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Isomorphism Problems and Groups of Automorphisms for Ore Extensions $K[x][y; f\frac{d}{dx}]$ (Prime Characteristic)

V. V. Bavula[1]

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Abstract

Let $\Lambda(f) = K[x][y; f\frac{d}{dx}]$ be an Ore extension of a polynomial algebra $K[x]$ over an arbitrary field K of characteristic $p > 0$ where $f \in K[x]$. For each polynomial f, the automorphism group of the algebras $\Lambda(f)$ is explicitly described. The automorphism group $\mathrm{Aut}_K(\Lambda(f)) = \mathbb{S} \rtimes G_f$ is a semidirect product of two explicit groups where G_f is the *eigengroup* of the polynomial f (the set of all automorphisms of $K[x]$ such that f is their common eigenvector). For each polynomial f, the eigengroup G_f is explicitly described. It is proven that every subgroup of $\mathrm{Aut}_K(K[x])$ is the eigengroup of a polynomial. It is proven that the Krull and global dimensions of the algebra $\Lambda(f)$ are 2. The prime, completely prime, primitive and maximal ideals of the algebra $\Lambda(f)$ are classified.

Keywords A skew polynomial ring · Automorphism · The eigengroup of a polynomial · A prime ideal · A completely prime ideal · A primitive ideal · A maximal ideal · Simple module · The Krull dimension · The global dimension · The centre · Localization · A left denominator set · An Ore set · A normal element

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1 Introduction

In this paper, module means a left module, K is a field of characteristic $p > 0$ and $\overline{K}$ is its algebraic closure, $K^\times := K\backslash\{0\}$, $K[x]$ be a polynomial algebra in the variable x over K, $\mathrm{Der}_K(K[x]) = K[x]\frac{d}{dx}$ is the set of all K-derivations of the algebra $K[x]$,

$$\Lambda := \Lambda(f) := K[x][y; \delta := f\frac{d}{dx}] = K\langle x, y \,|\, yx - xy = f\rangle = \bigoplus_{i\geq 0} K[x]y^i$$

is an Ore extension of the algebra $K[x]$ where $f = f(x) \in K[x]$. Given an algebra D and its derivation δ, the *Ore extension* of D, denoted $D[y; \delta]$, is an algebra generated by the algebra

Presented by: K Goodearl

✉ V. V. Bavula
v.bavula@sheffield.ac.uk

[1] Department of Pure Mathematics, University of Sheffield, Hicks Building, S3 7RH Sheffield, UK

D and y subject to the defining relations $yd - dy = \delta(d)$ for all $d \in D$. The algebra Λ is a Noetherian domain of Gelfand-Kirillov dimension 2.

The aim of the paper is for each polynomial f to give an explicit description of the automorphism group $\mathrm{Aut}_K(\Lambda(f))$ of the algebra $\Lambda(f)$.

We can assume that the polynomial f is *monic*, i.e. its leading coefficient is 1 provided $f \neq 0$ (by changing the generators from (x, y) to $(x, l^{-1}y)$ where l is the leading coefficient of the polynomial f). Then the algebras $\{\Lambda(f) \mid f \in K[x]\}$ as a class is a disjoint union of four subclasses: $f = 0$, $f = 1$, the polynomial f has only a *single* root in $\overline{K}$ and the polynomial f has at least two *distinct* roots in $\overline{K}$.

If $f = 0$ then the algebra $\Lambda(0) = K[x, y]$ is a polynomial algebra in two variables and its group of automorphisms is well-known [16]: The group $\mathrm{Aut}_K(K[x, y])$ is generated by the automorphisms:

$$\begin{aligned} t_\mu &: x \mapsto \mu x, & y &\mapsto y, \\ \Phi_{n,\lambda} &: x \mapsto x + \lambda y^n, & y &\mapsto y, \\ \Phi'_{n,\lambda} &: x \mapsto x, & y &\mapsto y + \lambda x^n, \end{aligned}$$

where $n \geq 0$, $\mu \in K^\times$, and and $\lambda \in K$.

If $f = 1$ then the algebra $\Lambda(1)$ is the (first) *Weyl algebra*

$$A_1 = K\langle x, \partial \mid \partial x - x\partial = 1\rangle \simeq K[x][y; \frac{d}{dx}].$$

In characteristic zero Dixmier [10], and in prime characteristic Makar-Limanov [13], gave an explicit set of generators for the automorphism group $\mathrm{Aut}_K(A_1)$ (see also [4] for more results on $\mathrm{Aut}_K(A_1)$): The group $\mathrm{Aut}_K(A_1)$ is generated by the automorphisms:

$$\begin{aligned} \Phi_{n,\lambda} &: x \mapsto x + \lambda y^n, & y &\mapsto y, \\ \Phi'_{n,\lambda} &: x \mapsto x, & y &\mapsto y + \lambda x^n, \end{aligned}$$

where $n \geq 0$ and $\lambda \in K$.

The first Weyl algebra A_1 belongs to a wide class of algebras - the class of generalized Weyl algebras. In [3], Bavula and Jordan found explicit generators for generalized Weyl algebras over a polynomial algebra in a single variable over a field of characteristic zero. Alev and Dumas [1] initiated the study of automorphisms of Ore extensions $\Lambda(f)$ in characteristic zero case. Their results were extended also to prime characteristic by Benkart, Lopes and Ondrus [6]. The algebra $\Lambda(x^2)$ (the Jordan plane) was studied by Shirikov [14], Cibils, Lauve, and [9], and Iyudu [11]. The example of the enveloping algebra of the nonabelian Lie algebra of dimension 2 studied by Martha K. Smith [15, Corollary 18]. Gadis [8] studied isomorphism problems for algebras on two generators that satisfy a single quadratic relation.

Isomorphism problems for the algebras $\Lambda(f)$. Theorem 1.1 is an isomorphism criterion for the algebras Λ.

Theorem 1.1 *Let $f, g \in K[x]$ be polynomials. Then $\Lambda(f) \simeq \Lambda(g)$ iff $g(x) = \lambda f(\alpha x + \beta)$ for some elements $\lambda, \alpha \in K^\times$ and $\beta \in K$.*

In characteristic zero, Theorem 1.1 was proven by Alev and Dumas [1, Proposition 3.6] (1997) and in prime characteristic – by Benkart, Lopes and Ondrus [6, Theorem 8.2] (2015).

Benkart, Lopes and Ondrus [6, Theorems 8.3 and 8.6] gave a description of the *set* of automorphisms groups of algebras $\Lambda(f)$ over arbitrary fields and if the automorphism group of $\Lambda(f)$ is *given* they presented information on the type of the polynomial f, [6, Corollary

8.7] (in general, if one fixes the type of the polynomial then the automorphism group is *larger* than the one which is naively expected). In this paper, we proceed in the opposite direction: if the polynomial f is *given* then the automorphism group $\mathrm{Aut}_K\,\Lambda(f)$ is explicitly described.

The eigengroup $G_f(K)$ of a polynomial $f \in K[x]$. Recall that $\mathrm{Aut}_K(K[x]) = \{\sigma_{\lambda,\mu} \,|\, \lambda \in K^\times, \mu \in K\}$ where $\sigma_{\lambda,\mu}(x) = \lambda x + \mu$.

Definition 1.2 [5] For a polynomial $f \in K[x]$,

$$G_f = G_f(K) := \{\sigma \in \mathrm{Aut}_K(K[x]) \,|\, \sigma(f) = \lambda_\sigma f \ \text{ for some } \ \lambda_\sigma \in K^\times\} \tag{1}$$

is called the *eigengroup* of the polynomial *f*.

Clearly, the set $G_f(K)$ is a subgroup of $\mathrm{Aut}_K(K[x])$, it is the largest subgroup of $\mathrm{Aut}_K(K[x])$ such that the polynomial *f* is their common eigenvector. For all scalars $\mu \in K$, $G_\mu = \mathrm{Aut}_K(K[x])$. For all scalars $\nu \in K^\times$, $G_f = G_{\nu f}$. So, in order to describe the eigengroup $G_f(K)$ we can assume that the polynomial *f* is a monic polynomial. It is proven that every subgroup of $\mathrm{Aut}_K(K[x])$ is the eigengroup of a polynomial (Theorem 4.35). For each subgroup G of $\mathrm{Aut}_K(K[x])$ all the polynomials $f \in K[x]$ with $G_f = G$ are explicitly described in the case when the field K is algebraically closed. The most interesting and difficult case is when the group G is a finite group. There are three types of finite subgroups in $\mathrm{Aut}_K(K[x])$ that are not the identity group. For each such group G, the polynomial f with $G_f = G$ has a unique form/presentation the, so-called, *eigenform* of *f*.

The eigengroup G_f has an isomorphic copy in the automorphism group $\mathrm{Aut}_K(\Lambda(f))$: The map

$$G_f(K) \to \mathrm{Aut}_K(\Lambda(f)), \sigma_{\lambda,\mu} \mapsto \sigma_{\lambda,\mu} : x \mapsto \lambda x + \mu, \quad y \mapsto \lambda^{\deg(f)-1} y \tag{2}$$

is a group monomorphism where $\deg(f)$ is the degree of the polynomial f. We identify the group $G_f(K)$ with its image in $\mathrm{Aut}_K(\Lambda(f))$. The group $G_f(K)$ is the most important/difficult part of the group $\mathrm{Aut}_K(\Lambda(f))$. Approximately half of the paper is about how to find it. If the group $G_f(K)$ is a finite group it is a semidirect product of two subgroups, $G_f(K) = \widetilde{G}_f(K) \rtimes \overline{G}_f(K)$ (Theorem 4.4).

There are four distinguish cases:

1. $\widetilde{G}_f \neq \{e\}, \overline{G}_f \neq \{e\}$,
2. $\widetilde{G}_f \neq \{e\}, \overline{G}_f = \{e\}$,
3. $\widetilde{G}_f = \{e\}, \overline{G}_f \neq \{e\}$,
4. $\widetilde{G}_f = \{e\}, \overline{G}_f = \{e\}$.

In the case when $K = \overline{K}$, Theorem 4.24, 4.27, 4.30 and 4.32 are criteria for each case to hold, respectively (see also Proposition 4.10). These four theorems are also explicit descriptions of the eigengroup $G_f(K)$. They also show that in each of four cases the polynomial *f* admits a *unique* presentation – *the eigenform* of *f* (introduced in the paper).

At the end of Section 4, a finite algorithm is given of finding the eigengroups $G_f(\overline{K})$ and $G_f(K)$, and the eigenform of *f*.

In the case when $K \neq \overline{K}$, similar results are obtained, see Theorem 4.33. Proposition 4.34 gives criteria for the groups $\widetilde{G}_f(K)$, $\overline{G}_f(K)$ and $G_f(K)$ to be $\{e\}$.

The group of automorphisms of the algebra $\Lambda(f)$. Given a group G, a normal subgroup N and a subgroup H. The group G is called the *semidirect product* of N and H, written $G = N \rtimes H$, if $G = NH := \{nh \,|\, n \in N, h \in H\}$ and $N \cap H = \{e\}$ where e is the identity of the group G.

The automorphism group $\mathrm{Aut}_K(\Lambda(f))$ of the algebra $\Lambda(f)$ contains an obvious subgroup

$$\mathbb{S} := \mathbb{S}(K) := \{s_p \mid p \in K[x]\} \simeq (K[x], +), \ s_p \mapsto p \ \text{ where } \ s_p(x) = x \text{ and } s_p(y) = y + p. \tag{3}$$

Theorem 1.3 *Suppose that $f \in K[x]$ is a monic non-scalar polynomial. Then*

$$\mathrm{Aut}_K(\Lambda(f)) = \mathbb{S}(K) \rtimes G_f(K).$$

The Krull and global dimensions of the algebra $\Lambda(f)$. It it proven that the Krull and global dimensions of the algebra $\Lambda(f)$ are 2 (Theorem 2.5).

Classifications of prime, completely prime, primitive and maximal ideals of the algebra $\Lambda(f)$. Theorem 2.3 classifies prime, completely prime, primitive and maximal ideals of the algebra $\Lambda(f)$. The height 1 prime ideals were classified in [6, Theorem 7.6]. It is proven that every nonzero ideal of the algebra $\Lambda(f)$ meets the centre of $\Lambda(f)$ (Corollary 2.2).

Classifications of simple $\Lambda(f)$-modules. In [2], simple modules were classified for all Ore extensions $A = D[x; \sigma, \partial]$ where D is a Dedekind domain, $\sigma \in$ is an automorphism of D and ∂ is a σ-derivation of D (for all $a, b \in D$, $\partial(ab) = \partial(a)b + \sigma(a)\partial(b)$). Recall that the ring A is generated by D and x subject to the defining relations: For all elements $d \in D$, $xd = \sigma(d)x + \partial(d)$. The algebras $\Lambda(f)$ is a a very special case of the rings A.

Theorem 2.4 classifies simple *left* $\Lambda(f)$-modules, see also [7]. For each simple left $\Lambda(f)$-module an explicit K-basis is given and the actions of the canonical generators x and y of the algebra $\Lambda(f)$ on the basis is explicitly described.

A classification of simple *right* $\Lambda(f)$-modules is obtained at once from the classification of simple left $\Lambda(f)$-modules by using the fact that the opposite algebra $\Lambda(f)^{op}$ of the algebra $\Lambda(f)$ is isomorphic to

$$\Lambda(f)^{op} \simeq \Lambda(-f). \tag{4}$$

Recall that the *opposite algebra* A^{op} of an algebra A coincides with the algebra A as vector space but the multiplication in A^{op} is given by the rule $a \cdot b = ba$. Every right A-module is a left A^{op}-module and vice versa.

2 Spectra, the Centre, the Krull and Global Dimensions of the Algebra Λ

In this section, K is a field of characteristic $p > 0$ (not necessarily algebraically closed) and $f = p_1^{n_1} \cdots p_s^{n_s}$ is a non-scalar polynomial of $K[x]$ where $p_1, \ldots, p_s$ are irreducible, co-prime divisors of f (i.e. $K[x]p_i + K[x]p_j = K[x]$ for all $i \neq j$). The aim of this section is to find the centre of the algebra $\Lambda(f)$; to classify simple $\Lambda(f)$-modules; to classify prime, completely prime, primitive and maximal ideals of the algebra $\Lambda(f)$; and to prove that the Krull and global dimension of the algebra $\Lambda(f)$ is 2.

The centre of the algebra $\Lambda(f)$. It follows from the direct sum

$$A_1 = \bigoplus_{i,j=0}^{p-1} K[x^p, \partial^p]x^i\partial^j$$

and the commutation relation $[\partial, x] = 1$, that the centre $Z(A_1)$ of the Weyl algebra is equal to $K[x^p, \partial^p]$, a polynomial algebra in two variables. Let $K(x^p, \partial^p)$ be the field of fractions of the polynomial algebra $K[x^p, \partial^p]$. The localization $\mathcal{A}_1$ of the Weyl algebra A_1 by the Ore set $K[x^p, \partial^p]\backslash\{0\}$ is a simple p^2-dimensional algebra

$$\mathcal{A}_1 = \bigoplus_{i,j=0}^{p-1} K(x^p, \partial^p)x^i\partial^j.$$

This follows from the relation $[\partial, x] = 1$. So, $Z(\mathcal{A}_1) = K(x^p, \partial^p)$. Hence, *every nonzero ideal of the Weyl algebra A_1 meets the centre of A_1.*

The polynomial algebra $K[x]$ is a left A_1-module where ∂ acts as the derivation $\frac{d}{dx}$. Furthermore, the kernel of the corresponding algebra homomorphism

$$A_1 \to \mathrm{End}_K(K[x]), \quad x \mapsto x, \quad \partial \mapsto \frac{d}{dx} \tag{5}$$

is generated by the central element ∂^p. So, $A_1/(\partial^p) = \bigoplus_{i=0}^{p-1} K[x]\partial^i \subseteq \mathrm{End}_K(K[x])$. The factor algebra $A_1/(\partial^p)$ is a subalgebra of the algebra $\mathcal{D}(K[x])$ of differential operators on the polynomial algebra $K[x]$ and the Weyl algebra A_1 is not. This is in sharp contrast with the characteristic zero case where $A_1 = \mathcal{D}(K[x])$.

The algebra $\Lambda = \Lambda(f)$ can be identified with a subalgebra of the Weyl algebra A_1 by the monomorphism:

$$\Lambda \to A_1, \quad x \mapsto x, \quad y \mapsto f\partial. \tag{6}$$

So, $\Lambda = K\langle x, y = f\partial\rangle \subset A_1$. Theorem 2.1 describes the centre of the algebra $\Lambda(f)$. It also gives explicit expressions for the p'th power of various elements of the algebras $\Lambda(f)$ and A_1 that are key facts in finding the centre of $\Lambda(f)$.

The fact that the centre of the algebra $\Lambda(f)$ is equal to $K[x^p, y^p - (\delta^{p-2}(f))'y]$ was proven by Benkart, Lopes and Ondrus, [6, Theorem 5.3,(2)]. Here we present a short proof of this fact.

Theorem 2.1 *Let $\delta = f\frac{d}{dx} \in \mathrm{Der}_K(K[x])$ where $f \in K[x]\backslash\{0\}$, and $g' := \frac{dg}{dx}$ where $g \in K[x]$. Then:*

1. $\delta^p = (\delta^{p-2}(f))'\delta \in \mathrm{Der}_K(K[x])$.
2. *In the algebra $\Lambda(f)$, $y^p = f^p\partial^p + (\delta^{p-2}(f))'y$. In particular, in the Weyl algebra A_1, $(f\partial)^p = f^p\partial^p + (\delta^{p-2}(f))'f\partial$.*
3. *The centre $Z(\Lambda(f))$ of the algebra $\Lambda(f)$ is the polynomial algebra*
$$K[x^p, y^p - (\delta^{p-2}(f))'y] = K[x^p, f^p\partial^p]$$
and $f^p\partial^p = y^p - (\delta^{p-2}(f))'y$.
4. *The algebra $\Lambda(f) = \bigoplus_{i,j=0}^{p-1} Z(\Lambda(f))x^iy^j$ is a free $Z(\Lambda(f))$-module of rank p^2.*
5. *The localization of the algebra $\Lambda(f)$ at the Ore set $Z(\Lambda(f))\backslash\{0\}$ is $\mathcal{A}_1$.*

Proof 1. Since $\delta^p \in \mathrm{Der}_K(K[x])$, we have that $\delta^p = g\frac{d}{dx}$ where $g = \delta^p(x) = \delta^{p-1}(f) = (\delta^{p-2}(f))'f$. Therefore,

$$\delta^p = (\delta^{p-2}(f))'f\frac{d}{dx} = (\delta^{p-2}(f))'\delta.$$

2. Notice that $y^p = (f\partial)^p = f^p\partial^p + \sum_{i=1}^{p-1} a_i\partial^i$ for some elements $a_i \in K[x]$. Recall that $A_1/(\partial^p) = \bigoplus_{i=0}^{p-1} K[x]\partial^i \subseteq \mathrm{End}_K(K[x])$. By statement 1,

$$(f\partial)^p \equiv \sum_{i=1}^{p-1} a_i\partial^i \equiv (\delta^{p-2}(f))'\delta \equiv (\delta^{p-2}(f))'f\partial \equiv (\delta^{p-2}(f))'y \mod (\partial^p),$$

and so $a_1 = (\delta^{p-2}(f))'f$ and $a_2 = \cdots = a_{p-1} = 0$.

3–5. By statement 2, $f^p\partial^p = y^p - (\delta^{p-2}(f))'y \in Z(\Lambda(f))$. Hence,

$$Z' := K[x^p, y^p - (\delta^{p-2}(f))'y] = K[x^p, f^p\partial^p] \subseteq Z(\Lambda(f))$$

and $\Lambda(f) = \bigoplus_{i,j=0}^{p-1} Z'x^iy^j$. Now,

$$(Z'\backslash\{0\})^{-1}\Lambda(f) = \bigoplus_{i,j=0}^{p-1} K(x^p, \partial^p)x^iy^j = \bigoplus_{i,j=0}^{p-1} K(x^p, \partial^p)x^i\partial^j = \mathcal{A}_1,$$

and so statement 5 is obvious and statements 3–4 follow. □

Let A be an algebra and $a \in A$. The map $\mathrm{ad}_a = [a, -] : A \to A, b \mapsto [a, b] := ab - ba$ is a derivation of the algebra A which is called the *inner derivation* of A associated with the element a.

Corollary 2.2 *Every nonzero ideal of the algebra $\Lambda(f)$ meets the centre of $\Lambda(f)$.*

Proof Let I be a nonzero ideal of the algebra $\Lambda(f)$. Fix a nonzero element of I, say $a = \sum_{i,j=0}^{p-1} z_{ij}x^i\partial^j$ for some elements $z_{ij} \in Z(\Lambda(f))$, by Theorem 2.1.(4). Then applying several times the inner derivation $\mathrm{ad}_x := [x, -]$ of the algebra $\Lambda(f)$, we obtain a nonzero element, say $b \in I \cap Z(\Lambda(f))[x]$. Then $0 \neq b^p \in I \cap Z(\Lambda(f))$. □

The prime, completely prime, primitive and maximal spectra of the algebra Λ. An ideal $\mathfrak{p}$ of a ring R is called a *completely prime ideal* if the factor ring $R/\mathfrak{p}$ is a domain. A completely prime ideal is a prime ideal. The sets of prime and completely prime ideals of the ring R are denoted by $\mathrm{Spec}(R)$ and $\mathrm{Spec}_c(R)$, respectively. The annihilator of a simple R-module is called a *primitive ideal* of R. Every primitive ideal is a prime ideal of R. The set of all primitive ideals is denoted by $\mathrm{Prim}(R)$. The set of all maximal ideals of R is denoted by $\mathrm{Max}(R)$. Clearly, $\mathrm{Max}(R) \subseteq \mathrm{Prim}(R) \subseteq \mathrm{Spec}(R)$.

An element a of an algebra A is called a *normal element* of A if $Aa = aA$. An element a of an algebra A is called a *regular element* if it is not a zero divisor. The set of all regular elements of the algebra A is denoted by $\mathcal{C}_A$. Each regular normal element a of the algebra A determines an automorphism of the algebra A given by the rule:

$$\omega_a : A \to A, \quad b \mapsto \omega_a(b) \text{ where } ab = \omega_a(b)a. \tag{7}$$

The elements $p_1, \ldots, p_s$ are regular normal elements of the algebra $\Lambda = \Lambda(f)$ (recall that $f = \prod_{i=1}^s p_i^{n_i}$) since

$$yp_i = p_i(y - p_i^{-1}f) \text{ and } xp_i = p_ix.$$

Therefore, $\omega_{p_i}(x) = x$ and $\omega_{p_i}(y) = y + p_i^{-1}f$.

For an ideal $\mathfrak{a}$ of an algebra A, we denote by $V(\mathfrak{a})$ the set of all prime ideals of A that contain the ideal $\mathfrak{a}$. Let $\min\mathfrak{a}$ be the *set of minimal primes* of $\mathfrak{a}$. These are the minimal elements of the set $V(\mathfrak{a})$ with respect to inclusion. Suppose that the set $S_a := \{a^i \mid i \geq 0\}$ is a left Ore set of a domain A. The algebra $A_a := S_a^{-1}A = \{a^{-i}b \mid i \geq 0, b \in A\}$ is called the localization of A at the powers of the element a.

For a commutative algebra C and a non-nilpotent element $s \in C$, the map

$$\mathrm{Spec}(C)\backslash V(s) \to \mathrm{Spec}(C_s), \quad \mathfrak{p} \mapsto S_s^{-1}\mathfrak{p}$$

is a bijection with the inverse map $\mathfrak{q} \mapsto C \cap \mathfrak{q}$. We identify the sets $\mathrm{Spec}(C)\backslash V(s)$ and $\mathrm{Spec}(C_s)$ via the bijection above, i.e. $\mathrm{Spec}(C)\backslash V(s) = \mathrm{Spec}(C_s)$.

Recall that the centre of the Weyl algebra $A_1 = K[x][\partial; \frac{d}{dx}]$ is the polynomial algebra $K[x^p, \partial^p]$ (the result is well-known). Then

$$f^p \in K[x^p] \subseteq Z(\Lambda(f)) = K[x^p, f^p\partial^p] \subseteq K[x^p, \partial^p] = Z(A_1)$$

and

$$\begin{aligned} L = \Lambda(f) \subseteq L_f = A_{1,f} = L_{f^p} = A_{1,f^p} &= \bigoplus_{i,j=0}^{p-1} Z(L)_{f^p} x^i y^i \\ &= \bigoplus_{i,j=0}^{p-1} Z(A_1)_{f^p} x^i \partial^i = \bigoplus_{i,j=0}^{p-1} K[x^p, \partial^p]_{f^p} x^i \partial^i. \end{aligned} \tag{8}$$

In particular, $Z(L)_{f^p} = Z(A_1)_{f^p} = K[x^p, \partial^p]_{f^p}$, and so we can write

$$\begin{aligned} \mathrm{Spec}(Z(L))\backslash V(f^p) &= \mathrm{Spec}(Z(L)_{f^p}) = \mathrm{Spec}(Z(A_1)_{f^p}) = \mathrm{Spec}(K[x^p, \partial^p]_{f^p}) \\ &= \mathrm{Spec}(Z(A_1))\backslash V(f^p) = \mathrm{Spec}(K[x^p, \partial^p])\backslash V(f^p). \end{aligned}$$

Theorem 2.3 gives explicit descriptions of the sets of prime, completely prime, primitive and maximal ideals of the algebra Λ. Let $\mathrm{Spec}_c(\Lambda, \mathrm{ht} = 1)$ be the set of completely prime ideals of height 1 of the algebra $\Lambda(f)$.

Theorem 2.3 *Let K be a field of characteristic $p > 0$, $\Lambda = K[x][y; \delta := f\frac{d}{dx}]$ where $f \in K[x]\backslash K$. Let $f = p_1^{n_1} \cdots p_s^{n_s}$ be a unique (up to permutation) product of irreducible polynomials of $K[x]$. Then:*

1. *The elements $p_1, \ldots, p_s$ are regular normal elements of the algebra Λ (i.e. p_i is a non-zero-divisor of Λ and $p_i\Lambda = \Lambda p_i$).*
2. $\min(f) = \{(p_1), \ldots, (p_s)\}$.
3. $\mathrm{Spec}_c(\Lambda) = \{0, \Lambda p_i, (p_i, q_i) \,|\, i = 1, \ldots, s;\ q_i \in \mathrm{Irr}_m(F_i[y])\}$ *where $F_i := K[x]/(p_i)$ is a field and $\mathrm{Irr}_m(F_i[y])$ is the set of monic irreducible polynomials of the polynomial algebra $F_i[y]$ over the field F_i in the variable y. If, in addition, $K = \overline{K}$ and $\lambda_1, \ldots, \lambda_s$ are the roots of the polynomial f then* $\mathrm{Spec}_c(\Lambda) = \{0, \Lambda(x - \lambda_i), (x - \lambda_i, y - \mu) \,|\, i = 1, \ldots, s;\ \mu \in K\}$.
4. $\mathrm{Spec}(\Lambda) = \mathrm{Spec}_c(\Lambda) \coprod \{\Lambda\mathfrak{p} \,|\, \mathfrak{p} \in \mathrm{Spec}(Z(\Lambda))\backslash\{(0), V(f^p)\}$.
5. *For all* $\mathfrak{p} \in \mathrm{Spec}(Z(\Lambda))\backslash V(f^p)$,

$$k(\mathfrak{p}) \otimes_{Z(\Lambda)} \Lambda/(\mathfrak{p}) \simeq \bigoplus_{i,j=0}^{p-1} k(\mathfrak{p}) x^i y^i = \bigoplus_{i,j=0}^{p-1} k(\mathfrak{p}) x^i \partial^i \simeq M_p(k(\mathfrak{p})),$$

   the algebra of $p \times p$ matrices over the field of fractions $k(\mathfrak{p})$ of the domain $Z(\Lambda)/\mathfrak{p}$.
6. $\mathrm{Max}(\Lambda) = \mathrm{Prim}(\Lambda) = \{(p_i, q_i), \Lambda\mathfrak{m} \,|\, i = 1, \ldots, s;\ q_i \in \mathrm{Irr}_m(F_i[y]), \mathfrak{m} \in \mathrm{Max}(Z(\Lambda))\backslash V(f^p)\}$.
7. $\mathrm{Spec}_c(\Lambda, \mathrm{ht} = 1) = \{(p_1), \ldots, (p_s)\}$. *If, in addition $K = \overline{K}$, then* $\mathrm{Spec}_c(\Lambda, \mathrm{ht} = 1) = \{(x - \lambda_1), \ldots, (x - \lambda_t)\}$ *where $\{\lambda_1, \ldots, \lambda_t\}$ is the set of roots of the polynomial f.*

Proof 1. Statement 1 is proven above.

2. Since

$$\Lambda/\Lambda p_i \simeq F_i[y] \tag{9}$$

is a polynomial algebra with coefficients in the field F_i (since $yx - xy = f \in \Lambda p_i$), the ideal Λp_i is a completely prime ideal of Λ. Now, statement 2 follows from the equality of ideals $(f) = (p_1)^{n_1} \cdots (p_s)^{n_s}$.

5. Since $\mathfrak{p} \in \mathrm{Spec}(Z(\Lambda))\backslash V(f^p)$, the element f^p is a unit in the field $k(\mathfrak{p})$. Now, the first isomorphism and the equality in statement 5 follows from Eq. 8. Then using the equalities,

$$[y, x^i] = ix^{i-1} \quad \text{and} \quad [y^i, x] = iy^{i-1},$$

and the fact that $k(\mathfrak{p})$ is a field, we see that the algebra $\bigoplus_{i,j=0}^{p-1} k(\mathfrak{p})x^i \partial^i$ is a simple, central $k(\mathfrak{p})$-algebra of dimension p^2 over the field $k(\mathfrak{p})$. Therefore, it is isomorphic to the matrix algebra $M_p(k(\mathfrak{p}))$.

3-4. The algebra Λ is a domain, hence $0 \in \mathrm{Spec}_c(\Lambda)$. We have seen in the proof of statement 2 that the ideals $(p_1), \dots, (p_s)$ of the algebra Λ are completely prime ideals. By Eq. 9,

$$V(f) = \{\Lambda p_i, (p_i, q_i) \mid i = 1, \dots, s;\ q_i \in \mathrm{Irr}_m(F_i[y])\} \subseteq \mathrm{Spec}_c(\Lambda).$$

Given a nonzero prime ideal P of Λ such that $P \notin V(f) = V(f^p)$. Then P_f is a nonzero prime ideal of the algebra $\Lambda_{f^p} = A_{1, f^p}$. By Corollary 2.2, the intersection $\mathfrak{p} := P \cap Z(\Lambda)$ is a nonzero prime ideal of the centre $Z(\Lambda)$ of the algebra Λ. By Theorem 2.1.(4), $\Lambda = \bigoplus_{i,j=0}^{p-1} Z(\Lambda)x^i y^i$. Now, by statement 5, $\Lambda\mathfrak{p} \in \mathrm{Spec}(\Lambda)\backslash V(f)$ and the prime ideal $\Lambda\mathfrak{p}$ is not completely prime. Now, statements 3 and 4 follows from statement 5.

6. Statement 6 follows from statement 4.

7. Statement 7 follows from statement 3. □

Classification of simple $\Lambda(f)$-modules For a Λ-module M, we denote by $\mathrm{ann}_\Lambda(M)$ the annihilator of the Λ-module M. For an algebra A, we denote by $\widehat{A}$, the set of isomorphism classes of (left) simple A-modules. An isomorphism class of a simple A-modules M is denoted by $[M]$. Let elements $a_1, \dots, a_n \in A$ be generators for a left ideal I of the algebra A. Then we write $I = A(a_1, \dots, a_n)$. Theorem 2.4 is a classification of simple $\Lambda(f)$-modules.

Theorem 2.4 *Let K be a field of characteristic $p > 0$, $\Lambda = K[x][y; \delta := f\frac{d}{dx}]$ where $f \in K[x]\backslash K$. Let $f = p_1^{n_1} \cdots p_s^{n_s}$ be a unique (up to permutation) product of irreducible polynomials of $K[x]$. Then:*

1. *The map*

$$\mathrm{Max}(\Lambda) \to \widehat{\Lambda}, \quad \mathfrak{m} \mapsto L(\mathfrak{m})$$

is a bijection with inverse $[M] \mapsto \mathrm{ann}_\Lambda(M)$ where $L(\mathfrak{m})$ is a unique (up to isomorphism) simple direct summand/submodule/factor module of the (simple) matrix algebra $\Lambda/\mathfrak{m}$. In particular, for all $\mathfrak{m} \in \mathrm{Max}(\Lambda)\backslash V(f^p)$, $\dim_K(L(\Lambda\mathfrak{m})) = p \cdot \dim_K(Z(\Lambda)/\mathfrak{m}) < \infty$.

2. *For each maximal ideal (p_i, q_i) of Λ, where $i = 1, \dots, s$ and $q_i \in \mathrm{Irr}_m(F_i[y])\}$,*

$$L(p_i, q_i) = \Lambda/(p_i, q_i) \simeq K[y]/(q_i)$$

and $\dim_K(L(p_i, q_i)) = \dim_K(K[y]/(q_i)) = \deg_y(q_i) < \infty$.

3. *Suppose that $K = \overline{K}$. For each maximal ideal $\Lambda\mathfrak{m}$ of Λ, where $\mathfrak{m} \in \mathrm{Max}(Z(\Lambda))\backslash V(f^p)$,*

$$L(\Lambda\mathfrak{m}) \simeq \Lambda/\Lambda(\mathfrak{m}, x - \sqrt[p]{\xi}) = \bigoplus_{i,j=0}^{p-1} Ky^i\bar{1}$$

$$\simeq A_{1,f^p}/A_{1,f^p}(\mathfrak{m}, x - \sqrt[p]{\xi}) = \bigoplus_{i,j=0}^{p-1} K\partial^i\hat{1},$$

where $x^p - \xi \in \mathfrak{m}$ for a unique element $\xi \in K$, $\bar{1} = 1 + \Lambda(\mathfrak{m}, x - \sqrt[p]{\xi})$ and $\hat{1} = 1 + A_{1,f^p}(\mathfrak{m}, x - \sqrt[p]{\xi})$, $\dim_K L(\Lambda\mathfrak{m}) = p < \infty$.

Proof 1. Statements 1 follows at once from Theorem 2.3.(5,6).

2. Statement 2 is obvious.

3. Notice that $(x - \sqrt[p]{\xi}))^p = x^p - \xi \in \mathfrak{m}$.

By Theorem 2.1.(4) and the choice of the ideal $\mathfrak{m}$, we have that

$$\Lambda/\Lambda\mathfrak{m} = \bigoplus_{i=0}^{p-1} Ky^i \otimes K[x]/(x^p - \xi) = \bigoplus_{i=0}^{p-1} Ky^i \otimes K[x]/((x - \sqrt[p]{\xi})^p),$$

direct sums of tensor products of vector spaces. Hence,

$$\Lambda/\Lambda(\mathfrak{m}, x - \sqrt[p]{\xi}) \simeq \bigoplus_{i=0}^{p-1} Ky^i \otimes K[x]/(x - \sqrt[p]{\xi}) \simeq \bigoplus_{i,j=0}^{p-1} Ky^i\bar{1}$$

is a p-dimensional Λ-module that is annihilated by the maximal ideal $\mathfrak{m}$. By statement 1, it must be $L(\Lambda\mathfrak{m})$. Since $\mathfrak{m} \notin V(f^p)$, the central element f^p acts as a bijection on the module $L(\Lambda\mathfrak{m})$. Therefore,

$$L(\Lambda\mathfrak{m}) = L(\Lambda\mathfrak{m})_{f^p} = \Lambda_{f^p}/\Lambda_{f^p}(\mathfrak{m}, x-\sqrt[p]{\xi}) \simeq A_{1,f^p}/A_{1,f^p}(\mathfrak{m}, x-\sqrt[p]{\xi}) = \bigoplus_{i,j=0}^{p-1} K\partial^i\hat{1}.$$

□

The action of the elements x and y on the K-basis $\{y\bar{1} \mid i = 0, \dots, p-1\}$ of the $\Lambda(f)$-module $L(\Lambda\mathfrak{m})$ of Theorem 2.4.(3) is given below:

$$x \cdot \bar{1} = \xi^{\frac{1}{p}}\bar{1},$$

$$x \cdot y^i\bar{1} = \xi^{\frac{1}{p}}y^i\bar{1} + \sum_{j=0}^{i-1}\binom{i}{j}\xi_{ij}y^j\bar{1} \quad \text{where} \quad \xi_{ij} = (-1)^{i-j}\phi_{ij}(\xi^{\frac{1}{p}}), \quad \phi_{ij} = \delta^{i-j-1}(f) \in K[x],$$

$$y \cdot y^i\bar{1} = y^{i+1}\bar{1} \quad \text{where} \quad 0 \le i \le p-2,$$

$$y \cdot y^{p-1}\bar{1} = \rho\bar{1} \quad \text{where} \quad y^p - \rho \in \mathfrak{m} \quad \text{for a unique element} \quad \rho \in K.$$

The Krull and global dimensions of the algebra $\Lambda(f)$.

Theorem 2.5 *Let K be a field of characteristic $p > 0$, $\Lambda = K[x][y; \delta := f\frac{d}{dx}]$ where $f \in K[x]\backslash K$. Then:*

1. *The Krull dimension of Λ is* $\text{K.dim}(\Lambda) = 2$.
2. *The global dimension of Λ is* $\text{gldim}(\Lambda) = 2$.

Proof Let $\Lambda = \Lambda(f)$.

1. By Theorem 2.1.(4), the algebra Λ is a finitely generated $Z(\Lambda)$-module. Therefore, the Krull dimension of the algebra Λ is equal to the Krull dimension of the polynomial algebra $Z(\Lambda)$ in two variables (Theorem 2.1.(3)), and statement 1 follows.

2. By [12, Theorem 7.5.3.(i)], $\text{gldim}(\Lambda) \le \text{gldim}(K[X]) + 1 = 1 + 1 = 2$.

Let $f = p_1^{n_1} \cdots p_s^{n_s}$ be a unique (up to permutation) product of irreducible polynomials of $K[x]$. By Eq. 9, $\text{gldim}(\Lambda/\Lambda p_i) = \text{gldim}(F_i[Y]) = 1 < \infty$. Now, by [12, Theorem 7.3.5.(i)],

$$\text{gldim}(\Lambda) \ge \text{gldim}(\Lambda/\Lambda p_i) + 1 \stackrel{(9)}{=} \text{gldim}(F_i[Y]) + 1 = 1 + 1 = 2.$$

Therefore, $\mathrm{gldim}(\Lambda) = 2$. □

3 Isomorphism Problems and Groups of Automorphisms for Ore Extensions $K[x][y; f\frac{d}{dx}]$

In this section, a proof Theorem 1.3 is given. It can be deduced from Theorem 1.1 but we give a different proof.

Let $K(x)$ be the field of rational functions in the variable x. Then the Ore extension $B_1 := K(x)[\partial; \frac{d}{dx}]$ is the localization $B_1 = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore set $S = K[x]\backslash\{0\}$ of A_1. The multiplicative set S is an Ore set of the algebra Λ such that $B_1 = S^{-1}\Lambda$, by Eq. 6.

Notice that

$$\mathbb{S}(K) \rtimes G_f(K) = \{\sigma_{\lambda,\mu,p} \,|\, \lambda \in K^\times, \mu \in K, p \in K[x]\} \tag{10}$$

where

$$\sigma_{\lambda,\mu,p}(x) = \lambda x + \mu \;\text{ and }\; \sigma_{\lambda,\mu,p}(y) = \lambda^{d-1} y + p$$

since $\sigma_{\lambda,\mu,p} = s_{\lambda^{-d+1}p}\sigma_{\lambda,\mu}$ where $d = \deg(f)$.

Proof of Theorem 1.3 Let σ be an automorphism of the K-algebra $\Lambda = \Lambda(f)$. It can be uniquely extended to a $\overline{K}$-automorphism, say σ, of the algebra $\overline{K} \otimes_K \Lambda$. Let $\lambda_1, \dots, \lambda_t$ be the roots of the polynomial f in $\overline{K}$. By Theorem 2.3.(7), the automorphism σ permutes the set

$$\mathrm{Spec}_c(\overline{K} \otimes_K \Lambda, \mathrm{ht} = 1) = \{(x - \lambda_1), \dots, (x - \lambda_t)\}$$

of height 1 completely prime ideals of the algebra $\overline{K} \otimes_K \Lambda$ that are generated by regular normal elements $x - \lambda_1, \dots, x - \lambda_t$ of the domain $\overline{K} \otimes_K \Lambda$ and the set $\overline{K}^\times$ is the group of units of the algebra $\overline{K} \otimes_K \Lambda$. So, we must have that

$$\sigma(x) = \lambda x + \mu$$

for some elements $\lambda \in \overline{K}^\times$ and $\mu \in \overline{K}$. Since $K[x] = \Lambda \cap \overline{K}[x]$, we must have that

$$\sigma(x) \in \sigma(\Lambda) \cap \sigma(\overline{K}[x]) = \Lambda \cap \overline{K}[x] = K[x],$$

and so $\lambda \in K^\times$ and $\mu \in K$. So, the automorphism σ respects the polynomial algebra $K[x]$. In particular, it respects the Ore set $S = K[x]\backslash\{0\}$ of the algebra Λ. The automorphism σ can be uniquely extended to an automorphism of the algebra $B_1 = S^{-1}\Lambda$. Then $\sigma(\partial) = \lambda^{-1}\partial + q$ for some element $q \in K(x)$. In particular,

$$\sigma(y) = \sigma(f\partial) = \sigma(f)(\lambda^{-1}\partial + q) = \lambda^{-1}\frac{\sigma(f)}{f} y + p \;\text{ where }\; p := \sigma(f)q \in K[x]$$

and $\sigma(f) = \gamma f$ for some element $\gamma \in K^\times$. Clearly, $\gamma = \lambda^d$ where $d = \deg(f)$ is the degree of the polynomial f (since $\sigma(x) = \lambda x + \mu$). So,

$$\sigma(x) = \lambda x + \mu \;\text{ and }\; \sigma(y) = \lambda^{d-1} y + p,$$

i.e. $\sigma \in \mathbb{S}(K) \rtimes G_f(K)$, as required. □.

By Theorem 1.3,

$$G_f(K) = \mathbb{S}(K) \rtimes G_f(K) = \{\sigma_{\lambda,\mu,p} \,|\, \lambda \in K^\times, \mu \in K, p \in K[x]\} \tag{11}$$

where the multiplication and the inversion in the group $G_f(K)$ are given by the rule (where $d = \deg(f)$):

$$\sigma_{\lambda_1,\mu_1,p_1}\sigma_{\lambda_2,\mu_2,p_2} = \sigma_{\lambda_1\lambda_2,\lambda_2\mu_1+\mu_2,\lambda_2^{d-1}p_1+p_2},$$
$$\sigma^{-1}_{\lambda,\mu,p} = \sigma_{\lambda^{-1},-\lambda^{-1}\mu,-\lambda^{-d+1}p}.$$

The algebra B_1 and its automorphism group The element f is a regular normal element of Λ (i.e. $\Lambda f = f\Lambda$) since

$$fy = yf - f'f = (y - f')f \quad \text{where} \quad f' = \frac{df}{dx}.$$

It determines the K-automorphism ω_f of the algebra Λ:

$$fu = \omega_f(u)f, \quad u \in \Lambda,$$
$$\omega_f : x \mapsto x, \quad y \mapsto y - f'.$$

We denote by Λ_f and $A_{1,f}$ the localizations of the algebras Λ and A_1 at the powers of the element f, i.e.

$$\Lambda_f = S_f^{-1}\Lambda \quad \text{and} \quad A_{1,f} = S_f^{-1}A_1 \quad \text{where} \quad S_f = \{f^i\}_{i\geq 0}.$$

By Eq. 6,

$$\Lambda \subset A_1 \subset \Lambda_f = A_{1,f} = K[x, f^{-1}][\partial; \frac{d}{dx}] \subset B_1. \tag{12}$$

Recall that $\mathrm{Aut}_K(K[x]) = \{\sigma_{\lambda,\mu} \mid \lambda \in K^\times, \mu \in K\}$, $\sigma_{\lambda,\mu}(x) = \lambda x + \mu$ and

$$\mathrm{Aut}_K(K(x)) = \{\sigma_M \mid M \in \mathrm{PGL}_2(K)\} \simeq \mathrm{PGL}_2(K), \ \sigma_M \to M \text{ where } \sigma_M(x) = \frac{ax+b}{cx+d},$$
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(K) := \mathrm{GL}_2(K)/K^\times E \text{ and } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The maps

$$\mathrm{Aut}_K(K[x]) \to \mathrm{Aut}_K(A_1), \quad \sigma_{\lambda,\mu} \mapsto \sigma_{\lambda,\mu} : x \mapsto \lambda x + \mu, \quad \partial \mapsto \lambda^{-1}\partial,$$
$$\mathrm{Aut}_K(K(x)) \to \mathrm{Aut}_K(B_1), \quad \sigma_M \mapsto \sigma_M : x \mapsto \frac{ax+b}{cx+d}, \quad \partial \mapsto \frac{1}{\sigma_M(x)'}\partial,$$

are group monomorphisms where $g' = \frac{dg}{dx}$ for $g \in K(x)$. We identify these groups with their images, i.e.

$$\mathrm{Aut}_K(K[x]) \subseteq \mathrm{Aut}_K(A_1) \quad \text{and} \quad \mathrm{Aut}_K(K(x)) \subseteq \mathrm{Aut}_K(B_1).$$

Notice that $\sigma_M(y) = \sigma_M(f\partial) = \frac{\sigma_M(f)}{f\sigma_M(x)'}f\partial = \frac{\sigma_M(f)}{f\sigma_M(x)'}y$. The automorphism group $\mathrm{Aut}_K(B_1)$ acts in the obvious way on the algebra B_1. Let

$$\mathbb{S}_1 := \mathrm{St}_{\mathrm{Aut}_K(B_1)}(x) := \{\sigma \in \mathrm{Aut}_K(B_1) \mid \sigma(x) = x\},$$

the stabilizer of the element $x \in B_1$ in $\mathrm{Aut}_K(B_1)$. Clearly,

$$\mathbb{S}_1 = \{s_q : \mid q \in K(x)\} \simeq (K(x), +), \quad s_q \mapsto q$$

where $s_q(x) = x$ and $s_q(\partial) = \partial + q$.

Lemma 3.1 *1.* $\mathrm{Aut}_K(B_1) = \mathbb{S}_1 \rtimes \mathrm{Aut}_K(K(x)) = \{\sigma_{M,q} := s_q\sigma_M \mid M \in \mathrm{PGL}_2(K), q \in K(x)\}$.

2. $\mathrm{Aut}_K(B_1, K[x]) := \{\sigma \in \mathrm{Aut}_K(B_1) \,|\, \sigma(K[x]) = K[x]\} = \mathbb{S}_1 \rtimes \mathrm{Aut}_K(K[x]) = \{\sigma_{l,\mu,q} \,|\, \lambda \in K^\times, \mu \in K, q \in K(x)\}$ *where* $\sigma_{l,\mu,q}(x) = \lambda x + \mu$ *and* $\sigma_{\lambda,\mu,q}(\partial) = \lambda^{-1}\partial + q$.

Proof 1. Since $\mathbb{S}_1 := \mathrm{St}_{\mathrm{Aut}_K(B_1)}(x)$, we must have $\mathbb{S}_1 \cap \mathrm{Aut}_K(B_1) = \{e\}$ and $\sigma\mathbb{S}_1\sigma^{-1} \subseteq \mathbb{S}_1$ for all automorphisms $\sigma \in \mathrm{Aut}_K(B_1)$. Hence, $\mathrm{Aut}_K(B_1) \supseteq \mathbb{S}_1 \rtimes \mathrm{Aut}_K(K(x))$.

To prove that the reverse inclusion holds we have to show that every element $\sigma \in \mathrm{Aut}_K(K(x))$ belongs to the group $\mathbb{S}_1 \rtimes \mathrm{Aut}_K(K(x))$. The group of units $K(x)^\times := K(x)\backslash\{0\}$ of the algebra B_1 is a σ-invariant set, i.e. $\sigma(K(x)^\times) = K(x)^\times$. Hence so is the field $K(x)$. Let τ be the restriction of the automorphism σ to the field $K(x)$. Then $\sigma_1 := \tau^{-1}\sigma \in \mathbb{S}_1$, and so $\sigma = \tau\sigma_1 \in \mathbb{S}_1 \rtimes \mathrm{Aut}_K(K(x))$, as required.

2. Statement 2 follows from statement 1. □

Below is a different proof of Theorem 1.1 is given.

Proof of Theorem 1.1 Let $\sigma : \Lambda(f) \to \Lambda(g)$ be an isomorphism of the K-algebras. It can be uniquely extended to a $\overline{K}$-isomorphism $\sigma : \overline{K} \otimes_K \Lambda(f) \to \overline{K} \otimes_K \Lambda(g)$. Let $\lambda_1, \dots, \lambda_s$ (resp., $\lambda'_1, \dots, \lambda'_t$) be the roots of the polynomial f (resp., g) in $\overline{K}$. By Theorem 2.3.(7), the automorphism σ maps bijectively the set $\{(x-\lambda_1), \dots, (x-\lambda_s)\}$ of height 1 completely prime ideals of the algebra $\overline{K} \otimes_K \Lambda(f)$ to the set $\{(x-\lambda'_1), \dots, (x-\lambda'_t)\}$ of height 1 completely prime ideals of the algebra $\overline{K} \otimes_K \Lambda(g)$. Therefore, $s = t$. Since the elements $x-\lambda'_1, \dots, x-\lambda'_t$ are regular normal elements of the domain $\overline{K} \otimes_K \Lambda(g)$ and the set $\overline{K}^\times$ is the group of units of the algebra $\Lambda(g)$, we must have that

$$\sigma(x) = \lambda x + \mu$$

for some elements $\lambda \in \overline{K}^\times$ and $\mu \in \overline{K}$. Since $K[x] = \Lambda(g) \cap \overline{K}[x]$, we must have that $\sigma(x) \in \sigma(\Lambda(f)) \cap \sigma(\overline{K}[x]) = \Lambda(g) \cap \overline{K}[x] = K[x]$, and so $\lambda \in K^\times$ and $\mu \in K$. So, the isomorphism σ respects the polynomial algebra $K[x]$ of the algebras $\Lambda(f)$ and $\Lambda(g)$. In particular, it respects the Ore sets $S = K[x]\backslash\{0\}$ of the algebras $\Lambda(f)$ and $\Lambda(g)$, i.e. $\sigma(S) = S$. The isomorphism σ can be uniquely extended to an isomorphism

$$\sigma : B_1 = S^{-1}\Lambda(f) \to B_1 = S^{-1}\Lambda(g).$$

Then $\sigma(\partial) = \lambda^{-1}\partial + q$ for some element $q \in K[x]$. In particular,

$$\sigma(y) = \sigma(f\partial) = \sigma(f)(\lambda^{-1}\partial + q) = \lambda^{-1}\frac{\sigma(f)}{g}y + p \quad \text{where} \quad p := \sigma(f)q \in K[x]$$

and $\sigma(f) = \gamma g$ for some element $0 \neq \gamma \in K[x]$. Applying the same argument for the isomorphism $\sigma^{-1} : \Lambda(g) \to \Lambda(f)$, we have that $\sigma^{-1}(g) = \gamma_1 f$ for some element $0 \neq \gamma_1 \in K[x]$. Therefore,

$$f = \sigma^{-1}\sigma(f) = \sigma^{-1}(\gamma g) = \sigma^{-1}(\gamma)\gamma_1 f,$$

and so $\gamma, \gamma_1 \in K^\times$, and $\gamma_1 = \gamma^{-1}$. Clearly, $\gamma = \lambda^d$ where $d = \deg(f)$ is the degree of the polynomial f (since $\sigma(x) = \lambda x + \mu$), and the theorem follows. Furthermore,

$$\sigma(x) = \lambda x + \mu \quad \text{and} \quad \sigma(y) = \lambda^{d-1}y + p.$$

□

4 The Eigengroup G_f of a Polynomial f

For each non-scalar monic polynomial $f(x)$, Proposition 4.10, Theorem 4.24, 4.27, 4.30 and 4.32 are explicit descriptions of the eigengroup $G_f(K)$ in the case when the field K is algebraically closed. The case of an arbitrary field is obtained from these results, Theorem 4.33. The aim of this section is to prove these results.

The eigengroup $G_U(K)$

Definition 4.1 [5] Given a group, a G-module V over a field K and a non-empty subset U of V. The *eigengroup* of the set U in G, denoted by $G_U(K)$, is the set of all elements of the group G such that the elements of the set U are eigenvectors of them with eigenvalues in the field K. Clearly, the eigengroup is a subgroup of G.

Clearly,

$$G_U = \bigcap_{u \in U} G_u$$

where $G_u := G_{\{u\}} = \{g \in G \mid gu = \lambda(g)u \text{ for some } \lambda(g) \in K\}$. If K is a subfield of a field K' then $G_U(K) \subseteq G_U(K')$ where U is a subset of the G-module $K' \otimes_K V$ over the field K'.

Finite subgroups of $\mathrm{Aut}_K(K[x])$ Let K be a field of prime characteristic $p > 0$, $\mathbb{F}_p = \mathbf{Z}/\mathbf{Z}p$ is the field that contains p elements, for each $n \geq 1$, $\mathbb{F}_{p^n}$ is the finite field that contains p^n elements, $\overline{\mathbb{F}}_p = \bigcup_{n \geq 1} \mathbb{F}_{p^n}$ is the algebraic closure of the field $\mathbb{F}_p$. Clearly, $\overline{\mathbb{F}}_p \subseteq \overline{K}$ and group of roots of 1 in the field $\overline{K}$ is $\overline{\mathbb{F}}_p^{\times} := \overline{\mathbb{F}}_p \backslash \{0\}$. The group $\mathrm{Aut}_K(K[x]) = \{\sigma_{\lambda,\mu} \mid \lambda \in K^{\times}, \mu \in K\}$ where $\sigma_{\lambda,\mu}(x) = \lambda x + \mu$ and

$$\mathrm{Aut}_K(K[x]) \simeq \mathrm{Sh}(K) \rtimes \mathbb{T} \simeq K \rtimes K^{\times}$$

where $\mathrm{Sh}(K) := \{\sigma_{1,\mu} \mid \mu \in K\} \simeq (K, +)$, $\sigma_{1,\mu} \mapsto \mu$ and $\mathbb{T} := \{\sigma_\lambda, 0\} \mid \lambda \in K^{\times}\} \simeq K^{\times}$, $\sigma_{\lambda,0} \mapsto \lambda$ is the the algebraic 1-dimensional torus.

The set $\mathbb{U} = \mathbb{U}(K) = K \cap \overline{\mathbb{F}}_p^{\times}$ is the group of roots of 1 of the field K. The map $\mathbb{U} \to \mathbb{T}$, $\lambda \mapsto \sigma_{\lambda,0}$ is a group monomorphism and we identify the group $\mathbb{U}$ with its image, i.e. $\mathbb{U} = \{\sigma_{\lambda,0} \mid \lambda \in \mathbb{U}\}$. Let $\mathrm{or}(g)$ be the *order* of an element g of a group G.

Lemma 4.2 *The group* $\mathrm{Sh}(K) \rtimes \mathbb{U}(K) = \{\sigma_{u,\mu} \mid u \in \mathbb{U}(K), \mu \in K\}$ *is the set of all finite order automorphisms of the group* $\mathrm{Aut}_K(K[x])$. *The order of the element* $\sigma_{u,\mu} = \sigma_{u,0}\sigma_{1,\mu}$ *is*

$$\mathrm{or}(\sigma_{\lambda,\mu}) = \begin{cases} \mathrm{or}(\lambda) & \text{if } \lambda \neq 1, \\ p & \text{if } \lambda = 1, \mu \neq 0, \\ 1 & \text{if } \lambda = 1, \mu = 0. \end{cases}$$

Proof For all $\lambda \in K^{\times} \backslash \{1\}$ and $\mu \in K$, $\sigma_{1,\mu}^i = \sigma_{1,i\mu}$ and $\sigma_{\lambda,\mu}^i = \sigma_{\lambda^i, \frac{1-\lambda^i}{1-\lambda}\mu}$, and statement 1 follows. □

By Lemma 4.2,

$$1 \to \mathrm{Sh}(K) \to \mathrm{Sh}(K) \rtimes \mathbb{U}(K) \stackrel{\varphi}{\to} \mathbb{U}(K) \to 1, \quad \text{where } \varphi(\sigma_{\lambda,\mu}) = \lambda, \tag{13}$$

is a short exact sequence of group homomorphisms.

Lemma 4.3 *If an element* $\lambda \in \mathbb{U}(\overline{K})$ *is a primitive* n*'th root of unity then* $\mathbb{F}_p(\lambda) = \mathbb{F}_{p^m}$ *where* $m = \min\{k \geq 1 \mid n | (p^k - 1)\}$ = *the degree of the minimal polynomial of* λ *over the field* $\mathbb{F}_p$.

Proof The element λ is algebraic over the field $\mathbb{F}_p$. Let φ be its minimal polynomial over $\mathbb{F}_p$. Then the field $\mathbb{F}_p(\lambda) \simeq \mathbb{F}_p[x]/(\varphi)$ is a finite field, and so $\mathbb{F}_p(\lambda) = \mathbb{F}_{p^m}$ for some $m \geq 1$. Now,

$$\deg(\varphi) = [\mathbb{F}_p(\lambda) : \mathbb{F}_p] = [\mathbb{F}_{p^m} : \mathbb{F}_p] = m.$$

Notice that the order of the group $\langle\lambda\rangle$, which is n (since λ is a primitive n'th root of unity) divides the order of the group $\mathbb{F}_{p^m}^\times$, which is $p^m - 1$. Clearly, $m \geq m' := \min\{k, \,|\, n|(p^k - 1)\}$. We claim that $m = m'$. Suppose that $m > m'$, wee seek a contradiction. Then $\lambda^{p^{m'}} = \lambda$, and so $\lambda \in \mathbb{F}_{p^{m'}}$, hence $F_{p^m} = \mathbb{F}_p(\lambda) \subseteq \mathbb{F}_{p^{m'}}$. Therefore, $m|m'$, a contradiction. □

The next theorem is a classification of all the finite subgroups of the automorphism group $\mathrm{Aut}_K(K[x])$.

Theorem 4.4 *Let G be a finite subgroup of $\mathrm{Aut}_K(K[x])$. Then:*

1. *$G = \widetilde{G} \rtimes \overline{G}$ where $\widetilde{G} = G \cap \mathrm{Sh}(K) = \{\sigma_{1,\mu} \,|\, \mu \in V\}$, $V \subseteq K$ is a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of K and $\overline{G} = \langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ is a cyclic group of order n where λ_n is a primitive n'th root of 1 and $\nu \in K$.*
   In particular, the order of the group G is np^l where $l = \dim_{\mathbb{F}_p}(V)$ such that $m|l$ where $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$ (Lemma 4.3).
2. *Conversely, given a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace V of K, an automorphism $\sigma_{\lambda_n,(1-\lambda_n)\nu}$ where $\lambda_n \in K$ is a primitive n'th root of unity and $\nu \in K$. Let $\widetilde{G} := \{\sigma_{1,\mu} \,|\, \mu \in V\}$ and $\overline{G} := \langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$. Then:*
   - *(a) The semidirect product $\widetilde{G} \rtimes \overline{G}$ is a finite subgroup of $\mathrm{Aut}_K(K[x])$ of order np^l where $l = \dim_{\mathbb{F}_p}(V)$ such that $m|l$.*
   - *(b) The element $\nu \in K$ is unique up to adding an arbitrary element of V, i.e.*

   $$\widetilde{G} \rtimes \langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle \simeq \widetilde{G} \rtimes \langle\sigma_{\lambda_n,(1-\lambda_n)\nu'}\rangle \;\text{ iff }\; \nu' - \nu \in V.$$

   Furthermore, $G = \{\sigma^i_{\lambda_n,(1-\lambda_n)\nu}\sigma_{1,v} \,|\, 0 \leq i \leq n-1, v \in V\}$ and

   $$\sigma^i_{\lambda_n,(1-\lambda_n)\nu}\sigma_{1,v} = \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu}\sigma_{1,v} : x \mapsto \lambda_n^i x + (1-\lambda_n^i)\nu + v.$$

Proof Let G be a finite subgroup of $\mathrm{Aut}_K(K[x])$. Then, by (13), the group $\varphi(G)$ is a finite subgroup of $\mathbb{U}(K)$ of order n, hence $\varphi(G) = \langle\lambda_n\rangle$ where λ_n is a primitive n'th root of 1. Fix an element, say $\sigma_{\lambda_n,(1-\lambda_n)\nu} \in G$ where $\nu \in K$, such that $\varphi(\sigma_{\lambda_n,(1-\lambda_n)\nu}) = \lambda_n$. Then $\overline{G} = \langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ is a cyclic group of order $n = |\overline{G}|$ since $\sigma^i_{\lambda_n,(1-\lambda_n)\nu} = \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu}$ for all $i \geq 1$. Therefore,

$$G = \widetilde{G} \rtimes \overline{G} \;\text{ where }\; \widetilde{G} := G \cap \mathrm{Sh}(K) = \{\sigma_{1,\mu} \,|\, \mu \in V\},$$

$V \subseteq K$ is a finite dimensional $\mathbb{F}_p$-subspace of K since $\sigma^i_{1,\mu} = \sigma_{1,i\mu}$ for all $i \geq 0$. Furthermore, $\lambda_n V \subseteq V$, i.e. the $\mathbb{F}_p$-vector space V is a $\mathbb{F}_p(\lambda)$-module since

$$\sigma^{-1}_{\lambda_n,(1-\lambda_n)\nu}\sigma_{1,\mu}\sigma_{\lambda_n,(1-\lambda_n)\nu} = \sigma_{1,\lambda\mu}.$$

Clearly, $|G| = |\widetilde{G}||\overline{G}| = p^l n$ and $m|l$ since V is a $\mathbb{F}_{p^m}$-module and $\dim_{\mathbb{F}_{p^m}}(V) = \frac{l}{m}$.
The converse, is obvious.

Clearly, $\widetilde{G} \rtimes \langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle \simeq \widetilde{G} \rtimes \langle\sigma_{\lambda_n,(1-\lambda_n)\nu'}\rangle$ iff there is natural number i such that $1 \leq i \leq n-1$ and a vector $v \in V$ such that

$$\sigma_{\lambda_n,(1-\lambda_n)\nu'} = \sigma^i_{\lambda_n,(1-\lambda_n)\nu}\sigma_{1,v} = \sigma_{\lambda_n^i,(1-\lambda_n^i)(\nu+(1-\lambda_n^i)^{-1}v)}$$

iff $i = 1$ and $\nu' = \nu + (1-\lambda_n^i)^{-1}v \in V$ iff $\nu' - \nu \in V$ since $(1-\lambda_n^i)^{-1}V = V$. □

The automorphism group $\mathrm{Aut}_K(K[x])$ acts on the set $\mathrm{Max}(K[x])$ of maximal ideals of $K[x]$ in the obvious way. If $K = \overline{K}$ then $\mathrm{Max}(K[x]) = \{(x-\gamma) \mid \gamma \in K\}$ and the action takes the form: For all $\sigma \in \mathrm{Aut}_K(K[x])$ and $\gamma \in K$,

$$\sigma((x-\gamma)) = (x - \sigma^{-1}(\gamma)) \ \text{ where } \ \sigma^{-1}(\gamma) := \sigma^{-1}(x)|_{x=\gamma}. \tag{14}$$

Let us identify the set $\mathrm{Max}(K[x])$ with K via $(x-\gamma) \mapsto \gamma$. Then the action of the group $\mathrm{Aut}_K(K[x])$ on $\mathrm{Max}(K[x]) = K$ is given below:

$$\mathrm{Aut}_K(K[x]) \times K \to K, \quad (\sigma, \gamma) \mapsto \sigma * \gamma := \sigma^{-1}(\gamma) = \sigma^{-1}(x)|_{x=\gamma}. \tag{15}$$

If $\sigma = \sigma_{\lambda,\mu}$ then $\sigma_{\lambda,\mu}^{-1} = \sigma_{\lambda^{-1},-\lambda^{-1}\mu}$ and $\sigma_{\lambda,\mu} * \gamma = \sigma_{\lambda,\mu}^{-1}(\gamma) = \sigma_{\lambda^{-1},-\lambda^{-1}\mu}(\gamma) = \lambda^{-1}\gamma - \lambda^{-1}\mu$.

Every automorphism $\sigma_{\lambda,\mu} \in \mathrm{Aut}_K(K[x])$ with $\lambda \neq 1$ can be uniquely written in the form $\sigma_{\lambda,(1-\lambda)\nu}$ where $\nu = (1-\lambda)^{-1}\mu$. Notice that

$$\sigma_{\lambda,(1-\lambda)\nu} * (\nu) = \nu.$$

Furthermore, the set $\{\nu\}$ is the only 1-element orbit in K of the cyclic group $\langle \sigma_{\lambda,(1-\lambda)\nu} \rangle$ generated by the automorphism $\sigma_{\lambda,(1-\lambda)\nu}$. The number of elements in any other orbit is equal to the order of the group $\langle \sigma_{\lambda,(1-\lambda)\nu} \rangle$ which is the order of the element λ in the group $(K^\times, \cdot)$.

Suppose that $K = \overline{K}$. Let $f \in K[x]$ be a non-scalar monic polynomial that has at least two distinct roots in $K = \overline{K}$. Recall that $\mathcal{R}(f)$ is the set of all roots of the polynomial f counted with multiplicity and $\mathcal{R}_d(f)$ be the set of all *distinct* roots of f (i.e. each root has multiplicity 1). Example. For $f = (x-1)^2(x-2)^3$, $\mathcal{R}(f) = \{1, 1, 2, 2, 2\}$ and $\mathcal{R}(f) = \{1, 2\}$.

The group G_f permutes the roots in $\mathcal{R}(f)$ and $\mathcal{R}_d(f)$ via the action Eq. 14. Let us stress that the action of G_f on $\mathcal{R}(f)$ respects the multiplicity. If the group G_f is finite then $G_f = \widetilde{G}_f \rtimes \overline{G}_f$ and $\widetilde{G}_f = \mathrm{Sh}_V$ is a normal subgroup of G_f. For a set $\mathcal{R} = \mathcal{R}(f), \mathcal{R}_d(f)$ and a group $G = G_f, \widetilde{G}_f, \overline{G}_f$, we denote by $\mathcal{R}/G$ the set of G-orbits in $\mathcal{R}$.

Invariants and eigenalgebras of finite subgroups of $\mathrm{Aut}_K(K[x])$ Notice that

$$x^p - x = \prod_{i \in \mathbb{F}_p} (x - i). \tag{16}$$

For each element $\mu \in K^\times$, let

$$f_\mu(x) := \prod_{i=0}^{p-1} \sigma_{1,\mu}^i(x) = \prod_{i=0}^{p-1} (x - i\mu) = \prod_{i \in \mathbb{F}_p} (x - i\mu) = x^p - \mu^{p-1}x. \tag{17}$$

The equality above follows at once from (16): $f_\mu(x) = \prod_{i=0}^{p-1}(x - i\mu) = \mu^p \prod_{i=0}^{p-1}(\mu^{-1} x - i) = \mu^p((\mu^{-1}x)^p - \mu^{-1}x) = x^p - \mu^{p-1}x$. For all $\alpha, \beta \in \mathbb{F}_p$:

$$f_\mu(\alpha x + \beta x') = \alpha f_\mu(x) + \beta f_\mu(x')$$

since $\gamma^p = \gamma$ for all $\gamma \in \mathbb{F}_p$. In particular, the map $K \to K, \lambda \mapsto f(\lambda)$ is a $\mathbb{F}_p$-linear map. Hence for all elements $\lambda \in K$,

$$\begin{aligned} x^p - \mu^{p-1}x - (\lambda^p - \mu^{p-1}\lambda) &= f_\mu(x) - f_\mu(\lambda) = f_\mu(x - \lambda) \\ &= \prod_{i=0}^{p-1} (x - \lambda - i\mu) = \prod_{i \in \mathbb{F}_p} (x - \lambda - i\mu). \end{aligned} \tag{18}$$

Given a K-algebra A, and a subgroup G of the automorphism group $\mathrm{Aut}_K(A)$. A group homomorphism $\chi : G \to K^\times$ is called a *character* of the group G in $K^\times$. Let $\widehat{G}(K)$ be the (multiplicative) *group of characters* of the group G in $K^\times$. The multiplication in the group $\widehat{G}(K)$ is given by the rule: For all $\chi, \psi \in \widehat{G}(K)$, $(\chi\psi)(g) = \chi(g)\psi(g)$ for all elements $g \in G$.

Definition 4.5 The direct sum of G-eigenspaces,

$$\mathbb{E}(A) = \mathbb{E}(A, G) := \bigoplus_{\chi\in\widehat{G}(K)} A^\chi, \quad \text{where} \quad A^\chi := \{a \in A \mid g(a) = \chi(g)a \ \text{ for all } \ g \in G\},$$

is called the *G-eigenalgebra* of A. The direct sum is a $\widehat{G}(K)$-graded algebra since $A^\chi A^\psi \subseteq A^{\chi\psi}$ for all $\chi, \psi \in \widehat{G}(K)$.

If e is the identity element of the character group $\widehat{G}(K)$ then $A^e = A^G$ is the *algebra of G-invariants*. In particular $A^G \subseteq \mathbb{E}(A, G)$.

The set $\mathrm{Supp}(A, G) := \{\chi \in \widehat{G}(K) \mid A^\chi \neq 0\}$ is called the *support* of G in A. For each character $\chi \in \mathrm{Supp}(A, G)$, the vector space A^χ is called the *χ-weight/eigenvalue subspace* of the algebra A. If the algebra $\mathbb{E}(A, G)$ is a domain (eg, the algebra A is a domain) then the support $\mathrm{Supp}(A, G)$ is a submonoid of $\widehat{G}(K)$ and the algebra $\mathbb{E}(A, G)$ is a $\mathrm{Supp}(A, G)$-graded algebra. If the algebra A is a commutative algebra then the *Frobenius* endomorphism $\mathrm{Fr} : A \to A, a \mapsto a^p$ is a $\mathbb{F}_p$-algebra endomorphism of A. It is a monomorphism if the algebra A is a domain. By the very definition, the Frobenius endomorphism commute with all endomorphisms of the ring A.

Lemma 4.6 *Let A be a commutative K-algebra and G be a subgroup of the automorphism group* $\mathrm{Aut}_K(A)$.

1. *The algebras $\mathbb{E}(A, G)$ and A^G are* Fr*-stable (that is $\mathrm{Fr}(\mathbb{E}(A, G)) \subseteq \mathbb{E}(A, G)$ and $\mathrm{Fr}(A^G) \subseteq A^G$).*
2. *Suppose that the algebra A is reduced and $\mathrm{Fr}(K) = K$. If $g(\mathrm{Fr}(a)) = \chi(g)\mathrm{Fr}(a)$ for all $g \in G$ then $g(a) = (\chi(g))^{\frac{1}{p}}a$. In particular, $\mathrm{Fr} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{E}(A, G))$ and $\mathrm{Fr} \in \mathrm{Aut}_{\mathbb{F}_p}(A^G)$.*

Proof 1. The Frobenius endomorphism commutes with all endomorphisms of the ring A, and statement 1 follows.

2. The equality $\mathrm{Fr}(K) = K$ implies that $\mathrm{Fr} \in \mathrm{Aut}_{\mathbb{F}_p}(K)$. Since the algebra A is reduced the Frobenius endomorphism A is a monomorphism. If $g(\mathrm{Fr}(a)) = \chi(g)\mathrm{Fr}(a)$ for all $g \in G$ then $(g(a) - (\chi(g))^{\frac{1}{p}}a)^p = 0$, and so $g(a) = (\chi(g))^{\frac{1}{p}}a$ for all $g \in G$, and statement 2 follows. □

Let $V \subseteq K$ be a $\mathbb{F}_{p^m}$-subspace of K. The subgroup $\mathrm{Sh}_V := \{\sigma_{1,v} \mid v \in V\}$ of $\mathrm{Aut}_K(K[x])$ is called the *shift group* that is determined by the $\mathbb{F}_{p^m}$-subspace V. Proposition 4.7 describes the algebra of invariants and the eigenalgebra of the shift group Sh_V.

Proposition 4.7 *Let $V \subseteq K$ be a nonzero $\mathbb{F}_{p^m}$-subspace of K. Then $\mathbb{E}(K[x], \mathrm{Sh}_V) = K[x]^{\mathrm{Sh}_V}$.*

1. *If $\dim_{\mathbb{F}_{p^m}}(V) = \infty$ then $K[x]^{\mathrm{Sh}_V} = K$.*
2. *If $l = \dim_{\mathbb{F}_{p^m}}(V) < \infty$ and $\{\mu_1, \dots, \mu_l\}$ is a basis of the vector space V over $\mathbb{F}_{p^m}$ then*
    (a) *the fixed algebra $K[x]^{\mathrm{Sh}_V} = K[f_V]$ is a polynomial algebra in $f_V := \prod_{v\in V}(x - v)$, the polynomial f_V is divisible by the polynomial $\prod_{i=1}^l f_{\mu_i}$,*

(b) *for all elements* $\alpha, \beta \in \mathbb{F}_{p^m}$ *and* $\lambda \in K$, $f_V(\alpha x + \beta\lambda) = \alpha f_V(x) + \beta f_V(\lambda)$. *In particular, the map* $K \to K$, $\lambda \mapsto f_V(\lambda)$ *is a* $\mathbb{F}_{p^m}$*-linear map,*
(c) *If* $V \subset V'$ *are distinct* $\mathbb{F}_{p^m}$*-subspaces of* K *then* $f_V | f_{V'}$ *and* $f_V \neq f_{V'}$,
(d) $\frac{df_V}{dx} \neq 0$.

3. *In particular, for all elements* $\mu \in K^\times$, $K[x]^{\sigma_{1,\mu}} = K[f_\mu]$.

Proof For all elements $\mu \in K$, the map

$$\sigma_{1,\mu} - 1 : K[x] \to K[x], \quad \psi(x) \mapsto \psi(x+\mu) - \psi(x)$$

is locally nilpotent map. Therefore, the element 1 is the only eigenvalue for the map $\sigma_{1,\mu}$, and so $\mathbb{E}(K[x], \mathrm{Sh}_V) = K[x]^{\mathrm{Sh}_V}$.

1. Statement 1 follows from statement 2. Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a family of $\mathbb{F}_{p^m}$-linearly independent elements of the vector space V and $V_i = \bigoplus_{j=1}^{i} \mathbb{F}_{p^m}\mu_i$. Then $V_1 \subset V_2 \subset \cdots \subset V_\infty := \bigoplus_{i\geq 1} \mathbb{F}_{p^m}\mu_i \subseteq V$. Hence,

$$K[x]^{\mathrm{Sh}_{V_1}} \supseteq K[x]^{\mathrm{Sh}_{V_2}} \supseteq \cdots \supseteq K[x]^{\mathrm{Sh}_{V_\infty}} = \bigcap_{i\geq 1} K[x]^{\mathrm{Sh}_{V_i}} = \bigcap_{i\geq 1} K[f_{V_i}] = K \supseteq K[x]^{\mathrm{Sh}_V} \supseteq K,$$

and so $K[x]^{\mathrm{Sh}_V} = K$.

2. (a,b). For all elements $v' \in V$,

$$\sigma_{1,v'}(f_V) = \prod_{v\in V}(x - v' - v) = f_V.$$

Therefore, $K[x]^{\mathrm{Sh}_V} \supseteq K[f_V]$. By Eq. 17, the polynomial $\prod_{i=1}^{l}\prod_{u\in\mathbb{F}_p}(x - u\mu_i) = \prod_{i=1}^{l} f_{\mu_i}$ is a divisor of the polynomial f_V. In particular, $f_V(0) = 0$.

First, we prove that the statements (a) and (b) hold in the case when $K = \overline{K}$ and then we deduce that the statements (a) and (b) hold for an arbitrary field K.

So, suppose that $K = \overline{K}$. Let $g(x) \in K[x]^{\mathrm{Sh}_V}$ be a non-scalar monic polynomial. Let γ be a root of $g(x)$. Then for all elements $v \in V$, the element $\gamma + v$ is also a root of the polynomial $g(x)$. So, the set of all roots of the polynomial $g(x)$ is a disjoint union of the sets $\coprod_{i=1}^{s}\{\gamma_i + V\}$ for some roots γ_i of $g(x)$.

Therefore, the polynomial

$$g(x) = \prod_{i=1}^{s}\prod_{v\in V}(x - \gamma_i - v) = \prod_{i=1}^{s} f_V(x - \gamma_i)$$

is a product of Sh_V-invariant polynomials $f_V(x-\gamma_i)$ of the same degree p^{lm}. In particular, every non-scalar Sh_V-invariant polynomial has degree at least p^{lm}. Therefore, for all elements $\lambda, \mu \in K$, the difference

$$c_{\lambda,\mu} := f_V(x+\lambda) - f_V(x+\mu)$$

of two monic Sh_V-invariant polynomials of degree p^{lm} must be a constant which is equal to $f_V(\lambda) - f_V(\mu)$. Therefore,

$$f_V(x+\lambda) - f_V(\lambda) = f_V(x+\mu) - f_V(\mu).$$

When $\mu = 0$, we have that

$$f_V(x+\lambda) = f_V(x) + f_V(\lambda) - f_V(0) = f_V(x) + f_V(\lambda)$$

since $f_V(0) = 0$. Since for all elements $u \in \mathbb{F}_{p^m}^{\times}$,

$$f_V(ux) = u^{p^{lm}} \prod_{v \in V}(x - u^{-1}v) = u \prod_{v \in V}(x - v) = u f_V(x),$$

and $f_V(0x) = 0 = 0 f_V(x)$, we see that for all elements $\xi \in \mathbb{F}_{p^m}$, $f_V(\xi x) = \xi f_V(x)$. Now, the statement (b) follows.

Now, the polynomial

$$g(x) = \prod_{i=1}^{s} f_V(x - \gamma_i) = \prod_{i=1}^{s}(f_V(x) - f_V(\gamma_i)) \in K[f_V(x)]$$

and the statement (a) follows.

Suppose that K is not necessarily algebraically closed field and $g(x) \in K[x]^{\mathrm{Sh}_V}$ be a non-scalar monic polynomial. Then $g(x) \in \overline{K}[f_V(x)]$. Since the $\overline{K} = K \oplus W$ for some K-subspace W of K and $f_V(x) \in K[x]$, we must have that $g(x) \in K[f_V(x)]$ since

$$\overline{K}[f_V(x)] = K[f_V(x)] \oplus \bigoplus_{i \geq 0} W f_V(x)^i \subseteq K[x] \oplus \bigoplus_{i \geq 0} W f_V(x)^i.$$

Now, the statements (a) and (b) hold for the field K.

(c) The statement (c) follows from the statement (a).

(d) WLOG we may assume that $K = \overline{K}$. Suppose that $\frac{df_V}{dx} = 0$. Then $f_V = g^p$ for some polynomial g. This is not possible as every root of f has multiplicity 1.

3. Statement 3 is a particular case of statement 2. $\square$

Notice that for all natural numbers $m \geq 1$,

$$x^{p^m} - x = \prod_{i \in \mathbb{F}_{p^m}} (x - i). \tag{19}$$

By Eq. 19, for each element $\mu \in K^{\times}$, let

$$f_{p^m,\mu}(x) := f_{\mathbb{F}_{p^m}\mu}(x) = \prod_{i \in \mathbb{F}_{p^m}} (x - i\mu) = x^{p^m} - \mu^{p^m-1}x. \tag{20}$$

For all $\alpha, \beta \in \mathbb{F}_{p^m}$:

$$f_{p^m,\mu}(\alpha x + \beta x') = \alpha f_{p^m,\mu}(x) + \beta f_{p^m,\mu}(x') \tag{21}$$

since $\gamma^{p^m} = \gamma$ for all $\gamma \in \mathbb{F}_{p^m}$. In particular, the map $K \to K$, $\lambda \mapsto f_{p^m,\mu}(\lambda)$ is a $\mathbb{F}_{p^m}$-linear map. Hence, for all elements $\lambda \in K$,

$$x^{p^m} - \mu^{p^m-1}x - (\lambda^{p^m} - \mu^{p^m-1}\lambda) = f_{p^m,\mu}(x) - f_{p^m,\mu}(\lambda) = f_{p^m,\mu}(x - \lambda) = \prod_{i \in \mathbb{F}_{p^m}} (x - \lambda - i\mu). \tag{22}$$

Theorem 4.8 describes the algebra of invariants and the eigenalgebra of a 'generic' finite subgroup of $\mathrm{Aut}_K(K[x])$.

Theorem 4.8 *Let $G = \widetilde{G} \rtimes \overline{G}$ be a finite subgroup of* $\mathrm{Aut}_K(K[x])$ (Theorem 4.4) *where $\overline{G} := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ and $\widetilde{G} := \{\sigma_{1,\mu} \mid \mu \in V\}$, $\lambda_n \in K$ is a primitive n'th root of unity, $n \geq 2$ and $\nu \in K$, V is a nonzero finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of K, and $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$* (Lemma 4.3). *Then:*

1. $\sigma_{\lambda_n,(1-\lambda_n)\nu}(f_V(x - \nu)) = \lambda_n f_V(x - \nu)$.

2. $K[x]^G = K[f_V^n(x-\nu)]$ *is a polynomial algebra in* $f_V^n(x-\nu) := \Big(f_V(x-\nu)\Big)^n$.
3. *The* G*-eigenvalue subalgebra of* $K[x]$ *is* $\mathbb{E}(K[x], G) = \bigoplus_{i=0}^{n-1} f_V^i(x-\nu)K[x]^G$*, a direct sum of distinct* G*-eigenspaces.*

Proof 1. Let $\sigma = \sigma_{\lambda_n,(1-\lambda_n)\nu}$. Suppose that $l = \dim_{\mathbb{F}_{p^m}}(V)$. Then $\deg(f_V(x)) = p^{lm}$. Now, by Proposition 4.7.(2b),

$$\sigma(f_V(x-\nu)) = f_V(\sigma(x-\nu)) = f_V(\lambda_n(x-\nu)) = \lambda_n f_V(x-\nu)$$

since $\lambda_n \in K(\lambda_n) = \mathbb{F}_{p^m}$.

2 and 3. By Proposition 4.7.(2b), the $\widetilde{G}$-eigenvalue subalgebra of $K[x]$ is the fixed algebra $K[x]^{\widetilde{G}} = K[f_V(x)] = K[f_V(x-\nu)]$ since

$$f_V(x) = f_V(x-\nu+\nu) = f_V(x-\nu) + f_V(\nu).$$

By statement 1, $\sigma(f_V(x-\nu)) = \lambda_n f_V(x-\nu)$, hence the $\widetilde{G}$-eigenvalue subalgebra of $K[x]$, $K[f_V(x-\nu)]$, is σ-invariant, and statements 2 and 3 follow. □

Proposition 4.9 describes the algebra of invariants and the eigenalgebra of the subgroup $G = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ of $\mathrm{Aut}_K(K[x])$ where $\lambda_n \in K$ is a primitive n'th root of unity, $n \geq 2$ and $\nu \in K$.

Proposition 4.9 *Let* $G = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ *be a finite subgroup of* $\mathrm{Aut}_K(K[x])$ *where* $\lambda_n \in K$ *is a primitive* n*'th root of unity,* $n \geq 2$ *and* $\nu \in K$*. Then:*

1. $\sigma_{\lambda_n,(1-\lambda_n)\nu}(x-\nu) = \lambda_n(x-\nu)$.
2. $K[x]^G = K[(x-\nu)^n]$ *is a polynomial algebra in* $(x-\nu)^n$.
3. $\mathbb{E}(K[x], G) = K[x] = \bigoplus_{i=0}^{n-1}(x-\nu)^i K[x]^G$ *is a direct sum of distinct* G*-eigenspaces.*

Proof 1. Statement 1 is obvious.

2. Statement 2 follows from statement 1 and the fact that λ_n is a primitive n'th root of unity.

3. Statement 3 follows from statement 2. □

The eigengroup $G_f(K)$ of a polynomial $f \in K[x]$ that has single root in $\overline{K}$ For an element $\nu \in K$, the subset

$$\mathbb{T}_\nu(K) := \{\sigma_{\lambda,(1-\lambda)\nu} \mid \lambda \in K^\times\} \tag{23}$$

of $\mathrm{Aut}_K(K[x])$ is a subgroup which is isomorphic to the algebraic torus $\mathbb{T} = (K^\times, \cdot)$ via

$$\mathbb{T}_\nu(K) \to \mathbb{T}, \quad \sigma_{\lambda,(1-\lambda)\nu} \mapsto \lambda$$

since $\sigma_{\lambda,(1-\lambda)\nu}\sigma_{\lambda',(1-\lambda')\nu} = \sigma_{\lambda\lambda',(1-\lambda\lambda')\nu}$ for all $\lambda, \lambda' \in K^\times$.

Suppose that a monic non-scalar polynomial $f \in \overline{K}[x]$ of degree d has single root, say $\nu \in \overline{K}$. Then $d = p^r d_1$ where for unique natural numbers r and d_1 such that $p \nmid d_1$. Then

$$f = (x-\nu)^d = (x-\nu)^{p^r d_1} = (x^{p^r} - \nu^{p^r})^{d_1} = x^{p^r d_1} - d_1\nu^{p^r}x^{p^r(d_1-1)} + \cdots. \tag{24}$$

Therefore, $f = (x-\nu)^d \in K[x]$ iff $\nu^{p^r} \in K$ (since $p \nmid d_1$).

The next proposition describes all the monic polynomial $f \in K[x]$ such that $G_f(K) = \mathbb{T}_\nu(K)$ for some $\nu \in K$. Furthermore, it describes the group G_f for all polynomial $f \in K[x]$ that has a single root in $\overline{K}$.

Proposition 4.10 *1. Let $f(x) \in \overline{K}[x]$ be a monic non-scalar polynomial of degree d. Then $f(x) = (x-\nu)^d$ for some $\nu \in \overline{K}$ iff $G_f(\overline{K}) = \mathbb{T}_\nu(\overline{K})$.*

2. Let $f(x) \in K[x]$ be a monic non-scalar polynomial of degree d that has a single root $\nu \in \overline{K}$. Then $G_f(K) = \begin{cases} \mathbb{T}_\nu(K) & \text{if } \nu \in K, \\ \{e\} & \text{if } \nu \notin K. \end{cases}$

3. Let $f(x) \in K[x]$ be a monic non-scalar polynomial of degree d that has a single root $\nu \in \overline{K}$. Then $f(x) = (x-\nu)^d$ for some $\nu \in K$ iff $G_f(K) = \mathbb{T}_\nu(K)$.

Proof 1. ($\Rightarrow$) Suppose that $f = (x-\nu)^d$. An automorphism $\sigma \in \mathrm{Aut}_K(K[x])$ belongs to the group $G_f(\overline{K})$ iff $\sigma(x-\nu) = \lambda(x-\nu)$ for some element $\lambda \in \overline{K}^\times$. The last equality is equivalent to the equality $\sigma = \sigma_{\lambda,(1-\lambda)\nu}$, or equivalently, $G_f = \mathbb{T}_\nu(\overline{K})$.

($\Leftarrow$) Suppose that $G_f = \mathbb{T}_\nu(\overline{K})$. Then $K[x] = \bigoplus_{i \geq 0} K(x-\nu)^i$ is a direct sum of the eigenspaces of the group $\mathbb{T}_\nu(\overline{K})$. Therefore, $f(x) = (x-\nu)^d$ for some $d \geq 1$.

2. Clearly, $G_f(K) = G_f(\overline{K}) \cap \mathrm{Aut}_K(K[x]) = \mathbb{T}_\nu(\overline{K}) \cap \mathrm{Aut}_K(K[x])$. By statement 1, $G_f(K) \neq \{e\}$ iff $e \neq \sigma_{\lambda,(1-\lambda)\nu} \in \mathbb{T}_\nu(\overline{K}) \cap \mathrm{Aut}_K(K[x])$ where $1 \neq \lambda \in K^\times$ and (necessarily) $\nu \in K^\times$ iff $G_f(K) = \mathbb{T}_\nu(K)$.

3. Statement 3 follows from statement 2. □

The eigengroup $G_f(K)$ of a polynomial $f \in K[x]$ that has at least two distinct roots in $\overline{K}$ For each non-scalar monic polynomial $f(x)$, Theorems 4.24, 4.27, 4.30 and 4.32 (resp., Theorem 4.33) are explicit descriptions of the eigengroup $G_f(K)$ in the case when the field K is algebraically closed (resp., in general case).

Lemma 4.11 is an explicit description of the roots of the polynomials of the type $g(f_V^n(x-\nu))$.

Lemma 4.11 *Suppose that $\lambda_n \in K$ is a primitive n'th root of unity, $n \geq 2$ and $\nu \in K$, V is a nonzero finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of K, and $K(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$* (Lemma 4.3). *Then:*

1. *For all elements $\rho \in K$,*

$$f_V^n(x-\nu) - f_V^n(\rho) = \prod_{i=0}^{n-1} \prod_{v \in V} (x - \nu - \lambda_n^i \rho - v).$$

2. *Let $g(x) = \prod_{j=1}^k (x - \xi_j) \in \overline{K}[x]$ where $\mathcal{R}(g) = \{\xi_1, \ldots, \xi_k\}$ is the set of roots of the polynomial $g(x)$ counted with multiplicity. Then $\xi_j = f_V^n(\rho_j)$ for some element $\rho_j \in \overline{K}$ and*

$$g(f_V^n(x-\nu)) = \prod_{j=1}^k \prod_{i=0}^{n-1} \prod_{v \in V} (x - \nu - \lambda_n^i \rho_j - v).$$

Proof 1. By Proposition 4.7.(2b), the map f_V is a $K(\lambda_n)$-linear map (since $K(\lambda_n) = \mathbb{F}_{p^m}$), and the result follows:

$$f_V^n(x-\nu) - f_V^n(\rho) = \prod_{i=0}^{n-1} (f_V(x-\nu) - \lambda_n^i f_V(\rho)) = \prod_{i=0}^{n-1} f_V(x - \nu - \lambda_n^i \rho)$$
$$= \prod_{i=0}^{n-1} \prod_{v \in V} (x - \nu - \lambda_n^i \rho - v).$$

2. Notice that $g(x) = \prod_{j=1}^k (f_V^n(x-\nu) - f_V^n(\rho_j))$, and statement 2 follows from statement 1. □

Theorem 4.12 *Suppose that a monic polynomial $f(x) \in K[x]$ has at least two distinct roots in $\overline{K}$. Then the group $G_f(K)$ is a finite group, $G_f(K) = \widetilde{G}_f(K) \rtimes \overline{G}_f(K)$ where $\overline{G}_f(K) = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ and $\widetilde{G}_f(K) = \mathrm{Sh}_V(K)$, $\lambda_n \in K$ is a primitive n'th root of unity, $\nu \in K$, V is a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of K, and $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$* (Lemma 4.3).

Proof Since $G_f(K) = G_f(\overline{K}) \cap \mathrm{Aut}_K(K[x])$, it suffices to prove the theorem in the case when the field K is an algebraically closed field. So, we assume that $K = \overline{K}$. Recall that $\mathcal{R}_d$ be the set of *distinct* roots of the polynomial f. The subgroup $\widetilde{G}_f = G_f \cap \mathrm{Sh}(K)$ is equal to a group Sh_V where V is a finite dimensional vector space over the field $\mathbb{F}_p$ since $|V| \leq |\mathcal{R}_d|$ (as $\mathcal{R}_d + V \subseteq \mathcal{R}_d$).

If G is a finite subgroup of G_f that contains the group Sh_V then $G = \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ where λ_n is a primitive n'th root of unity and $\nu \in K$. The cyclic group $\langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ of order n acts on the field K and on the set $\mathcal{R}_d$, see Eq. 14. The point ν is the only fixed point of the action and the orbit of every element $\lambda \neq \nu$ contains precisely n elements. The polynomial f contains at least two distinct roots. Therefore $n \leq |\mathcal{R}_d|$. Then, by Eq. 13, the group $\varphi(G_f)$ is equal to $\langle \sigma_{\lambda_{n'},(1-\lambda_{n'})\nu'} \rangle$ where $\lambda_{n'}$ is a primitive n''th root of unity and $\nu' \in K$. By Theorem 4.4, $G_f = \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_{n'},(1-\lambda_{n'})\nu'} \rangle$ is a finite group. □

Corollary 4.13 is a criterion for the group G_f to be an infinite group.

Corollary 4.13 *Let $f(x) \in K[x]$ be a non-scalar monic polynomial. Then the following statement are equivalent:*

1. *The group G_f is an infinite group.*
2. *$f(x) = (x - \nu)^d$ for some $\nu \in K$.*
3. *$G_f(K) = \mathbb{T}_\nu(K)$ (see Eq. 23 for the definition of the group $\mathbb{T}_\nu(K)$).*

Proof The corollary follows from Proposition 4.10.(3) and Theorem 4.12. □

Recall that the field K is an algebraically closed field, $f(x) \in K[x]$ is a monic polynomial that has at least two distinct roots, and $\mathcal{R}(f)$ is the set of all roots of f counted with multiplicity. Recall that (Theorems 4.12 and 4.4) the group $G_f = \widetilde{G}_f \rtimes \overline{G}_f$ is a finite subgroup of $\mathrm{Aut}_K(K[x])$ where $\overline{G}_f := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ provided $\overline{G}_f \neq \{e\}$ and $\widetilde{G}_f := \{\sigma_{1,\mu} \mid \mu \in V\}$, $\lambda_n \in K$ is a primitive n'th root of unity, $\nu \in K$, V is a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of K, and $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$ (Lemma 4.3).

There are four distinguish cases:

1. $\widetilde{G}_f \neq \{e\}$, $\overline{G}_f \neq \{e\}$,
2. $\widetilde{G}_f \neq \{e\}$, $\overline{G}_f = \{e\}$,
3. $\widetilde{G}_f = \{e\}$, $\overline{G}_f \neq \{e\}$,
4. $\widetilde{G}_f = \{e\}$, $\overline{G}_f = \{e\}$.

Below, in the case of $K = \overline{K}$, for the polynomial f criteria are given in terms of its roots for each case to hold.

A description of the group $\widetilde{G}_f(K)$ and a criterion for $\widetilde{G}_f(K) \neq \{e\}$.

Definition 4.14 Let $f(x) \in K[x]$ be a non-scalar polynomial. Two distinct roots $\lambda, \lambda' \in \overline{K}$ of the polynomial $f(x)$ are called a *K-shift pair* of $f(x)$ if

$$\lambda - \lambda' \in K \quad \text{and} \quad \mathbb{F}_p(\lambda - \lambda') + \mathcal{R}(f) \subseteq \mathcal{R}(f) \tag{25}$$

where $\mathcal{R}(f)$ is the set of all roots in $\overline{K}$ of the polynomial $F(x)$ counted with multiplicity.

The set of all K-shift pairs of the polynomial $f(x)$ is denoted by $\mathrm{SP}(f, K)$. The vector space over K,

$$V(f,K) = \begin{cases} \sum_{\{\lambda,\lambda'\}\in \mathrm{SP}(f,K)} \mathbb{F}_p(\lambda-\lambda') & \text{if } \mathrm{SP}(f,K)\neq\emptyset, \\ 0 & \text{if } \mathrm{SP}(f,K)=\emptyset. \end{cases} \tag{26}$$

is called the K-*shift vector space* of the polynomial $f(x)$.

For fields $K \subseteq L \subseteq \overline{K}$, we have $\mathrm{SP}(f, K) \subseteq \mathrm{SP}(f, L) \subseteq \mathrm{SP}(f, \overline{K})$ and $V(f, K) \subseteq V(f, L) \subseteq V(f, \overline{K})$.

Proposition 4.15 gives an explicit description of the group $\widetilde{G}_f(K)$ and a criterion for $\widetilde{G}_f(K) \neq \{e\}$.

Proposition 4.15 *Let $f(x) \in K[x]$ be a non-scalar monic polynomial. Then:*

1. *$\widetilde{G}_f(K) = \mathrm{Sh}_V$ where $V = V(f, K)$ is the K-shift vector space of f.*
2. *$\widetilde{G}_f(K) = \{e\}$ iff $\mathrm{SP}(f, K) = \emptyset$ iff for all distinct roots $\lambda, \lambda' \in \overline{K}$ of the polynomial f such that $\lambda - \lambda' \in K$ (if they exist), $\mathbb{F}_p(\lambda - \lambda') + \mathcal{R}(f) \nsubseteq \mathcal{R}(f)$.*
3. *$\widetilde{G}_f(\overline{K}) = \{e\}$ iff $\mathrm{SP}(f, \overline{K}) = \emptyset$ iff for all distinct roots $\lambda, \lambda' \in \overline{K}$ of the polynomial f, $\mathbb{F}_p(\lambda - \lambda') + \mathcal{R}(f) \nsubseteq \mathcal{R}(f)$.*

Proof 1. It follows from the description of the group $\widetilde{G}_f(K)$ as a shift group, $\widetilde{G}_f(K) = \mathrm{Sh}_V$, that $V = V(f, K)$.

2 and 3. Statement 2 follows from statement 1 and statement 3 is a particular case of statement 2. □

Definition 4.16 Let V be a nonzero finite dimensional $\mathbb{F}_p$-subspace of K. The largest finite field, denoted $\mathbb{F}_{p^e}$, where $e = e(V) \geq 1$, such that $\mathbb{F}_{p^e} V \subseteq V$ is called the *multiplier field* of V. The natural number $e = e(V)$ is called the p-*exponent* of V.

The multiplier field $\mathbb{F}_{p^e}$ is the composite of all finite fields $\mathbb{F}_{p^m}$ such that $\mathbb{F}_{p^m} V \subseteq V$. If $\dim_{\mathbb{F}_p}(V) = n$ then $p^m \leq |V| = p^n$, and so $m \leq n$.

Let λ_{p^e-1} be a primitive $p^e - 1$'st root of unity (a generator of the cyclic group $\mathbb{F}_{p^e}^{\times}$). Since $\lambda_{p^e-1} \in \mathbb{F}_{p^e}$ and $\mathbb{F}_{p^e} V \subseteq V$, we have that $\lambda_{p^e-1} V \subseteq V$.

For each finite dimensional $\mathbb{F}_p$-subspace V of the field K, Lemma 4.17 describes all the roots of unity λ_n such that $\lambda_n V \subseteq V$.

Lemma 4.17 *Let V be a nonzero finite dimensional $\mathbb{F}_p$-subspace of the field K, $\mathbb{F}_{p^e}$ be its multiplier field.*

1. *Suppose that λ_n is a primitive n'th root of unity. Then $\lambda_n V \subseteq V$ iff $n | p^e - 1$.*
2. *$|\mathbb{F}_{p^e}^{\times}| = 1$ iff $(p, e) = (2, 1)$ iff $\lambda_n V \subseteq V$ (where λ_n is a primitive n'th root of unity) implies $\lambda_n = 1$.*

Proof 1. $\lambda_n V \subseteq V$ iff $\lambda_n \in \mathbb{F}_{p^e}^{\times}$ iff $n || \mathbb{F}_{p^e}^{\times}|$ iff $n | p^e - 1$.

2. Statement 2 follows from statement 1. □

Classification of subgroups G of $\mathrm{Aut}_K(K[x])$ which are maximal satisfying the property $G \cap \mathrm{Sh}(K) = \mathrm{Sh}_V$.

Corollary 4.18 *Let V be a nonzero finite dimensional $\mathbb{F}_p$-subspace of the field K, $\mathbb{F}_{p^e}$ be its multiplier field and λ_{p^e-1} be a primitive $p^e - 1$'st root of unity. Then the finite groups $G_{V,\nu} := \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_{p^e-1}, (1-\lambda_{p^e-1})\nu} \rangle$, where $\nu \in K/V$, are the maximal subgroups G of the group $\mathrm{Aut}_K(K[x])$ that satisfy the property that $G \cap \mathrm{Sh}(K) = \mathrm{Sh}_V$.*

Proof Recall that for all elements $\lambda \in K^\times$ and $\mu, v \in K$, $\sigma_{\lambda,\mu}\sigma_{1,v}\sigma_{\lambda,\mu}^{-1} = \sigma_{1,\lambda^{-1}v}$. Hence the groups $G_{V,\nu}$ are well defined and every subgroup H of $\mathrm{Aut}_K(K[x])$ such that $H \cap \mathrm{Sh}(K) = \mathrm{Sh}_V$ is a finite group.

Given a finite subgroup G' of $\mathrm{Aut}_K(K[x])$ such that $G' \cap \mathrm{Sh}(K) = \mathrm{Sh}_V$. By Theorem 4.4, $G' = \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ for some $\nu \in K$ and a primitive n'th root of unity λ_n such that $n|p^e - 1$, by Lemma 4.17.(1), and so $G' \subseteq G_{V,\nu}$. Since the groups $\{G_{V,\nu}\}_{\nu\in K}$ are distinct, the corollary follows ($G_{V,\nu} = G_{V,\nu'}$ iff $\sigma_{\lambda,(1-\lambda)\nu'} = \sigma^i_{\lambda,(1-\lambda)\nu}\sigma_{1,v}$ for some natural number i such that $1 \le i < p$ and $\gcd(i, p) = 1$ and an element $v \in V$ where $\lambda = \lambda_{p^e-1}$ iff $\nu' = \nu + (1-\lambda^i)^{-1}v$ since $\sigma^i_{\lambda,(1-\lambda)\nu}\sigma_{1,v} = \sigma_{\lambda^i,(1-\lambda^i)(\nu+(1-\lambda^i)^{-1}v)}$ iff $\nu' \equiv \nu \mod V$ since $\mathbb{F}_p(\lambda)V = \mathbb{F}_{p^e}V = V$). □

Criterion for $\widetilde{G}_f \neq \{e\}$ and $\overline{G}_f \neq \{e\}$.

Lemma 4.19 *Suppose that K is an algebraically closed field, $f(x) \in K[x]$ is monic non-scalar polynomial that has at least two distinct roots, $\widetilde{G}_f = \mathrm{Sh}_V \neq \{e\}$ and $\overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle \neq \{e\}$ where V is a nonzero $\mathbb{F}_p(\lambda_n)$-subspace of the field K and λ_n is primitive n'th root of unity. Then:*

1. *$\lambda_n \in \mathbb{F}_{p^e}$ where $\mathbb{F}_{p^e}$ is the multiplier field of the $\mathbb{F}_p$-subspace V of K, or, equivalently, $n|p^e - 1$.*
2. *The group $G_f = \mathrm{Sh}_V \rtimes \overline{G}_f$ is a subgroup of $\mathrm{Sh}_V \rtimes \langle \sigma_{\lambda,(1-\lambda)\nu}\rangle$ where λ is a cyclic generator of the group $\mathbb{F}^\times_{p^e}$, i.e. $\lambda = \lambda_{p^e-1}$ is primitive $p^e - 1$'st root of unity.*

Proof 1. Since $\lambda_n V \subseteq V$, $\lambda_n \in \mathbb{F}_{p^e}$ and the lemma follows from Corollary 4.18. □

Definition 4.20 Let $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 \in \overline{K}[x]$ be a monic polynomial of degree $d \ge 1$ where $a_i \in \overline{K}$ are the coefficients of the polynomial $f(x)$. Then the natural number

$$\gcd(f(x)) := \gcd\{i \ge 1 \mid a_i \neq 0\}$$

is called the *exponent* of $f(x)$.

Clearly, the exponent of $f(x)$ is the largest natural number $m \ge 0$ such that $f(x) = g(x^m)$ for some polynomial $g(x) \in K[x]$.

Definition 4.21 For a non-scalar polynomial $f \in K[x]$, we have the unique product

$$\gcd(f) = p^s \gcd{}_p(f) \quad \text{where} \quad s \ge 0, \quad \gcd{}_p(f) \in \mathbb{N} \quad \text{and} \quad p \nmid \gcd{}_p(f). \tag{27}$$

Proposition 4.22 *Suppose that $f(x) = f_V^i(x - \nu)$ for some nonzero finite dimensional $\mathbb{F}_p$-subspace V of K, $\nu \in K$ and a natural number $i \ge 1$ (i.e. $\mathcal{R}_d(f) = \nu + V$ and each root of $f(x)$ has multiplicity i). Let $\mathbb{F}_{p^e}$ be the multiplier field of V, $\mathbb{F}^\times_{p^e} = \langle\lambda_n\rangle$ where $n = p^e - 1$. Then*

$$G_f = \begin{cases} \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle \neq \mathrm{Sh}_V & \text{if } (p, e) \neq (2, 1), \\ \mathrm{Sh}_V & \text{if } (p, e) = (2, 1). \end{cases}$$

Proof Since $|\mathcal{R}_d(f)| = |\nu + V| = |V| \ge 2$, the group $G_f = \widetilde{G}_f \rtimes \overline{G}_f$ is a finite group. Clearly, $\mathrm{Sh}_V \subseteq \widetilde{G}_f$. In fact, $\mathrm{Sh}_V = \widetilde{G}_f$ since $\mathcal{R}_d(f) = \nu + V$.

Suppose that $(p, e) \neq (2, 1)$. Recall that $n = p^e - 1 > 1$ and $\mathbb{F}^\times_{p^e} = \langle\lambda_n\rangle$, the multiplier field of V. In particular, $\lambda_n V \subseteq V$. Therefore, $\overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$, by Lemma 4.17.(1).

The case $(p, e) = (2, 1)$ is obvious, see Lemma 4.17.(2). □

Lemma 4.23 *Suppose that* $\mathrm{Fr}(K) = K$ *(where* $\mathrm{Fr}(a) = a^p$*, the Frobenius endomorphism). Then* $G_{f^{p^n}}(K) = G_f(K)$ *for all polynomials* $f \in K[x]$ *and all natural numbers* $n \geq 1$.

Proof (i) $G_{f^{p^n}}(K) \supseteq G_f(K)$: If $\sigma \in G_f(K)$ then $\sigma(f) = \lambda f$ for some $\lambda \in K$, and so $\sigma(f^{p^n}) = \lambda^{p^n} f^{p^n}$. This means that $\sigma \in G_{f^{p^n}}(K)$.

(ii) $G_{f^{p^n}}(K) \subseteq G_f(K)$: If $\tau \in G_{f^{p^n}}(K)$ then $\tau(f^{p^n}) = \mu f^{p^n}$ for some $\mu \in K$, and so $(\tau(f) - \mu^{\frac{1}{p^n}} f)^{p^n} = 0$, i.e. $\tau(f) = \mu^{\frac{1}{p^n}} f$. This means that $\tau \in G_f(K)$. $\square$

Let $f(x) = \sum a_i x^i \in K[x]$ be a monic non-scalar polynomial and $\gcd(f) = p^s \gcd_p(f)$. Suppose that $K = \overline{K}$. Then there is a unique monic non-scalar polynomial $f_1(x) \in K[x]$ such that

$$f(x) = f_1^{p^s}(x). \tag{28}$$

Clearly, $\gcd(f_1) = \gcd_p(f)$, $\deg(f) = p^s \deg(f_1)$ and $f_1' \neq 0$ (the derivative of f_1).

Theorem 4.24 is a criterion for the group $G_f = \widetilde{G}_f \rtimes \overline{G}_f$ to have nontrivial subgroups $\widetilde{G}_f$ and $\overline{G}_f$, it also gives an explicit description of the group G_f.

Theorem 4.24 *Suppose that the field* K *is an algebraically closed field and* $f(x) \in K[x]$ *is a monic polynomial that has at least two distinct roots,* $\gcd(f) = p^s \gcd_p(f)$ *and* $f(x) = f_1^{p^s}(x)$ *for a unique monic non-scalar polynomial* $f_1(x) \in K[x]$*, see* Eq. 28. *Suppose that* $V \neq 0$ *is a finite dimensional* $\mathbb{F}_p$*-subspace of* K *and* $\mathbb{F}_{p^e}$ *is the multiplier field of* V*. Then the following statements are equivalent:*

1. $\widetilde{G}_f = \mathrm{Sh}_V \neq \{e\}$ *and* $\overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle \neq \{e\}$.
2. *There is a primitive* n*'th root of unity* $\lambda_n \neq 1$ *(in particular,* $n \geq 2$*) such that* $\lambda_n V \subseteq V$*, and an element* $\nu \in K$ *such that either* $f(x) = f_V^i(x-\nu)$ *for some natural number* $i \geq 1$ *and* $n = p^e - 1$ *(in this case,* $(p, e) \neq (2, 1)$*,* $\widetilde{G}_f = \mathrm{Sh}_V$ *and* $\overline{G}_f = \langle \sigma_{\lambda_{p^e-1},(1-\lambda_{p^e-1})\nu} \rangle$ *where* $\mathbb{F}_{p^e}^{\times} = \langle \lambda_{p^e-1} \rangle$*) or otherwise* $f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$ *for some natural number* $i \geq 0$ *and a monic nonn-scalar polynomial* $g(x) \in K[x]$ *such that* $g(0) \neq 0$*, and the following two conditions hold:*
   (a) $n \geq 2$ *and* $\gcd(\frac{p^e-1}{n}, \gcd_p(g)) = 1$*, and*
   (b) $\mathcal{R}(f) + \mathbb{F}_p(\lambda_n)(\lambda - \lambda') \nsubseteq \mathcal{R}(f)$ *for all distinct roots* λ *and* λ' *of the polynomial* f *such that* $\lambda - \lambda' \notin V$.

Suppose that statement 1 holds. Then:

- *the natural number* i *and the polynomial* $g(x)$ *in statement 2 are unique,*
- *the element* ν *is unique up to adding an arbitrary element of* V *(i.e.* ν *can be replaced by* $\nu + v$ *for any element* $v \in V$*),*
- *the equality* $f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$ *is unique (since* $f_V(x-\nu-v) = f_V(x-\nu)$ *for all* $v \in V$*). Furthermore,* $\sigma_{\lambda_n,(1-\lambda_n)\nu}(f) = \lambda_n^i f$*, and* $f \in K[x]^{G_f}$ *iff* $n|i$.
- *In the second case, i.e.* $f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$,

$$n = \gcd(p^e - 1, \gcd_p(h)$$

where $h(x) \in K[x]$ *is a unique polynomial such that* $f(x) = f_V^i(x-\nu) h(f_V(x-\nu))$ *(i.e.* $h(x) = g(x^n)$*), and either* ν *is a root of* $f(x)$ *(i.e.* $i \geq 1$*) or otherwise (i.e.* $i = 0$*)* ν *is a root of* $f_1'(x)$ *(the derivative of* $f_1(x)$*).*

Proof (1 $\Rightarrow$ 2) Suppose that statement 1 holds. By Theorem 4.12, $\widetilde{G}_f = \mathrm{Sh}_V$ and $\overline{G}_f = \langle \sigma_{\lambda_n, (1-\lambda_n)\nu} \rangle$ for a nonzero $\mathbb{F}_p$-subspace V of K, a primitive n'th root of unity $\lambda_n \neq 1$ (in particular, $n \geq 2$) such that $\lambda_n V \subseteq V$ and an element $\nu \in K$. Then, by Theorem 4.8.(3), either $f(x) = f_V^i(x-\nu)$ for some natural number $i \geq 1$ or otherwise

$$f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$$

for some natural number $i \geq 0$, $n \geq 2$, and a monic non-scalar polynomial $g(x) \in K[x]$ such that $g(0) \neq 0$.

In the first case, by Proposition 4.22, $\widetilde{G}_f = \mathrm{Sh}_V \neq \{e\}$ and $\overline{G}_f = \langle \sigma_{\lambda_{p^e-1}, (1-\lambda_{p^e-1})\nu} \rangle \neq \{e\}$.

Now let us consider the second case. By Lemma 4.17, $n | p^e - 1$ (since $\lambda_n \in \mathbb{F}_{p^e}$). Suppose that $l := \gcd(\frac{p^e-1}{n}, \gcd_p(g)) > 1$. Then $\sigma_{\lambda_{ln}, (1-\lambda_{ln})\nu} \in \overline{G}_f = \langle \sigma_{\lambda_n, (1-\lambda_n)\nu} \rangle$ where λ_{ln} is a primitive ln'th root of unity, a contradiction (since the order of the element $\sigma_{\lambda_{ln}, (1-\lambda_{ln})\nu}$ is $ln > n = |\overline{G}_f|$). Therefore, the statement (a) holds.

Suppose that the condition (b) does not holds, i.e. there are two distinct roots λ and λ' of the polynomial $f(x)$ such that $v' := \lambda - \lambda' \notin V$ and $\mathcal{R}(f) + \mathbb{F}_p(\lambda_n)(\lambda - \lambda') \subseteq \mathcal{R}(f)$. Then

$$V' := V + \mathbb{F}_p(\lambda_n) v'$$

is a $\mathbb{F}_p(\lambda_n)$-submodule of K that properly contains the $\mathbb{F}_p(\lambda_n)$-module V. Then $\mathrm{Sh}_{V'} \subsetneq \mathrm{Sh}_V$, a contradiction.

(2 $\Rightarrow$ 1) In the first case, i.e. $f(x) = f_V^i(x-\nu)$, the implication follows from Proposition 4.22. In the second case, i.e. $f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$,

$$\overline{G}_f \supseteq \langle \sigma_{\lambda_n, (1-\lambda_n)\nu} \rangle \neq \{e\} \text{ and } \widetilde{G}_f \supseteq \{\sigma_{1,\mu} \,|\, \mu \in V\} \neq \{e\}.$$

The conditions (a) and (b) imply that the inclusions above are equalities, see the proof of the implication (1 $\Rightarrow$ 2).

Suppose that statement 1 holds. Then statement 2 holds and vice versa. So, we have the equality

$$f(x) = f_V^i(x-\nu) g(f_V^n(x-\nu))$$

in statement 2 (the case $g = 1$ corresponds to the first case). The polynomial $f_V(x-\nu)$ is $\widetilde{G}_f$-invariant, i.e. for all elements $v \in V$, $f_V(x-\nu) = \sigma_{1,-v}(f_V(x-\nu)) = f_V(x-(\nu+v))$. Therefore, for all elements $v \in V$,

$$f(x) = f_V^i(x-(\nu+v)) g(f_V^n(x-(\nu+v))),$$

i.e. the element ν can be replaced by the element $\nu + v$ for any element $v \in V$. This is the only freedom for the choice of the element ν. Indeed, $G_f = \{\sigma^j_{\lambda_n, (1-\lambda_n)\nu} \sigma_{1,v} \,|\, 0 \leq j \leq n-1, v \in V\}$. Since

$$\sigma^j_{\lambda_n, (1-\lambda_n)\nu} \sigma_{1,v} = \sigma_{\lambda_n^j, (1-\lambda_n^j)\nu} \sigma_{1,v} = \sigma_{\lambda_n^j, (1-\lambda_n^j)(\nu + (1-\lambda_n^j)^{-1} v)} \quad \text{for } 1 \leq j \leq n-1,$$

it follows that the only freedom in choosing the generator $\sigma_{\lambda_n, (1-\lambda_n)\nu}$ in Theorem 4.8 is an element of the type

$$\sigma_{\lambda_n^j, (1-\lambda_n^j)(\nu + (1-\lambda_n^j)^{-1} v)}$$

where j is a natural number such that $1 \leq j \leq n-1$, $\gcd(j, n) = 1$ and v is an arbitrary element of V. Now, by Theorem 4.8, ν is a unique (up to addition) element of V, and the elements i and $g(x)$ are unique.

Clearly, $\sigma_{\lambda_n,(1-\lambda_n)\nu}(f)=\lambda_n^i f$ (Theorem 4.8.(1)), and so $f\in K[x]^{G_f}$ iff $n|i$ (Theorem 4.8.(2,3)).

Suppose that $g(x)\neq 1$. Clearly, ν is a root of $f(x)$ iff $i\geq 1$. Suppose that ν is not a root of $f(x)$, i.e. $i=0$ and $f(x)=g(f_V^n(x-\nu))$. Recall that $f(x)=f_1^{p^s}(x)$ and $G_f=G_{f_1}$, by Lemma 4.23. Then ν is not a root of $f_1(x)$, i.e. $f_1(x)=g_1(f_V^n(x-\nu))$ for a unique polynomial $g_1(x)\in K[x]$ such that $g=g_1^{p^s}$. Hence, ν is a root of the polynomial $f_1(x)$ since

$$0\neq f_1'(x)=nf_V^{n-1}(x-\nu)f_V'(x-\nu)g_1'(f_V^n(x-\nu)),$$

$n\geq 2$ and $f_V(0)=0$.

In the second case, i.e. $f(x)=f_V^i(x-\nu)g(f_V^n(x-\nu))$, $n=\gcd(p^e-1,\gcd_p(h)$, by the statement (a). $\square$

Definition 4.25 The unique presentation of the polynomial $f(x)$,

$$f(x)=f_V^i(x-\nu)\ \text{ or }\ f(x)=f_V^i(x-\nu)g(f_V^n(x-\nu)),$$

in Theorem 4.24.(2) is called the *eigenform* or the *eigenpresentation* of the polynomial $f(x)$. The scalars $\nu+V$ and the natural number $i\geq 0$ are called the *eigenroots* of $f(x)$ and their *multiplicity*, respectively. The natural number $n\geq 2$ and the monic polynomial $g(x)$ are called the *eigenorder* and the *eigenfactor* of $f(x)$. In the second case, the eigenroots may not be roots of the polynomial $f(x)$. They are iff $i\neq 0$.

Corollary 4.26 *Suppose that the field K is an algebraically closed field and $f(x)\in K[x]$ is a monic polynomial that has at least two distinct roots, $gcd(f)=p^s\gcd_p(f)$ and $f(x)=f_1^{p^s}(x)$ for a unique monic non-scalar polynomial $f_1(x)\in K[x]$. Suppose that the polynomial f satisfies the assumption of* Theorem 4.24, *and $f_1(x)=f_V^j(x-\nu)$ or $f_1(x)=f_V^j(x-\nu)g_1(f_V^n(x-\nu))$, is the eigenform of the polynomial $f_1(x)$. Then $f(x)=f_V^{p^sj}(x-\nu)$ or $f(x)=f_V^{p^sj}(x-\nu)g_1^{p^s}(f_V^n(x-\nu))$, is the eigenform of the polynomial $f_1(x)$.*

Proof The statement follows from the facts that $f(x)=f_1^{p^s}(x)$, $G_f=G_{f_1}$ (Lemma 4.23) and the uniqueness of the eigenform (Theorem 4.24). $\square$

Criterion for $\widetilde{G}_f=\{e\}$ and $\overline{G}_f\neq\{e\}$. Theorem 4.27 is a criterion for the group $G_f=\widetilde{G}_f\rtimes\overline{G}_f$ to be equal to $\overline{G}_f\neq\{e\}$.

Theorem 4.27 *Suppose that the field K is an algebraically closed field, $f(x)\in K[x]$ is a monic polynomial that has at least two distinct roots, $gcd(f)=p^s\gcd_p(f)$ and $f(x)=f_1^{p^s}(x)$ for a unique monic non-scalar polynomial $f_1(x)\in K[x]$, see* Eq. 28. *Then the following statements are equivalent:*

1. *$\widetilde{G}_f=\{e\}$ and $\overline{G}_f=\langle\sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle\neq\{e\}$ where λ_n is a primitive n'th root of unity and $\nu\in K$.*
2. *$f(x)=(x-\nu)^ig((x-\nu)^n)$ for some natural number $i\geq 0$ and a monic non-scalar polynomial $g(x)\in K[x]$ such that $g(0)\neq 0$,*

   (a) $n\geq 2$, $p\nmid n$ and $\gcd_p(g(x))=1$, and
   (b) $\mathcal{R}(f)+\mathbb{F}_p(\lambda-\lambda')\nsubseteq\mathcal{R}(f)$ for all distinct roots λ and λ' of the polynomial f.

Suppose that statement 1 holds. Then:

- *The presentation* $f(x) = (x-\nu)^i g((x-\nu)^n)$ *is unique, i.e. the triple* $(\nu, i, g(x))$ *is unique.*
- *Either* ν *is a root of* $f(x)$ *(i.e.* $i \geq 1$*) or otherwise (i.e.* $i = 0$*)* ν *is a root of* $f_1'(x)$ *(the derivative of* $f_1(x)$*).*
- *If* ν *is a root of* $f(x)$ *then* $n = \gcd_p(x^{-i} f(x+\nu))$.
- *If* ν *is not a root of* $f(x)$ *then* $n = \gcd_p(f(x+\nu))$.
- $\sigma_{\lambda_n,(1-\lambda_n)\nu}(f) = \lambda_n^i f$*, and* $f \in K[x]^{G_f}$ *iff* $n|i$.

Proof By Proposition 4.15.(3), $\widetilde{G}_f = \{e\}$ iff the condition (b) holds.

(1 $\Rightarrow$ 2) Suppose that statement 1 holds. Then, by Theorem 4.9.(2,3),

$$f(x) = (x-\nu)^i g((x-\nu)^n)$$

for some natural number $i \geq 0$ and a monic *non-scalar* polynomial $g(x) \in K[x]$ such that $g(0) \neq 0$ (since $|\mathcal{R}_d(f)| \geq 2$). Clearly, $n \geq 2$ and $p \nmid n$.

Suppose that $l := \gcd_p(g(x)) > 1$. Then $\sigma_{\lambda_{ln},(1-\lambda_{ln})\nu} \in \overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ where λ_{ln} is a primitive ln'th root of unity, a contradiction (since the order of the element $\sigma_{\lambda_{ln},(1-\lambda_{ln})\nu}$ is $ln > n = |\overline{G}_f|$). Therefore, the statement (a) holds.

(2 $\Rightarrow$ 1) By the statement (b), $\widetilde{G}_f = \{e\}$. Since $f(x) = (x-\nu)^i g((x-\nu)^n)$ for some natural number $i \geq 0$, $n \geq 2$, $p \nmid n$ and a monic non-scalar polynomial $g(x) \in K[x]$ such that $g(0) \neq 0$,

$$\overline{G}_f \supseteq \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle \neq \{e\}.$$

The condition (a) implies that the inclusion above is the equality, see the proof of the implication (1 $\Rightarrow$ 2).

Suppose that statement 1 holds. Then statement 2 holds and vice versa. So, we have the equality $f(x) = (x-\nu)^i g((x-\nu)^n)$ as in statement 2. To prove uniqueness of this presentation it suffices to show that the element ν is unique. The set of cyclic generators for the group $G_f = \overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ is equal to $\{\sigma^j_{\lambda_n,(1-\lambda_n)\nu} \mid 1 \leq j \leq n-1, \gcd(j,n) = 1\}$. Since $\sigma^j_{\lambda_n,(1-\lambda_n)\nu} = \sigma_{\lambda_n^j,(1-\lambda_n^j)\nu}$, the element ν is unique.

Clearly, ν is a root of $f(x)$ iff $i \geq 1$. Suppose that ν is not a root of $f(x)$, i.e. $i = 0$ and $f(x) = g((x-\nu)^n)$, then $f_1(x) = h((x-\nu)^n)$ for a unique monic non-scalar polynomial $h(x) := g^{\frac{1}{p^s}}(x) \in K[x]$ (since $p \nmid n$). Hence, ν is a root of the polynomial $f_1(x)$ since

$$0 \neq f_1'(x) = n(x-\nu)^{n-1} h'((x-\nu)^n)$$

and $n \geq 2$.

If ν is a root of $f(x)$ then $n = \gcd_p(x^{-i} f(x+\nu))$ (since $f(x+\nu) = x^i g(x^n)$). If ν is not a root of $f(x)$ then $n = \gcd_p(f(x+\nu))$ (since $f(x+\nu) = g(x^n)$).

Clearly, $\sigma_{\lambda_n,(1-\lambda_n)\nu}(f) = \lambda_n^i f$, and so $f \in K[x]^{G_f}$ iff $n|i$. □

Definition 4.28 The unique presentation of the polynomial $f(x)$,

$$f(x) = (x-\nu)^i g((x-\nu)^n),$$

in Theorem 4.27.(2) is called the *eigenform* or the *eigenpresentation* of the polynomial $f(x)$. The scalar ν and the natural number $i \geq 0$ are called the *eigenroot* of $f(x)$ and its *multiplicity*, respectively. The natural number $n \geq 2$ and the monic polynomial $g(x)$ are called the *eigenorder* and the *eigenfactor* of $f(x)$. In general, the eigenroot may not be a root of the polynomial $f(x)$. It is iff $i \neq 1$.

Criterion for $\widetilde{G}_f \neq \{e\}$ **and** $\overline{G}_f = \{e\}$**.**

Lemma 4.29 *Let $g(x) \in K[x]$ be a monic non-scalar polynomial such that $g' \neq 0$ (the derivative of g), and V be an $\mathbb{F}_p$-subspace of K and $\mathbb{F}_{p^e}$ be its multiplier field (for $V = 0$, $f_V(x) = x$). Then:*

1. *$g(f_V(x))' \neq 0$.*
2. *For each $\nu \in K$, $g(f_V(x)) = f_V^{i(\nu)}(x-\nu)g_\nu(f_V(x-\nu))$ for a unique natural number $i(\nu) \geq 0$ and a unique monic polynomial $g_\nu(x) \in K[x]$ such that $g_\nu(0) \neq 0$; $i(\nu) \neq 0$ iff ν is a root of the polynomial $g(f_V(x))$. If $n = \gcd(p^e-1, \mathrm{gcd}_p(g_\nu(x))) \geq 2$ then $e \neq \sigma_{\lambda_n,(1-\lambda_n)\nu} \in \overline{G}_{g(f_V(x))}(\overline{K})$ where $\lambda_n \in \overline{K}$ is a primitive n'th root of unity. If, in addition, $\lambda_n \in K$ then $e \neq \sigma_{\lambda_n,(1-\lambda_n)\nu} \in \overline{G}_{g(f_V(x))}(K)$.*
3. *Suppose that ν is not a root of the polynomial $g(f_V(x))$, i.e. $i(\nu) = 0$ and $g(f_V(x)) = g_\nu(f_V(x-\nu))$, and $\mathrm{gcd}_p(g_\nu(x)) \neq 1$ then ν is a root of the derivative $g(f_V(x))'$ of the polynomial $g(f_V(x))$.*

Proof 1. $g(f_V(x))' = g'(f_V(x))f_V'(x) \neq 0$, by Proposition 4.7.(2d).

2. $g(f_V(x)) = g(f_V(x-\nu+\nu)) = g(f_V(x-\nu)+f_V(\nu)) = f_V^{i(\nu)}(x-\nu)g_\nu(f_V(x-\nu))$ for a unique natural number $i(\nu) \geq 0$ and a unique monic polynomial $g_\nu(x) \in K[x]$ such that $g_\nu(0) \neq 0$. Since $f_V(0) = 0$ and $g_\nu(0) \neq 0$, we see that $i(\nu) \neq 0$ iff ν is a root of the polynomial $g(f_V(x))$.

If $n \geq 2$ then $e \neq \sigma_{\lambda_n,(1-\lambda_n)\nu} \in \overline{G}_{g(f_V(x))}(\overline{K})$, by Theorem 4.8.(1). If, in addition, $\lambda_n \in K$ then $e \neq \sigma_{\lambda_n,(1-\lambda_n)\nu} \in \overline{G}_{g(f_V(x))}(K)$.

3. Suppose that ν is not a root of the polynomial $g(f_V(x))$ and $m = \mathrm{gcd}_p(g_\nu(x)) \neq 1$, i.e. $g(f_V(x)) = g_\nu(f_V(x-\nu)) = h_\nu(f_V^m(x-\nu))$ for some monic non-scalar polynomial $h_\nu(x) \in K[x]$. Then

$$g(f_V(x))' = h_\nu(f_V^m(x-\nu))' = mf_V^{m-1}(x-\nu)f_V'(x-\nu)h_\nu'(f_V^m(x-\nu)),$$

and so ν is a root of the polynomial $g(f_V(x))'$. □

Given monic non-scalar polynomials $f(x), h(x) \in K[x]$. *If $f(x) = g(h(x))$ for some polynomial $g(x) \in K[x]$ then the polynomial $g(x)$ is unique and necessarily monic.* (Proof. If $f(x) = g(h(x))$ then the polynomial $g(x)$ is monic, $\deg(f) = \deg(g)\deg(h)$, $K[h] \ni f_1 := f - h^{\deg(g)}$ and $\deg(f_1) < \deg(f)$. Now, the induction on $\deg(f)$ completes the proof).

Theorem 4.30 is a criterion for the group $G_f = \widetilde{G}_f \rtimes \overline{G}_f$ to be equal to $\widetilde{G}_f \neq \{e\}$.

Theorem 4.30 *Suppose that the field K is an algebraically closed field, $f(x) \in K[x]$ is a monic non-scalar polynomial that has at least two distinct roots, $gcd(f) = p^s\,\mathrm{gcd}_p(f)$ and $f(x) = f_1^{p^s}(x)$ for a unique monic non-scalar polynomial $f_1(x) \in K[x]$. Suppose that $V \neq 0$ is a finite dimensional $\mathbb{F}_p$-subspace of K and $\mathbb{F}_{p^e}$ is the multiplier field of V. Then the following statements are equivalent:*

1. *$\widetilde{G}_f = \mathrm{Sh}_V \neq \{e\}$ and $\overline{G}_f = \{e\}$.*
2. *$f_1(x) = g(f_V(x))$ for a (unique) monic polynomial $g(x) \in K[x]$ such that*
   - *(a) either $|\mathcal{R}_d(g)| = 1$ and $(p, e) = (2, 1)$ or otherwise $|\mathcal{R}_d(g)| \geq 2$ and $\gcd(p^e-1, \mathrm{gcd}_p(g_\nu(x))) = 1$ for all roots $\nu \in \mathcal{R}_d(f_1(x)) \cup \mathcal{R}_d(f_1(x)')$ where g_ν is as in Lemma 4.29.(2) (i.e. $f_1(x) = f_V^{i(\nu)}(x-\nu)g_\nu(f_V(x-\nu))$), and*
   - *(b) $\mathcal{R}(f_1) + \mathbb{F}_p(\lambda-\lambda') \nsubseteq \mathcal{R}(f_1)$ for all distinct roots λ and λ' of the polynomial f_1 such that $\lambda-\lambda' \notin V$ ($\Leftrightarrow \mathcal{R}(f) + \mathbb{F}_p(\lambda-\lambda') \nsubseteq \mathcal{R}(f)$ for all distinct roots λ and λ' of the polynomial f such that $\lambda-\lambda' \notin V$).*

Suppose that statement 1 holds. Then $f, f_1 \in K[x]^{G_f} = K[x]^{G_{f_1}}$.

Remark By Lemma 4.23, $G_f = G_{f_1^{p^s}} = G_{f_1}$. This explains why statement 2 is given via properties of the polynomial f_1 rather than f.

Proof By Proposition 4.7 and Proposition 4.15.(1), $\widetilde{G}_f = \widetilde{G}_{f_1} = \mathrm{Sh}_V(\neq \{e\})$ iff $f_1(x) = g(f_V(x))$ for a (unique) monic non-scalar polynomial $g(x) \in K[x]$ such that the condition 2(b) holds.

Suppose that $|\mathcal{R}_d(g)| = 1$, i.e. $\mathcal{R}_d(g) = \{\rho\}$ and let $i(\rho)$ be the multiplicity of the root ρ. Fix an element $\nu' \in K = \overline{K}$ such that $f_V(\nu') = \rho$. Then $f_1(x) = (f_V(x) - f_V(\nu'))^{i(\rho)} = f_V^{i(\rho)}(x - \nu')$, by Proposition 4.7.(2b). By Proposition 4.22, $G_{f_1} = \widetilde{G}_{f_1} = \mathrm{Sh}_V$ iff $(p, e) = (2, 1)$.

Suppose that $|\mathcal{R}_d(g)| \geq 2$. By Lemma 4.29.(2),

$$f_1(x) = g(f_V(x)) = f_V^{i(\nu)}(x - \nu)g_\nu(f_V(x - \nu))$$

for a unique monic *non-scalar* polynomial $g_\nu(x) \in K[x]$ such that $g_\nu(0) \neq 0$ where $i(\nu) \geq 0$ is the multiplicity of the root ν (if $g_\nu(x) = 1$ then $f_1(x) = g(f_V(x)) = f_V^{i(\nu)}(x - \nu) = (f_V(x) - f_V(\nu))^{i(\nu)}$, and so $|\mathcal{R}_d(g)| = 1$, a contradiction).

By Theorem 4.8 and Lemma 4.29.(2), $\overline{G}_{f_1} = \{e\}$ iff $\gcd(p^e - 1, \gcd_p(g_\nu(x))) = 1$ for all $\nu \in K$ iff $\gcd(p^e - 1, \gcd_p(g_\nu(x))) = 1$ for all $\nu \in \mathcal{R}_d(f_1(x)) \cup \mathcal{R}_d(f_1(x)')$, Lemma 4.29.(2,3).

Clearly, $f, f_1 \in K[x]^{G_f} = K[x]^{G_{f_1}}$ (Proposition 4.7 and Lemma 4.23). □

Definition 4.31 The unique presentation $f(x) = g^{p^s}(f_V(x))$ in Theorem 4.30 (where $\gcd(f) = p^s \gcd_p(f)$) is called the *eigenform* or *eigenpresentation* of the polynomial $f(x)$ and the polynomial $g(x)$ is called the *eigenfactor* of $f(x)$.

Criterion for $G_f = \{e\}$**.** Given a monic non-scalar polynomial $g(x) \in K[x]$ with $g'(x) \neq 0$. By Lemma 4.29.(2) (where $V = 0$), for each $\nu \in K$,

$$g(x) = (x - \nu)^{i(\nu)} g_\nu(x - \nu) \tag{29}$$

for a natural number $i(\nu) \geq 0$ and a unique monic polynomial $g_\nu(x) \in K[x]$ such that $g_\nu(0) \neq 0$. Clearly, $i(\nu) \neq 0$ iff ν is a root of the polynomial $g(x)$.

Theorem 4.32 is a criterion for $G_f = \{e\}$.

Theorem 4.32 *Suppose that the field K is an algebraically closed field, $f(x) \in K[x]$ is a monic polynomial that has at least two distinct roots,* $\gcd(f) = p^s \gcd_p(f)$ *and* $f(x) = g^{p^s}(x)$ *for a unique monic non-scalar polynomial $g(x) \in K[x]$, see* Eq. 28. *The following statements are equivalent:*

1. $G_f = \{e\}$.
2. *(a) For each root ν of the polynomial $g(x)$,* $\gcd_p(g_\nu(x)) = 1$ *where the polynomial $g_\nu(x)$ is defined in* Eq. 29,
   (b) for each root ν' of the derivative $g'(x)$ of the polynomial $g(x)$ such that $g(\nu') \neq 0$, $\gcd_p(g_\nu(x)) = 1$, *and*
   (c) $\mathcal{R}(g) + \mathbb{F}_p(\lambda - \lambda') \nsubseteq \mathcal{R}(g)$ for all distinct roots λ and λ' of the polynomial g ($\Leftrightarrow$ $\mathcal{R}(f) + \mathbb{F}_p(\lambda - \lambda') \nsubseteq \mathcal{R}(f)$ for all distinct roots λ and λ' of the polynomial f).

Proof Notice that $G_f = G_{g^{p^s}} = G_g$. The condition (c) is equivalent to the condition that $\widetilde{G}_g = \{e\}$ (Proposition 4.15.(3)). It remains to show that provided $\widetilde{G}_g = \{e\}$ the condition

$\overline{G}_g = \{e\}$ is equivalent to the conditions (a) and (b). Equivalently, $\widetilde{G}_g = \{e\}$ and $\overline{G}_g \neq \{e\}$ iff one of the conditions (a) or (b) does not hold and the condition (c) holds. This follows from Theorem 4.27. Indeed, by Theorem 4.27, $\widetilde{G}_g = \{e\}$ and $\overline{G}_g \neq \{e\}$ iff the condition (c) holds and $g(x) = (x-\nu)^i g_\nu(x-\nu)$ for a unique $\nu \in K$, a natural number $i \geq 0$ and a monic non-scalar polynomial $g_\nu(x)$ such that $g_\nu(0) \neq 0$ (since $|\mathcal{R}_d(f)| \geq 2$) and $\gcd_p(g_\nu(x)) \geq 2$. We have two options either ν is a root of the polynomial $g(x)$ or not. If ν is not a root of the polynomial $g(x)$, i.e. $i = 0$, then $g(x) = g_\nu(x-\nu)$, and so ν is a root of $g'(x)$ since $\gcd_p(g_\nu(x)) \geq 2$. Now, it follows that statements 1 and 2 are equivalent. $\square$

Theorem 4.33 describes the group $G_f(K)$ in terms of the group $G_f(\overline{K})$.

Theorem 4.33 *Suppose that the field K is not necessarily algebraically closed and $f(x) \in K[x]$ is a monic polynomial that has at least two distinct roots in $\overline{K}$. Recall that (*Theorems 4.12 and 4.4*) the group $G_f(\overline{K}) = \widetilde{G}_f(\overline{K}) \rtimes \overline{G}_f(\overline{K})$ where $\overline{G}_f(\overline{K}) := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ and $\widetilde{G}_f(\overline{K}) := \{\sigma_{1,\mu} \mid \mu \in \overline{V}\}$, $\lambda_n \in \overline{K}$ is a primitive n'th root of unity provided $\overline{G}_f(\overline{K}) \neq \{e\}$, $\nu \in \overline{K}$, $\overline{V}$ is a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of $\overline{K}$, and $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$ (*Lemma 4.3*). Then*

$$G_f(K) = \widetilde{G}_f(K) \rtimes \overline{G}_f(K) = \mathrm{Aut}_K(K[x]) \cap G_f(\overline{K}), \quad \widetilde{G}_f(K) = \mathrm{Sh}_V$$

where $V := K \cap \overline{V}$ and if $\overline{G}_f(K) \neq \{e\}$ then $\overline{G}_f(K) = \langle \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu+\overline{v}}\rangle$ where $i = \min\{i' = 1, \dots, n-1 \mid i'|n, \lambda_n^{i'} \in K, (1-\lambda_n^{i'})\nu \in \overline{V} + K\}$ and $\overline{v} \in \overline{V}$ is any (fixed) element such that $(1-\lambda_n^i)\nu + \overline{v} \in K$.

Proof It is obvious that $G_f(K) = \mathrm{Aut}_K(K[x]) \cap G_f(\overline{K})$. By Theorem 4.4, $G_f(K) = \widetilde{G}_f(K) \rtimes \overline{G}_f(K)$. It is obvious that $\widetilde{G}_f(K) = \mathrm{Sh}_V$ where $V := K \cap \overline{V}$. By Theorem 4.4, $\overline{G}_f(K) = \langle \sigma_{\lambda_{n'},(1-\lambda_{n'})\nu'}\rangle$ where $\lambda_{n'} \in K$ is a primitive n''th root of unity and $\nu' \in K$ provided $\overline{G}_f(K) \neq \{e\}$. Notice that

$$\sigma_{\lambda_{n'},(1-\lambda_{n'})\nu'} = \sigma^i_{\lambda_n,(1-\lambda_n)\nu}\sigma_{1,\overline{v}} = \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu}\sigma_{1,\overline{v}} = \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu+\overline{v}}$$

for unique elements i and $\overline{v} \in \overline{V}$ such that $0 \leq i \leq n-1$. So, the elements i can be chosen such that

$$\begin{aligned} i &= \min\{i' = 1, \dots, n-1 \mid i'|n,\ \lambda_n^{i'} \in K,\ (1-\lambda_n^{i'})\nu + \overline{v} \in K \ \text{ for some element } \ \overline{v} \in \overline{V}\} \\ &= \min\{i' = 1, \dots, n-1 \mid i'|n,\ \lambda_n^{i'} \in K,\ (1-\lambda_n^{i'})\nu \in \overline{V} + K\} \end{aligned}$$

and $\overline{v} \in \overline{V}$ is any (fixed) element such that $(1-\lambda_n^i)\nu + \overline{v} \in K$. $\square$

Proposition 4.34 gives criteria for the groups $\widetilde{G}_f(K)$, $\overline{G}_f(K)$ and $G_f(K)$ to be $\{e\}$.

Proposition 4.34 *Suppose that the field K is not necessarily algebraically closed and $f(x) \in K[x]$ is a monic polynomial that has at least two distinct roots in $\overline{K}$. Recall that* (Theorems 4.12 and 4.4) *the group $G_f(\overline{K}) = \widetilde{G}_f(\overline{K}) \rtimes \overline{G}_f(\overline{K})$ where $\overline{G}_f(\overline{K}) := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ and $\widetilde{G}_f(\overline{K}) := \{\sigma_{1,\mu} \mid \mu \in \overline{V}\}$, $\lambda_n \in \overline{K}$ is a primitive n'th root of unity provided $\overline{G}_f(\overline{K}) \neq \{e\}$, $\nu \in \overline{K}$, $\overline{V}$ is a finite dimensional $\mathbb{F}_p(\lambda_n)$-subspace of $\overline{K}$, and $\mathbb{F}_p(\lambda_n) = \mathbb{F}_{p^m}$ for some $m \geq 1$* (Lemma 4.3). *Then:*

1. *$\widetilde{G}_f(K) = \{e\}$ iff $\overline{V} \cap K = 0$.*
2. *$\overline{G}_f(K) = \{e\}$ iff $\overline{G}_f(\overline{K}) = \{e\}$ or otherwise $\overline{G}_f(\overline{K}) := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu}\rangle$ and there is no a natural number i' such that $1 \leq i' \leq n-1$ such that $i'|n$, $\lambda_n^{i'} \in K$ and $(1-\lambda_n^{i'})\nu \in \overline{V}+K$.*

3. $G_f(K) = \{e\}$ *iff* $G_f(\overline{K}) = \{e\}$ *or otherwise* $\overline{V} \cap K = 0$, $\overline{G}_f(\overline{K}) := \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ *and there is no a natural number* i' *such that* $1 \leq i' \leq n-1$, $i'|n$, $\lambda_n^{i'} \in K$ *and* $(1-\lambda_n^{i'})\nu \in \overline{V} + K$.

Proof Statements 1 and 2 follow at once from Theorem 4.33. Then statement 3 follows from statements 1 and 2. □

Every subgroup of $\mathrm{Aut}_K(K[x])$ **is of type** G_f. Theorem 4.35 shows that all subgroups of $\mathrm{Aut}_K(K[x])$ are eigengroups of polynomials.

Theorem 4.35 *Let* K *be an arbitrary field of characteristic* $p > 0$. *Then for each subgroup* H *of* $\mathrm{Aut}_K(K[x])$ *there is a monic polynomial* f_H *such that* $G_{f_H} = H$:

1. *For* $H = \{e\}$, $f_H = x(x+1)^2$.
2. *For* $H = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ *where* $\lambda_n \in K$ *is a primitive* n*'th root of unity and* $\nu \in K$, $f_H = (x-\nu)^n - 1$.
3. *For* $H = \mathrm{Sh}_V$ *where* V *is a nonzero* $\mathbb{F}_p$*-subspace of* K,
   (a) *if* $K = \mathbb{F}_{p^n}$ *then* $f_H(x) = f_V(x-\nu) - \rho$ *where* ρ *is any element of* $\mathbb{F}_{p^n}$ *that does not belong to the image of the map* $f_V(x-\nu) : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$, $x \mapsto f_V(x-\nu)$ *(the map* $f_V(x-\nu)$ *is not a surjection since the set* $\nu + V$ *is mapped to* 0*).*
   (b) *If* $|K| = \infty$ *then* $f_H(x) = f_V(x) f_V^2(x-\nu)$ *where* $\nu \in K\backslash V$.
4. *For* $H = \mathrm{Sh}_V \rtimes \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ *where* V *is a nonzero* $\mathbb{F}_p$*-subspace of* K, $\mathbb{F}_{p^e}$ *is its multiplier field, and* λ_n *is primitive* n*'th root of unity such that* $\lambda_n V \subseteq V$, $f_H(x) = \begin{cases} f_V(x-\nu) & \text{if } n = p^e - 1, \\ f_V^n(x-\nu) + 1 & \text{if } n < p^e - 1. \end{cases}$
5. *For* $H = \mathbb{T}_\nu(K) = \{\sigma_{\lambda,(1-\lambda)\nu} \,|\, \lambda \in K^\times\}$, $f_H = x - \nu$.
6. *For* $H = \mathrm{Aut}_K(K[x])$, $f_H = \begin{cases} 1 & \text{if } |K| = \infty, \\ x^{p^n} - x & \text{if } K = \mathbb{F}_{p^n}. \end{cases}$

Proof 1. If $\sigma \in G_{f_H}$ then the maximal ideals (x) and $(x+1)$ of $K[x]$ are σ-stable, hence $\sigma = e$.

2. Clearly, $G_{f_H} \supseteq H$.

(i) $\widetilde{G}_{f_H} = \{e\}$: The polynomial f_H has n distinct roots, namely, $\{\nu + \lambda_n^i \,|\, i = 0, 1, \ldots, n-1\}$, and $p \nmid n$. Suppose that $\widetilde{G}_{f_H} = \mathrm{Sh}_V \neq \{e\}$ for some nonzero $\mathbb{F}_p$-subspace V of K. Then $p||V|$. Since $V + \mathcal{R}(f) \subseteq \mathcal{R}(f)$, we must have $|V|||\mathcal{R}(f)|$, i.e. $|V||n$, and so $p|n$, a contradiction.

(ii) $G_{f_H} = H$: Let $\sigma = \sigma_{\lambda_n,(1-\lambda_n)\nu}$. By the statement (i), $G_{f_H} = \overline{G}_{f_H} = \langle \sigma' \rangle$ where $\sigma = \sigma_{\lambda_m,(1-\lambda_m)\nu'}$ for some primitive m'th root of unity λ_m and $\nu' \in K$. Since $H \subseteq \overline{G}_{f_H}$, we must have $\nu' = \nu$ and $n|m$ (since $(x-\nu')$ is the only $\langle \sigma' \rangle$-invariant maximal ideal of $K[x]$, $(x-\nu)$ is the only $\langle \sigma \rangle$-invariant maximal ideal of $K[x]$ and $\langle \sigma \rangle \subseteq \langle \sigma' \rangle$).

The polynomial f_H is an eigenvector for the automorphism σ' with (necessarily) eigenvalue λ_m^n since $n = \deg(f_H)$. Now,

$$\lambda_m^n f_H = \lambda_m^n((x-\nu)^n - 1) = \sigma'(f_H) = \lambda_m^n (x-\nu)^n - 1.$$

Therefore, $\lambda_m^n = 1$, and so $\langle \sigma \rangle = \langle \sigma' \rangle$. This means that $\overline{G}_{f_H} = H$, and the statement (ii) follows from the statement (i).

3(a). Clearly, $G_{f_H} \supseteq H = \mathrm{Sh}_V$.

(i) $\widetilde{G}_{f_H} = H$: The statement follows at once from the fact that the polynomial f_H has $|V|$ distinct roots in $\overline{K}$ (if $\widetilde{G}_{f_H} = \mathrm{Sh}_{V'}$ for some $\mathbb{F}_p$-subspace V' of K that properly contains V then the polynomial f_H contains at least $|V'|$ distinct roots in $\overline{K}$, a contradiction).

(ii) $\overline{G}_{f_H} = \{e\}$: Suppose that $\overline{G}_{f_H} \neq \{e\}$. Then $\overline{G}_{f_H} = \langle \sigma \rangle$ where $\sigma = \sigma_{\lambda_n,(1-\lambda_n)\nu'}$, $1 \neq \lambda_n \in K$ is a primitive n'th root of unity such that $\lambda_n V \subseteq V$ and $\nu' \in K$. Notice that $\sigma(f_H) = \lambda_n^{|V|} f_H$ and

$$f_H(x) = f_V(x-\nu'-(\nu-\nu'))-\rho = f_V(x-\nu')-f_V(\nu-\nu')-\rho = f_V(x-\nu')+f_V(\nu'-\nu)-\rho.$$

By Theorem 4.8.(1), $\sigma(f_V(x-\nu')) = \lambda_n f_V(x-\nu')$. Let $a = f_V(\nu'-\nu)-\rho$. Now,

$$\lambda_n^{|V|}(f_V(x-\nu')+a) = \lambda_n^{|V|} f_H = \sigma(f_H) = \sigma(f_V(x-\nu')+a) = \lambda_n f_V(x-\nu')+a.$$

Hence, $\lambda_n^{|V|} = \lambda_n \neq 1$ and $(\lambda_n - 1)a = 0$, i.e. $a = 0$. The last equality implies that $\rho \in \mathrm{im}\, f_V(x-\nu)$, a contradiction, and the statement (ii) follows.

(b). Clearly, $G_{f_H} \supseteq H = \mathrm{Sh}_V$.

(i) $\widetilde{G}_{f_H} = H$: The statement follows at once from the fact that $\mathcal{R}(f) = V \coprod (\nu + V)^2$ where the upper index '2' means that the multiplicity of each root in $\nu + V$ is 2.

(ii) $\overline{G}_{f_H} = \{e\}$: Suppose that $e \neq \sigma \in \overline{G}_{f_H}$. Then $\sigma \in \overline{G}_{f_V(x)} \cap \overline{G}_{f_V(x-\nu)}$. Notice that $\overline{G}_{f_V(x-\nu)} = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ for some primitive n'th root of unity λ_n such that $\mathbb{F}_p(\lambda_n)V \subseteq V$. Then there is a natural number i such that $\sigma = \sigma^i_{\lambda_n,(1-\lambda_n)\nu} = \sigma_{\lambda_n^i,(1-\lambda_n^i)\nu} \in \overline{G}_{f_V(x)}$. In particular, $V \ni \sigma^{-1} * (0) = (1-\lambda_n^i)\nu$, and so $\nu \in (1-\lambda_n^i)^{-1} V = V$, a contradiction ($1-\lambda_n^i \neq 0$ since $\sigma \neq e$).

4. Statement 4 follows from Theorem 4.24.

5 and 6. Statements 5 and 6 are obvious.

□

Algorithm of finding the eigengroup $G_f(\overline{K})$ and the eigenform of f. The algorithm consists of finitely many steps and is based on Prpoposition 4.10, Theorems 4.24, 4.27, 4.30 and 4.32. We assume that $K = \overline{K}$.

Step 1. If $|\mathcal{R}_d(f)| = 1$ then apply Proposition 4.10 to find G_f.

From this moment on we assume that $|\mathcal{R}_d(f)| \geq 2$.

Step 2. Use Theorem 4.32 to check whether $G_f = \{e\}$ or $G_f \neq \{e\}$.

From this moment on we assume that $G_f \neq \{e\}$.

Step 3. By Proposition 4.15.(1), the group $\widetilde{G}_f = \mathrm{Sh}_V$ can be found.

Step 4. Suppose that $\widetilde{G}_f = \{e\}$. Then necessarily $\overline{G}_f \neq \{e\}$, and using Theorem 4.27 the group $\overline{G}_f$ is found. In more detail, we know that $\overline{G}_f = \langle \sigma_{\lambda_n,(1-\lambda_n)\nu} \rangle$ and that $f(x) = (x-\nu)^i g((x-\nu)^n)$ for a unique $\nu \in \mathcal{R}_d(f) \cup \mathcal{R}_d(f_1')$ and $n \geq 2$ such that if ν is a root of $f(x)$ then $n = \gcd_p(x^{-i} f(x+\nu))$, and if ν is not a root of $f(x)$ then $n = \gcd_p(f(x+\nu))$.

From this moment on we assume that $\widetilde{G}_f = \mathrm{Sh}_V \neq \{e\}$, the $\mathbb{F}_p$-subspace V of the field K is non-zero. Let $\mathbb{F}_{p^e}$ be the multiplier field of V. It can be easily found since the multiplier field $\mathbb{F}_{p^e}$ is the largest among finite fields $\mathbb{F}_{p^m}$ such that $\mathbb{F}_{p^m} V \subseteq V$ and $m \leq \dim_{\mathbb{F}_p}(V)$.

Step 5. Now, we check whether the conditions of Theorem 4.30 hold or not. If they do then $\overline{G}_f = \{e\}$.

If they do not then necessarily $\overline{G}_f \neq \{e\}$ and hence the conditions of Theorem 4.24 hold. Using Theorem 4.24 the group $\overline{G}_f$ and the eigenform of f are found in finitely many steps.

□

Algorithm of finding the eigengroup $G_f(K)$ where $K \neq \overline{K}$.

Step 1. Using the algorithm above the group $G_f(\overline{K})$ is found.

Step 2. The group $G_f(K)$ is found by using Theorem 4.33. □

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