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RESEARCH ARTICLE

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On K-stability of $\mathbb{P}^3$ blown up along a (2,3) complete intersection

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Abstract

We prove K-stability of every smooth member of the Fano 3-fold family 2.15 of the Mori, Mukai and Iskovskikh classification.

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1 | INTRODUCTION

The existence of a Kähler–Einstein metric on a compact manifold X is a foundational problem in complex geometry. In the seminal series of papers [6–8, 21], the authors prove that such an existence has an algebro-geometric characterisation, known as K-polystability, and hence, solve the famous Yau–Tian–Donaldson conjecture. From then on, a great deal of work has been carried out to verify K-stability of Fano manifolds. Of particular importance is the work of Abban and Zhuang [2] where the authors introduce a new powerful inductive framework. These new techniques have been most notably used in [1] where the 105 families of smooth Fano 3-folds have been analysed. Despite the extensive investigation, the K-stability of all smooth members of certain families is still yet to be described making this one of the most trending problems of recent times. While there has been quite some progress made in this regard (see [4, 5, 9–12, 15–17, 19]), our paper makes an in-depth analysis of the K-stability of every smooth member of Fano 3-folds belonging to family 2.15.

This is particularly unique as compared to the previous cases, since members of this family belong to two distinct structural scenarios (see Section 3 for a detailed description). That is, each smooth member of family 2.15 is a 3-fold of Picard number 2 obtained as the blow-up of $\mathbb{P}^3$ in a (2,3)-complete intersection, see [1, Section 4.4] and references therein. In particular, this unique quadric Q containing the blown-up curve can be smooth (Subsection 3.1) or singular (Subsection 3.3). Each one of these cases involve techniques that are starkly different, with the geometry of singular quadric having to be exploited carefully to obtain the required result (using [15, Corollary 4.18 (2)]). Our main result is the following.

Theorem 1.1 (**Main theorem**. See Theorem 3.15). *Every smooth member of the Fano family 2.15, which is the blow-up of $\mathbb{P}^3$ in a curve given by the complete intersection of a quadric and a cubic, is K-stable.*

1.2 | Structure of the paper

In Section 2, we recall the preliminaries and the result from Abban–Zhuang theory that we use to prove the main result. In Section 3, after a brief presentation of the smooth members of the family, we show the main theorem by estimating the local stability threshold δ_p. The computations are split according to the position of the point p. Particular care has to be taken when the unique quadric containing the blown-up curve is singular.

2 | ABBAN–ZHUANG THEORY

In this section, we recall the definition of K-stability and the main results used in order to prove Theorem 3.15.

Definition 2.1. Let Δ be an effective $\mathbb{Q}$-divisor on a normal projective variety X for which $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that (X, Δ) is a log Fano pair if (X, Δ) is klt and $-(K_X + \Delta)$ is ample. If $\Delta = 0$, we call $(X, 0)$ a Fano variety and denote it by X.

We recall the notion of stability threshold (or δ-invariant) introduced in [13].

Definition 2.2. Let (X, Δ) be a log Fano pair, and let $f : Y \to X$ be a projective birational morphism such that Y is normal and let E be a prime divisor on Y. Let L be an ample $\mathbb{Q}$-Cartier divisor on X. We set

$$A_{X,\Delta}(E) = 1 + \mathrm{ord}_E(K_Y - f^*(K_X + \Delta)), \quad S_L(X) = \frac{1}{L^n}\int_0^\infty \mathrm{vol}(f^*(L) - uE)du.$$

We define the stability threshold as

$$\delta(X,\Delta;L) = \inf_{E/X} \frac{A_{X,\Delta}(E)}{S_L(X)},$$

where the infimum runs over all prime divisors over X. For a point $p \in X$, we define the local stability threshold as

$$\delta_p(X,\Delta;L) = \inf_{\substack{E/X \\ p\in C_X(E)}} \frac{A_{X,\Delta}(E)}{S_L(X)},$$

where the infimum runs over all prime divisors over X whose centres on X contain p.

It is proved in [3, 13, 18] that the following equivalence holds:

$$\delta(X) > 1 \iff X \text{ is K-stable.}$$

We will, in fact, take this to be our definition of K-stability of a Fano variety. Moreover,

$$\delta(X,\Delta;L) = \inf_{p\in X} \delta_p(X,\Delta;L).$$

Definition 2.3 [14, Definition 1.1]. Let Δ be an effective $\mathbb{Q}$-divisor on X and (X,Δ) be a klt pair. A prime divisor Y over X is said to be of plt-type over (X,Δ) if there is a projective birational morphism $\mu : \widetilde{X} \to X$ with $Y \subset \widetilde{X}$ such that $-Y$ is a μ-ample $\mathbb{Q}$-Cartier divisor on $\widetilde{X}$ for which $(\widetilde{X}, \widetilde{\Delta} + Y)$ is a plt pair where the $\mathbb{Q}$-divisor $\widetilde{\Delta}$ is defined by

$$K_{\widetilde{X}} + \widetilde{\Delta} + (1 - A_{X,\Delta}(Y))Y = \mu^*(K_X + \Delta).$$

Remark 2.4. The morphism μ is completely determined by Y and it is called the *plt-blow-up* associated to Y.

In the following, we study K-(semi)stability of certain Fano 3-folds X. We do this by employing the Abban–Zhuang theory developed in [2] to estimate the local stability threshold δ_p for every point in X. We recall the main results we need by referring to the book [1].

Given a smooth Fano 3-fold X, so that, in particular, $\mathrm{Nef}(X)=\overline{\mathrm{Mov}}(X)$ by [20], and a point $p \in X$ we consider flags $p \in Z \subset Y \subset X$ where:

- Y is an irreducible surface with at most Du Val singularities;
- Z is a non-singular curve such that (Y, Z) is plt.

We denote by Δ_Z the different of the log pair (Y, Z).

For $u \in \mathbb{R}$, we consider the divisor class $-K_X - uY$ and we denote by $\tau = \tau(Y)$ its pseudo-effective threshold, that is, the largest number for which $-K_X - uY$ is pseudo-effective. For $u \in [0, \tau]$, let $P(u)$ (respectively, $N(u)$) be the positive (respectively, negative) part of its Zariski decomposition. Since $Y \not\subset \mathrm{Supp}(N(u))$, we can consider the restriction $N(u)|_Y$ and define $N'_Y(u)$ to be its part not supported on Z, that is, $N'_Y(u)$ is the effective $\mathbb{R}$-divisor such that $Z \not\subset \mathrm{Supp}(N'_Y(u))$

defined by:

$$N(u)|_Y = d(u)Z + N'_Y(u),$$

where $d(u) := \mathrm{ord}_Z(N(u)|_Y)$.

We consider then for every $u \in [0, \tau]$ the restriction $P(u)|_Y$ and denote by $t(u)$ the pseudo-effective threshold of the divisor $P(u)|_Y - vZ$, by $P(u,v)$ and $N(u,v)$ the positive and negative part of its Zariski decomposition. Let $V^Y_{\bullet,\bullet}$ and $W^{Y,Z}_{\bullet,\bullet,\bullet}$ be the multigraded linear series defined in [1, Page 57].

Finally, we can state the main tool we use to estimate the local δ-invariant.

Theorem 2.5 [1, Theorem 1.112].

$$\delta_p(X) \geqslant \min\left\{\frac{1-\mathrm{ord}_p\Delta_Z}{S\left(W^{Y,Z}_{\bullet,\bullet,\bullet};p\right)}, \frac{1}{S\left(V^Y_{\bullet,\bullet};Z\right)}, \frac{1}{S_X(Y)}\right\},$$

where

$$S\left(V^Y_{\bullet,\bullet};Z\right) = \frac{3}{(-K_X)^3}\int_0^\tau (P(u)^2\cdot Y)\cdot \mathrm{ord}_Z(N(u)|_Y)du + \frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \mathrm{vol}(P(u)|_Y - vZ)dvdu, \tag{1}$$

and

$$S\left(W^{Y,Z}_{\bullet,\bullet,\bullet};p\right) = \frac{3}{(-K_X)^3}\int_0^\tau\int_0^{t(u)} (P(u,v)\cdot Z)^2 dvdu + F_p\left(W^{Y,Z}_{\bullet,\bullet,\bullet}\right), \tag{2}$$

with

$$F_p\left(W^{Y,Z}_{\bullet,\bullet,\bullet}\right) = \frac{6}{(-K_X)^3}\int_0^\tau\int_0^{t(u)} (P(u,v)\cdot Z)\cdot \mathrm{ord}_p(N'_Y(u)|_Z + N(u,v)|_Z)dvdu. \tag{3}$$

The theorem above admits a slight generalisation which allows to consider not only flags of varieties in X, but also over X. In particular, let X and Y be as above, in order to estimate δ_p for $p \in Y$, it turns out to be useful to consider curves over Y. For this, let $\sigma : \widetilde{Y} \to Y$ be a plt blow-up of Y in p and denote by $\widetilde{Z}$ its exceptional divisor. We consider the linear system $\sigma^*(P(u)|_Y) - v\widetilde{Z}$ and denote by $\widetilde{t}(u)$ its pseudo-effective threshold, that is,

$$\widetilde{t}(u) = \max\{v \in \mathbb{R}_{\geqslant 0} \ : \ \sigma^*(P(u)|_Y) - v\widetilde{Z} \text{ is pseudo} - effective\}.$$

For every $v \in [0, \widetilde{t}(u)]$, we denote by $\widetilde{P}(u,v)$ and $\widetilde{N}(u,v)$ the positive and negative part of its Zariski decomposition. We also denote by $N'_{\widetilde{Y}}(u)$ the strict transform of the divisor $N(u)|_Y$.

Theorem 2.6 [1, Remark 1.113].

$$\delta_p(X) \geqslant \min\left\{ \min_{q\in\widetilde{Z}} \frac{1-\mathrm{ord}_q \Delta_{\widetilde{Z}}}{S\left(W^{Y,\widetilde{Z}}_{\bullet,\bullet,\bullet};q\right)}, \frac{A_Y(\widetilde{Z})}{S\left(V^Y_{\bullet,\bullet};\widetilde{Z}\right)}, \frac{1}{S_X(Y)} \right\},$$

where

$$S\left(V^Y_{\bullet,\bullet};\widetilde{Z}\right) = \frac{3}{(-K_X)^3}\int_0^\tau \sigma^*(P(u)|Y)^2 \cdot \mathrm{ord}_{\widetilde{Z}}(\sigma^*(N(u)|_Y))du + \frac{3}{(-K_X)^3}\int_0^\tau \int_0^\infty \mathrm{vol}\left(\sigma^*(P(u)|_Y) - v\widetilde{Z}\right)dvdu, \tag{4}$$

and

$$S\left(W^{Y,\widetilde{Z}}_{\bullet,\bullet,\bullet};q\right) = \frac{3}{(-K_X)^3}\int_0^\tau \int_0^{\tilde{t}(u)} (\widetilde{P}(u,v)\cdot\widetilde{Z})^2 dvdu + F_q\left(W^{Y,\widetilde{Z}}_{\bullet,\bullet,\bullet}\right), \tag{5}$$

with

$$F_q\left(W^{Y,\widetilde{Z}}_{\bullet,\bullet,\bullet}\right) = \frac{6}{(-K_X)^3}\int_0^\tau \int_0^{\tilde{t}(u)} (\widetilde{P}(u,v)\cdot\widetilde{Z})\cdot \mathrm{ord}_q(N'_{\widetilde{Y}}(u)|_{\widetilde{Z}} + \widetilde{N}(u,v)|_{\widetilde{Z}})dvdu. \tag{6}$$

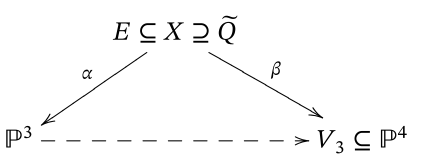
3 | K-STABILITY OF THE FAMILY 2.15

We briefly review the geometry of a smooth Fano 3-fold in the family 2.15.

Let $\mathscr{C} \subset \mathbb{P}^3$ be the complete intersection of a quadric $Q = (f_2 = 0)$ and a cubic $S_3 = (f_3 = 0)$. We are interested in the K-stability of the blow-up $X := \mathrm{Bl}_{\mathscr{C}}\,\mathbb{P}^3$. We stress the fact that the quadric Q can be either smooth or a quadric cone. Let $\alpha : X \to \mathbb{P}^3$ be the projection, E the exceptional divisor and $\widetilde{Q}$ the strict transform of Q. The linear system of cubics vanishing along $\mathscr{C}$ gives a rational map:

$$\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$$
$$[x : y : z : w] \mapsto [xf_2 : yf_2 : zf_2 : wf_2 : f_3].$$

with indeterminacy locus $\mathscr{C}$. The blow-up X is a resolution of indeterminacy of φ fitting in the diagram



where β contracts $\widetilde{Q}$ to a point and maps X to a cubic 3-fold V_3 singular only at the point $\beta(\widetilde{Q}) = [0 : 0 : 0 : 0 : 1]$. It is an ordinary double point if Q is smooth and an A_2 singularity if Q is a cone.

We denote by $H \in \mathrm{NS}(X)$ the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ along α. The Neron–Severi group of X is generated by H and E and its anti-canonical divisor is given by

$$-K_X = 4H - E = 2H + \widetilde{Q} = 2\widetilde{Q} + E,$$

where we used the equality $\widetilde{Q} = 2H - E$. We denote by $f_1 \in N_1(X)$ the class of the fibre of the restriction $\alpha|_E : E \to \mathscr{C}$ and by $f_2 \in N_1(X)$ the class of a ruling of $\widetilde{Q}$ so that the Mori cone is $\overline{NE}(X) = \mathbb{R}_{\geqslant 0} f_1 + \mathbb{R}_{\geqslant 0} f_2$. The intersection numbers are as follows:

$$\begin{aligned}
&E \cdot f_1 = \widetilde{Q} \cdot f_2 = -1, \quad E \cdot f_2 = 3\\
&H \cdot f_2 = \widetilde{Q} \cdot f_1 = 1, \quad H \cdot f_1 = 0,\\
&H^3 = 1, \quad H \cdot E^2 = -6, \quad H^2 \cdot E = 0, \quad \text{and}\\
&E^3 = -\deg N_{\mathscr{C}|\mathbb{P}^3} = -2g + 2 + K_{\mathbb{P}^3} \cdot \mathscr{C} = -30.
\end{aligned}$$

3.1 | Estimate of δ_p for p in $\widetilde{Q}$ when Q is a smooth quadric

In this section, we estimate the K-stability threshold δ_p for a point $p \in \widetilde{Q}$ by applying Theorem 2.5 to a specific flag.

Proposition 3.2. *If p is a point in $\widetilde{Q}$ and not in E, then*

$$\delta_p(X) = \frac{44}{37}$$

and it is computed by the divisor $\widetilde{Q}$ in X. If $p \in E \cap \widetilde{Q}$, then

$$\delta_p(X) \geqslant \frac{8}{7}.$$

Proof. Given a point $p \in \widetilde{Q}$, we consider the flag

$$p \in L \subset \widetilde{Q} \subset X,$$

where L is a line of $\widetilde{Q}$ through p which is not tangent to the curve $E \cap \widetilde{Q}$ at p, or equivalently, whose image under the map α is not tangent to $\mathscr{C}$ at $\alpha(p)$.

We start by computing $S_X(\widetilde{Q})$. For this, we consider the linear system $K_X - u\widetilde{Q} = E + (2-u)\widetilde{Q}$ for $u \in \mathbb{R}$. Clearly, its pseudo-effective threshold is $\tau = 2$. The Zariski decomposition is given by:

$$P(u) = \begin{cases} (4-2u)H + (u-1)E & \text{if } u \in [0,1], \\ (4-2u)H & \text{if } u \in [1,2], \end{cases} \quad \text{and} \quad N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)E & \text{if } u \in [1,2]. \end{cases}$$

Therefore, the volume can be computed to be:

$$\mathrm{vol}(-K_X - u\widetilde{Q}) = (P(u))^3 = \begin{cases} 22 - 6u - 6u^2 - 2u^3 & \text{if } u \in [0,1], \\ 64 - 96u + 48u^2 - 8u^3 & \text{if } u \in [1,2]. \end{cases}$$

Hence, we get

$$S_X(\widetilde{Q}) = \frac{1}{(-K_X)^3}\int_0^{\tau(\widetilde{Q})} \mathrm{vol}\big(-K_X - u\widetilde{Q}\big)du = \frac{37}{44}. \tag{7}$$

We move on to compute the value $S(V^{\widetilde{Q}}_{\bullet,\bullet};L)$. For this, let ℓ_1, ℓ_2 the classes of the rulings of $\widetilde{Q}$ so that the class of L is ℓ_1, we consider for $v \in \mathbb{R}_{\geqslant 0}$ the linear system:

$$P(u)|_{\widetilde{Q}} - vL = \begin{cases} (1+u-v)\ell_1 + (1+u)\ell_2 & \text{if } u \in [0,1], \\ (4-2u-v)\ell_1 + (4-2u)\ell_2 & \text{if } u \in [1,2]. \end{cases}$$

The nefness and bigness of the above linear system is readily checked and its Zariski decomposition is given by

$$P(u,v) = \begin{cases} (1+u-v)\ell_1 + (1+u)\ell_2 & \text{if } u \in [0,1],\ v \in [0,1+u] \\ (4-2u-v)\ell_1 + (4-2u)\ell_2 & \text{if } u \in [1,2],\ v \in [0,4-2u], \end{cases} \quad N(u,v) = \begin{cases} 0 \\ 0. \end{cases}$$

Hence,

$$\mathrm{vol}(P(u)|_{\widetilde{Q}} - vL) = \begin{cases} 2(1+u-v)(1+u) & \text{if } u \in [0,1],\ v \in [0,1+u] \\ 4(4-2u-v)(2-u) & \text{if } u \in [1,2],\ v \in [0,4-2u]. \end{cases}$$

We note that the restriction of the divisor E to $\widetilde{Q}$ consists of an irreducible curve which is isomorphically mapped to $\mathscr{C}$ by the blow-up morphism α. In particular, we see that $E|_{\widetilde{Q}}$ has no support on L and the negative part $N(u)$ does not contribute in the formula (1) and we get:

$$S\Big(V^{\widetilde{Q}}_{\bullet,\bullet};L\Big) = \frac{69}{88}. \tag{8}$$

We move on now to compute $S\left(W^{\widetilde{Q},L}_{\bullet,\bullet,\bullet};p\right)$.

If the point $p \in \widetilde{Q}\setminus E$, then the order of $E|_{\widetilde{Q}}$ at p is trivial; hence, the value $F_p\left(W^{\widetilde{Q},L}_{\bullet,\bullet,\bullet}\right)$ of (3) is zero. A direct computation gives the value of (2):

$$S\Big(W^{\widetilde{Q},L}_{\bullet,\bullet,\bullet};p\Big) = \frac{69}{88}. \tag{9}$$

On the other hand, if the point p is in $\widetilde{Q}\cap E$, the value F_p in (3) is not trivial. First of all, we notice that L is not contained in $E|_{\widetilde{Q}}$, so we have $N(u) = N'_{\widetilde{Q}}(u)$. Secondly, since in the choice of the flag, we assumed that L intersects $E\cap\widetilde{Q}$ transversely we have $\mathrm{ord}_p(N'_{\widetilde{Q}}(u)|_L) = u-1$ if $u \in [1,2]$. For the value in (3), we therefore get:

$$F_p = \frac{1}{11}. \tag{10}$$

If $p \notin E$, the values $S_X(\widetilde{Q})$, $S(V^{\widetilde{Q}}_{\bullet,\bullet};L)$ and $S(W^{\widetilde{Q},L}_{\bullet,\bullet,\bullet};p)$ are computed in the formulas (7)–(9), so that:

$$\frac{44}{37} = \frac{1}{S_X(\widetilde{Q})} \geqslant \delta_p(X) \geqslant \min\left\{\frac{44}{37}, \frac{88}{69}, \frac{88}{69}\right\} = \frac{44}{37}.$$

If the point p is in E, the value $S(W^{\widetilde{Q},L}_{\bullet,\bullet,\bullet};p)$ is obtained by summing up also F_p, which is computed in (10) and one gets:

$$\delta_p(X) \geqslant \min\left\{\frac{44}{37}, \frac{88}{69}, \frac{8}{7}\right\} = \frac{8}{7}.$$

This concludes the proof. □

3.3 | Estimate of δ_p for p in $\widetilde{Q}$ when Q is a quadric cone

We divide the computations in two separate cases: These are when p is the vertex of the quadric cone or p is away from it.

3.3.1 | p is the vertex of the quadric cone

Let $\pi : \widehat{X} \to X$ be the blow-up of X at p with exceptional divisor $G \simeq \mathbb{P}^2$. Let $\widehat{Q}$ be the strict transform of $\widetilde{Q}$ in $\widehat{X}$. Since $\widehat{Q} = \pi^*\widetilde{Q} - 2G$ and $-K_X = 2\widetilde{Q} + E$, we have

$$\pi^*(-K_X) - uG = 2\widehat{Q} + \widehat{E} + (4-u)G, \tag{11}$$

where $\widehat{E} \simeq E$ is the strict transform of E in $\widehat{X}$.

Lemma 3.4. *The pseudo-effective threshold τ of the linear system $\pi^*(-K_X) - uG$ is $\tau = 4$.*

Proof. From Equation (11), we clearly we have that $\tau \geqslant 4$. In order to prove the equality, it is enough to show that the divisor $2\widehat{Q} + \widehat{E}$ is not big. For this, let $\gamma : \widehat{X} \to \mathrm{Bl}_{\alpha(p)}\mathbb{P}^3$ be the divisorial contraction of $\widehat{E}$. Since the pushforward of a big divisor along a birational morphism is big, in order to show the claim, it is enough to show that $\gamma_*\widehat{Q}$ is not big. For this, notice that $\mathrm{Bl}_{\alpha(p)}\mathbb{P}^3$ is the resolution of indeterminacy of the projection from $\alpha(p)$ and is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, $h : \mathrm{Bl}_{\alpha(p)}\mathbb{P}^3 \to \mathbb{P}^2$, which contracts $\gamma(\widehat{Q})$ to a conic. In particular, $\gamma(\widehat{Q}) \equiv h^*\mathcal{O}_{\mathbb{P}^2}(2)$ is not big. The claim is proven. □

Let l, f_G and f_E be the ruling of $\widehat{Q}$, the class in Pic(G) of a line of G and a fibre of E, respectively. We have the following intersection numbers:

	l	f_G	f_E
$\widehat{Q}$	−3	2	1
G	1	−1	0
$\widehat{E}$	3	0	−1

Moreover,

$$\widehat{Q}^2 \cdot \widehat{E} = -6, \quad \widehat{Q} \cdot G^2 = -2, \quad \widehat{Q}^2 \cdot G = 4, \quad G^2 \cdot \widehat{E} = G \cdot \widehat{E}^2 = 0,$$
$$\widehat{E}^3 = -30 \quad \widehat{Q} \cdot \widehat{E}^2 = 18, \quad \widehat{Q}^3 = -6, \quad \widehat{Q} \cdot \widehat{E} \cdot G = 0, \quad G^3 = 1.$$

Proposition 3.5. *If p is the vertex of the quadric cone $\widetilde{Q}$, then*

$$\delta_p(X) = \frac{11}{10},$$

and it is computed by the exceptional divisor G corresponding to the ordinary blow-up of X at p.

Proof. By [Corollary 4.18 (2)], we have

$$\frac{A_X(G)}{S_X(G)} \geqslant \delta_p(X) \geqslant \min\left\{ \frac{A_X(G)}{S_X(G)}, \inf_{q\in G} \delta_q\left(G, \Delta_G; V^G_{\bullet,\bullet}\right)\right\}. \tag{12}$$

We compute first $\frac{A_X(G)}{S_X(G)}$ and then show that this is the bound given by the right-hand side of the second inequality of (12). Let $P(u)$ and $N(u)$ be the positive and negative parts of $\pi^*(-K_X) - uG$. We have

$$P(u) = \begin{cases} 2\widehat{Q} + \widehat{E} + (4-u)G & \text{if } u \in [0,1], \\ \frac{7-u}{3}\widehat{Q} + \widehat{E} + (4-u)G & \text{if } u \in [1,4], \end{cases} \text{ and } N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ \frac{(u-1)}{3}\widehat{Q} & \text{if } u \in [1,4]. \end{cases}$$

A direct computation gives

$$\frac{A_X(G)}{S_X(G)} = \frac{11}{10}.$$

We now compute $\inf_{q\in G} \delta_q(G, \Delta_G; V^G_{\bullet,\bullet})$.

- Suppose $q \notin \widehat{Q}|_G$.

For every such point, we choose a flag $q \in L \subset G$, where L is a line in G. Then, by [2, Theorem 3.2],

$$\delta_q\left(G, \Delta_G; W^G_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{1}{S\left(W^G_{\bullet,\bullet}; L\right)}, \frac{1-\mathrm{ord}_q\Delta_L}{S\left(W^{G,L}_{\bullet,\bullet,\bullet}; q\right)}\right\}.$$

Let $P(u,v)$ and $N(u,v)$ be the positive and negative parts of $P(u)|_G - vL$. These are given by

$$P(u,v) = \begin{cases} (u-v)L & \text{if } u \in [0,1],\ v \in [0,u], \\ \left(\frac{2+u}{3} - v\right)L & \text{if } u \in [1,4],\ v \in [0, \frac{2+u}{3}], \end{cases} \quad \text{and} \quad N(u,v) = 0.$$

Notice that $\mathrm{ord}_L(N(u)|_G) = 0$ since $\widehat{Q}|_G$ is not supported on L and $\mathrm{ord}_q(N'_G(u)|_L + N(u,v)|_L) = 0$ since $q \notin \widehat{Q}|_G$. Hence,

$$\frac{1}{S\left(W^G_{\bullet,\bullet};L\right)} = \frac{1-\mathrm{ord}_q\Delta_L}{S\left(W^{G,L}_{\bullet,\bullet,\bullet};q\right)} = \frac{44}{23}.$$

- Suppose $q \in \widehat{Q}|_G$.

We denote by $\eta : \widehat{G} \to G$ the (1,2)-weighted blow-up of q with exceptional divisor $F \simeq \mathbb{P}(1,2)$. By [15, Corollary 4.18 (1)], we have

$$\delta_q\left(G,\Delta_G;W^{\widehat{G}}_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{A_G(F)}{S\left(V^{\widehat{G}}_{\bullet,\bullet};F\right)}, \inf_{\substack{q'\in F\\ \eta(q')=q}} \frac{A_{F,\Delta_F}(q')}{S\left(W^{\widehat{G},F}_{\bullet,\bullet,\bullet};q'\right)}\right\}. \tag{13}$$

The surface $\widehat{G}$ has an A_1 singular point q_0 lying on F. Denote by C the conic $\widehat{Q}|_G$ and by ℓ_T the line tangent to C at q. Their strict transforms $\widetilde{C}$ and $\widetilde{\ell_T}$ intersect F at a regular point of $\widehat{G}$. We have

$$\widetilde{C} = \eta^*C - 2F, \qquad \widetilde{\ell_T} = \eta^*\ell_T - 2F, \text{ and}$$

$$\widetilde{\ell_T}^2 = -1, \quad \widetilde{C}^2 = 2, \quad F^2 = -\frac{1}{2}, \quad \widetilde{\ell_T}\cdot F = 1.$$

We consider the linear system

$$\eta^*(P(u)|_G) - vF = \begin{cases} u\widetilde{\ell_T} + (2u-v)F & \text{if } u \in [0,1],\\ \frac{2+u}{3}\widetilde{\ell_T} + \left(\frac{2}{3}(2+u) - v\right)F & \text{if } u \in [1,4].\end{cases}$$

Then, its Zariski decomposition has positive part

$$\widetilde{P}(u,v) = \begin{cases} u\widetilde{\ell_T} + (2u-v)F & \text{if } u \in [0,1]\ v \in [0,u]\\ (2u-v)(\widetilde{\ell_T}+F) & \text{if } u \in [0,1]\ v \in [u,2u]\\ \frac{2+u}{3}\widetilde{\ell_T} + \left(\frac{4+2u}{3} - v\right)F & \text{if } u \in [1,4]\ v \in \left[0,\frac{2+u}{3}\right]\\ \frac{4+2u}{3}(\widetilde{\ell_T}+F) & \text{if } u \in [1,4]\ v \in \left[\frac{2+u}{3},\frac{4+2u}{3}\right],\end{cases}$$

and negative part

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } u \in [0,1] v \in [0,u]\\ (v-u)\widetilde{\ell_T} & \text{if } u \in [0,1] v \in [u,2u]\\ 0 & \text{if } u \in [1,4]\ v \in [0,\frac{2+u}{3}]\\ (v-\frac{2+u}{3})\widetilde{\ell_T} & \text{if } u \in [1,4]\ v \in \left[\frac{2+u}{3},\frac{4+2u}{3}\right].\end{cases}$$

Notice that

$$\operatorname{ord}_F(\eta^* N(u)|_G) = \begin{cases} 0 & \text{if } u \in [0,1] \\ \operatorname{ord}_F\left(\frac{u-1}{3}\eta^* C\right) & \text{if } u \in [1,4] \end{cases} = \begin{cases} 0 & \text{if } u \in [0,1], \\ \frac{2}{3}(u-1) & \text{if } u \in [1,4]. \end{cases}$$

A direct computation gives

$$\frac{A_G(F)}{S\left(V^{\widehat{G}}_{\bullet,\bullet};F\right)} = \frac{11}{10}.$$

We now compute the second term in formula (13). For $u \in [0,1]$,

$$\begin{aligned} \operatorname{ord}_{q'}(\eta^*(N'_{\widehat{G}}(u)|_F + N(u,v)|_F)) &= \operatorname{ord}_{q'}(\eta^* N(u,v)|_F) \\ &= \operatorname{ord}_{q'}((v-u)\widetilde{\ell_T}|_F) \\ &= \begin{cases} 0 & \text{if } q' \notin \widetilde{\ell_T}, \\ v-u & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, for $u \in [1,4]$,

$$\begin{aligned} \operatorname{ord}_{q'}(\eta^*(N'_{\widehat{G}}(u)|_F + N(u,v)|_F)) &= \operatorname{ord}_{q'}\left(\frac{u-1}{3}\widetilde{C}|_F + \left(v - \frac{2+u}{3}\right)\widetilde{\ell_T}|_F\right) \\ &= \begin{cases} 0 & \text{if } q' \notin \widetilde{\ell_T} \cup \widetilde{C}, \\ \frac{u-1}{3} & \text{if } q' \in \widetilde{C}, \\ v - \frac{2+u}{3} & \text{if } q' \in \widetilde{\ell_T}. \end{cases} \end{aligned}$$

Then,

$$S(W^{\widetilde{G},F}_{\bullet,\bullet,\bullet};q') = \begin{cases} \frac{23}{88} & \text{if } q' \notin \widetilde{\ell_T} \cup \widetilde{C}, \\ \frac{37}{44} & \text{if } q' \in \widetilde{C}, \\ \frac{23}{44} & \text{if } q' \in \widetilde{\ell_T}. \end{cases}$$

Moreover, $A_{F,\Delta_F}(q') = 1$ for every $q' \in \widetilde{F}$ except when q' is the A_1 singularity introduced by η, in which case it is $\frac{1}{2}$. Hence,

$$\begin{aligned} \inf_{\substack{q' \in F \\ \eta(q')=q}} \frac{A_{F,\Delta_F}(q')}{S(W^{\widehat{G},F}_{\bullet,\bullet,\bullet};q')} &= \min\left\{\frac{1}{23/88}, \frac{1/2}{23/88}, \frac{1}{23/44}, \frac{1}{37/44}\right\} \\ &= \min\left\{\frac{88}{23}, \frac{88}{46}, \frac{44}{23}, \frac{37}{44}\right\} \\ &= \frac{44}{37}. \end{aligned}$$

Therefore,

$$\delta_q\left(G, \Delta_G; W^{\widehat{G}}_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{11}{10}, \frac{44}{37}\right\} = \frac{11}{10}$$

for $q \in C$.

Putting together the cases, $q \notin C$ and $q \in C$, we have indeed

$$\delta_q\left(G, \Delta_G; W^{\widehat{G}}_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{11}{10}, \frac{44}{23}\right\} = \frac{11}{10}.$$

Hence,

$$\delta_p(X) \geqslant \frac{11}{10},$$

and the claim follows. □

3.3.1 | The point p is away from the vertex of the quadric cone

Let p be any point in $\widetilde{Q}$ such that $\alpha(p)$ is not the vertex of Q. Let $\Pi \subset \mathbb{P}^3$ be a general hyperplane containing the point $\alpha(p)$. Its strict transform S in X is isomorphic to the blow-up of Π in the six points $p_1, \ldots, p_6$ given by the intersection of $\Pi \cap \mathscr{C}$. In particular, the points p_i lie on the conic $C = Q \cap \Pi$.

We consider the blow-up $\sigma : \widetilde{S} \to S$ in the point p with exceptional divisor F. We denote by $\widetilde{C}$ the strict transform of C in $\widetilde{S}$, by $E_1, \ldots, E_6$ the curves lying over the points $p_1, \ldots, p_6$ and by L_j the strict transform of the line through the points $\alpha(p)$ and p_j for $j = 1, \ldots, 6$. Finally, let h denote the pullback of a line in $\mathbb{P}^3$ lying in the hyperplane section considered.

Proposition 3.6. *Assume that Q is a quadric cone. Let $p \in X$ be a point such that $\alpha(p) \in Q$ is away from the vertex. Then,*

$$\delta_p(X) \geqslant \frac{44}{43}.$$

Proof. The result follows from applying Theorem 2.6 to the flag consisting of the strict transform S of a hyperplane in $\mathbb{P}^3$, the exceptional curve F in $\widetilde{S}$.

We consider the linear system $-K_X - uS$. Its Zariski decomposition is then given by

$$P(u) = \begin{cases} (4-u)H - E & \text{if } u \in [0,1], \\ (6-3u)H + (u-2)E & \text{if } u \in [1,2], \end{cases} \text{ and } N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)\widetilde{Q} & \text{if } u \in [1,2]. \end{cases}$$

A direct computation gives

$$\frac{A_X(S)}{S_X(S)} = \frac{44}{23}. \tag{14}$$

We consider then the linear system

$$D = \sigma^*(P(u)|_S) - vF = \begin{cases} (4-u)h - \sum_{i=1}^6 E_i - vF & \text{if } u \in [0,1], \\ (6-3u)h - (2-u)\sum_{i=1}^6 E_i - vF & \text{if } u \in [1,2]. \end{cases}$$

Its Zariski decomposition for $u \in [0,1]$ is given by

$$P = \begin{cases} D & \text{if } v \in [0, 2-2u], \\ D - a\widetilde{C} & \text{if } v \in [2-2u, 3-u], \\ D - a\widetilde{C} - b\sum_{i=1}^6 L_j & \text{if } v \in [3-u, \frac{1}{4}(14-5u)]. \end{cases} \quad \text{and}$$

$$N = \begin{cases} 0 & \text{if } v \in [0, 2-2u], \\ a\widetilde{C} & \text{if } v \in [2-2u, 3-u], \\ a\widetilde{C} + b\sum_{i=1}^6 L_j & \text{if } v \in [3-u, \frac{1}{4}(14-5u)], \end{cases}$$

where $a = \frac{1}{3}(v + 2u - 2)$ and $b = v - 3 + u$. For $u \in [1,2]$, it is given by

$$P = \begin{cases} D - a\widetilde{C} & \text{if } v \in [0, 4-2u], \\ D - a\widetilde{C} - b\sum_{j=1}^6 L_j & \text{if } v \in [4-2u, \frac{1}{4}(18-9u)] \end{cases} \quad \text{and}$$

$$N = \begin{cases} a\widetilde{C} & \text{if } v \in [0, 4-2u], \\ a\widetilde{C} + b\sum_{j=1}^6 L_j & \text{if } v \in [4-2u, \frac{1}{4}(18-9u)], \end{cases}$$

where $a = \frac{v}{3}$ and $b = v - 4 + 2u$. Hence, for $u \in [0,1]$, the volume of the divisor D is

$$\mathrm{vol}(D) = P^2 = \begin{cases} u^2 - v^2 - 8u + 10 & \text{if } v \in [0, 2-2u], \\ \frac{1}{3}(7u^2 + 4uv - 2v^2 - 32u - 4v + 34) & \text{if } v \in [2-2u, 3-u], \\ \frac{1}{3}(5u + 4v - 14)^2 & \text{if } v \in [3-u, \frac{1}{4}(14-5u)] \end{cases}$$

and for $u \in [1,2]$

$$\mathrm{vol}(D) = P^2 = \begin{cases} u^2 - v^2 - 8u + 10 & \text{if } v \in [0, 4-2u], \\ \frac{1}{3}(7u^2 + 4uv - 2v^2 - 32u - 4v + 34) & \text{if } v \in [4-2u, \frac{1}{4}(18-9u)]. \end{cases}$$

We note that for $u \in [1,2]$, the contribution of the negative part in (4) is $\mathrm{ord}_F(\sigma^* N(u)|_S) = \mathrm{ord}_F((u-1)(\widetilde{C} + F)) = u - 1$. So, the value can be computed

$$\frac{A_S(F)}{S(V^S_{\bullet,\bullet}; F)} = \frac{8}{7}. \tag{15}$$

We now compute $S(W^{S,F}_{\bullet,\bullet,\bullet}; q)$. Since $\widetilde{S}$ is smooth, the different Δ_F is trivial, while the value of $F_q(W^{S,F}_{\bullet,\bullet,\bullet})$ depends on the position of q in F. We split thus into following three cases:

- $q \notin \widetilde{C} \cup \bigcup_{j=1}^{6} L_j$, so that $\mathrm{ord}_q(N'_{\widetilde{S}}(u)|_F + \widetilde{N}(u,v)|_F) = 0$ and $F_q = 0$. And one has:

$$\frac{1 - \mathrm{ord}_q \Delta_F}{S\left(W^{Y,F}_{\bullet,\bullet,\bullet}; q\right)} = \frac{22}{15}.$$

- $q = \widetilde{C} \cap F$ so that

$$\mathrm{ord}_q(N'_{\widetilde{S}}(u)|_F + \widetilde{N}(u,v)|_F) = \begin{cases} \frac{1}{3}(v + 2u - 2) & \text{if } u \in [0,1] \text{ and } v \in [2-2u, \frac{1}{4}(14-5u)], \\ u - 1 + \frac{v}{3} & \text{if } u \in [1,2] \text{ and } v \in [0, \frac{1}{4}(18-9u)], \\ 0 & \text{otherwise.} \end{cases}$$

From which, one can compute:

$$F_q\left(W^{S,F}_{\bullet,\bullet,\bullet}\right) = \frac{13}{44} \quad \text{and} \quad \frac{1 - \mathrm{ord}_q \Delta_F}{S\left(W^{Y,F}_{\bullet,\bullet,\bullet}; q\right)} = \frac{44}{43}.$$

- $q = F \cap L_j$ for some $j = 1, \dots, 6$ so that

$$\mathrm{ord}_q(N'_{\widetilde{S}}(u)|_F + \widetilde{N}(u,v)|_F) = \begin{cases} v + u - 3 & \text{if } u \in [0,1] \text{ and } v \in [3-u, \frac{1}{4}(14-5u)], \\ v - 4 + 2u & \text{if } u \in [1,2] \text{ and } v \in [4-2u, \frac{1}{4}(18-9u)], \\ 0 & \text{otherwise.} \end{cases}$$

From which, one can compute:

$$F_q\left(W^{S,F}_{\bullet,\bullet,\bullet}\right) = \frac{1}{66} \quad \text{and} \quad \frac{1 - \mathrm{ord}_q \Delta_F}{S\left(W^{Y,F}_{\bullet,\bullet,\bullet}; q\right)} = \frac{33}{23}.$$

Therefore,

$$\min_{q \in F} \frac{1 - \mathrm{ord}_q \Delta_F}{S\left(W^{S,F}_{\bullet,\bullet,\bullet}; q\right)} = \min\left\{\frac{22}{15}, \frac{44}{43}, \frac{33}{23}\right\} = \frac{44}{43}. \tag{16}$$

Finally, by combining Equations (14)–(16), we get

$$\delta_p(X) \geqslant \min\left\{\frac{44}{23}, \frac{8}{7}, \frac{44}{43}\right\} = \frac{44}{43}. \qquad \square$$

3.4 | Estimate of δ_p for a point p off E and $\widetilde{Q}$

In this section, we estimate $\delta_p(X)$ for a point $p \in X \setminus (E \cup \widetilde{Q})$. Roughly speaking, we consider the flag given by the general hyperplane section of V_3 containing $\beta(p)$ and the curve given by its tangent hyperplane section. The precise flag depends though on the singularity of the latter.

Lemma 3.8. *Let S be the strict transform of a hyperplane section of V_3 not containing the singular point of $\beta(\widetilde{Q})$. Then,*

$$S_X(S) = \frac{14}{33}.$$

Proof. The linear system $-K_X - uS$ can be written as

$$-K_X - uS = \left(2 - \frac{3}{2}u\right)\widetilde{Q} + \left(1 - \frac{u}{2}\right)E = (4-3u)H + (u-1)E.$$

Thus, its pseudo-effective threshold is $\tau(u) = \frac{4}{3}$ and its Zariski decomposition is given by

$$P(u) = \begin{cases} (4-3u)H + (u-1)E & \text{if } u \in [0,1], \\ (4-3u)H & \text{if } u \in [1, \frac{4}{3}]. \end{cases} \text{ and } N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)E & \text{if } u \in [1, \frac{4}{3}]. \end{cases}$$

Therefore, $\mathrm{vol}(-K_X - uS) = \begin{cases} 22 - 36u + 18u^2 - 3u^3 & \text{if } u \in [0,1], \\ 64 - 144u + 108u^2 - 27u^3 & \text{if } u \in [1, \frac{4}{3}]. \end{cases}$

Hence,

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^{\tau(S)} \mathrm{vol}(-K_X - uS) du = \frac{14}{33}. \tag{17}$$

□

We consider a hyperplane section S of V_3 containing the point $\beta(p)$ and not containing the point $\beta(\widetilde{Q})$, so that S is a smooth cubic surface. We study the singularities of its tangent hyperplane section, because the relevant flag we use to estimate δ_p depends on them.

For an appropriate choice of coordinates, $\beta(p) = (0,0,0,0) \in \mathbb{C}^4_{x,y,z,t}$ in a chart of $\mathbb{P}^4$ and the surface S is given by

$$S = \begin{cases} x + f_2(x,y,z,t) + f_3(x,y,z,t) = 0, \\ y = 0, \end{cases}$$

where f_2 (respectively, f_3) is a homogeneous polynomial of degree 2 (respectively of degree 3). Recall that an n-dimensional quadric $f_2(x_0, \dots, x_{n+1}) = 0 \subset \mathbb{P}^{n+1}$ has a rank which is the rank of the associated hessian matrix. We denote it by $\mathrm{rk}(f_2)$.

By considering a suitable change of variables, we might assume that no monomials containing x appear in the expression of f_2, and so, we have

$$\mathrm{rk}(f_2|_{(y=0)}) \in \{0,1,2\}.$$

The tangent hyperplane section of S is the curve C given by

$$(x=0) \cap S = \begin{cases} x = y = 0, \\ f_2(0,z,t) + f_3(0,0,z,t) = 0. \end{cases}$$

Therefore, the curve C consists of

- a rational curve with a node at $\beta(p)$ if $\mathrm{rk}(f_2|_{(y=0)}) = 2$;
- a rational curve with a cusp at $\beta(p)$ if $\mathrm{rk}(f_2|_{(y=0)}) = 1$;
- three lines intersecting at $\beta(p)$ if $\mathrm{rk}(f_2|_{(y=0)}) = 0$.

We note that when $\mathrm{rk}(f_2|_{(y=0)}) \geqslant 1$ by generality of S, the curve C can be taken to be irreducible.

In each of these cases, we use a different flag. Since we are assuming that S does not contain the point $\beta(\widetilde{Q})$, the surface S is isomorphic to its strict transform in X, and so, is C. In what follows we slightly abuse notation and use the symbols S and C for the strict transforms as well.

3.4.8 | Nodal curve

Suppose the point p on X is such that the curve C on V_3 is a curve with a node at $\beta(p)$. In order to estimate δ_p, we make use of Theorem 2.6. Let $\sigma : \widehat{S} \to S$ be the blow-up of S in p with exceptional curve G. We denote by $\widehat{C}$ the strict transform of C in $\widehat{S}$. We have the following intersection numbers:

$$G^2 = -1, \quad G \cdot \widehat{C} = 2, \quad \widehat{C}^2 = -1.$$

Proposition 3.9. *Suppose that $p \in X \backslash (\widetilde{Q} \cup E)$ is such that $\beta(p)$ is the node of the tangent hyperplane section to the general hyperplane section of V_3 containing $\beta(p)$, then*

$$\delta_p(X) \geqslant \frac{176}{161}.$$

Proof. We apply Theorem 2.6 to the flag consisting of p, the exceptional curve G and the strict transform of the general hyperplane section of V_3 through $\beta(p)$. For this, we consider the linear system

$$\sigma^*(P(u)|_S) - vG = \begin{cases} (2-u)\widehat{C} + (4-2u-v)G & \text{if } u \in [0,1], \\ (4-3u)\widehat{C} + (8-6u-v)G & \text{if } u \in [1, \frac{4}{3}]. \end{cases}$$

Its Zariski decomposition is given by

$$\widetilde{P}(u,v) = \begin{cases} (2-u)\widehat{C} + (4-2u-v)G & \text{if } u \in [0,1]\ v \in \left[0, 3-\frac{3u}{2}\right]; \\ (4-2u-v)(2\widehat{C}+G) & \text{if } u \in [0,1]\ v \in \left[3-\frac{3u}{2}, 4-2u\right] \\ (4-3u)\widehat{C} + (8-6u-v)G & \text{if } u \in [1, \frac{4}{3}]\ v \in \left[0, 6-\frac{9u}{2}\right] \\ (8-6u-v)(2\widehat{C}+G) & \text{if } u \in [1, \frac{4}{3}]\ v \in \left[6-\frac{9u}{2}, 8-6u\right]. \end{cases}$$

and by

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } u \in [0,1]\ v \in \left[0, 3-\frac{3u}{2}\right]; \\ (2v+3u-6)\widehat{C} & \text{if } u \in [0,1]\ v \in \left[3-\frac{3u}{2}, 4-2u\right] \\ 0 & \text{if } u \in [1, \frac{4}{3}]\ v \in [0, 6-\frac{9u}{2}] \\ (2v+9u-12)\widehat{C} & \text{if } u \in \left[1, \frac{4}{3}\right]\ v \in \left[6-\frac{9u}{2}, 8-6u\right]. \end{cases}$$

Its volume can be directly computed to be

$$\operatorname{vol}(\sigma^*(P(u)|_S) - vG) = \begin{cases} 3u^2 - v^2 - 12u + 12 & \text{if } u \in [0,1],\ v \in [0, 3 - \frac{3u}{2}], \\ 12u^2 + 12uv + 3v^2 - 48u - 24v + 48 & \text{if } u \in [0,1],\ v \in \left[3 - \frac{3u}{2}, 4 - 2u\right], \\ 27u^2 - v^2 - 72u + 48 & \text{if } u \in \left[1, \frac{4}{3}\right],\ v \in \left[0, 6 - \frac{9u}{2}\right], \\ 108u^2 + 36uv + 3v^2 - 288u - 48v + 192 & \text{if } u \in \left[1, \frac{4}{3}\right],\ v \in \left[6 - \frac{9u}{2}, 8 - 6u\right]. \end{cases} \tag{18}$$

We note that

$$\operatorname{ord}_p N(u)|_S = \begin{cases} 0 & \text{if } u \in [0,1], \\ \operatorname{ord}_p (u-1)E|_S & \text{if } u \in [1, \frac{4}{3}], \end{cases}$$

and therefore, $\operatorname{ord}_p N(u)|_S = 0$ since p is not in E by assumption. Thus,

$$S(V^S_{\bullet,\bullet}; G) = \frac{161}{88}. \tag{19}$$

Since $A_S(G) = 1 + \operatorname{ord}_G(K_{\widehat{S}} - \sigma^*(K_S)) = 2$, we have that $\frac{A_S(G)}{S(V^S_{\bullet,\bullet};G)} = \frac{176}{161}$.

Next, we compute $S(W^{S,G}_{\bullet,\bullet,\bullet}; q)$. Straightforward computations using the intersection numbers give us the first summand in (5)

$$\frac{3}{(-K_X)^3} \int_0^\tau \int_0^{\tilde{t}(u)} (\widetilde{P}(u,v) \cdot G)^2 dv du = \begin{cases} \frac{135}{176} & \text{if} u \in [0,1], \\ \frac{3}{176} & \text{if} u \in [1, \frac{4}{3}]. \end{cases}$$

For $u \in [0,1]$ since $N_S(u) = 0$, we have that $N'_{\widetilde{S}}(u) = 0$. When $u \in [1, \frac{4}{3}]$, $N_{\widetilde{S}}(u) = (u-1)\widetilde{E|}_S$, where $\widetilde{E|}_S$ is the strict transform of the curve $E|_S$ on $\widehat{S}$. Since by assumption $p \notin E$, we have $N_{\widetilde{S}}(u)|_G = 0$. We have different cases depending on the position of the point q.

If $q \in G \cap \widehat{C}$,

$$F_q\left(W^{S,G}_{\bullet,\bullet,\bullet}\right) = \begin{cases} 0 & \text{if} u \in [0,1], v \in [0, 3 - \frac{3u}{2}], \\ \frac{45}{352} & \text{if} u \in [0,1], v \in \left[3 - \frac{3u}{2}, 4 - 2u\right], \\ 0 & \text{if} u \in [1, \frac{4}{3}], v \in \left[0, 6 - \frac{9u}{2}\right], \\ \frac{1}{352} & \text{if} u \in \left[1, \frac{4}{3}\right], v \in \left[6 - \frac{9u}{2}, 8 - 6u\right]. \end{cases}$$

If $q \in G \backslash \widehat{C}$,

$$F_q\left(W^{S,G}_{\bullet,\bullet,\bullet}\right) = 0.$$

The value in (5) is then given by

$$S\left(W^{S,G}_{\bullet,\bullet,\bullet};q\right) = \frac{161}{176} \text{ when } q \in G \cap \widehat{C} \text{ and}$$
$$S\left(W^{S,G}_{\bullet,\bullet,\bullet};q\right) = \frac{69}{88} \text{ when } q \in G \backslash \widehat{C}.$$

Since the surface $\widehat{S}$ is smooth, the different Δ_G is trivial and we get

$$\min_{q\in G} \frac{1 - \mathrm{ord}_q \Delta_G}{S\left(W^{S,G}_{\bullet,\bullet,\bullet};p\right)} = \frac{176}{161}. \tag{20}$$

In conclusion, combining Lemma 3.8 and Equations (19) and (20), we get

$$\delta_p(X) \geqslant \min\left\{\frac{176}{161}, \frac{176}{161}, \frac{33}{14}\right\} = \frac{176}{161}.$$ □

3.4.9 | Cuspidal curve

Suppose the point p on X is such that $C \subset S$ is cuspidal at the point $\beta(p)$. Similar to the previous subsection, we use Theorem 2.6 to obtain an estimate to $\delta_p(X)$.

Let $\sigma : \widehat{S} \to S$ be the (2,3)-weighted blow-up of S at the point p with exceptional divisor G. The strict transform $\widehat{C}$ of C in $\widehat{S}$ intersects the exceptional curve G in one regular point. The following hold:

$$\widehat{C} = \sigma^*(C) - 6G, \qquad K_{\widehat{S}} = \sigma^*(K_S) + 4G, \text{ and}$$
$$G^2 = -\frac{1}{6}, \qquad \widehat{C} \cdot G = 1, \qquad \widehat{C}^2 = -3.$$

We note that G has two singular points, we denote by p_0 the one of type $\frac{1}{2}(1,1)$ and by p_1 the one of type $\frac{1}{3}(1,1)$. In particular, the different Δ_G defined by:

$$\left(K_{\widehat{S}} + G\right)|_G = K_G + \Delta_G \quad \text{is given by} \quad \Delta_G = \frac{1}{2}p_0 + \frac{2}{3}p_1.$$

Proposition 3.10. *Suppose the point $p \in X \backslash (\widetilde{Q} \cup E)$ is a cusp of the tangent hyperplane section to the general hyperplane section of V_3 containing $\beta(p)$, then*

$$\delta_p(X) \geqslant \frac{220}{207}.$$

Proof. We apply Theorem 2.6 to the flag consisting of p, the exceptional curve G and the strict transform of the general hyperplane section of V_3 through $\beta(p)$.

We start by computing $S(V^{\widehat{S}}_{\bullet,\bullet}, G)$. We consider the linear system

$$\sigma^*(P(u)|_S) - vG = \begin{cases} (2-u)\widehat{C} + (12-6u-v)G & \text{if } u \in [0,1], \\ (4-3u)\widehat{C} + (24-18u-v)G & \text{if } u \in [1,\frac{4}{3}]. \end{cases}$$

Its Zariski decomposition has positive part given by

$$\widetilde{P}(u,v) = \begin{cases} (2-u)\widehat{C} + (12-6u-v)G & \text{if } u \in [0,1], v \in [0, 6-3u]; \\ (12-6u-v)\left(\frac{1}{3}\widehat{C} + G\right) & \text{if } u \in [0,1], v \in [6-3u, 12-6u]; \\ (4-3u)\widehat{C} + (24-18u-v)G & \text{if } u \in \left[1,\frac{4}{3}\right], v \in [0, 12-9u]; \\ \left(8-6u-\frac{v}{3}\right)\left(\widehat{C} + 3G\right) & \text{if } u \in \left[1,\frac{4}{3}\right], v \in [12-9u, 24-18u] \end{cases}$$

and negative given by

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } u \in [0,1], v \in [0, 6-3u]; \\ \left(u-2+\frac{v}{3}\right)\widehat{C} & \text{if } u \in [0,1], v \in [6-3u, 12-6u]; \\ 0 & \text{if } u \in [1,\frac{4}{3}], v \in [0, 12-9u]; \\ \left(\frac{v}{3}+3u-4\right)\widehat{C} & \text{if } u \in [1,\frac{4}{3}], v \in [12-9u, 24-18u]. \end{cases}$$

Note that $N_S(u)|_S = 0$ for $u \in [0,1]$ and $\mathrm{ord}_G(N_S(u)|_S) = \mathrm{ord}_G((u-1)E|_S) = 0$ for $u \in [1,\frac{4}{3}]$ since by assumption $p \notin E$. Therefore, the value in Equation (4)

$$S\left(V^S_{\bullet,\bullet}, G\right) = \frac{207}{44}, \quad \text{and thus} \quad \frac{A_S(G)}{S\left(V^S_{\bullet,\bullet}, G\right)} = \frac{220}{207}, \tag{21}$$

since $A_S(G) = 1 + \mathrm{ord}_G(K_S - \sigma^*(K_S)) = 5$.

We now compute $S(W^{S,G}_{\bullet,\bullet,\bullet}; q)$ for various points $q \in G$. To compute the value in formula (5), we notice that the first term is independent of the position of $q \in G$, while $F_q := F_q(W^{S,G}_{\bullet,\bullet,\bullet})$ varies, so we split in cases. We notice that $\mathrm{ord}_q(N'_{\widehat{S}}(u)|_G) = 0$ for any point $q \in G$, since $N(u)|_S$ is a multiple of E and $p \notin E$ by assumption. Also, $\widetilde{N}(u,v)$ is a multiple of $\widehat{C}$, hence $F_q \neq 0$ only for $q = G \cap \widehat{C}$. We have $S(W^{S,G}_{\bullet,\bullet,\bullet}; q) = \frac{23}{88} + F_q$ and we get the cases:

- $q = p_0$, so $\mathrm{ord}_{p_0}(\Delta_G) = \frac{1}{2}$ and

$$\frac{1 - \mathrm{ord}_q(\Delta_G)}{S(W^{S,G}_{\bullet,\bullet,\bullet}; q)} = \left(1 - \frac{1}{2}\right) \cdot \frac{88}{23} = \frac{44}{23};$$

- $q = p_1$, so $\mathrm{ord}_{p_1}(\Delta_G) = \frac{2}{3}$ and

$$\frac{1 - \mathrm{ord}_q(\Delta_G)}{S(W^{S,G}_{\bullet,\bullet,\bullet}; q)} = \left(1 - \frac{2}{3}\right) \cdot \frac{88}{23} = \frac{88}{69};$$

- $q = \widehat{C} \cap G$, so $\mathrm{ord}_q(\Delta_G) = 0$, $F_q = \frac{23}{88}$ and

$$\frac{1 - \mathrm{ord}_q(\Delta_G)}{S(W^{S,G}_{\bullet,\bullet,\bullet}; q)} = \frac{1}{\frac{23}{88} + \frac{23}{88}} = \frac{44}{23};$$

- $q \notin \{p_0, p_1, \widehat{C} \cap G\}$, so $\mathrm{ord}_{p_q}(\Delta_G) = 0$ and

$$\frac{1 - \mathrm{ord}_q(\Delta_G)}{S(W^{S,G}_{\bullet,\bullet,\bullet}; q)} = \frac{88}{23}.$$

Therefore,

$$\min_{q \in G} \frac{1 - \mathrm{ord}_q \Delta_G}{S\left(W^{S,G}_{\bullet,\bullet,\bullet}; q\right)} = \min\left\{\frac{88}{23}, \frac{44}{23}, \frac{88}{69}, \frac{44}{23}\right\} = \frac{88}{69}. \tag{22}$$

In conclusion, by Lemma 3.8 and Equations (21) and (22), we have

$$\delta_p(X) \geqslant \min\left\{\frac{33}{14}, \frac{220}{207}, \frac{88}{69}\right\} = \frac{220}{207}.$$

□

3.4.10 | Three lines

Suppose the point $p \in X$ is such that the curve $C \subset S$ containing $\beta(p)$ is a union of three lines that intersect at $\beta(p)$. Then, unlike the previous two cases, blowing up the surface S in X does not prove useful in giving a good estimate to $\delta_p(X)$, and therefore, we will use the notion of infinitesimal flags over X.

Let $\pi : \widehat{X} \to X$ be the blow-up of the 3-fold X at the point p, with the exceptional divisor given by G. We consider the surface $V_3 \cap (x = 0)$ and its strict transforms S_x in X and $\widehat{S}_x$ in $\widehat{X}$. In particular, S_x is isomorphic to a cubic cone in $\mathbb{P}^3$. Since $-K_X = 2S_x - \widetilde{Q}$ and $-K_{\widehat{X}} = \pi^*(-K_X) - 2G$, the divisor

$$\pi^*(-K_X) - uG = \frac{4}{3}\widehat{S}_x + \frac{1}{3}\widehat{E} + (4-u)G \tag{23}$$

where we also use $\widehat{S}_x = \pi^*(S_x) - 3G$ and $\widetilde{Q} = \frac{2}{3}S_x - \frac{1}{3}E$.

Lemma 3.11. *The pseudo-effective threshold τ of the linear system $\pi^*(-K_X) - uG$ is $\tau = 4$.*

Proof. From Equation (23), we clearly have that $\tau \geqslant 4$. In order to prove the equality, we show that the divisor $4\widehat{S}_x + \widehat{E}$ is not big. For this, let $\gamma : \widehat{X} \to \mathrm{Bl}_{\alpha(p)}\,\mathbb{P}^3$ be the divisorial contraction of $\widehat{E}$. Since the pushforward of a big divisor along a birational morphism is big, in order to show the claim, it is enough to show that $\gamma_*\widehat{S}_x$ is not big. For this, notice that $\mathrm{Bl}_{\alpha(p)}\,\mathbb{P}^3$ is the resolution of indeterminacy of the projection from $\alpha(p)$ and is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, $h : \mathrm{Bl}_{\alpha(p)}\,\mathbb{P}^3 \to \mathbb{P}^2$, which contracts $\gamma(\widehat{S}_x)$ to an elliptic curve. In particular, $\gamma(\widehat{S}_x) = h^*\mathcal{O}_{\mathbb{P}^2}(3)$ is not big. The claim is proven. $\square$

Proposition 3.12. *Suppose $p \in X\backslash(\widetilde{Q} \cup E)$ is such that p is the Eckardt point of curve C given by the tangent hyperplane section to the general hyperplane section of V_3 containing $\beta(p)$. Then*

$$\delta_p(X) = \frac{22}{17},$$

and it is computed by the exceptional divisor G corresponding to the ordinary blow-up of X at p.

Proof. By [15, Corollary 4.18 (2)], we have

$$\frac{A_X(G)}{S_X(G)} \geqslant \delta_p(X) \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{q\in G} \delta_q\left(G, \Delta_G; V^G_{\bullet,\bullet}\right)\right\}, \tag{24}$$

where the infimum runs over all points $q \in G$.

We first compute the left-hand side of inequality (24) and prove that the right-hand side is bounded below by $\frac{A_X(G)}{S_X(G)}$. From the proof of Lemma 3.11, we know that $\widehat{S}_x$ is a cone over an elliptic curve. Let L be the class of a ruling in $\widehat{S}_x$, then

$$G \cdot L = 1 \quad \widehat{E} \cdot L = 2 \quad \text{and} \quad \widehat{S}_x \cdot L = -2.$$

Moreover,

$$\widehat{S}_x^2 \cdot E = 6, \quad \widehat{S}_x \cdot G^2 = -3, \quad \widehat{S}_x^2 \cdot G = 9, \quad G^2 \cdot \widehat{E} = G \cdot \widehat{E}^2 = 0,$$
$$\widehat{E}^3 = -30 \quad \widehat{S}_x \cdot E^2 = 12, \quad \widehat{S}_x^3 = -24, \quad \widehat{S}_x \cdot \widehat{E} \cdot G = 0, \quad G^3 = 1.$$

Let $P(u)$ and $N(u)$ be the positive and negative part of $\pi^*(-K_X) - uG$. We have

$$P(u) = \begin{cases} \frac{1}{3}\widehat{E} + \frac{4}{3}\widehat{S}_x + (4-u)G & \text{if } u \in [0,2], \\ \frac{1}{3}\widehat{E} + \left(\frac{7}{3} - \frac{u}{2}\right)\widehat{S}_x + (4-u)G & \text{if } u \in [2,4], \end{cases} \text{ and } N(u) = \begin{cases} 0 & \text{if } u \in [0,2], \\ \frac{(u-2)}{2}\widehat{S}_x & \text{if } u \in [2,4], \end{cases}$$

and since $-K_{\widehat{X}} = \pi^*(-K_X) - 2G$, we have that $A_X(G) = 3$ and $S_X(G) = \frac{51}{22}$, so that

$$\frac{A_X(G)}{S_X(G)} = \frac{22}{17}.$$

We now estimate $\inf_{q\in G} \delta_q(G, \Delta_G; V^G_{\bullet,\bullet})$. For every point $q \in G$, we choose the flag $q \in L \subset G$, where L is a line in G intersecting $\widehat{S}|_G$ transversely. Then, by [2, Theorem 3.2],

$$\delta_q\left(G, \Delta_G; W^G_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{1}{S\left(W^G_{\bullet,\bullet}; L\right)}, \frac{1 - \mathrm{ord}_q \Delta_L}{S\left(W^{G,L}_{\bullet,\bullet,\bullet}; q\right)}\right\}.$$

Let $P(u,v)$ and $N(u,v)$ be the positive and negative parts of $P(u)|_G - vL$. These are given by

$$P(u,v) = \begin{cases} (u-v)L & \text{if } u \in [0,2],\ v \in [0,u], \\ \left(\frac{6-u}{2} - v\right)L & \text{if } u \in [2,4],\ v \in [0, \frac{6-u}{2}], \end{cases} \quad \text{and} \quad N(u,v) = 0.$$

Notice that $\mathrm{ord}_L(N(u)|_G) = 0$ since $\widehat{S}_x|_G$ is not supported on L. Then,

$$\frac{1}{S\left(W^G_{\bullet,\bullet};L\right)} = \frac{44}{23}.$$

Let Z be the elliptic curve $\widehat{S}_x|_G$. Then,

$$\mathrm{ord}_q(N'_G(u)|_L + N(u,v)|_L) = \mathrm{ord}_q(N'_G(u)|_L) = \mathrm{ord}_q\left(\frac{u-2}{2}Z|_L\right)$$
$$= \begin{cases} 0 & \text{if } q \notin Z|_L, \\ \frac{u-2}{2} & \text{otherwise.} \end{cases}$$

Then,

$$S\left(W^{G,L}_{\bullet,\bullet,\bullet};q\right) = \begin{cases} \frac{23}{44} & \text{if } q \notin Z|_L, \\ \frac{17}{22} & \text{if } q \in Z|_L. \end{cases}$$

Hence,

$$\inf_{q\in G} \delta_q\left(G, \Delta_G; V^G_{\bullet,\bullet}\right) \geqslant \min\left\{\frac{44}{23}, \min\left\{\frac{44}{23}, \frac{22}{17}\right\}\right\} = \frac{22}{17}.$$

The claim follows. □

3.13 | Estimate of δ_p for a point p in E

We now estimate $\delta_p(X)$ where $p \in E$. Let $\Pi \subseteq \mathbb{P}^3$ be a general hyperplane containing $\alpha(p)$. Then, recall that Π intersects the curve $\mathscr{C}$ in six points, which we denote $p_1 := \alpha(p), p_2, \dots, p_6$, lying on the conic $C := Q \cap \Pi$. Let S be the strict transform of Π. The morphism $S \to \Pi$ is the blow-up of $\Pi \simeq \mathbb{P}^2$ in the six points $p_1, p_2, \dots, p_6$, and we denote by E_i the associated exceptional divisors. Let l be the strict transform of a line in Π.

Proposition 3.14. *If $p \in E \cap \widetilde{Q}$*[†]*, then $\delta_p(X) \geqslant \frac{132}{131}$. If $p \in E \setminus \widetilde{Q}$, then $\delta_p(X) \geqslant \frac{66}{65}$.*

Proof. We apply Theorem 2.5 to the flag:

$$p \in E_1 \subset S \subset X.$$

[†] Note that a (better) bound for a point $p \in E \cap \widetilde{Q}$ is provided in Proposition 3.2.

To compute $S_X(S)$, we consider the linear system $-K_X - uS$ for $u \in \mathbb{R}_{\geqslant 0}$ which, in terms of the generators of $\mathrm{Eff}(X)$, is given by

$$\left(2-\frac{u}{2}\right)\widetilde{Q}+\left(1-\frac{u}{2}\right)E.$$

Hence, its pseudo-effective threshold is $\tau = 2$. Consider the Zariski decomposition of $-K_X - uS$:

$$P(u)=\begin{cases}(4-u)H-E & \text{if } u\in[0,1],\\ (6-3u)H+(u-2)E & \text{if } u\in[1,2],\end{cases} \text{ and } N(u)=\begin{cases}0 & \text{if } u\in[0,1],\\ (u-1)\widetilde{Q} & \text{if } u\in[1,2].\end{cases}$$

Recall that this is the same as in Proposition 3.6, from which we have that $S_X(S)=\frac{23}{44}$ (14). We now compute the value $S(V^S_{\bullet,\bullet};E_1)$. Consider the linear system

$$D=P(u)|_S-vE_1=\begin{cases}(4-u)l-\sum_{i=i}^6 E_i - vE_1 & \text{if } u\in[0,1],\\ (6-3u)l-(2-u)\sum_{i=i}^6 E_i - vE_1 & \text{if } u\in[1,2].\end{cases}$$

We denote by $L_{i,j}$ the strict transform of the line through the points p_i, p_j. Its Zariski decomposition for $u \in [0,1]$ is

$$P=\begin{cases}D & \text{if } v\in[0,2-2u],\\ D-a\widetilde{C} & \text{if } v\in[2-2u,2-u], \text{ and}\\ D-a\widetilde{C}-b\sum_{j=2}^6 L_{1,j} & \text{if } v\in[2-u,\frac{8-4u}{3}]\end{cases}$$

$$N=\begin{cases}0 & \text{if } v\in[0,2-2u],\\ a\widetilde{C} & \text{if } v\in[2-2u,2-u],\\ a\widetilde{C}+b\sum_{j=2}^6 L_{1,j} & \text{if } v\in[2-u,\frac{8-4u}{3}],\end{cases}$$

where $a=\frac{v}{2}+u-1$ and $b=v+u-2$ and for $u\in[1,2]$ is

$$P=\begin{cases}D-a\widetilde{C} & \text{if } v\in[0,2-u],\\ D-a\widetilde{C}-b\sum_{j=2}^6 L_{1,j} & \text{if } v\in[2-u,\frac{8-4u}{3}]\end{cases} \text{ and}$$

$$N=\begin{cases}a\widetilde{C} & \text{if } v\in[0,2-u],\\ a\widetilde{C}+b\sum_{j=2}^6 L_{1,j} & \text{if } v\in[2-u,\frac{8-4u}{3}],\end{cases}$$

where $a=\frac{v}{2}$ and $b=v+u-2$. Hence, the volume of the divisor D for $u\in[0,1]$ is

$$\mathrm{vol}(D)=P^2=\begin{cases}u^2-v^2-8u-2v+10 & \text{if } v\in[0,2-2u],\\ 12-4v-12u-\frac{1}{2}v^2+2vu+3u^2 & \text{if } v\in[2-2u,2-u],\\ \frac{1}{2}(4u+3v-8)^2 & \text{if } v\in\left[2-u,\frac{8-4u}{3}\right],\end{cases}$$

and for $u \in [1,2]$ is

$$\operatorname{vol}(D) = P^2 = \begin{cases} 12 - 4v - 12u - \frac{1}{2}v^2 + 2vu + 3u^2 & \text{if } v \in [0, 2-u], \\ \frac{1}{2}(4u + 3v - 8)^2 & \text{if } v \in \left[2-u, \frac{8-4u}{3}\right]. \end{cases}$$

We note that $\widetilde{Q}|_S = \widetilde{C}$ which has no support on E_1. Hence, the negative part $N_S(u)$ does not contribute in the formula (1), and we get

$$S(V^S_{\bullet,\bullet}; E_1) = \frac{65}{66}. \tag{25}$$

We now compute $S(W^{S,E_1}_{\bullet,\bullet,\bullet}; p)$. Notice that the hyperplane Π can be chosen so that p does not lie in any of the $L_{1,j}$. Then,

$$\operatorname{ord}_p(N'_S(u)|_{E_1} + N|_{E_1}) = \operatorname{ord}_p\left(\left(u - 1 + \frac{v}{2}\right)\widetilde{C}|_{E_1}\right) = \begin{cases} u - 1 + \frac{v}{2} & \text{if } p \in \widetilde{C}, \\ 0 & \text{if } p \notin \widetilde{C}, \end{cases}$$

since E_1 is transversal to $\widetilde{C}$. By (3), we have,

$$F_p\left(W^{S,E_1}_{\bullet,\bullet,\bullet}\right) = \begin{cases} \frac{5}{33} & \text{if } p \in \widetilde{C}, \\ 0 & \text{if } p \notin \widetilde{C}. \end{cases}$$

By direct application of (2), we have

$$S\left(W^{S,E_1}_{\bullet,\bullet,\bullet}; p\right) = \begin{cases} \frac{131}{132} & \text{if } p \in \widetilde{C}, \\ \frac{37}{44} & \text{if } p \notin \widetilde{C}. \end{cases} \tag{26}$$

By Theorem 2.5, we have

$$\delta_p(X) \geqslant \begin{cases} \min\left\{\frac{44}{23}, \frac{66}{65}, \frac{132}{131}\right\} = \frac{132}{131} & \text{if } p \in \widetilde{C}, \\ \min\left\{\frac{44}{23}, \frac{66}{65}, \frac{44}{37}\right\} = \frac{66}{65} & \text{if } p \notin \widetilde{C}. \end{cases} \tag{27}$$

□

We can finally prove our main theorem.

Theorem 3.15. *Let X be a smooth member of the Fano family 2.15, which is the blow-up of $\mathbb{P}^3$ in a curve given by the intersection of a quadric and a cubic. Then,*

$$\delta(X) \geqslant \frac{66}{65}.$$

In particular, X is K-stable.

Proof. The local stability threshold is estimated for every point $p \in X$. In particular, by Propositions 3.2, 3.5, 3.6, 3.9, 3.10, 3.12 and 3.14, one has

$$\delta(X) \geqslant \min\left\{\frac{8}{7}, \frac{11}{10}, \frac{44}{43}, \frac{176}{161}, \frac{220}{207}, \frac{22}{17}, \frac{66}{65}\right\} = \frac{66}{65}. \qquad \square$$

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