c_n(x, x_1 \cdots, x_{n-1}; \theta, \theta_1 \cdots, \theta_{n-1}) \prod_{i=1}^{n-1} \rho(x_i, \theta_i) \mathrm{d}x_i \mathrm{d}\theta_i \tag{3.2}$$

its functional derivative, so that $F[\rho + \epsilon] - F[\rho] = \int \mathrm{d}x \mathrm{d}\theta \, F'[\rho](x, \theta) \epsilon(x, \theta) + O(\epsilon^2)$ (higher-order functional derivatives are defined analogously). Note that $F'[\rho](x, \theta)$ is of type $(0, 0)$ (it is a scalar): this can be seen by the fact that the infinitesimal perturbation $\epsilon(x, \theta)$ must be a phase-space density, of type $(1, 1)$. The algebra $\mathfrak{U}$ includes polynomials in ρ—such series where only a finite number of c_n's are nonzero. Given the form (3.1), the algebra product is naturally obtained from the Cauchy product of infinite series; see "Appendix E.1" for a partial analysis of $\mathfrak{U}$.

This choice of functionals $F[\rho]$, written as series that do not explicitly involve derivatives of the field ρ, is inspired by the theory of *hydrodynamic-type systems* (in the sense of Lax [37]) and comes from the notion of *functionals of hydrodynamic type* that we will recall in Sect. 4. Note however that, by choosing c_n to be total derivatives in some variables, integration by parts brings derivatives of ρ in (3.1). Thus the absence of derivatives of the field is not as big as a constraint as it may initially appear, but would become stronger when considering dynamical spectral phase-space Λ.

Naturally, one may also extend the set of observables to all functionals of ρ that are "smooth" on an appropriate space of functions (using the theory of Fréchet differentiability or more general frameworks, see e.g. [38]), instead of considering explicit power series (3.1). It would be interesting to develop this in future works.

3.2. Definition of the Poisson Bracket

We may now define a Poisson structure on the algebra of observables $\mathfrak{U}$. Let $F[\rho]$ and $G[\rho]$ be two observables, we propose the following Poisson bracket:

$$\{F, G\} = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n\, \left[F'_x\, (G'_\theta)^{\mathrm{dr}} - G'_x\, (F'_\theta)^{\mathrm{dr}}\right]. \tag{3.3}$$

Here we kept the dependences on ρ and on (x, θ) implicit and used the shorthand notation $f_x(x,\theta) = \partial_x f(x,\theta)$ and $f_\theta(x,\theta) = \partial_\theta f(x,\theta)$, as above. Note how the integrand is of type $(1, 1)$; thus, the integral is well-defined. The Poisson bracket (3.3) is clearly anti-symmetric and bilinear. We show in "Appendix E" that it satisfies the Leibniz rule and the Jacobi identity. Hence, this is a well-defined Poisson bracket which can be used to define Hamiltonian flows as in classical field theory.

In general the Poisson bracket maps linear functionals to nonlinear functionals. Another important aspect of the Poisson bracket (3.3) is the fact that the Poisson structure it defines is degenerate, as is the case for example with the KdV Poisson structure [39]. For example, here the annihilator contains the admissible observable

$$\mathcal{N} = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\ \rho(x,\theta)\ , \tag{3.4}$$

which Poisson-commutes with all observables since $\mathcal{N}'_x = \mathcal{N}'_\theta = 0$. Note that $\mathcal{N}$ is not the only Casimir invariant of the Poisson bracket (3.3). There are, in fact, infinitely many, as we shall discuss in Sect. 3.5.

For free systems, where $\psi = 0$, the Poisson bracket simplifies

$$\{F, G\} = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{\Omega_{x\theta}}\, \rho \left(F'_x\, G'_\theta - G'_x\, F'_\theta\right) \qquad \text{(free)}. \tag{3.5}$$

This bracket, or rather its normalised version in terms of the normal density (3.12) defined below which amounts to setting the spectral volume density $\Omega_{x\theta}$ to 1, will be discussed in Sect. 5.2 in relation to brackets of hydrodynamic type.

For any real smooth function $f(x,\theta)$ on $\mathbb{R}^2$, we denote the associated *linear functional* as

$$Q_f = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, f(x,\theta)\rho(x,\theta) \in \mathfrak{U}. \tag{3.6}$$

This is the total charge for the quantity that takes value $f(x,\theta)$ on every quasi-particle[6]. The Poisson bracket of total charges is

$$\{Q_f, Q_g\} = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n\left(f_x {g_\theta}^{\mathrm{dr}} - g_x {f_\theta}^{\mathrm{dr}}\right). \tag{3.7}$$

Rewriting the result in terms of the dynamical variables themselves in distributional form, this gives (1.5). In the case of conventional GHD, Eq. (2.10), the spatial part factorises out of the dressing, and we obtain (1.3).

In the free case (3.5), the Poisson bracket maps linear functionals to linear functionals, and we have simply

$$\{\rho(x,\theta), \rho(x',\theta')\} = \delta'(x-x')\delta'(\theta-\theta')\left(\frac{\rho(x,\theta')}{\Omega_{x\theta'}(x,\theta')} - \frac{\rho(x',\theta)}{\Omega_{x'\theta}(x',\theta)}\right) \quad \text{(free)}. \tag{3.8}$$

Note how this equation is indeed invariant under reparametrisations (2.16) as ρ is of type $(1,1)$. In the case of flat volume element $\Omega_{x\theta} = 1$, the Poisson bracket (3.8) also arises from the classical mechanics of N uninteracting particles, for the empirical density

$$\begin{aligned} \rho(x,\theta) &= \sum_{i=1}^{N} \delta(x-x_i)\delta(\theta-\theta_i), \\ \{x_i, \theta_j\} &= \delta_{ij} \quad \text{(classical mechanics, } \Omega_{x\theta} = 1). \end{aligned} \tag{3.9}$$

In fact, the Poisson bracket (1.5) arises in a similar way from the classical particle models discussed in Appendix A.

3.3. Linearisation of the Poisson Bracket

In general, the Poisson bracket simplifies when expressed in terms of appropriate "normal" densities. Following and adapting [24], we define a new coordinate[7] $y = Y(x,\theta)$ as

$$\mathrm{d}y = \Omega_{x\theta}^{\mathrm{dr}}\mathrm{d}x = 2\pi\rho_{\mathrm{s}}\,\mathrm{d}x \qquad (\theta \text{ fixed, arbitrary}). \tag{3.10}$$

[6]For instance, if the underlying system is a quantum Bethe ansatz integrable system and $f(x,\theta) = f(\theta)$ is independent of x, the associated operator $\hat{Q}_f$ has one-particle eigenvalues $\hat{Q}_f|\theta\rangle = f(\theta)|\theta\rangle$. If the underlying system is a soliton gas for a field $u(x,t)$, $\hat{Q}_f[u(\cdot,t)]$ represents one of the conserved quantities (mass, momentum, energy, etc.) associated with the underlying microscopic model (KdV, NLS, Boussinesq, etc.); then, given the single-soliton solution with parameter θ, $u_\theta(x,t)$, $f(\theta)$ is obtained as $\hat{Q}_f[u_\theta(\cdot,t)] = f(\theta)$ (e.g. in the case of the KdV equation, the mass is defined as $\int_{\mathbb{R}} \mathrm{d}x\, u(x,t)$, such that the associated function is $f(\theta) = \int_{\mathbb{R}} \mathrm{d}x\, u_\theta(x,t) = \theta$).

[7]This is slightly different from that defined in [24], where the factor was instead $2\pi\rho_{\mathrm{s}}/P_\theta$ in order to keep transparent transformation properties. The present transformation is however more convenient and more general.

The explicit solution can be obtained in terms of a different dressing operation, denoted $^{\mathrm{dR}}$:

$$Y(x,\theta) = \Omega_\theta{}^{\mathrm{d}R}(x,\theta)\,, \tag{3.11}$$

see "Appendix D". This change of coordinate transforms the x direction in order to "trivialise" the volume element $P_\theta{}^{\mathrm{dr}}\mathrm{d}x\mathrm{d}\theta = \Omega_{x\theta}^{\mathrm{dr}}\mathrm{d}x\mathrm{d}\theta = \mathrm{d}y\mathrm{d}\theta$. With interaction, the volume element depends on the fluid density ρ, because of the dressing. Hence this is a coordinate transformation that depends on the dynamical variable itself. Without interaction, the transformation does not depend on the dynamical variable, but it still is a non-identity transformation in general: then it simply trivialises the fixed volume element $\Omega_{x\theta}\mathrm{d}x\mathrm{d}\theta$.

We define the *normal density* as the density $\hat\rho(y,\theta)$ for this coordinate. It is simply related to the occupation function (which are the normal modes for the fluid, as explained above): given the Definition (2.19) and the identity (2.25) we have

$$\hat\rho(y,\theta)\,\mathrm{d}y = \rho(x,\theta)\,\mathrm{d}x \quad\Rightarrow\quad \hat\rho(Y(x,\theta),\theta) = \frac{n(x,\theta)}{2\pi}. \tag{3.12}$$

Note that the normal density is in fact a *scalar* in x,θ (type $(0,0)$), while the coordinate $Y(x,\theta)$ is scalar in x and a spectral vector field (type $(0,1)$). Thus, under transformations of the spatial coordinate x, the y coordinate and normal density are unchanged—this is because we have chosen them specifically in order to trivialise the volume element. We show in Appendix E that in terms of the normal density, the Poisson bracket (3.3) *takes the free form* (3.8) with unit spectral volume density,

$$\{\hat\rho(y,\theta),\hat\rho(y',\theta')\} = \delta'(y-y')\delta'(\theta-\theta')\left(\hat\rho(y,\theta')-\hat\rho(y',\theta)\right)\quad\text{(linearised)}. \tag{3.13}$$

Clearly, this equation is not invariant under reparametrisation, as, again, we have chosen a given, convenient coordinate system.

Remark 3.1. If $\Omega_{x\theta}>0$ (as is the case for the delta-Bose and hard rods gases), then in the context of "Appendix B", $\Omega_{x\theta}^{\mathrm{dr}}>0$. Then the change of coordinates (3.10) is orientation-preserving. If $\inf_{x\in\mathbb{R}}\Omega_{x\theta}^{\mathrm{dr}}>0$ then the set of all values of y is $\hat{\mathscr{L}}_\theta=\mathbb{R}$ (this is the typical case); otherwise it may be an open subset $\hat{\mathscr{L}}_\theta\subset\mathbb{R}$, which may depend on θ. If $\Omega_{x\theta}^{\mathrm{dr}}$ is not strictly positive, then the resulting space $\hat{\mathscr{L}}_\theta$ in which y lies is not $\mathbb{R}$, but an open subset of a multiple cover of $\mathbb{R}$. In all examples of interest, this space is *independent of the fluid density* $\rho(\cdot,\cdot)$, and we will assume so here (if this were not so, new terms would arise, since partial derivatives with respect to x, y, or θ would also act on the bounds of integration in the y space, and our derivations in "Appendix E" would fail). It is by considering $Y(x,\theta)\in\hat{\mathscr{L}}_\theta$ that (3.12) holds in general; that is

$$Y(\cdot,\theta):\mathbb{R}\to\hat{\mathscr{L}}_\theta. \tag{3.14}$$

Remark 3.2. The change of variables (3.10) is not the only one to linearise the Poisson bracket (3.3). Indeed one could keep the physical space as it is and introduce a new coordinate $p = \kappa(x,\theta)$ in order to contract the spectral space instead

$$\mathrm{d}p = \Omega_{x\theta}^{\mathrm{dr}}\mathrm{d}\theta = 2\pi\rho_{\mathrm{s}}\,\mathrm{d}\theta \qquad (x \text{ fixed, arbitrary}). \tag{3.15}$$

In this case, an explicit solutions can be obtained in terms of yet another dressing operation (c.f. "Appendix D"), and the result is the "physical momentum" commonly discussed in the Thermodynamic Bethe Ansatz [40], the momentum the system gains upon adding a quasi-particle of spectral parameter θ at x,

$$\kappa(x,\theta) = {\Omega_x}^{\mathrm{Dr}}(x,\theta). \tag{3.16}$$

Defining an alternative normal density $\tilde{\rho}(x,p)$ as

$$\tilde{\rho}(x,p)\mathrm{d}p = \rho(x,\theta)\mathrm{d}\theta, \tag{3.17}$$

the Poisson bracket (3.3) again takes the free from (3.5)

$$\{\tilde{\rho}(x,p),\tilde{\rho}(x',p')\} = \delta'(x-x')\delta'(p-p')\left(\tilde{\rho}(x,p') - \tilde{\rho}(x',p)\right). \tag{3.18}$$

This follows immediately from the "spectral crossing symmetry" of our construction, the structural symmetry under exchange of spatial and spectral variables, x and θ. We chose to highlight the change of variables (3.10) as it is more standard when it comes to GHD, since it trivialises the force-less GHD equation in which $E(x,\theta) = E(\theta)$ (thus $a^{\mathrm{eff}}(x,\theta) = 0$) [24]. However, the alternative transformation (3.15) also trivialises the GHD equations, this time when $E(x,\theta) = E(x)$ (thus $v^{\mathrm{eff}}(x,\theta) = 0$), a situation which has only been marginally considered. A priori, there is no reason why one should restrict oneself to only contracting either physical or spectral space, and there may be a transformation involving both that trivialises the more general GHD equation. Since this would probably require a more general theory of reparametrisation than the one discussed in Sect. 2.5, we leave this for future work.

3.4. Hamiltonian Flows

A Hamiltonian H is a functional of the fluid density ρ, $H \in \mathfrak{U}$, and, in physics, the Hamiltonian is often associated with the energy of the system of interest. Given the phase-space energy function $E(x,\theta)$, the total energy of a system in which particles are distributed according to the fluid density $\rho(x,\theta)$ is simply the linear functional Q_E. One of our main results is that the intuitive Hamiltonian

$$H = Q_E = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, E(x,\theta)\rho(x,\theta), \tag{3.19}$$

is indeed the Hamiltonian whose flow with respect to the Poisson structure (3.3) gives the appropriate GHD equation, given the external force field described by $E(x,\theta)$.

Let Q_f be a linear functional of type (3.6). From the specialised Poisson bracket (3.7), by using the symmetry relation (2.28) and integration by parts, we may write

$$\{Q_f, Q_E\} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta \left(- f\partial_x(n{E_\theta}^{\mathrm{dr}}) + f\partial_\theta(n{E_x}^{\mathrm{dr}})\right), \tag{3.20}$$

and, in particular, if $Q_f = Q_\delta = \rho$, this yields

$$\{\rho(x,\theta), Q_E\} + \partial_x(v^{\mathrm{eff}}\rho) + \partial_\theta(a^{\mathrm{eff}}\rho) = 0, \tag{3.21}$$

where we have used the relations (2.29), along with the definition of the occupation function (2.19), which we recall here for convenience: $v^{\mathrm{eff}} = E_\theta{}^{\mathrm{dr}}/\Omega_{x\theta}^{\mathrm{dr}}$, $a^{\mathrm{eff}} = -E_x{}^{\mathrm{dr}}/\Omega_{x\theta}^{\mathrm{dr}}$, and $n = 2\pi\rho/\Omega_{x\theta}^{\mathrm{dr}}$. With $\partial_t\rho(x,\theta) = \{\rho(x,\theta), Q_E\}$, this indeed *yields the spatially extended GHD equation* (2.7) *as a Hamiltonian system*:

$$\partial_t\rho(x,\theta) = \{\rho(x,\theta), Q_E\} = -\partial_x(v^{\mathrm{eff}}\rho) - \partial_\theta(a^{\mathrm{eff}}\rho). \tag{3.22}$$

As such, thanks to the Poisson structure (3.3), GHD can be seen as a classical Hamiltonian field theory in two dimensions (in spectral phase space x, θ). This is the main result of this paper.

From this result, it is immediate to recover the result of [24], generalised here to inhomogeneous momenta and spatially extended interaction kernel: that, in the force-less case $E_x = 0$, the normal density $\hat\rho(y,\theta)$ satisfies the Liouville equation. Indeed, in the force-less case, using (3.12), the Hamiltonian is written as

$$Q_E = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, E(\theta)\rho(x,\theta) = \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, E(\theta)\hat\rho(y,\theta) \tag{3.23}$$

and hence, the linearised Poisson bracket (3.13) with $\partial_t\hat\rho(y,\theta) = \{\hat\rho(y,\theta), Q_E\}$ gives

$$\partial_t\hat\rho(y,\theta) + E_\theta\partial_y\hat\rho(y,\theta) = 0 \qquad (E_x = 0). \tag{3.24}$$

Likewise, according to Remark 3.2, in the case $E_\theta = 0$ (the “kinetic-less” case), the density $\tilde\rho(x,p)$ satisfies

$$\partial_t\tilde\rho(x,p) - E_x\partial_p\tilde\rho(x,p) = 0 \qquad (E_\theta = 0). \tag{3.25}$$

This last case is mostly formal and, a priori, not (yet) physically relevant. Rather, we mention it to highlight the symmetry between x and θ in our construction, which naturally appears as we introduce spatial inhomogeneity in the interaction kernel ψ.

Clearly, any linear functional Q_h of type (3.6) generates its own GHD-like flow. If $E_x = 0$, then any linear functional Q_h such that $h_x = 0$ are conserved quantities of the GHD equations and are in involution with respect to the (specialised) Poisson bracket (3.7)

$$\{Q_h, Q_E\} = 0 \qquad (E_x = 0,\ h_x = 0). \tag{3.26}$$

This corresponds to the case of conventional GHD describing the evolution of an integrable many-body system without external potential. From the point of view of the microscopic model underlying the GHD equation, the functionals Q_h are the higher conserved charges of the hierarchy, i.e. these are the commuting flows of the hierarchy. Similarly, if $E_\theta = 0$, linear functionals Q_g such that $g_\theta = 0$ are, again, conserved and in involution

$$\{Q_g, Q_E\} = 0 \qquad (E_\theta = 0,\ g_\theta = 0). \tag{3.27}$$

From the point of view of the microscopic model, these flows correspond, this time, to dynamics generated by Hamiltonians that are inhomogeneous in space, but that couple to a single conserved quantity of the hierarchy, which is the total particle density.

In both cases above, we see that the GHD equation admits a large number of conserved quantities in involution, which are extensive in phase space. However, this is not sufficient to claim integrability: as GHD is a two-dimensional field theory, we would expect a family of conserved quantities specified by functions of two variables, in effect fixing "every point" of phase space. We will now discuss this.

Remark 3.3. Note that the Liouville equation (3.24) we obtain is slightly different from the result of [24] (assuming the condition (2.3) is met)

$$\partial_t \hat{\rho} + v^{\mathrm{gr}}(\theta)\partial_y \hat{\rho} = 0\ , \tag{3.28}$$

where $v^{\mathrm{gr}} = E_\theta/\Omega_{x\theta}$. This is because the change of metric introduced in [24] to trivialise the conventional GHD equation is not the one defined in Eq. (3.10); rather, it is

$$\mathrm{d}y = \frac{\Omega^{\mathrm{dr}}_{x\theta}}{\Omega_{x\theta}}\mathrm{d}x. \tag{3.29}$$

As such, in the context of [24], in the absence of interaction we have $\mathrm{d}y = \mathrm{d}x$, which is not the case in this manuscript, as remarked upon in Sect. 3.3, following Eq. (3.11).

3.5. Extensive Conserved Quantities in Involution

It is known (see e.g. [41] Appendix C[8]) that, for general force field $E(x,\theta)$, given any continuously differentiable function of the normal modes $f(n)$, functionals of the type

$$\tilde{Q}_f = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, f(n(x,\theta))\rho_{\mathrm{s}}(x,\theta), \tag{3.30}$$

are invariant under the GHD evolution

$$\{\tilde{Q}_f, Q_E\} = 0. \tag{3.31}$$

Indeed, this follows immediately from the GHD equation (2.7), its diagonalisation or normal mode decomposition (2.30), and the definition of the occupation function (2.19). This includes, in particular, the entropy function of the underlying microscopic model, conserved under the GHD equation [11] (for instance for classical particle systems, one chooses $f(n) = -n[\log n - 1]$), and the total number of particles ($f(n) = n$, which is $\tilde{Q}_f = Q_1$).

Given the result (3.31), it is now clear that, in both cases considered previously, (3.26) and (3.27), there are more conserved quantities besides those

[8]To make upcoming computations simpler, we use a definition (3.30) of $\tilde{Q}_f$ which is slightly different from that of [41] Eq. (17), where our f corresponds to their f divided by n.

written there. Indeed, if $E_x = 0$, given any function of two variables $f(n,\theta)$ the quantity

$$\tilde{Q}_f = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, f(n(x,\theta),\theta)\rho_{\mathrm{s}}(x,\theta) \qquad (\text{case } E_x = 0), \tag{3.32}$$

is invariant under the GHD evolution. This includes all the quantities previously highlighted by (3.26), with $\tilde{Q}_f = Q_h$ for $f(n,\theta) = nh(\theta)$, and notably the Hamiltonian with $h(\theta) = E(\theta)$. Moreover, if instead $E_\theta = 0$, given any function $f(x,n)$ the quantity

$$\tilde{Q}_f = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, f(x,n(x,\theta))\rho_{\mathrm{s}}(x,\theta) \qquad (\text{case } E_\theta = 0), \tag{3.33}$$

is invariant under the GHD evolution. This includes those highlighted by (3.27), with $\tilde{Q}_f = Q_g$ for $f(x,n) = ng(x)$, and, once again, the Hamiltonian with $g(x) = E(x)$.

In general, quantities $\tilde{Q}_f$ of type (3.30) are nonlinear functionals of ρ. In fact, if the interaction kernel has finite range, short range, or long range with a quick enough algebraic decay (see Remark 2.1), then they are "extensive", with a density $f(n(x,\theta))\rho_{\mathrm{s}}(x,\theta)$ "supported in a neighbourhood of (x,θ)": it depends only on the dynamical variable $\rho(x',\theta')$ at spectral phase-space points (x',θ') near to (x,θ). Extensivity of conserved quantities is known to be an important concept in many-body physics. However, it is not clear to us how much it plays a role in the present construction, and in some examples (hard rods, the KdV equation) the interaction appears to be of infinite range in spectral space. Notwithstanding this, in the general case, the GHD equation possesses an infinite number of (extensive) conserved quantities parametrised by a function of a single variable, while in the special cases $E_x = 0$ and $E_\theta = 0$, they are parametrised by a function of two variables.

The above discussion suggests that GHD, seen as a classical field theory, may be integrable. However, to claim the GHD equation is integrable (in some accepted field theoretic version of Liouville theorem that has been established for several $(1+1)-$dimensional systems, see [3]), another crucial point is that the *conserved quantities must be non-trivially in involution, that is not Casimirs.* This last point is particularly important. It highlights the fact that conservation of functionals of type (3.30) cannot be used as an argument to suggest integrability of the GHD equation since they are, in fact, Casimir elements of our Poisson bracket (3.3). This can be easily checked by using the change of coordinates (3.10) which linearises the Poisson bracket; then, Eq. (3.30) becomes

$$\tilde{Q}_f = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, f(2\pi\hat{\rho}(y,\theta)), \tag{3.34}$$

where the space $\hat{\mathcal{L}}_\theta$ has been introduced in Remark 3.1. Note that in terms of $\hat{\rho}(y,\theta)$, these are now truly extensive, with an ultra-local density. Hence, we have

$$\frac{\delta \tilde{Q}_f}{\delta \hat{\rho}(y,\theta)} = f_n(2\pi\hat{\rho}(y,\theta)), \tag{3.35}$$

where as usual we use the index notation for derivatives, with in particular $f_n(n,\theta) = \partial f(n,\theta)/\partial n$. Using the normal form of the Poisson bracket (3.13)—or the equivalent form (E.14) from Appendix E.3—we can therefore evaluate for any $G \in \mathfrak{U}$

$$\begin{aligned}\{\tilde{Q}_f, G\} &= \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat{\rho}\left[(f_n)_y \left(\frac{\delta G}{\delta\hat{\rho}}\right)_\theta - \left(\frac{\delta G}{\delta\hat{\rho}}\right)_y (f_n)_\theta\right] \\ &= 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \frac{\delta G}{\delta\hat{\rho}} \left[(\hat{\rho} f_{nn}\hat{\rho}_\theta)_y - (\hat{\rho} f_{nn}\hat{\rho}_y)_\theta\right] \\ &= 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \frac{\delta G}{\delta\hat{\rho}} (f_{nn} + f_{nnn}\hat{\rho})\left[\hat{\rho}_y\hat{\rho}_\theta - \hat{\rho}_\theta\hat{\rho}_y\right] = 0\end{aligned} \tag{3.36}$$

When the energy function depends both on x and θ, we were not able to find non-Casimir conserved quantities in involution and we do not expect GHD to be integrable. This is consistent with the atomic reduction of GHD (discussed in "Appendix A") which corresponds to Hamiltonian systems of particles that can not be expected to be integrable if the energy is both $x-$ and $\theta-$dependent, while they are integrable if it only depends on x or θ.

Functionals of type (3.32) (respectively, of type (3.33)) are not Casimirs and, when the energy function is such that $E_x = 0$ (respectively, $E_\theta = 0$), they are indeed in involution. In the following, we only discuss the family associated with $E_x = 0$, that is functionals of type (3.32); involution of functionals of type (3.33) follows from a similar derivation. Again, to make things simpler, we make use of the change of coordinates (3.10)

$$\begin{aligned}\tilde{Q}_f &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, f(2\pi\hat{\rho}(y,\theta),\theta)\ , \\ \frac{\delta\tilde{Q}_f}{\delta\hat{\rho}(y,\theta)} &= f_n(2\pi\hat{\rho}(y,\theta),\theta).\end{aligned} \tag{3.37}$$

Using the normal form of the Poisson bracket (3.13), we therefore obtain

$$\begin{aligned}\{\tilde{Q}_f, \tilde{Q}_g\} &= \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat{\rho}\left[(f_n)_y (g_n)_\theta - (g_n)_y (f_n)_\theta\right] \\ &= 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat{\rho}\left[f_{nn}\hat{\rho}_y(2\pi g_{nn}\hat{\rho}_\theta + g_{n\theta}) - g_{nn}\hat{\rho}_y(2\pi f_{nn}\hat{\rho}_\theta + f_{n\theta})\right], \\ &= 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat{\rho}\hat{\rho}_y,\left[f_{nn}g_{n\theta} - g_{nn}f_{n\theta}\right]\end{aligned} \tag{3.38}$$

where $(f_n)_\theta = \partial_\theta\big(f_n(2\pi\hat{\rho}(y,\theta),\theta)\big)$ while $f_{n\theta} = \partial_\theta f_n(n,\theta)\big|_{n=2\pi\hat{\rho}(y,\theta)}$, etc. Note that $f_{nn}g_{n\theta} - g_{nn}f_{n\theta}$ is a function of $\hat{\rho}(y,\theta)$ and θ only (that is, its y dependence comes solely from $\hat\rho(y,\theta)$); we denote it as $h(\hat\rho(y,\theta),\theta)$. Then we obtain

$$\{\tilde Q_f, \tilde Q_g\} = 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat\rho\hat\rho_y, h(\hat\rho,\theta) = 2\pi \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \partial_y j(\hat\rho,\theta), \tag{3.39}$$

where $j(\hat\rho,\theta) = \int_0^{\hat\rho} \mathrm{d}\hat\rho'\, \hat\rho' h(\hat\rho',\theta)$ is a primitive of $\hat\rho h(\hat\rho,\theta)$ with respect to $\hat\rho$ (at θ fixed). Hence the result is, using the fact that $\hat\rho(y)$ at the asymptotic boundaries of $\hat{\mathcal{L}}_\theta$ takes the values $n(\pm\infty,\theta)/2\pi = 0$ thanks to the mapping (3.14),

$$\{\tilde Q_f, \tilde Q_g\} = 2\pi \int_{\mathbb{R}} \mathrm{d}\theta\, (j(0,\theta) - j(0,\theta)) = 0. \tag{3.40}$$

This shows that, under appropriate conditions on the energy function, quantities of either type (3.32) or (3.33) are all invariant under the GHD evolution *and* in involution with respect to the Poisson bracket (3.3).

One last requirement for integrability is that the family of conserved quantities should be "large enough". Although this is not a fully accurate concept, for field theories lying in two dimensions of space—here spectral phase space (x,θ)— intuitively one would need a family of conserved quantities that explores the full two dimensions. In the case $E_x = 0$, for instance, one may see θ as parametrising an infinite set of coupled equations (the GHD equations at fixed θ) for fields lying in one dimension of space x. Then, one may use $f(n,\theta) = f(n)\delta(\theta-\theta_0)$ to construct functionals $\tilde Q_f$ of type (3.32), which provide a full family of extensive conserved quantities for each parameter θ_0; indeed, the family appears to be large enough. Such an approach has in fact been considered before, we will discuss it in Sect. 4, and we shall see we indeed have additional reasons to presume the GHD equation is integrable in either cases $E_x = 0$ or $E_\theta = 0$.

4. Overview of the Previously Known Structures of the GHD Equation

Our work is not the first one to deal with the Hamiltonian structure and/or integrability of the GHD equations. However, in most cases, previous works have not dealt with the GHD equations directly, focusing instead on particular reductions. Moreover, their language was often not that of GHD, but that of differential geometry. In this section we provide a brief overview of the state of the art, showcasing previous known results regarding reductions of the GHD equations and introducing the theoretical framework in which they were derived, in order to better highlight what our construction brings to the table.

4.1. Reduced GHD Equations as Systems of Hydrodynamic Type

Investigation into the integrability of the GHD equations (2.7–2.8–2.9) can be traced back to the work of El, Kamchatnov, Pavlov, and Zykov in [19]. In that work, the authors studied the so-called kinetic equations of the KdV soliton gas, i.e. a special case of the conventional and homogeneous GHD equation: $E(x,\theta) = E(\theta)$, $P(x,\theta) = P(\theta)$, $a^{\text{eff}}(x,\theta) = 0$, with the choice of data given by (2.10) and (2.15). They established integrability of the kinetic equations via generalised hodograph transform [42] under the "cold gas" (a.k.a "polychromatic gas" [43]) reduction

$$\rho(x,\theta,t) = \sum_{i=1}^{m} \rho^i(x,t)\delta(\theta - \theta^i), \tag{4.1}$$

which generically transforms the GHD equations into an m-component, linearly degenerate, hyperbolic system of hydrodynamic type for the propagation of the weights $\{\rho^i\}_{i=1}^m$ according to their effective velocities $v^i(x,t) \equiv v^{\text{eff}}(x,\theta^i,t)$. In the particular case they investigated, the resulting system takes the form

$$\partial_t \rho^i + \partial_x \left(v^i \rho^i\right) = 0, \tag{4.2}$$

$$v^i = 4(\theta^i)^2 + \frac{1}{\theta^i} \sum_{k\neq i} \log\left|\frac{\theta^i - \theta^k}{\theta^i + \theta^k}\right| (v^i - v^k). \tag{4.3}$$

It is important to note that the resulting system admits a Riemann invariants representation

$$\partial_t r^i + v^i \partial_x r^i = 0, \tag{4.4}$$

$$r^i = \frac{1}{\rho^i}\left(1 + \sum_{k\neq i} \frac{1}{\theta^i} \log\left|\frac{\theta^i - \theta^k}{\theta^i + \theta^k}\right| \rho^k\right) = \frac{\pi}{4\theta^i n(x,\theta^i,t)}, \tag{4.5}$$

the GHD analogue of which is the "diagonalised" Eq. (2.30).[9] This aspect is crucial in establishing integrability: to show that a diagonal system of hydrodynamic type is integrable it is sufficient to show that it is either Hamiltonian or semi-Hamiltonian [44,45]. We will now discuss those two notions and introduce the framework through which reduced GHD equations have been studied.

Remark 4.1. We highlight the fact that the effective velocities v_i in the system of reduced GHD equations (4.2) are also the characteristic velocities that appear in the Riemann invariants representation (4.4). This is generically not the case when it comes to systems of hydrodynamic type but is a specificity of reduced GHD equations. The equivalent situation occurs with the effective velocity and acceleration in the full GHD equation, see (2.7), (2.30). In the force-less case of conventional GHD, this plays an important role in [46].

4.2. Integrable Systems of Hydrodynamic Type

The theory of Hamiltonian and semi-Hamiltonian systems of hydrodynamic type is well-developed, we provide here some basic elements and, for more information, we refer the interested reader to the reviews [42,47]. In this section we consider an m-component system of hydrodynamic type

$$\partial_t \rho^i + \partial_x \left[v^i(\rho^1, \cdots \rho^m)\rho^i\right] = 0 \quad i = 1, \cdots m, \tag{4.6}$$

---

[9] Note that, given the transport equation (4.4), any function of a Riemann invariant $f(r^i)$ is also a Riemann invariant. This is also the main reason why relation (3.31) holds.

similar to the reduced GHD equations (4.4) but with arbitrary velocities. Given this, an integral is said to be of hydrodynamic type if it is of form

$$I = \int_{\mathbb{R}} \mathrm{d}x \ P(\rho^1, \cdots \rho^m), \tag{4.7}$$

where the densities P are functions of the components $\{\rho^i\}_{i=1}^m$ only and do not depend on their derivatives. This notion is what informed our choice regarding the algebra of observables $\mathfrak{U}$ in the previous section. A system of hydrodynamic type is then said to be Hamiltonian if there exists an integral of hydrodynamic type H such that

$$\partial_t \rho^i = \{\rho^i, H\}, \tag{4.8}$$

where the Poisson bracket (summing over repeated indices)

$$\{I, J\} = \int_{\mathbb{R}} \mathrm{d}x \frac{\delta I}{\delta \rho^i(x)} A^{ij} \frac{\delta J}{\delta \rho^j(x)}, \tag{4.9}$$

is of Dubrovin–Novikov type [48]

$$A^{ij} = g^{ij}(\rho^1, \cdots \rho^n)\partial_x - g^{is}(\rho^1, \cdots \rho^n)\Gamma^j_{sk}(\rho_1, \cdots \rho_n)\partial_x \rho^k. \tag{4.10}$$

Here the variational derivatives are defined from

$$I[\rho^1, \cdots \rho^i + \delta\rho^i, \cdots \rho^m] - I[\rho^1, \cdots \rho^m] = \int_{\mathbb{R}} \mathrm{d}x \frac{\delta I}{\delta \rho^i(x)} \delta\rho^i(x) + o(\delta\rho^i). \tag{4.11}$$

Moreover, we require that the coefficients g_{ij} define a contravariant, flat, pseudo-Riemannian metric (notably $\det g_{ij} \neq 0$ and $g_{ij} = g_{ji}$), and that Γ^i_{jk} are Christoffel symbols of the associated Levi-Civita connection. These conditions ensure that the Poisson bracket is skew-symmetric and satisfies both the Leibniz rule and the Jacobi identity.

If a Hamiltonian system of hydrodynamic type admits a Riemann invariants representation[10]

$$\partial_t r^i + v^i \partial_x r^i = 0, \tag{4.12}$$

then, under this change of variable, the metric becomes diagonal $g^{ij}(r^1, \cdots r^m) = g^{ij}(r^1, \cdots r^m)\delta^{ij}$ and the Christoffel symbols simplify [44]

$$\Gamma^k_{ij} = 0, \quad \Gamma^i_{ij} = \partial_{r^j} \log \sqrt{g_{ii}} = \frac{\partial_{r^j} v^i}{v^j - v^i} \quad i \neq j \neq k. \tag{4.13}$$

In particular, this last expression implies the relation

$$\partial_{r^j} \frac{\partial_{r^k} v^i}{v^k - v^i} = \partial_{r^k} \frac{\partial_{r^j} v^i}{v^j - v^i}, \quad i, j = 1, \cdots m, \quad i \neq j \neq k, \tag{4.14}$$

known as the *semi-Hamiltonian* property [45]. Note that a diagonalisable system of hydrodynamic type may possess this property without being Hamiltonian, examples of this include the equations of isotachophoresis [49], ideal

[10] Again, in general, the characteristic velocities v_i are not necessarily the same as in Eq. (4.6). However, we limit ourselves to this situation since it is generically the case when it comes to polychromatic reductions of the GHD equation.

chromatography [50,51], or an extended Born-Infeld equation written in hydrodynamic form [45,52].

A semi-Hamiltonian system of hydrodynamic type is also integrable: it possesses infinitely many linearly independent integrals of hydrodynamic type that are invariant under the dynamics (4.4), are in involution and generate commuting Hamiltonian flows including the system Hamiltonian, and form a complete set in the sense of [42]. Moreover such a system is (implicitly) solvable via the generalised hodograph transform developed by Tsarev [44]

$$x + v^i t = w_i^{(\tau)}, \tag{4.15}$$

where the functions $\{w_i^{(\tau)}\}_{i=1}^m$ solve the overdetermined linear system

$$\frac{\partial_{r^j} w_i^{(\tau)}}{w_j^{(\tau)} - w_i^{(\tau)}} = \frac{\partial_{r^j} v^i}{v^j - v^i}, \quad i, j = 1, \cdots m, \quad i \neq j, \tag{4.16}$$

and specify the commuting flows

$$\partial_\tau r^i + w_i^{(\tau)} \partial_x r^i = 0, \tag{4.17}$$

for the time (group parameter) τ, such that $\partial_\tau \left(\partial_t r^i\right) = \partial_t \left(\partial_\tau r^i\right)$.

Admittedly, the generalised hodograph method may seem rather unwieldy: assuming one manages to solve the over-determined system (4.16), one would then need to invert the transform (4.15). However, there exist (relevant) circumstances, that we highlight here briefly for reference, under which this procedure simplifies. In particular, this is the case if the system is linearly degenerate

$$\partial_{r^i} v^i = 0, \quad \text{for any } i = 1, \cdots m\,, \tag{4.18}$$

a condition that also implies the system does not support classical shocks [50]. Ferapontov showed in [53] that, of the infinitely many linearly independent commuting flows possessed by a linearly degenerate semi-Hamiltonian system, only m of them are linearly degenerate as well (including the trivial ones $w_i^{(\tau_0)} = 1$ and $w_i^{(\tau_1)} = v^i$); he then provided a method to solve (4.16) by quadrature. Further simplification can also be obtained if the system belongs to the Egorov class [54], i.e. if it features a unique pair of conservation laws such that

$$\partial_t A(r^1, \cdots r^m) + \partial_x B(r^1, \cdots r^m) = 0, \quad \partial_t B(r^1, \cdots r^m) + \partial_x C(r^1, \cdots r^m) = 0, \tag{4.19}$$

which essentially expresses the fact that the system is Galilean invariant. The Egorov property implies that the metric is *potential*: there exists a function $\mathcal{G}(r^1, \cdots r^m)$ such that $g_{ii} = \partial_{r^i} \mathcal{G}$, Pavlov then proved that the potential is in fact the first conserved density $\mathcal{G}(r^1, \cdots r^m) = A(r^1, \cdots r^m)$ [55,56]. Furthermore, Egorov systems lie in the isomorphism class derived in [19] in which Riemann invariants can be computed explicitly.

4.3. Structure of GHD Equations: State of the Art

In [19] the authors showed that the cold gas reduction of the kinetic equations of the KdV soliton gas is linearly degenerate, Egorov and indeed possesses the semi-Hamiltonian property for any m. Moreover, they provided explicit solutions (in terms of the Riemann invariants $\{r^i\}_{i=1}^m$ and of the weights $\{\rho^i\}_{i=1}^m$) for $m = 3$, and they found the following solution by quadrature for arbitrary m

$$x + 4(\theta^i)^2 t = \int^{r^i} \mathrm{d}\xi \, \frac{\xi \phi^i(\xi)}{\zeta(\xi)} + \sum_{j \neq i} \frac{1}{\theta^i \theta^j} \log \left| \frac{\theta^i - \theta^j}{\theta^i + \theta^j} \right| \int^{r^j} \mathrm{d}\xi \, \frac{\phi^j(\xi)}{\zeta(\xi)} \,, \tag{4.20}$$

where the ϕ^i's and ζ play the role of functional degrees of freedom[11]. The Hamiltonian structure of the cold gas reduction was then explored in [57] and [20]. The cold gas reduction was extended in [58] to account for non-constant θ_i's, in which case the diagonal system (4.4) now takes the form

$$\partial_t r^i + v^i \partial_x r^i + p^i \partial_x \theta^i = 0, \tag{4.21}$$

where the p^i's are functions of both the Riemann invariants and the θ^i's, and are supplemented by m more equations describing the propagations of the θ^i's

$$\partial_t \theta^i + v^i \partial_x \theta^i = 0. \tag{4.22}$$

More recently it was shown in [59] that the extended system (4.21–4.22), whose matrix is composed of m 2×2 Jordan blocks, is linearly degenerate (using the same definition (4.18) as before, which does not involve the p^i's or θ^i's) and integrable via an extension of the generalised hodograph method[12]. Hamiltonian structure of this extended system (along with other ones corresponding to different models like Lieb-Liniger, sinh-Gordon, DNLS, etc.) was established in [21,22]. In addition, a Hamiltonian formulation was proposed for the (non-reduced) conventional GHD of the hard rods model, represented as an infinite integrable hydrodynamic chain (see [60] for a review of that topic) associated with the spectral moments of the fluid density ρ. We will comment on this point further in the next section.

At least when it comes to standard (conventional and homogeneous) GHD, continuum generalisations of some of the aforementioned aspects have been discussed in [4,8,20,23,24]. In particular the GHD equations are diagonalisable (for constant domain Λ), linearly degenerate

$$\frac{\delta v^{\mathrm{eff}}(\theta)}{\delta n(\theta)} = 0 \quad \forall \theta, \tag{4.23}$$

[11] For instance, quasi-periodic (finite-gap like) solutions are obtained by imposing that the ϕ^i's are polynomials of degree less than m and that $\zeta = \prod_{k=1}^K (\xi - \lambda_k)$, where the λ_k's are real constants and where $K = 2m+1$ if m is odd or $2m+2$ if it is even.

[12] Note that the Jordan block reducible equations (4.21–4.22) do not correspond to reduced versions of the spatially extended GHD equations. They rather are reduced versions of conventional GHD equations on a dynamical spectral phase space $\Lambda(x,t)$, a case we allude to in Sect. 5.1, but do not directly address in this paper.

and the existence of a self-conserved current (i.e. the Egorov property) was used in [61,62] to justify the collision rate ansatz yielding the effective velocity (2.29). Moreover, GHD equations satisfy the semi-Hamiltonian property

$$\int \mathrm{d}\nu \left[\frac{\delta}{\delta n(\nu)}\left(\frac{\delta v^{\mathrm{eff}}(\eta)/\delta n(\mu)}{v^{\mathrm{eff}}(\mu)-v^{\mathrm{eff}}(\eta)}\right)\right] = \int \mathrm{d}\mu \left[\frac{\delta}{\delta n(\mu)}\left(\frac{\delta v^{\mathrm{eff}}(\eta)/\delta n(\nu)}{v^{\mathrm{eff}}(\nu)-v^{\mathrm{eff}}(\eta)}\right)\right], \quad \mu \neq \nu \neq \eta. \tag{4.24}$$

This means we expect integrability to hold, at least when it comes to standard GHD, beyond the previously considered reductions. In fact, as we have shown in Sect. 3, even in the present case of spatially extended GHD, the constitutive system of equations (2.7–2.8–2.9) is diagonalisable *and* Hamiltonian, and in the homogeneous (force-less) case possesses a seemingly sufficiently large family of extensive conserved quantities in involution, suggesting GHD may be integrable even in its more general, spatially extended form.

Lastly, a geometric approach to conventional GHD was developed in [24], based on the change of metric (3.10). As mentioned in this change of metric trivialises the conventional GHD equation, Eq. (3.24), viz.

$$\partial_t \hat{\rho}(y,\theta,t) + v^{\mathrm{gr}}(\theta)\partial_y \hat{\rho}(y,\theta,t) = 0, \tag{4.25}$$

where we introduced the group velocity $v^{\mathrm{gr}} := E_\theta$, which is the (bare) velocity at which a quasi-particle of parameter θ moves in the absence of interactions. Importantly, the group velocity is only a function of θ and does not involve the dressing operation,[13] contrary to the effective velocity. This allows for an implicit solution by the method of characteristics, indeed

$$\hat{\rho}(y,\theta,t) = \hat{\rho}(y - v^{\mathrm{gr}}(\theta)t, \theta, 0), \tag{4.26}$$

is solution of (4.25). As such, recalling the identity (3.12), we have

$$n(x,\theta,t) = n(U(x,\theta,t),\theta,0), \tag{4.27}$$

where the function U is determined by

$$Y(U(x,\theta,t),\theta,0) = Y(x,\theta,t) - v^{\mathrm{gr}}(\theta)t, \tag{4.28}$$

or, equivalently, according to the identity (3.10)

$$2\pi\left(\int_{-\infty}^{x} \mathrm{d}z \rho_s(z,\theta,t) - \int_{-\infty}^{U(x,\theta,t)} \mathrm{d}z \rho_s(z,\theta,0)\right) = v^{\mathrm{gr}}(\theta)t. \tag{4.29}$$

Note that time only appears as a parameter in equations (4.27)–4.29), and a solution at arbitrary time can be obtained directly by solving these equations iteratively [24]; there is no need to solve the GHD equations using, for instance, a finite element method.

[13] Recall that in force-less GHD $E(x,\theta) = E(\theta)$.

4.4. Hamiltonian Structure of the Hard Rod Gas: Comparison with Vergallo–Ferapontov

While this manuscript was in preparation, Vergallo and Ferapontov (VF) posted the preprint [22], where two Hamiltonian structures for the polychromatic reductions of the GHD equation (or soliton gas kinetic equation) were constructed. Most interestingly, they took the continuous (non-reduced) limit of their construction for the special case of the hard rod gas, obtaining Hamiltonian structures for the corresponding conventional GHD equation. We now verify that one of the VF Hamiltonian structures for the hard rod gas (the one that is "local"[14]) is indeed a special case of our construction. We note that it is a non-trivial matter to take the continuous limit in the general setup discussed in [22], although we expect such a limit to reproduce our more general construction, in its force-less and spatially local specialisation.

The VF construction holds for the extended polychromatic reductions of the conventional GHD equation (4.21–4.22), under the constraint that one may choose a parametrisation θ such that the scattering shift takes the form $\varphi(\theta,\theta') = a(\theta)a(\theta')g(b(\theta)-b(\theta'))$ for arbitrary functions a, b and even function g. By contrast, our construction works directly for the non-reduced GHD equation, and under the constraint that $\varphi(\theta,\theta') = \partial_\theta\phi(\theta,\theta')$ where $\phi(\theta,\theta') = -\phi(\theta',\theta)$ is odd. Notwithstanding the reduction, the constraints of the GHD data appear to be non-related; however, it is a simple matter to check that all special cases considered in [22] are also covered by our construction.

The hard rod gas corresponds to the special choice of GHD data (2.14). In section 3.2 of their paper, VF present the Hamiltonian structure of the infinite hydrodynamic chain generated by the spectral moments of the fluid density (keeping their notation)

$$A^m(x) = \int \mathrm{d}\theta\,\theta^m \rho(x,\theta). \tag{4.30}$$

Note that they construct their Hamiltonian structure with respect to the Hamiltonian density $h_{\mathrm{VF}} = -A^2/2$. Traditionally, in physics, the Hamiltonian of a system is associated with its energy, the density of which is $h = A^2/2$ in the hard rod case. As we went with the conventions of physics, in order for our and VF's construction to agree, the two proposed Poisson brackets must differ by a sign.

We will now show, by direct computation, that our Poisson bracket specialises to the one proposed by VF in [22] in the case of conventional GHD with choice of data (2.14). To that end, we compute the Poisson bracket between two arbitrary spectral moments: using the fundamental bracket (1.3) of conventional GHD, it is a simple matter to write

$$\{A^m(x), A^n(x')\} = B^{mn}(x)\delta(x-x'), \tag{4.31}$$

[14] In this instance, "local" is to be interpreted as referring to an operator of Dubrovin–Novikov type (4.10), while "non-local" refers to a generalisation of this type of operators, as the one introduced in [63,64], in which, notably, the coefficients g_{ij} define a metric of constant curvature.

where $B^{mn}(x)$ is the following differential operator

$$B^{mn}(x) = -\frac{1}{2\pi}\Bigg[\partial_x \int \mathrm{d}\theta\, h_m^{\mathrm{dr}}(x,\theta)\partial_\theta h_n(\theta)\, n(x,\theta) + \int \mathrm{d}\theta\, \partial_\theta h_m(\theta) h_n^{\mathrm{dr}}(\theta)\, n(x,\theta)\, \partial_x\Bigg], \tag{4.32}$$

and $h_m(\theta) = \theta^m$. In the case of the hard rod gas, it is well known that the quantities of interest simplify as follows:

$$2\pi\rho_{\mathrm{s}}(x,\theta) = 1 - A^0(x), \quad f^{\mathrm{dr}}(x,\theta) = f(x,\theta) - a\int_{\mathbb{R}} \mathrm{d}\theta f(x,\theta)\rho(x,\theta), \tag{4.33}$$

and, therefore, $h_m^{\mathrm{dr}}(x,\theta) = \theta^m - aA^m(x)$. Putting all of these together, we end up with an explicit expression for the Hamiltonian operator in terms of the moments

$$B^{mn}(x) = -\Bigg[\partial_x \frac{n}{1-aA^0}\left(A^{m+n-1} - aA^m A^{n-1}\right) + \frac{m}{1-aA^0}\left(A^{m+n-1} - aA^n A^{m-1}\right)\partial_x\Bigg]. \tag{4.34}$$

Conversely, in their paper, VF obtain the Poisson bracket

$$\{A^m(x), A^n(x')\}_{\mathrm{VF}} = B_{\mathrm{VF}}^{mn}(x)\delta(x-x'), \tag{4.35}$$

with the Hamiltonian operator (indices being encoded in matrix form)

$$B_{\mathrm{VF}}(x) = J\Big(\frac{1}{1-aA^0}U\partial_x + \partial_x U^T \frac{1}{1-aA^0}\Big)J^T + \frac{a}{1-aA^0}\left(PA_x^T - A_x P^T\right), \tag{4.36}$$

where

$$\begin{aligned} U^{mn} &= mA^{m+n-1}, \\ J^{mn} &= \delta_{mn} - aA^m\delta_{n,0}, \\ P^m &= mA^{m-1}. \end{aligned} \tag{4.37}$$

In order to compare the operators B and B_{VF}, we bring both in the standard form (4.34) and find that, indeed,

$$\begin{aligned} B_{\mathrm{VF}}^{mn} &= \frac{1}{1-aA^0}\left((m+n)A^{m+n-1} - amA^{m-1}A^n - anA^{n-1}A^m\right)\partial_x \\ &\quad + \frac{n}{1-aA^0}\left(A_x^{m+n-1} - aA^m A_x^{n-1} - aA_x^m A^{n-1}\right) \\ &\quad + \frac{anA_x^0}{(1-aA^0)^2}\left(aA^m A^{n-1} - A^{m+n-1}\right) \\ &= -B^{mn}. \end{aligned} \tag{4.38}$$

Hence, the Hamiltonian structure associated with the hard rod gas VF constructed corresponds to a special (conventional) case of our general result on the Hamiltonian structure of the GHD equation. That is arguable, of course, but we believe our formulation to be slightly simpler than VF's, provided one is comfortable with the dressing formalism. Our field theoretic formulation also seems to be more direct, as it deals explicitly with the GHD equation, rather than with the infinite hydrodynamic chain characterised by the spectral moments of the fluid density. It would be interesting to understand the relation between the two constructions more generally in conventional GHD, beyond the hard rod gas, and to construct the analogue of the second Hamiltonian structure they presented.

5. Discussion and Perspectives

In this paper, we introduced a new, more general (spatially extended) form of the GHD equations (2.7–2.8–2.9), which accounts for space-dependent external potentials, space-dependent momenta, and spatially extended scattering kernels. We showed that, even in its spatially extended form, the GHD equations constitute a Hamiltonian system (technically, under the sufficient but not necessary condition (2.3) regarding the functional space of fluid densities). We constructed the associated Poisson structure on an appropriate algebra of observables $\mathfrak{U}$, for which we took inspiration from the notion of functional of hydrodynamic type [47]. We further showed that the GHD equations admit an infinite set of linearly independent (extensive) conserved quantities that are in mutual involution in the sense of Liouville. Although this is no proof, as we currently do not know if the Hamiltonian flows generated by the conserved quantities we highlight are complete, this strongly suggests that the GHD equations might be integrable. And, in fact, integrability of the conventional GHD equations had been previously investigated under the so-called polychromatic reduction, for which a bi-Hamiltonian structure has been evinced [21]. As such, we provided a brief overview of the previous results and of the framework in which they were obtained: the theory of systems of hydrodynamic type. Clarifying the precise connection between systems of hydrodynamic type and GHD, beyond its polychromatic reductions, would be useful; especially since the formalism used by those two theories is fairly different. We believe that we have taken a first step towards this goal by showing that our results agree with those presented in the recent preprint [22].

In this section, to conclude our discussion, we go over the potential extensions of our construction and further reflect on the connection between GHD and systems of hydrodynamic type. For clarity, we will examine those two aspects in different subsections.

5.1. More General Dynamical Spaces

In Sect. 2.1, we introduced the spectral phase space $\Lambda = \mathscr{L} \times \mathscr{P}$, but in the rest of the text we restricted ourselves to the case $\Lambda = \mathbb{R}^2$. Moreover we have assumed that the fluid density quickly vanishes in unbounded directions, i.e. $\mathcal{M} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$. However, even discarding boundary conditions in space, there are many physically relevant situations in which $\mathscr{P}$ may be a more general manifold that may include many disconnected components (representing, for instance, many particle types; see the GHD description of this situation in [11]), be of higher dimension [35], or have the topology of the circle as in quantum spin chains (see e.g. [65]). A natural extension of our work would be to consider more general dynamical spaces $\mathcal{M}$. For instance, one may consider the additional cases

(i) the fluid density is periodic in physical space but quickly vanishes in spectral space:

$$\mathscr{L} = [x_0, x_0 + \mathsf{L}], \quad \mathscr{P} = \mathbb{R}, \quad \rho(x,\theta) = \rho(x+\mathsf{L},\theta) \quad \lim_{|\theta|\to\infty} \rho(x,\theta) = 0; \tag{5.1}$$

(ii) the fluid density quickly vanishes in physical space but is periodic in spectral space:

$$\mathscr{L} = \mathbb{R}, \quad \mathscr{P} = [\theta_0, \theta_0 + \mathsf{P}], \quad \rho(x,\theta) = \rho(x,\theta+\mathsf{P}), \quad \lim_{|x|\to\infty} \rho(x,\theta) = 0; \tag{5.2}$$

(iii) the fluid density is periodic in both physical and spectral space:

$$\Lambda = [x_0, x_0 + \mathsf{L}] \times [\theta_0, \theta_0 + \mathsf{P}], \quad \rho(x,\theta) = \rho(x+\mathsf{L},\theta), \quad \rho(x,\theta) = \rho(x,\theta+\mathsf{P}) \ , \tag{5.3}$$

where we give periodic directions the topology of the circle and where the condition (2.3) becomes

$$\mathcal{M} = \{|\rho| \le \rho_* : \rho \in \mathcal{C}^\infty_c(\Lambda)\}. \tag{5.4}$$

One may even consider that the fluid density lies on a domain $\Lambda \subset \mathbb{R}^2$ (which may not be fully connected), taking arbitrary values on its boundary. This last case is useful in GHD, when considering zero-entropy states [66], and in soliton gases [7], notably when dealing with the so-called soliton condensates [34]. This may be seen as a singular limiting case of the situation discussed in the main text of this paper, where the fluid density on $\mathbb{R}^2$ has less and less regularity at the boundary of Λ. Importantly, in this case, the domain Λ is in general dynamical (in the sense that it depends on time).

Note that, even in the cases (i), (ii), and (iii) we do not require the GHD data to be periodic. In fact, we may rather assume $E \in \mathcal{C}^\infty(\Lambda)$ and Ω smooth on the universal cover of Λ such that its mixed derivative $\Omega_{x\theta}$ is continuous on Λ (hence periodic in compact directions in cases (i)–(iii)). Using the notation $\partial_x^{-1}\mathcal{C}^\infty(\Lambda)$, $\partial_\theta^{-1}\mathcal{C}^\infty(\Lambda)$, etc., in order to represent pre-images of derivatives on the universal cover of Λ, we require

$$\Omega \in \partial_x^{-1}\partial_\theta^{-1}\mathcal{C}^\infty(\Lambda) = \{f \in \mathcal{C}^\infty(\mathbb{R}^2) : f_{x\theta}\big|_\Lambda \in \mathcal{C}^\infty(\Lambda)\}. \tag{5.5}$$

Similarly, the condition (2.6) on the interaction kernel should be changed into, respectively, for the different cases $u = \psi,\ \psi_{\tilde{x}},\ \psi_{\tilde{\theta}}$ or $\psi_{\tilde{x}\tilde{\theta}}$,

$$\int_\Lambda \mathrm{d}x'\mathrm{d}\theta'\, u(\cdot,\cdot;x',\theta') f(x',\theta') \ \in \partial_x^{-1}\partial_\theta^{-1}\mathcal{C}^\infty(\Lambda),\ \partial_\theta^{-1}\mathcal{C}^\infty(\Lambda),\ \partial_x^{-1}\mathcal{C}^\infty(\Lambda), \text{ or } \mathcal{C}^\infty(\Lambda)\ , \tag{5.6}$$

and for every $f \in \mathcal{C}_c^\infty(\Lambda)$.

Our construction still holds in the cases (i) and (ii): our derivation is based on the change of metric giving the normal fluid density, either in real space Eq. (3.10) or in spectral space Eq. (3.15). In the case (i), we may use Eq. (3.15), while in the case (ii), we may use Eq. (3.10), and the derivation is unchanged. Thus (3.3) still is a valid Poisson bracket and (3.19) still generates the GHD equation, with integrations over Λ instead of $\mathbb{R}^2$.

However, in the case (iii) our derivation appears to have technical problems. The change of metric may still be done in periodic directions. In particular, the property (5.6) of the interaction kernel guarantees that the change of coordinates (3.10) introduced in Sect. 3 remains well-defined. Indeed, $Y \in \partial_x^{-1}\mathcal{C}^\infty(\Lambda)$—at least in the setup of Eq. (5.4), see Appendix D—which is the right space for a change of coordinates. As before, under this change of coordinates, the space $\mathcal{L}$ is mapped onto $\hat{\mathcal{L}}_\theta$ which depends on θ (and also, in general, on the fluid density), and thus (y,θ) lies on $\hat\Lambda = \bigsqcup_{\theta\in\mathcal{P}} \hat{\mathcal{L}}_\theta$. If $\Omega_{x\theta} > 0$, $\rho_{\rm s}$ is strictly positive and $Y(x,\theta)$ is monotonic in x, so it maps $\mathcal{L}$ onto $\hat{\mathcal{L}}_\theta$ as a subset of $\mathbb{R}$ (or all of $\mathbb{R}$), which is then given the induced topology; on this subset its inverse $X(\cdot,\theta)$ exists

$$X : \hat\Lambda \to \Lambda,\ (y,\theta) \mapsto X(y,\theta) \quad : \quad Y(X(y,\theta),\theta) = y. \tag{5.7}$$

Otherwise, Y is smooth but not necessarily monotonic, and $\hat{\mathcal{L}}_\theta$ is, as before, taken as a multiple cover of a subset of $\mathbb{R}$; then $Y(\cdot,\theta)$ is a smooth diffeomorphism $\mathcal{L} \to \hat{\mathcal{L}}_\theta$, with $X(\cdot,\theta) : \hat{\mathcal{L}}_\theta \to \mathcal{L}$ its inverse. As before, one must require $\hat{\mathcal{L}}_\theta$ not to depend on the fluid density (see Remark 3.1).

However, the lack of periodicity of the interaction kernel affects the adjoint operator $\mathcal{O}^\dagger$ in "Appendix F", as this requires integration by part. Our derivation fails at that point. Similar technical issues would occur when trying to show the conserved quantities of the GHD are in involution in Sect. 3.5, Eqs. (3.39–3.40) in particular. We believe (3.3) still is a valid Poisson bracket, but we leave a more in depth analysis for future work.

5.2. GHD as a System of Hydrodynamic Type

Our Hamiltonian formulation of the spatially extended GHD equation led us to consider the linearised Poisson bracket (3.13). On functionals, it is the specialisation to $\Omega_{x\theta} = 1$ and $\Psi = 0$ of (3.3) and thus can be written[15] as

$$\begin{aligned}\{F,G\} &= \int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}\theta\mathrm{d}y'\mathrm{d}\theta'\, \frac{\delta F}{\delta\hat\rho(y,\theta)}\{\hat\rho(y,\theta),\hat\rho(y',\theta')\}\frac{\delta G}{\delta\hat\rho(y',\theta')} \\ &= \int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}\theta\, \hat\rho(y,\theta)\left(\partial_y \frac{\delta F}{\delta\hat\rho(y,\theta)}\partial_\theta\frac{\delta G}{\delta\hat\rho(y,\theta)} - \partial_\theta\frac{\delta F}{\delta\hat\rho(y,\theta)}\partial_y\frac{\delta G}{\delta\hat\rho(y,\theta)}\right).\end{aligned} \tag{5.8}$$

This is a type of non-canonical Poisson bracket known as a Lie–Poisson bracket [67], which is rather well known [15]. It is a special case of a Poisson bracket first proposed in [68] for the Maxwell–Vlasov equations. The Jacobi identity

[15] Here we assume $\hat{\mathcal{L}}_\theta = \mathbb{R}$ for simplicity; see Remark 3.1.

for the general bracket of [68] did not hold (this was fixed in [69], see also the comment [70]), but it did anyway for the special case which is of interest for us here. In any case, we prove the Jacobi identity directly in "Appendix E.4". Other contexts in which the same Poisson bracket also appears include the Benney equation [71,72] (see also [73] for a study of its reductions via the theory of hydrodynamic-type systems), the two-dimensional ideal fluid [74], or the Kida vortex [75]. Even some stochastic extensions have been considered, see for instance the recent paper [76].

Note that we can also write

$$\{F, G\} = \int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}\theta\, \mathcal{K}(y,\theta)\, \frac{\delta F}{\delta\hat{\rho}(y,\theta)} \otimes \frac{\delta G}{\delta\hat{\rho}(y,\theta)}\ , \tag{5.9}$$

in terms of the Hamiltonian operator

$$\mathcal{K}(y,\theta) = \hat{\rho}(y,\theta)\, \partial_y \wedge \partial_\theta. \tag{5.10}$$

In this last form, it is of course reminiscent of the Dubrovin–Novikov hydrodynamic bracket [48], Eq. (4.10), and its various generalisations [77–79]. Despite the fact recalled above that this Poisson bracket has been known for a long time, to the best of our knowledge, it seems that its Hamiltonian operator falls outside the many investigations of Hamiltonian operators for PDEs of hydrodynamic type. The various generalisations of the original Dubrovin–Novikov bracket can be labelled by three integers: N for the number of fields, n for the number of independent variables, and d for the highest order of the differential operators involved. The case $n = d = 1$, N arbitrary corresponds to the original works by Dubrovin and Novikov. The cases N and n arbitrary have only been considered and (partially) classified when $d = 1$, and the cases N arbitrary and $d > 1$ have only been considered when $n = 1$, see e.g. [78,80–83] and references therein. The case at hand here corresponds to $N = 1$, $n = 2$, and $d = 2$. We introduced it with a concrete motivation in mind, and it appears in a rather special form, but we believe it could stimulate research on the case $d \geq 2$ (with $n,\ N$ arbitrary). In fact, there is an alternative, tantalising way of visualising (3.13) as a large N limit of a $n = 1$, $d = 1$ Jordan block-type Hamiltonian operator as described in [21,22,58,59]: first because of the equivalence highlighted in Sect. 4.4, second because the $\delta'(\theta - \theta')$ terms in the Poisson brackets could be interpreted as coupling nearest neighbours in spectral space, in which case we would expect g_{ij} in Eq. (4.10) to be block diagonal. At this point, this is admittedly rather formal, but we believe properly investigating the connection between those interpretations may prove fruitful.

5.3. Lagrangian (Multiform) Formulation

Systems of hydrodynamic type are known to have profound connections with differential geometry. It is therefore not surprising that they are amenable to Lagrangian formulations which offer variational counterparts of their Hamiltonian formulation (when they have one). For the linearised Poisson bracket (3.13) and a linear Hamiltonian as in (3.19), a Lagrangian formulation is obtainable from results of [84]; this can in principle be rewritten in terms of the (interacting) fluid density ρ. However, such traditional variational formulations know nothing of possible integrable structures. There have been attempts at capturing integrability in a Lagrangian formulation by emulating the bi-Hamiltonian picture and proposing the notion of bi-Lagrangian systems, see [79]. More recently, the notion of Lagrangian multiforms, originally introduced in [85], has emerged as a variational framework for integrability. It has been developed for almost all incarnations of integrable systems—discrete and continuous, finite and infinite dimensional—but not yet for systems of hydrodynamic type. Lagrangian multiform theory for continuous infinite dimensional systems in $1+1$d [86–90] and $2+1$d [91] is relatively well understood, so it is natural and desirable to try to cast the GHD equation treated in this paper (which lives in $2+1$d), and some of its reductions (which live in $1+1$d), into this framework. We believe this alternative approach could shed additional light on the integrability features of GHD.

The Hamiltonian formulation we have provided, and also potential Lagrangian formulations (conventional or multiform), should help shed light on a number of other problems related to GHD, such as its quantisation (e.g. is there a dressed Moyal bracket?), and the characterisation of random GHD configurations as may occur in possible turbulent regimes [92].

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A. Hamiltonian System of Particles and Its Hydrodynamic Limit

This Appendix generalises the construction made in [28,29] to space-dependent energy functions (that is, to the addition of force terms due to external space-varying fields coupled to conserved densities) and to more general interaction kernels. The derivation of the spatially extended GHD equation is also based on a different scaling of the interaction kernel that we present below.

Although we omit the explicit calculation, we mention that the models we present in fact correspond to the *atomic reduction* of the (spatially extended) GHD equation, making the assumption (3.9) with $x_i = x_i(t)$ and $\theta_i = \theta_i(t)$ in the GHD equation (2.7).

A.1. Definition of the System

We define a Hamiltonian system on phase space $\mathbb{R}^{2N}$ as follows. Consider the function

$$\Phi(\boldsymbol{x},\boldsymbol{\theta}) = \sum_{i=1}^{N} \Omega(x_i,\theta_i) + \frac{1}{2}\sum_{i\neq j} \psi(x_i,\theta_i;x_j,\theta_j) \tag{A.1}$$

as a generating function for a canonical transformation between two sets of canonical coordinates on $\mathbb{R}^{2N}$

$$(\boldsymbol{x},\boldsymbol{p}) \leftrightarrow (\boldsymbol{y},\boldsymbol{\theta}), \tag{A.2}$$

where we used a bold font to denote vectors, e.g. $\boldsymbol{x} = (x_i)_{i=1}^N$. Given the generating function Φ our new set of canonical coordinates takes the form

$$y_i = \partial_{\theta_i}\Phi = \Omega_\theta(x_i,\theta_i) + \sum_{j\neq i} \psi_\theta(x_i,\theta_i;x_j,\theta_j) \tag{A.3}$$

$$p_i = \partial_{x_i}\Phi = \Omega_x(x_i,\theta_i) + \sum_{j\neq i} \psi_x(x_i,\theta_i;x_j,\theta_j), \tag{A.4}$$

where we used the symmetry of ψ, Eq. (2.5). Here and below we use indices x and θ to denote derivatives with respect to the spatial and spectral variable, respectively. The two sets of coordinates $(\boldsymbol{x},\boldsymbol{p})$ and $(\boldsymbol{y},\boldsymbol{\theta})$ being canonical, we naturally have the fundamental relations

$$\{x_i,x_j\} = \{p_i,p_j\} = \{y_i,y_j\} = \{\theta_i,\theta_j\} = 0, \tag{A.5}$$

and

$$\{x_i,p_j\} = \{y_i,\theta_j\} = \delta_{ij}. \tag{A.6}$$

Additionally, we define our Hamiltonian, in terms of the energy function $E \in \mathcal{C}^\infty(\mathbb{R}^2)$, as

$$H = \sum_{i=1}^{N} E(x_i, \theta_i). \tag{A.7}$$

The coordinates $(\boldsymbol{x}, \boldsymbol{p})$ represent the real particle coordinates, while, following the nomenclature of [28,29], $(\boldsymbol{y}, \boldsymbol{\theta})$ can be referred to as "asymptotic coordinates". However, in general, the asymptotic analysis of [28,29] does not hold here, as the Hamiltonian is not solely a function of $\boldsymbol{\theta}$. Further, the model (A.7) is not necessarily integrable. However, if $E(x, \theta) = E(\theta)$ is independent of x, then we see that each θ_i is conserved and y_i evolves linearly in time. In this case, with appropriate asymptotic properties on ψ, $\boldsymbol{\theta}$ indeed represent the asymptotic momenta and $\boldsymbol{y}$ are simply related to impact parameters (in the sense of [4]). Then, the model is integrable, by an extension of the analysis made in [28,29] (which we omit here).

A.2. Properties of the Generating Function Φ

In order for Eq. (A.7) to be a well-defined function on the real phase space $(\boldsymbol{x}, \boldsymbol{p})$, and for (A.3) and (A.4) to give rise to a well-defined canonical transformation, Eqs. (A.3) and (A.4) need to be intertible for $\boldsymbol{x}$ and $\boldsymbol{\theta}$, respectively. This holds if the following matrix is positive definite for all $\boldsymbol{x}, \boldsymbol{\theta}$:

$$\begin{aligned}\partial_{x_i}\partial_{\theta_j}\Phi = \Gamma_{ij}(\boldsymbol{x}, \boldsymbol{\theta}) = &\left(\Omega_{x\theta}(x_i, \theta_i) + \sum_{k\neq i} \psi_{x\theta}(x_i, \theta_i; x_k, \theta_k)\right)\delta_{ij} \\ &+\psi_{x\theta'}(x_i, \theta_i; x_j, \theta_j)(1-\delta_{ij})\end{aligned} \tag{A.8}$$

where, following the notations of the main text, here and below we use

$$\psi_{x\theta'}(a, d; c, d) = \partial_x \partial_{\theta'} \psi(x, \theta; x', \theta')|_{(a,b;c,d)}, \text{ etc.} \tag{A.9}$$

Note that the matrix Γ can be interpreted as an analogue of the Gaudin matrix in the theory of quantum integrable systems, the determinant of which is the Jacobian of the transformation that maps the quasi-momenta to the quantum numbers (c.f. Sections 4.3.2 and 4.3.3 of [93]). The positive definite condition $\sum_{ij} \Gamma_{ij} v_i v_j > 0$ (for all $\boldsymbol{v} \in \mathbb{R}^N$ with $|\boldsymbol{v}| = 1$) amounts to

$$\begin{aligned}\sum_{i=1}^{N} v_i^2 \Omega_{x\theta}(x_i, \theta_i) \; &+ \frac{1}{2}\sum_{j\neq i}\left[(v_i^2 + v_j^2)\psi_{x\theta}(x_i, \theta_i; x_j, \theta_j) + 2v_i v_j \psi_{x\theta'}(x_i, \theta_i; x_j, \theta_j)\right] \\ &> 0,\end{aligned} \tag{A.10}$$

thanks to the symmetry (2.5). For instance, this holds if $\Omega_{x\theta},\ \psi_{x\theta} > 0$ and $|\psi_{x\theta'}| \leq \psi_{x\theta}$, which includes the cases considered in [28,29]. The result is a smooth diffeomorphism of $\mathbb{R}^2$ if Ω and ψ are smooth. Alternatively, a more compact way to write this condition is

$$\sum_{i=1}^{N} \hat{v}_i^x \hat{v}_i^\theta \Omega(x_i, \theta_i) + \frac{1}{2}\sum_{j\neq i}(\hat{v}_i^x + \hat{v}_j^x)(\hat{v}_i^\theta + \hat{v}_j^\theta)\psi(x_i, \theta_i; x_j, \theta_j) > 0, \tag{A.11}$$

where $\hat{v}_i^x = v_i \partial_{x_i}$ and $\hat{v}_i^\theta = v_i \partial_{\theta_i}$. A full analysis of the set of Ω and ψ giving invertibility of (A.3) and (A.4) would be interesting, but is beyond the scope of this paper. Below we assume that the transformation (A.3), (A.4) is a smooth diffeomorphism of $\mathbb{R}^2$.

A.3. Equations of Motion

Denote $\dot{y}_i|_{\boldsymbol{\theta}} = \sum_j \partial_{x_j}\partial_{\theta_i}\Phi\, \dot{x}_j$ (that is, the time derivative of the right-hand side of (A.3) keeping $\boldsymbol{\theta}$ fixed), and $\dot{p}_i|_{\boldsymbol{x}} = \sum_j \partial_{x_i}\partial_{\theta_j}\Phi\, \dot{\theta}_j$. Then we show below that the equations of motion generated by H, that is $\dot{x}_i = \{x_i, H\}$, $\dot{p}_i = \{p_i, H\}$, and $\dot{y}_i = \{y_i, H\}$, $\dot{\theta}_i = \{\theta_i, H\}$, are equivalent to

$$\dot{y}_i|_{\boldsymbol{\theta}} = E_\theta(x_i, \theta_i), \quad \dot{p}_i|_{\boldsymbol{x}} = -E_x(x_i, \theta_i). \tag{A.12}$$

Indeed, by definition of H (A.7)

$$\dot{y}_i = \{y_i, H\} = E_\theta(x_i, \theta_i) + \sum_{j=1}^{N} \{y_i, x_j\} E_x(x_j, \theta_j) \tag{A.13}$$

Then, recalling the definition (A.3), $y_i = \partial_{\theta_i}\Phi(\boldsymbol{x}, \boldsymbol{\theta})$, we have

$$\{y_i, x_j\} = \sum_{l=1}^{N} \partial_{\theta_i}\partial_{\theta_l}\Phi(\boldsymbol{x}, \boldsymbol{\theta})\{\theta_l, x_j\}, \tag{A.14}$$

and therefore

$$\begin{aligned} \sum_{j=1}^{N} \{y_i, x_j\} E_x(x_j, \theta_j) &= \sum_{l=1}^{N} \partial_{\theta_i}\partial_{\theta_l}\Phi(\boldsymbol{x}, \boldsymbol{\theta})\{\theta_l, H\} \\ &= \sum_{l=1}^{N} \partial_{\theta_i}\partial_{\theta_l}\Phi(\boldsymbol{x}, \boldsymbol{\theta})\dot{\theta}_l \end{aligned} \tag{A.15}$$

As $\dot{y}_i = \sum_j \partial_{x_j}\partial_{\theta_i}\Phi\, \dot{x}_j + \sum_j \partial_{\theta_j}\partial_{\theta_i}\Phi\, \dot{\theta}_j$, we find the first equation of (A.12). A similar derivation starting with

$$\dot{p}_i = \{p_i, H\} = -E_x(x_i, \theta_i) + \sum_{j=1}^{N} \{p_i, \theta_j\} E_\theta(x_j, \theta_j) \tag{A.16}$$

gives the second equation of (A.12). As we assumed that the transformation of coordinates $(\boldsymbol{x}, \boldsymbol{p}) \leftrightarrow (\boldsymbol{y}, \boldsymbol{\theta})$ is smooth and invertible, then Eqs. (A.17) (consequence of the first equation of (A.12)) can be inverted for $\dot{x}_i$, and hence that equation along with the second equation of (A.12) fully fixes the dynamics. Hence, the inverse statement immediately holds.

Given (A.12), we may eventually write the relations

$$E_\theta(x_i, \theta_i) = \Omega_{x\theta}(x_i, \theta_i)\dot{x}_i + \sum_{j \neq i} \left(\psi_{x\theta}(x_i, \theta_i; x_j, \theta_j)\dot{x}_i + \psi_{x'\theta}(x_i, \theta_i; x_j, \theta_j)\dot{x}_j\right), \tag{A.17}$$

$$-E_x(x_i, \theta_i) = \Omega_{x\theta}(x_i, \theta_i)\dot{\theta}_i + \sum_{j \neq i} \left(\psi_{x\theta}(x_i, \theta_i; x_j, \theta_j)\dot{\theta}_i + \psi_{x\theta'}(x_i, \theta_i; x_j, \theta_j)\dot{\theta}_j\right), \tag{A.18}$$

which are reminiscent of Eqs. (2.8)–(2.9).

A.4. Hydrodynamic Limit

We now argue that, in an appropriate hydrodynamic limit, we recover the spatially extended GHD equation (2.7). For this purpose, we introduce a large parameter L, such that in the limit $L \to \infty$ one recovers the hydrodynamic equation. The data $E,\ \Omega,\ \psi$ depend on L as follows:

$$\begin{aligned} E(x,\theta) = \bar{E}(x/L,\theta), \quad \Omega(x,\theta) = L\bar{\Omega}(x/L,\theta), \\ \psi(x,\theta;x',\theta') = \bar{\psi}(x/L,\theta;x'/L,\theta'), \end{aligned} \tag{A.19}$$

where $\bar{E}, \bar{\Omega}, \bar{\psi}$ are independent of L. Consider the empirical density

$$\rho(\bar{x},\theta) = L^{-1}\sum_{i=1}^{N}\delta(\bar{x} - x_iL^{-1})\delta(\theta - \theta_i), \tag{A.20}$$

With $\bar{t} = tL^{-1}$, this satisfies the continuity equation

$$\begin{aligned} \partial_{\bar{t}}\rho(\bar{x},\theta) + \partial_{\bar{x}}\left(L^{-1}\sum_{i=1}^{N}\dot{x}_i\delta(\bar{x} - x_iL^{-1})\delta(\theta - \theta_i)\right) \\ +\partial_\theta\left(L^{-1}\sum_{i=1}^{N}\dot{\theta}_iL\,\delta(\bar{x} - x_iL^{-1})\delta(\theta - \theta_i)\right) = 0. \end{aligned} \tag{A.21}$$

Note that under this scaling, the two terms in the generating function Φ scale uniformly,

$$\begin{aligned} \Phi \to L^2\bar{\Phi}[\rho] \ &= L^2\left(\int_{\mathbb{R}^2}\mathrm{d}\bar{x}\mathrm{d}\theta\,\rho(\bar{x},\theta)\bar{\Omega}(\bar{x},\theta)\right. \\ &\quad \left.+\int_{\mathbb{R}^2}\mathrm{d}\bar{x}\mathrm{d}\theta\mathrm{d}\bar{x}'\mathrm{d}\theta'\,\rho(\bar{x},\theta)\rho(\bar{x}',\theta')\bar{\psi}(\bar{x},\theta;\bar{x}',\bar{\theta}')\right), \end{aligned} \tag{A.22}$$

(the term with $i = j$ in the sum on the second term is a subleading correction, scaling as $O(L)$).

In the limit $L \to \infty$, we assume that: (i) the number of particles increases proportionally, $N \propto L$; (ii) the positions are spread on a region of length $\propto L$; and (iii) the asymptotic momenta condense on a region of order $O(1)$. Thus, the interparticle distance scales as $O(1)$, while the separation between nearby momenta scales as $O(L^{-1})$. Then the empirical density (A.20) tends, weakly, to a finite, nonzero function. Now let $\bar{x}_i = x_i/L$ and assume that $\dot{x}_i$ can be written as a smooth function $v^{\mathrm{eff}}(\bar{x}_i,\theta_i)$ of $\bar{x}_i,\theta_i$ with a uniformly smooth limit as $L \to \infty$, and likewise $\dot{\theta}_iL$ can be written as a smooth function $a^{\mathrm{eff}}(\bar{x}_i,\theta_i)$ (both functionals of ρ). This should be the case for good enough particle distributions under the above scaling (see for example [1]). Then Eq. (A.17) becomes

$$\bar{E}_\theta(\bar{x}_i,\theta_i) = \bar{\Omega}_{x\theta}(\bar{x}_i,\theta_i)v^{\rm eff}(\bar{x}_i,\theta_i) + \int_{\mathbb{R}^2} {\rm d}\bar{x}'{\rm d}\theta'\,\rho(\bar{x}',\theta')\left[\bar{\psi}_{x\theta}(\bar{x}_i,\theta_i;\bar{x}',\theta')v^{\rm eff}(\bar{x}_i,\theta_i) + \bar{\psi}_{x'\theta}(\bar{x}_i,\theta_i;\bar{x}',\theta')v^{\rm eff}(\bar{x}',\theta')\right] . \tag{A.23}$$

As the set of $(\bar{x}_i,\theta_i)$'s becomes dense, this should hold for all $(\bar{x}_i,\theta_i)=(\bar{x},\theta)\in\mathbb{R}^2$. This reproduces (2.8) (omitting the over-bars). Similarly, (A.18) becomes

$$\bar{E}_x(\bar{x}_i,\theta_i) = \bar{\Omega}_{x\theta}(\bar{x}_i,\theta_i)a^{\rm eff}(\bar{x}_i,\theta_i) + \int_{\mathbb{R}^2} {\rm d}\bar{x}'{\rm d}\theta'\,\rho(\bar{x}',\theta')\left[\bar{\psi}_{x\theta}(\bar{x}_i,\theta_i;\bar{x}',\theta')a^{\rm eff}(\bar{x}_i,\theta_i) + \bar{\psi}_{x'\theta}(\bar{x}_i,\theta_i;\bar{x}',\theta')a^{\rm eff}(\bar{x}',\theta')\right], \tag{A.24}$$

and we recover (2.9). Together with (A.21), we find that the hydrodynamic equation (2.7) indeed emerges under this scaling.

B. Analysis of the Dressing Operation

The dressing operation is defined in (2.21). Here we show that there exists a non-negative function ρ_* on $\mathbb{R}^2$, such that for any smooth function $f\in\mathcal{C}^\infty(\mathbb{R}^2)$, the dressed quantity $f^{\rm dr}$ (with all possible transformation types) is unique, well-defined and smooth on $\mathbb{R}^2$, for all $\rho\in\mathcal{C}_c^\infty(\mathbb{R}^2)$ with $|\rho|\le\rho_*$ (i.e. for every $\rho\in\mathcal{M}$ with the dynamical space $\mathcal{M}$ defined as in Eq. (2.3)). This holds under certain technical conditions on Ω and on the interaction kernel ψ: Eqs. (B.1), (B.2), (B.4), and (B.12). The techniques we use are based on (and slightly extend) those of [31].

As already mentioned in Remark 2.2, the conditions above for the dressed function to exist and be smooth are in no way necessary. They are fulfilled in the Lieb–Liniger gas (2.13) and the hard rod gas (2.14), as well as in the GHD of the family of models of "Appendix A", but not in the KdV soliton gas (2.15). Notably, different sets of sufficient conditions in the Lieb–Liniger model have been fully worked out in [31]. Further, in the KdV soliton gas, even though the conditions are not met, the existence and uniqueness of $\Omega_{x\theta}^{\rm dr}$ and of $E_\theta^{\rm dr}$ have been proven in [36].

B.1. Existence of Dressed Functions

In order for dressed functions to exist, one needs to ensure the occupation function n, defined by Eq. (2.19), exists and remains bounded. To that end, we first require that

$$\Omega_{x\theta}(x,\theta)\neq 0,\quad (x,\theta)\in\mathbb{R}^2 , \tag{B.1}$$

and we also require that there exists a non-negative, locally integrable function μ on $\mathbb{R}^2$, such that the following bounds hold for all $(x,\theta)\in\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} {\rm d}x'{\rm d}\theta'\,|\psi_{x\theta}(x,\theta;x',\theta')|\,\mu(x',\theta') < |\Omega_{x\theta}(x,\theta)|. \tag{B.2}$$

In this equation, the absolute value $|\psi_{x\theta}|$ of the mixed derivative of the interaction kernel ψ is involved; recall that we had only required (2.6) (along with (2.5)). Thus, in particular, we require here that $|\psi_{x\theta}|$ is also an integral kernel. In fact, we also ask that this is the case for derivatives with respect to both $\tilde{x} = x,\, x'$ and both $\tilde{\theta} = \theta,\, \theta'$: for every positive continuous, compactly supported $\nu \in \mathcal{C}_c(\mathbb{R}^2)_+ = \{\nu \in \mathcal{C}_c(\mathbb{R}^2) : \nu(x,\theta) > 0\, \forall\, (x,\theta) \in \mathbb{R}^2\}$, there is $V \in \mathcal{C}(\mathbb{R}^2)$ with

$$\inf_{(x,\theta)\in\mathbb{R}^2} V(x,\theta) > 0, \tag{B.3}$$

such that

$$\int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta', |\psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')|\, \nu(x',\theta') < V(x,\theta). \tag{B.4}$$

The function μ and the pairs (ν, V) are not part of the data for the GHD equation itself, but influence the space $\mathcal{M}$ in which the fluid density ρ must lie. We note that the bound (B.2) is invariant under re-parametrisation, with μ of type $(1,1)$.

As we work under the assumption (2.6), for all $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with $|\rho| \le \mu$, we have smooth $\rho_{\rm s}(x,\theta) \neq 0\ \forall (x,\theta) \in \mathbb{R}^2$; hence, the occupation function n, defined in (2.19), is also in $\mathcal{C}_c^\infty(\mathbb{R}^2)$. Moreover, we can show that, for any non-negative function h on $\mathbb{R}^2$, there exists a non-negative function ρ_* such that $|n| \le h$ for all $|\rho| \le \rho_*$. Indeed, let

$$m(x,\theta) = \frac{1}{|\Omega_{x\theta}(x,\theta)|} \int \mathrm{d}x'\mathrm{d}\theta', |\psi_{x\theta}(x,\theta;x',\theta')|\, \mu(x',\theta'), \tag{B.5}$$

where μ is the function defined in the first condition of (B.2). The function m exists and satisfies $0 \le m(x,\theta) < 1,\ \forall (x,\theta) \in \mathbb{R}^2$ by definition. Then, take for instance

$$\rho_* = \rho_*[h] := \frac{1-m}{2\pi} \frac{|\Omega_{x\theta}|}{\sqrt{1+\Omega_{x\theta}^2}} \frac{\mu}{\sqrt{1+\mu^2}} \frac{h}{\sqrt{1+h^2}} < \mu, \tag{B.6}$$

which implies $2\pi|\rho_{\rm s}| \ge |\Omega_{x\theta}|(1-m)$, and therefore $|n| \le h$ for any $|\rho| \le \rho_*$.

Now, consider the dressing of a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, and suppose n is supported on a compact $C \in \mathbb{R}^2$. One can then multiply the dressing equation satisfied by f, Eq. (2.21), by any continuous, positive function $w \in \mathcal{C}(\mathbb{R}^2)_+ := \{w \in \mathcal{C}(\mathbb{R}^2) : w(x,\theta) > 0\, \forall\, (x,\theta) \in \mathbb{R}^2\}$, and look for a solution for $wf^{\rm dr}$ as a Liouville–Neumann series

$$wf^{\rm dr} = \sum_{k=0}^{\infty} \left(w\, \Psi_{\tilde{x}\tilde{\theta}}\, \frac{n}{w} \right)^k wf \tag{B.7}$$

where we recall that Ψ is an integral operator defined in Eq. (2.27). One can see that, if

$$q := \sup_{(x,\theta)\in C} w(x,\theta) \int_{\mathbb{R}^2} \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi w(x',\theta')}\, |\psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')|\, n(x',\theta') < 1\ , \tag{B.8}$$

then the Liouville–Neumann series is absolutely convergent on C provided

$$||f||_{w,C} = \sup_{(x,\theta)\in C} w(x,\theta)|f(x,\theta)| < \infty\ , \tag{B.9}$$

since each term is then bounded by $q^k||f||_{w,C}$. Thus Eq. (2.21) has, on C, a unique solution on the Banach space $\mathcal{B}_w(C)$ of functions f (on $\mathbb{R}^2$) such that wf is bounded on C, with norm $||f||_{w,C}$. Then the dressing operation is indeed well-defined and gives the same function (on C) for any $0 < w' \le w$ such that (B.8) holds (with w' in place of w). As C is compact, $||f||_{w,C} < \infty$ for every $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, and thus under condition (B.8), the dressing equation (2.21) has a unique solution for every $f \in \mathcal{C}^\infty(\mathbb{R}^2)$. By the bound (B.4) and the result expressed around (B.6), we see that with

$$h(x,\theta) = \frac{2\pi\chi_C(x,\theta)\nu(x,\theta)}{V(x,\theta)} \qquad w(x,\theta) = \frac{1}{V(x,\theta)}, \tag{B.10}$$

where χ_C is the characteristic function of the set C; the condition (B.8) is satisfied for every $|n| \le h$. Therefore, the dressing of any smooth f exists on C for every $|\rho| \le \rho_*[2\pi\chi_C\nu/V]$. As C is arbitrary, the dressing exists and gives a unique function on $\mathbb{R}^2$ for every $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $|\rho| \le \rho_*$ with

$$\rho_* = (1-m)\frac{|\Omega_{x\theta}|}{\sqrt{1+\Omega_{x\theta}^2}}\frac{\mu}{\sqrt{1+\mu^2}}\frac{\nu}{\sqrt{V^2+(2\pi\nu)^2}} \tag{B.11}$$

where we recall that m is defined in (B.5) and μ, ν, V in (B.2). This gives the explicit space $\mathcal{M}$, Eq. (2.3).

B.2. Smoothness of Dressed Functions

The result of the dressing $f^{\rm dr}(x,\theta)$ of a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ is, in general, a locally integrable function. For smoothness, we note that each term of the Liouville–Neumann series is smooth; hence, we must simply show convergence of the series of derivatives. For simplicity, we analyse only the first derivatives in x and θ (a similar analysis holds for higher derivatives), and we consider four situations:

$$\begin{aligned} &\text{either (i) } |\partial_x\psi_{\tilde x\tilde\theta}| \text{ is locally integrable,}\\ &\quad\text{or (ii) } \partial_x\psi_{\tilde x\tilde\theta} = -\partial_{x'}\psi_{\tilde x\tilde\theta};\\ &\text{either (iii) } |\partial_\theta\psi_{\tilde x\tilde\theta}| \text{ is locally integrable,}\\ &\quad\text{or (iv) } \partial_\theta\psi_{\tilde x\tilde\theta} = -\partial_{\theta'}\psi_{\tilde x\tilde\theta}. \end{aligned} \tag{B.12}$$

If $\psi_{\tilde x\tilde\theta}$ is a smooth function, then the first and third alternatives hold; if it is a distribution then, in general, they do not. However, in many examples, at least alternatives (ii) and/or (iv) hold. For instance, for local GHD, Eq. (2.10), in the examples of the Lieb–Liniger gas (2.13) and hard rod gas (2.14), alternatives (i) and (iv) hold. For the spatially extended GHD arising from the models of "Appendix A", alternatives (i) and (iii) hold. The conditions (B.12) are sufficient, but as mentioned not necessary, in order to have differentiability.

Consider the x derivative for definiteness. On a generic term of the Liouville–Neumann series for $f^{\rm dr}$ this gives

$$a_k = \partial_x \Psi_{\tilde{x}\tilde{\theta}} n (\Psi_{\tilde{x}\tilde{\theta}} n)^k f \tag{B.13}$$

where we assume that n is supported on a compact $C \subset \mathbb{R}^2$. In the case (i) this is bounded as

$$\begin{aligned} |a_k(x,\theta)| &\leq \int_{\mathbb{R}^2} \frac{{\rm d}x'{\rm d}\theta'}{2\pi} |\partial_x \psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')| \frac{n(x',\theta')}{w(x',\theta')} ||(\psi_{\tilde{x}\tilde{\theta}} n)^k f||_{w,C} \\ &\leq \int_{\mathbb{R}^2} \frac{{\rm d}x'{\rm d}\theta'}{2\pi} |\partial_x \psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')| \frac{n(x',\theta')}{w(x',\theta')} q^k ||f||_{w,C} \end{aligned}, \tag{B.14}$$

using (B.8). Since $n/w \in \mathcal{C}_c(\mathbb{R}^2)$, the result of the integral is finite by (B.4) and, because $0 \leq q < 1$ under condition (B.8), the series converges.

In the case (ii), we have

$$a_k = \sum_{j=0}^{k} (\Psi_{\tilde{x}\tilde{\theta}} n)^j \Psi_{\tilde{x}\tilde{\theta}} \partial_{x'} n (\Psi_{\tilde{x}\tilde{\theta}} n)^{k-j} f + (\Psi_{\tilde{x}\tilde{\theta}} n)^k \partial_{x'} f \tag{B.15}$$

The last term gives rise to a convergent series by the results already established, because $\partial_{x'} f(x',\theta')$ is a smooth function of x',θ'. We bound the first term for $(x,\theta) \in C$ as

$$\begin{aligned} |w(x,\theta) a_k(x,\theta)| \leq \sum_{j=0}^{k} q^j &\left(\sup_{(x,\theta)\in C} w(x,\theta) \int_{\mathbb{R}^2} \frac{{\rm d}x'{\rm d}\theta'}{2\pi} |\psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta') \right. \\ &\left. | \frac{|\partial_{x'} n(x',\theta')|}{w(x',\theta')} \right) q^{k-j} \, ||f||_w . \end{aligned} \tag{B.16}$$

By (B.3) and (B.10), the function w is upper bounded, as is, in fact, $\partial_x n/w \in \mathcal{C}_c(\mathbb{R}^2)$ by (B.10). Then the condition (B.4) implies that the result inside the parentheses is a finite quantity, say b. Therefore $|a_k(x,\theta)| \leq k q^k b ||f||_w / |w(x,\theta)|$, and again the series converge.

B.3. Dressing of Distributions

As per Remark 2.3, it is also convenient to define the dressing of distributions. In fact, dressed distributions are to be interpreted in terms of their effect when integrated against a compactly supported smooth test function g. For instance, when it comes to the fundamental Poisson bracket (1.5), we can then make use of the symmetry relation (2.28),

$$\int_{\mathbb{R}^2} {\rm d}x'{\rm d}\theta' \, \left[\delta'(\cdot - x)\delta'(\cdot - \theta)\right]^{\rm dr} (x',\theta') n(x',\theta') f(x',\theta') = -\partial_x \partial_\theta \left[n(x,\theta) f^{\rm dr}(x,\theta) \right], \tag{B.17}$$

and, in this case, the right-hand side is indeed well-defined. But, more generally, looking at the definition of the dressing (2.21), let f be a distribution and g, again, a compactly supported smooth function, we may write

$$\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, g(x,\theta) f^{\mathrm{dr}}(x,\theta) = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, g(x,\theta) f(x,\theta)$$
$$+ \int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta' \left(\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, \psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta') g(x,\theta) \right) n(x',\theta')\, f^{\mathrm{dr}}(x',\theta')$$
$$= \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta,\, g(x,\theta) f(x,\theta)$$
$$+ \int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta' \left(\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta \psi_{\tilde{x}\tilde{\theta}}(x,\theta;\cdot,\cdot) g(x,\theta) \right)^{\mathrm{dr}} (x',\theta')\, n(x',\theta')\, f(x',\theta'). \tag{B.18}$$

Because of the symmetry (2.5) and the condition (2.6) on the integral kernel ψ, everything in (B.18) is well-defined.

C. Diagonalisation of the Spatially Extended GHD Equation

In this section we show that the occupation function n diagonalises the GHD equation (2.7). To that end, we recall the definition of the occupation function (2.19), along with the relations (2.25) and (2.29), from which we may write the following identities

$$\rho = \frac{1}{2\pi} n {\Omega_{x\theta}}^{\mathrm{dr}}, \qquad v^{\mathrm{eff}}\rho = \frac{1}{2\pi} n {E_\theta}^{\mathrm{dr}}, \qquad a^{\mathrm{eff}}\rho = -\frac{1}{2\pi} n {E_x}^{\mathrm{dr}}. \tag{C.1}$$

The derivation works for $\Lambda = \mathbb{R}^2$, Eq. (2.1), as well as for Λ as in the cases (i), (ii), and (iii) of Sect. 5.1.

As per (2.17), both Ω and E are of type $(0,0)$, meaning $\Omega_{x\theta}$ is of type $(1,1)$, E_θ of type $(0,1)$, and E_x of type $(1,0)$. From the dressing operation (2.21), if $f \in C^\infty(\mathbb{R}^2)$ is of type $(0,1)$, we have

$$\partial_x f^{\mathrm{dr}} = f_x + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi} (-)\psi_{xx'\theta}\, n(x',\theta')\, f^{\mathrm{dr}}(x',\theta')$$
$$= f_x + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, \psi_{x\theta} \left(n(x',\theta')\, \partial_{x'} f^{\mathrm{dr}}(x',\theta') + \partial_{x'} n(x',\theta')\, f^{\mathrm{dr}}(x',\theta') \right), \tag{C.2}$$

and therefore, in operatorial form,

$$\partial_x f^{\mathrm{dr}} = (1 - \Psi_{x\theta}\, n)^{-1} \left(f_x + \Psi_{x\theta}\, n_x\, f^{\mathrm{dr}} \right) = {f_x}^{\mathrm{dr}} + (1 - \Psi_{x\theta}\, n)^{-1} \Psi_{x\theta}\, n_x\, f^{\mathrm{dr}}. \tag{C.3}$$

If f is instead of type $(1,0)$, then similarly

$$\partial_\theta f^{\mathrm{dr}} = {f_\theta}^{\mathrm{dr}} + (1 - \Psi_{x\theta}\, n)^{-1} \Psi_{x\theta}\, n_\theta\, f^{\mathrm{dr}}. \tag{C.4}$$

Finally, if f is of type $(1,1)$, then

$$\partial_t f^{\mathrm{dr}} = \partial_t f + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}, \psi_{x\theta} \Big(n(x',\theta')\, \partial_t f^{\mathrm{dr}}(x',\theta') + \partial_t n(x',\theta')\, f^{\mathrm{dr}}(x',\theta') \Big), \tag{C.5}$$

thus

$$\partial_t f^{\mathrm{dr}} = (\partial_t f)^{\mathrm{dr}} + (1-\Psi_{x\theta}\, n)^{-1} \Psi_{x\theta}\, \partial_t n\, f^{\mathrm{dr}}. \tag{C.6}$$

Hence, substituting the identities (C.1) in the GHD equation (2.7), and noting that $\partial_t \Omega_{x\theta} = 0$, we obtain

$$\begin{aligned} 0 + nE_{x\theta}^{\mathrm{dr}} - nE_{x\theta}^{\mathrm{dr}} + \big((1-\Psi_{x\theta}\, n)^{-1}\Psi_{x\theta} + 1\big) \Big(\Omega_{x\theta}^{\mathrm{dr}}\, \partial_t n + {E_\theta}^{\mathrm{dr}}\, \partial_x n - {E_x}^{\mathrm{dr}}\, \partial_\theta n \Big) \\ = 0, \end{aligned} \tag{C.7}$$

which leads to the diagonalised Eq. (2.30) by inversion of the operator $(1-\Psi_{x\theta}\, n)^{-1}\Psi_{x\theta} + 1$.

D. Alternative Dressing Formulations

Following and extending the general idea and notation introduced in [94, Suppl Mat II.A], it is possible to define different types of dressing operations that will prove useful in "Appendix E". First recall the usual "**dr**essing" operation (2.21)

$$f^{\mathrm{dr}}(x,\theta) = f(x,\theta) + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, \psi_{\tilde{x}\tilde{\theta}}(x,\theta;x',\theta')\, n(x',\theta')\, f^{\mathrm{dr}}(x',\theta')\ , \tag{D.1}$$

where $\tilde{x} = x$ or $-x'$ and $\tilde{\theta} = \theta$ or $-\theta'$ as per the transformation type of f:

$$\psi_{\tilde{x}\tilde{\theta}} = \begin{cases} \psi_{x'\theta'} & (f \text{ is of type } (0,0)) \\ -\psi_{x'\theta} & (f \text{ is of type } (0,1)) \\ -\psi_{x\theta'} & (f \text{ is of type } (1,0)) \\ \psi_{x\theta} & (f \text{ is of type } (1,1)). \end{cases} \tag{D.2}$$

Similarly, we define the following alternative operations.

- "**Dr**essing" is

$$f^{\mathrm{Dr}}(x,\theta) = f(x,\theta) + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, \psi_{\tilde{x}}(x,\theta;x',\theta')\, n(x',\theta'), \partial_{\theta'} f^{\mathrm{Dr}}(x',\theta')\ , \tag{D.3}$$

  for f of type either $(0,0)$ or $(1,0)$ (with $\tilde{x} = -x'$ or $\tilde{x} = x$, resp.), that is, scalar in the spectral variable;
- "**dR**essing" is

$$f^{\mathrm{dR}}(x,\theta) = f(x,\theta) + \int_\Lambda \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, \psi_{\tilde{\theta}}(x,\theta;x',\theta')\, n(x',\theta'), \partial_{x'} f^{\mathrm{dR}}(x',\theta')\ , \tag{D.4}$$

  for f of type either $(0,0)$ or $(0,1)$ (with $\tilde{\theta} = -\theta'$ or $\tilde{\theta} = \theta$, resp.), that is, scalar in the spatial variable;

- "**DR**essing" is

$$f^{\rm DR}(x,\theta) = f(x,\theta) + \int_\Lambda \frac{{\rm d}x'{\rm d}\theta'}{2\pi}\,\psi(x,\theta;x',\theta'),n(x',\theta')\,\partial_{x'}\partial_{\theta'} f^{\rm DR}(x',\theta')\ , \tag{D.5}$$

for f of type $(0,0)$, that is, scalar in both variables.

Alternatively, in terms of integral operators, we may write

$$\begin{aligned} &f^{\rm Dr} = (1-\partial_{\tilde x}\Psi\, n\,\partial_\theta)^{-1} f, \quad f^{\rm dR} = (1-\partial_{\tilde\theta}\Psi\, n\,\partial_x)^{-1} f,\\ &f^{\rm DR} = (1-\Psi\, n\,\partial_x\partial_\theta)^{-1} f. \end{aligned} \tag{D.6}$$

All dressing operations preserve the transformation type (i,j) of the function. Further, as per the requirements on ψ (see Eq.(2.6)), they act as follows:

$$\begin{aligned} {}^{\rm dr}:&\quad \mathcal{C}^\infty(\Lambda)\to\mathcal{C}^\infty(\Lambda)\\ {}^{\rm Dr}:&\quad \partial_\theta^{-1}C^\infty(\Lambda)\to\partial_\theta^{-1}\mathcal{C}^\infty(\Lambda),\\ {}^{\rm dR}:&\quad \partial_x^{-1}C^\infty(\Lambda)\to\partial_x^{-1}\mathcal{C}^\infty(\Lambda),\\ {}^{\rm DR}:&\quad \partial_x^{-1}\partial_\theta^{-1}C^\infty(\Lambda)\to\partial_x^{-1}\partial_\theta^{-1}\mathcal{C}^\infty(\Lambda), \end{aligned} \tag{D.7}$$

or rather, they do so at least in the context of $\mathcal{M} = \{|\rho|\le\rho_* : \rho\in\mathcal{C}_c^\infty(\Lambda)\}$, with a domain Λ either as in Eq. (2.1), or which falls under one of the categories discussed in Sect. 5.1. Moreover, the following basic properties are immediate from the definitions (D.6)

$$\begin{aligned} &{f_\theta}^{\rm dr} = (f^{\rm Dr})_\theta, \quad {f_x}^{\rm dr} = (f^{\rm dR})_x, \quad {f_\theta}^{\rm dR} = (f^{\rm DR})_\theta, \quad {f_x}^{\rm Dr} = (f^{\rm DR})_x,\\ &{f_{x\theta}}^{\rm dr} = (f^{\rm DR})_{x\theta}. \end{aligned} \tag{D.8}$$

These, along with the symmetry property (2.28), allow us to write the Poisson bracket (3.3) in various ways, which are nonetheless equivalent, involving different types of dressing operations

$$\begin{aligned} \{F,G\} &= \int_\Lambda \frac{{\rm d}x{\rm d}\theta}{2\pi}\, n\left[F'_x\left({G'}^{\rm Dr}\right)_\theta - G'_x\left({F'}^{\rm Dr}\right)_\theta\right]\\ &= \int_\Lambda \frac{{\rm d}x{\rm d}\theta}{2\pi}\, n\left[(F'_x)^{\rm dr}\, G'_\theta - (G'_x)^{\rm dr}\, F'_\theta\right]\\ &= \int_\Lambda \frac{{\rm d}x{\rm d}\theta}{2\pi}\, n\left[\left({F'}^{\rm dR}\right)_x G'_\theta - \left({G'}^{\rm dR}\right)_x F'_\theta\right]. \end{aligned} \tag{D.9}$$

Note that in the case of conventional GHD, in which the interaction kernel takes the form (2.10), both **dr**essing and **Dr**essing specialise to integral operators on spectral space θ only[16]

$$f^{\rm dr} = (1-\partial_{\tilde\theta}\Theta\, n)^{-1} f, \quad f^{\rm Dr} = (1-\Theta\, n\,\partial_\theta)^{-1} f \qquad \text{(conventional GHD with homogeneous coupling)} \tag{D.10}$$

where $\Theta = \phi/(2\pi)$. In this case, the **Dr**essing operation $f\mapsto f^{\rm Dr}$ is in fact a standard operation in the context of the Thermodynamic Bethe Ansatz, and

[16]The other dressing operations still make sense, but act on the full spectral phase space.

one writes it as

$$f^{\mathrm{Dr}}(\theta) = f(\theta) - \int_{\mathscr{P}} \mathrm{d}\theta' n(\theta') F(\theta'|\theta) \partial_{\theta'} f(\theta') \tag{D.11}$$

in terms of the "shift function" or "backflow function" [40, Chap 1], defined as

$$F(\theta|\alpha) = \frac{\phi(\cdot, \alpha)^{\mathrm{dr}}(\theta)}{2\pi}. \tag{D.12}$$

Since the **Dr**essing operation defined as such is known to satisfy the first equation of (D.8), we show the equivalence with our formulation (D.3) as follows

$$\begin{aligned} f^{\mathrm{Dr}}(\theta) &= f(\theta) - \int_{\mathscr{P}} \frac{\mathrm{d}\theta'}{2\pi} n(\theta') \phi(\cdot, \theta)^{\mathrm{dr}}(\theta') \partial_{\theta'} f(\theta') \\ &= f(\theta) - \int_{\mathscr{P}} \frac{\mathrm{d}\theta'}{2\pi} n(\theta') \phi(\theta', \theta) {f_\theta}^{\mathrm{dr}}(\theta') \\ &= f(\theta) - \int_{\mathscr{P}} \frac{\mathrm{d}\theta'}{2\pi} n(\theta') \phi(\theta', \theta) \partial_{\theta'} f^{\mathrm{Dr}}(\theta') \\ &= f(\theta) - \int_{\mathscr{P}} \frac{\mathrm{d}x' \mathrm{d}\theta'}{2\pi} \partial_{\tilde{x}} \frac{1}{2} \operatorname{sgn}(x - x') \phi(\theta', \theta) n(\theta') \partial_{\theta'} f^{\mathrm{Dr}}(\theta') \end{aligned} \tag{D.13}$$

where in the second line we used the symmetry (2.28), and in the third we used the first equation of (D.8). Note that the same result is obtained for $\tilde{x} = x$ or $\tilde{x} = -x'$. This indeed reproduces (D.3).

E. Properties of the Poisson Bracket

For operation (3.3) to define a valid Poisson bracket it must be bilinear, skew-symmetric, and satisfy both Leibniz's rule and the Jacobi identity. While the first two of these properties are trivially verified, the latter two are not. This appendix provides the proof that operation (3.3) indeed defines a valid Poisson bracket. On top of that, we show the linearised form (3.13) under the change of metric (3.10), which proves useful in establishing the involution (3.40).

The strategy is as follows. We prove the Leibniz property of the general Poisson bracket (3.3) directly in Sect. E.2. For the Jacobi identity, we proceed in two steps which take advantage of the fact that the validity of the Jacobi identity is independent of the choice of coordinates. In Sect. E.3, we show that the linearised Poisson bracket (E.14) in terms of $\hat{\rho}$ and the general Poisson bracket (3.3) in terms of ρ are equivalent under a change of fluid density coordinate. This also relies on a technical result proved in "Appendix F". Then the Jacobi identity for (E.14) is proved directly in Sect. E.4.

We start with some clarifications on the algebra of observables in Sect. E.1.

E.1. Algebra of Observables

Observables are appropriate functionals of $\rho \in \mathcal{M}$. One may wish to specify a general algebra of functionals on $\mathcal{M}$ and use, say, Fréchet differentiability theory for their functional derivatives. In particular, the product rule and chain rule hold for Fréchet derivatives. Instead, we take the more straightforward series form of observables described in Sect. 3.1, which we repeat for convenience:

$$F[\rho] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{2n}} c_n(x_1, \cdots, x_n; \theta_1, \cdots, \theta_n) \prod_{i=1}^{n} \rho(x_i, \theta_i) \mathrm{d}x_i \mathrm{d}\theta_i, \tag{E.1}$$

$c_0 \in \mathbb{R}$, $c_n \in \mathcal{D}m(\mathbb{R}^{2n})$, with the functional derivatives defined by (3.2). The product rule is shown to hold in the following but, to give a complete picture, we would have to show that functional derivatives are elements of $\mathcal{D}m(\mathbb{R}^2)$, as well as their dressing, and that the Poisson bracket (3.3) gives a functional within the required space. This is beyond the scope of this paper.

We will now make the space $\mathfrak{U}$ slightly more precise and show that it forms an algebra, both in the *abstract setup*, and in the *concrete setup* of Eq. (2.3) (see Sect. 2.1). We also show that functional derivatives, as defined in (3.2), satisfy the product rule, which is important for the Leibniz property of the Poisson bracket (Sect. E.2). We do not show the chain rule (used in Sect. E.3), which should nevertheless hold (as it does for instance for Fréchet derivatives). In the abstract setup, it is then relatively simple to show that the Poisson bracket (3.3) maps $\mathfrak{U} \otimes \mathfrak{U} \to \mathfrak{U}$. In the concrete setup, this should hold for ρ_* in (2.3) small enough (possibly smaller than (B.11)).

It is simpler for the discussion not to assume c_n in (E.1) to be symmetric. Instead, we consider equivalence classes $[c_n] = \{\tilde{c}_n : \tilde{c}_n \sim c_n\}$ of the functions c_n in (E.1) whereby two functions are related if they differ by a function which is antisymmetric under at least one element of the symmetric group S_n acting on the variables: $c_n \sim \tilde{c}_n$ if and only if there exists $f_n \in \mathcal{D}m(\mathbb{R}^{2n})$ and $\tau \in S_n$ such that

$$\begin{aligned} c_n &= \tilde{c}_n + f_n, \;\; f_n(x_1, \cdots, x_n; \theta_1, \cdots, \theta_n) \\ &= -f_n(x_{\tau(1)}, \cdots, x_{\tau(n)}; \theta_{\tau(1)}, \cdots, \theta_{\tau(n)}). \end{aligned} \tag{E.2}$$

It is easy to see that if the functionals $F[\rho]$, $\widetilde{F}[\rho]$ associated to c_n, $\tilde{c}_n$, $n \geq 0$, are such that $c_n \sim \tilde{c}_n$ for all $n \geq 0$, then $F[\rho] = \widetilde{F}[\rho]$ (in both setups). We denote $[\mathcal{D}m(\mathbb{R}^{2n})] = \{[c_n] : c_n \in \mathcal{D}m(\mathbb{R}^{2n})\}$ the space of equivalence classes of c_n. The concatenation of functions gives rise to a well-defined product, which we denote $\cdot$, and which acts as $\cdot : [\mathcal{D}m(\mathbb{R}^{2n})] \times [\mathcal{D}m(\mathbb{R}^{2\,m})] \to [\mathcal{D}m(\mathbb{R}^{2(n+m)})]$. On representatives it is

$$\begin{aligned} &(c_n \cdot c_m)(x_1, \ldots, x_{n+m}; \theta_1, \ldots, \theta_{n+m}) \\ &\quad = c_n(x_1, \ldots, x_n; \theta_1, \ldots, \theta_n) c_m(x_{n+1}, \ldots, x_{n+m}; \theta_{n+1}, \ldots, \theta_{n+m}), \end{aligned} \tag{E.3}$$

and we define[17] $[c_n] \cdot [c_m] = [c_n \cdot c_m]$.

In the *abstract setup*, $\mathfrak{U}$ is the algebra of formal functional series in ρ, that is, an observable is a sequence of equivalence classes of functions $F[\rho] = ([c_0], [c_1], [c_2], \ldots)$ and $\mathfrak{U} = \mathbb{R} \times [\mathcal{D}m(\mathbb{R}^2)] \times [\mathcal{D}m(\mathbb{R}^4)] \times \cdots$. The algebra product

[17]The fact the product $\cdot$ is well-defined follows naturally: for every f_n, f_m anti-symmetric, under some $\tau_n \in S_n$, $\tau_m \in S_m$, respectively, $(c_n + f_n) \cdot (c_m + f_m) = c_n \cdot c_m + f_{n+m}$ for some f_{n+m} which is anti-symmetric under some $\tau_{n+m} \in S_{n+m}$.

is obtained from the Cauchy product of series, which gives, for every $F[\rho] = (c_n)_n$, $G[\rho] = (d_n)_n \in \mathfrak{U}$,

$$(FG)[\rho] = \Big(\sum_{k=0}^{n}[c_k]\cdot[d_{n-k}]\Big)_n \in \mathfrak{U}. \tag{E.4}$$

The Cauchy product and concatenation of functions are commutative and associative operations, thus

$$FG = GF, \quad (F(GH)) = ((FG)H). \tag{E.5}$$

For the *concrete setup of Eq.* (2.3), an element of the algebra $\mathfrak{U}$ is an infinite series (E.1), with the algebra product taken as point-wise multiplication

$$(FG)[\rho] = F[\rho]G[\rho]. \tag{E.6}$$

We note that for every $\rho \in \mathcal{M}$, each integral in (E.1) exists and, in addition, we require that: (i) the series for $F[\rho]$ be absolutely convergent; (ii) the series for all functional derivatives of $F[\rho]$ (see (3.2) for the first derivative) and all their x, θ derivatives, be, as distributions (e.g. integrated against compactly supported smooth functions in the context of Eq. (2.3)), absolutely convergent. Absolute convergence guarantees that point-wise multiplication (E.6) is equivalent to the Cauchy product, hence to the algebra product defined as (E.4); absolute convergence is preserved under the algebra product.

Concerning functional derivatives, we may be more precise as follows. Recall that $F'[\rho](x,\theta)$ are given by Eq. (3.2) for the symmetric choice of representative c_n. We may define the operation $|_{(x,\theta)} : \mathcal{D}_{\mathcal{M}}(\mathbb{R}^{2n}) \to \mathcal{D}_{\mathcal{M}}(\mathbb{R}^{2(n-1)})$ as

$$c_n|_{(x,\theta)} = \sum_{k=1}^{n} c_n\left(x_1,\ldots,\underbrace{x}_{k^{\text{th}}\text{ position}},\ldots,x_{n-1};\theta_1,\ldots,\underbrace{\theta}_{k^{\text{th}}\text{ position}},\ldots,x_{n-1}\right). \tag{E.7}$$

One can show that this acts well on equivalence classes, so we can define $[c_n]|_{(x,\theta)} = [c_n|_{(x,\theta)}]$. We note that under concatenation, this operation acts as a *differentiation*,

$$(c_n \cdot c_m)|_{(x,\theta)} = c_n|_{(x,\theta)} \cdot c_m + c_n \cdot c_m|_{(x,\theta)}. \tag{E.8}$$

Then Eq. (3.2) is equivalent to

$$F'[\rho](x,\theta) = \sum_{n=1}^{\infty}\int_{\mathbb{R}^{2(n-1)}} c_n|_{(x,\theta)} \prod_{i=1}^{n-1}\rho(x_i,\theta_i)\mathrm{d}x_i\mathrm{d}\theta_i. \tag{E.9}$$

In the concrete setup, for every $F[\rho] \in \mathfrak{U}$ and every $\rho \in \mathcal{M}$, the series defining $F'[\rho](x,\theta)$ results in a function of (x,θ) that lies in $\mathcal{D}_{\mathcal{M}}(\mathbb{R}^2)$, as per our conditions. In the abstract setup, each coefficient in the series is, as a function of (x,θ), an element of $\mathcal{D}_{\mathcal{M}}(\mathbb{R}^2)$. This viewpoint is taken, for instance, when dressing the functional derivative, $(F'[\rho])^{\mathrm{dr}}(x,\theta)$ (dressing smooth functions or more generally distributions, see "Appendix B"), using linearity of the dressing operation in the abstract setup in order to apply it on each term of the series.

The functional derivative is also a map

$$\mathbb{R}^2 \to \mathfrak{U} : (x, \theta) \mapsto F'[\rho](x, \theta). \tag{E.10}$$

That is, for every $(x, \theta) \in \mathbb{R}^2$, the series for $F'[\rho](x, \theta)$ gives rise, as distribution, to an element of $\mathfrak{U}$. This is immediate in the abstract setup and follows from the conditions stated in the concrete setup. Therefore, the products $F'[\rho](x, \theta)G[\rho]$, etc., are well-defined.

The product rule then follows immediately from the Cauchy product and the differentiation-like property (E.8):

$$\begin{aligned}
(F[\rho]G[\rho])'(x, \theta) &= \left(\left(\sum_{k=0}^{n} [c_k] \cdot [d_{n-k}] \right)_n \right)' (x, \theta) \\
&= \left(\sum_{k=0}^{n} \left([c_k] \cdot [d_{n-k}] \right)|_{(x,\theta)} \right)_n \\
&= \left(\sum_{k=0}^{n} [c_k]|_{(x,\theta)} \cdot [d_{n-k}] + [c_k] \cdot [d_{n-k}]|_{(x,\theta)} \right)_n \\
&= F'[\rho](x, \theta)G[\rho] + F[\rho]G'[\rho](x, \theta) \; .
\end{aligned} \tag{E.11}$$

For $f \in \mathscr{D}_{\mathfrak{M}}(\mathbb{R}^2)$, its dressing $f^{\mathrm{dr}}(x, \theta)$ gives rise to an element of $\mathfrak{U}$ (again as a distribution). This follows from the Liouville-Neumann series; it is immediate in the abstract setup, but would necessitate a more accurate analysis in the concrete setup, which we omit here. A similar analysis can be done for $(F'[\rho])^{\mathrm{dr}}(x, \theta)$.

E.2. Leibniz Rule

We now show that the Poisson bracket (3.3) satisfies the Leibniz's rule

$$\{F_1F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}, \tag{E.12}$$

with the algebra product on $\mathfrak{U}$ defined as in Sect. E.1. Indeed, by writing (3.3), via integration by part and the symmetry property (2.28), as

$$\{F, G\} = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi} \left(F'\partial_x \left(nG'^{\,\mathrm{dr}}_\theta\right) - \partial_\theta \left(G'^{\,\mathrm{dr}}_x n\right) F' \right), \tag{E.13}$$

the product rule (E.11) immediately gives (E.12).

E.3. Equivalence of the Normal-Density and Fluid-Density Formulations

The Poisson bracket for functionals of the normal density is determined by the fundamental linearised bracket (3.13) along with Leibniz's rule, viz. Eq. (5.8) which we write here for generic $\hat{\mathcal{L}}_\theta$ (Remark 3.1):

$$\{F, G\} = \int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y \, \hat{\rho}(y, \theta) \left(\partial_y \frac{\delta F}{\delta\hat{\rho}(y, \theta)} \partial_\theta \frac{\delta G}{\delta\hat{\rho}(y, \theta)} - \partial_y \frac{\delta G}{\delta\hat{\rho}(y, \theta)} \partial_\theta \frac{\delta F}{\delta\hat{\rho}(y, \theta)} \right). \tag{E.14}$$

We now show that (E.14) implies (3.3); the steps can be retraced backwards to show the implication in the opposite direction. For our purposes, we will need to make use of the following identity

$$\frac{\delta F}{\delta\hat\rho(y,\theta)} = {F'}^{\mathrm{dR}}\left(X(y,\theta),\theta\right), \tag{E.15}$$

that expresses the variational derivative of a functional of type (3.1) in terms of the **dR**essing (D.4), and that we derive in "Appendix F". From this we may evaluate the first half of the Poisson bracket (E.14)

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}\theta \int_{\hat{\mathcal{L}}_\theta} \mathrm{d}y\, \hat\rho(y,\theta)\partial_y \frac{\delta F}{\delta\hat\rho(y,\theta)}\,\partial_\theta \frac{\delta G}{\delta\hat\rho(y,\theta)} \\
&\quad = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, \rho(x,\theta) \\
&\qquad \left(\partial_y \frac{\delta F}{\delta\hat\rho(y,\theta)}\right)\Big|_{y=Y(x,\theta)} \left(\partial_\theta \frac{\delta G}{\delta\hat\rho(Y(x,\theta),\theta)} - 2\pi\rho_{\mathrm{s}}(x,\theta)\left(\partial_y \frac{\delta G}{\delta\hat\rho(y,\theta)}\right)\Big|_{y=Y(x,\theta)}\right) \\
&\quad = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n(x,\theta)\,\partial_x \frac{\delta F}{\delta\hat\rho(Y(x,\theta),\theta)}\,\partial_\theta \frac{\delta G}{\delta\hat\rho(Y(x,\theta),\theta)} \\
&\qquad - 2\pi \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, \rho(x,\theta)\rho_{\mathrm{s}}(x,\theta)\left(\partial_y \frac{\delta F}{\delta\hat\rho(y,\theta)}\,\partial_y \frac{\delta G}{\delta\hat\rho(y,\theta)}\right)\Big|_{y=Y(x,\theta)}.
\end{aligned} \tag{E.16}$$

Similarly, we may also evaluate the second half of the bracket (E.14), obtained by exchanging $F \leftrightarrow G$, with the opposite sign. But we note that the second term on the right-hand side of the last equality is symmetric under $F \leftrightarrow G$ and hence cancels out in (E.14). Thus, omitting it and using (E.15), we find

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n(x,\theta)\,\partial_x {F'}^{\mathrm{dR}}(x,\theta)\partial_\theta {G'}^{\mathrm{dR}}(x,\theta) = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n(x,\theta)\, {F'_x}^{\mathrm{dr}}(x,\theta)\partial_\theta {G'}^{\mathrm{dR}}(x,\theta). \tag{E.17}$$

We use the integral-operator formulation $g^{\mathrm{dR}} = g + \Psi_{-\theta'} n\partial_x g^{\mathrm{dR}}$ in order to write the general identity

$$\partial_\theta g^{\mathrm{dR}} = g_\theta - \Psi_{\theta\theta'} {g_x}^{\mathrm{dr}}. \tag{E.18}$$

Therefore we obtain

$$\begin{aligned}
&= \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi}\, n(x,\theta)\, {F'_x}^{\mathrm{dr}}(x,\theta) G'_\theta(x,\theta) \\
&\quad + \int_{\mathbb{R}^4} \frac{\mathrm{d}x\mathrm{d}\theta\mathrm{d}x'\mathrm{d}\theta'}{(2\pi)^2}\, n(x,\theta)n(x',\theta')\,\psi_{\theta\theta'}(x,\theta;x',\theta'), {F'_x}^{\mathrm{dr}}(x,\theta){G'_x}^{\mathrm{dr}}(x',\theta').
\end{aligned} \tag{E.19}$$

As $\psi_{\theta\theta'}(x,\theta;x',\theta')$ is symmetric under exchange $(x,\theta) \leftrightarrow (x',\theta')$, in (E.14), where again we anti-symmetrise under $F \leftrightarrow G$, the double integral terms (i.e. the second term on the right-hand side of (E.19) and its companion from the other half of (E.14)) cancel out. The result is (3.3).

E.4. Jacobi Identity

Thanks to the equivalence shown in Sect. E.3, it is sufficient to show the Jacobi identity in the normal density formulation. Moreover, thanks to Leibniz's rule and to the fact that, in the normal density formulation, the space of linear functionals is preserved, it is sufficient to show that (3.13) satisfies

$$\sum_{\text{cyclic permutations } 1\to 2\to 3} \{\{\hat\rho(x_1,\theta_1),\hat\rho(x_2,\theta_2)\},\hat\rho(x_3,\theta_3)\} = 0. \tag{E.20}$$

That this implies the Jacobi identity for arbitrary functionals of $\hat\rho$ is then a standard result (see e.g. [95] Section 8.1).

We evaluate (in a short-hand notation)

$$\begin{aligned}&\{\{\hat\rho(x_1,\theta_1),\hat\rho(x_2,\theta_2)\},\hat\rho(x_3,\theta_3)\}\\ &\quad= \delta'_{x_{12}}\delta'_{\theta_{12}}\left(\delta'_{x_{13}}\delta'_{\theta_{23}}\left(\hat\rho(x_1,\theta_3)-\hat\rho(x_3,\theta_2)\right)-\delta'_{x_{23}}\delta'_{\theta_{13}}\left(\hat\rho(x_2,\theta_3)-\hat\rho(x_3,\theta_1)\right)\right).\end{aligned} \tag{E.21}$$

For convenience, we will refer to each term in the right-hand side as, from left to right, (i),(ii), (iii), and (iv). In every such term, we extract the derivatives with respect to the four variables that are not repeated[18]. For each term (i)–(iv), two new terms arise: a term of type I where the derivatives are applied to the full expression, and a term of type II where one of the derivatives is applied to the normal density $\hat\rho(\cdot,\cdot)$—the derivative that acts non-trivially on it. More explicitly, in the case of term (i), we have

$$\begin{aligned}&\text{Type I term:} && -\partial_{x_2}\partial_{\theta_1}\partial_{x_3}\partial_{\theta_3}\left[\left(\prod\delta\right)\hat\rho(x_1,\theta_3)\right],\\ &\text{Type II term:} && \partial_{x_2}\partial_{\theta_1}\partial_{x_3}\left[\left(\prod\delta\right)\partial_{\theta_3}\hat\rho(x_1,\theta_3)\right],\end{aligned} \tag{E.22}$$

where we introduced the notation

$$\left(\prod\delta\right) = \delta_{x_{12}}\delta_{\theta_{12}}\delta_{x_{13}}\delta_{\theta_{23}}. \tag{E.23}$$

In the terms of type I, there is a product of delta functions that imposes $x_1 = x_2 = x_3$ and $\theta_1 = \theta_2 = \theta_3$ (for instance $(\prod\delta) = \delta_{x_{12}}\delta_{\theta_{12}}\delta_{x_{13}}\delta_{\theta_{23}}$). Each such product is equivalent and, thus, in each term of type I, for (i)–(iv), we can set the normal density to $\hat\rho(x_1,\theta_1)$. Since terms (i) and (ii) have the same set of extracted derivatives and opposite sign, they cancel; similarly for terms (iii) and (iv). In the end, only terms of type II remain; by using the delta functions, in such terms we have either $\partial_{x_1}\hat\rho(x_1,\theta_1)$ or $\partial_{\theta_1}\hat\rho(x_1,\theta_1)$. Hence, we are left with

$$\begin{aligned}&\{\{\hat\rho(x_1,\theta_1),\hat\rho(x_2,\theta_2)\},\hat\rho(x_3,\theta_3)\}\\ &\quad= \left(\partial_{x_2}\partial_{\theta_1}\partial_{x_3}-\partial_{x_1}\partial_{\theta_2}\partial_{x_3}\right)\left[\left(\prod\delta\right)\partial_{\theta_1}\rho(x_1,\theta_1)\right]\\ &\qquad+\left(\partial_{x_1}\partial_{\theta_2}\partial_{\theta_3}-\partial_{x_2}\partial_{\theta_1}\partial_{\theta_3}\right)\left[\left(\prod\delta\right)\partial_{x_1}\rho(x_1,\theta_1)\right].\end{aligned} \tag{E.24}$$

[18] For instance, for terms (i) and (ii), with the factors $\delta'_{x_{12}}\delta'_{\theta_{12}}\delta'_{x_{13}}\delta'_{\theta_{23}}$, these are ∂_{x_2}, ∂_{θ_1}, ∂_{x_3}, ∂_{θ_3}.

The sum of cyclic permutations of the first line and of the second line both vanish, which shows the Jacobi identity (E.20).

F. Variational Derivative with Respect to the Normal Density $\hat{\rho}$

In this appendix, we derive the identity (E.15), expressing the variational derivative of a functional of type (3.1) in terms of the **dR**essing Eq. (D.4). We recall the identity for the sake of convenience:

$$\frac{\delta F}{\delta\hat{\rho}(y,\theta)} = F'^{\mathrm{dR}}(X(y,\theta),\theta) \tag{F.1}$$

where $X(y,\theta)$ is defined in (5.7).

For our purposes, we must first recall the relation between the spectral fluid density and its normal counterpart (3.12), viz.

$$\rho(x,\theta) = 2\pi\rho_{\mathrm{s}}(x,\theta)\hat{\rho}(Y(x,\theta),\theta). \tag{F.2}$$

We may now evaluate the functional derivative $\delta\rho(x,\theta)/\delta\hat{\rho}(y,\alpha)$, first by computing

$$\frac{\delta(2\pi\rho_{\mathrm{s}}(x,\theta))}{\delta\hat{\rho}(y,\alpha)} = \int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta'\,\partial_x\partial_\theta\psi(x,\theta;x',\theta')\frac{\delta\rho(x',\theta')}{\delta\hat{\rho}(y,\alpha)}, \tag{F.3}$$

and then, using the definition of ρ_s (2.25), and that of Y (3.11) in terms of the **dR**essing (D.4), along the dressing relations (D.8), yields

$$\frac{\delta Y(x,\theta)}{\delta\hat{\rho}(y,\alpha)} = \int_{\mathbb{R}^2} \mathrm{d}x'\mathrm{d}\theta'\,\partial_\theta\psi(x,\theta;x',\theta')\frac{\delta\rho(x',\theta')}{\delta\hat{\rho}(y,\alpha)}. \tag{F.4}$$

Putting all of this together, we eventually obtain

$$\begin{aligned}
\frac{\delta\rho(x,\theta)}{\delta\hat{\rho}(y,\alpha)} &= 2\pi\rho_{\mathrm{s}}(x,\theta)\,\delta(y-Y(x,\theta))\delta(\alpha-\theta) \\
&\quad + \left(\hat{\rho}(Y(x,\theta),\theta)\int_{\mathbb{R}^2}\mathrm{d}x'\,\mathrm{d}\theta'\,\partial_x\partial_\theta\psi(x,\theta;x',\theta')\right. \\
&\quad \left. + 2\pi\rho_{\mathrm{s}}(x,\theta)\partial_y\hat{\rho}(Y(x,\theta),\theta)\int_{\mathbb{R}^2}\mathrm{d}x'\mathrm{d}\theta'\,\partial_\theta\psi(x,\theta;x',\theta')\right)\frac{\delta\rho(x',\theta')}{\delta\hat{\rho}(y,\alpha)} \\
&= \delta(x-X(y,\alpha))\delta(\theta-\alpha) \\
&\quad + \left(n(x,\theta)\int_{\mathbb{R}^2},\frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\,\partial_x\partial_\theta\psi(x,\theta;x',\theta')\right. \\
&\quad \left. + \partial_x n(x,\theta)\int_{\mathbb{R}^2},\frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\,\partial_\theta\psi(x,\theta;x',\theta')\right)\frac{\delta\rho(x',\theta')}{\delta\hat{\rho}(y,\alpha)} \\
&\equiv \delta(x-X(y,\alpha))\delta(\theta-\alpha) + \left(\mathcal{O}\frac{\delta\rho(\cdot_1,\cdot_2)}{\delta\hat{\rho}(y,\alpha)}\right)(x,\theta),
\end{aligned} \tag{F.5}$$

where X is defined by (5.7), and where $\mathcal{O}$ is the operator

$$(\mathcal{O}f)(x,\theta) = \partial_x \left(n(x,\theta) \int_{\mathbb{R}^2} \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, \partial_\theta \psi(x,\theta;x',\theta') f(x',\theta') \right).$$

Its adjoint is given by

$$(\mathcal{O}^\dagger f)(x,\theta) = \int_{\mathbb{R}^2}, \frac{\mathrm{d}x'\mathrm{d}\theta'}{2\pi}\, n(x',\theta')\partial_{-\theta'}\psi(x,\theta;x',\theta')\partial_{x'} f(x',\theta'),$$

which is the operator appearing in the **dR**essing (for a function of type $(0,0)$). Therefore

$$\begin{aligned} \frac{\delta F}{\delta \hat{\rho}(y,\alpha)} &= \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, F'[\rho](x,\theta) \Big[(1-\mathcal{O})^{-1}\delta(\cdot_1 - X(y,\alpha))\delta(\cdot_2 - \alpha)\Big](x,\theta) \\ &= \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, ((1-\mathcal{O}^\dagger)^{-1}F'[\rho])(x,\theta)\delta(x - X(y,\alpha))\delta(\theta-\alpha) \\ &= \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, F'[\rho]^{\mathrm{dR}}(x,\theta)\, \delta(x - X(y,\alpha))\delta(\theta-\alpha) \\ &= F'[\rho]^{\mathrm{dR}}(X(y,\alpha),\alpha) \end{aligned} \tag{F.6}$$

which indeed yields the identity (E.15).

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