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Boundary Quantum Field Theories Perturbed by $T\bar{T}$: Towards a Form Factor Program

Olalla A. Castro-Alvaredo[♥], Stefano Negro[◇] and Fabio Sails[♣]

[♣] [♥] Department of Mathematics, City St George's, University of London,
10 Northampton Square EC1V 0HB, UK

[◇] Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

Abstract

Our understanding of irrelevant perturbations of integrable quantum field theories has greatly expanded over the last decade. In particular, we know that, from a scattering theory viewpoint at least, their effect is realised as a modification the two-body scattering amplitudes by a CDD factor. While this sounds like a relatively small change, this CDD factor incorporates a non-trivial dependence on the perturbation parameter(s) and alters substantially the high-energy physics of the model. This occurs through the introduction of a natural length scale and is associated with phenomena such as the Hagedorn transition. In this paper we discuss how all these features extend to boundary integrable quantum field theories and propose a construction for the building blocks of matrix elements of local fields. We show that the same type of building blocks are also found in the sinh-Gordon model with Dirichlet boundary conditions.

Keywords: $T\bar{T}$ Perturbations, Integrable Quantum Field Theory, Boundary Theories, Form Factors

[♥] o.castro-alvaredo@city.ac.uk
[◇] stefano.negro@york.ac.uk
[♣] fabio.sails@city.ac.uk

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1 Introduction

Given an integrable quantum field theory, it has been known since 2016 that a perturbation by $T\bar{T}$ and higher spin versions thereof introduces a deformation of the two-body scattering matrix [1, 2]. If the theory is diagonal, that is there is no back-scattering, then the deformation takes the form

$$S_{ab}^{\alpha}(\theta) := S_{ab}(\theta)\Phi_{ab}^{\alpha}(\theta) \quad \text{with} \quad \log \Phi_{ab}^{\alpha}(\theta) = -i \sum_{s \in \mathcal{S}} m_a^s m_b^s \alpha_s \sinh(s\theta), \quad (1)$$

where $S_{ab}(\theta)$ is the original S -matrix associated to the process $a + b \mapsto a + b$, with a, b particle quantum numbers. The masses of these particles are denoted by m_a, m_b and α_s are couplings such that the combinations $m_a^s m_b^s \alpha_s$ are dimensionless. The values of s are drawn from the set \mathcal{S} of (integer) spins of local conserved quantities in the IQFT. The bold symbol α indicates the set of parameters α_s in the sum.

Starting from this S -matrix, various techniques commonly associated with integrable quantum field theories (IQFTs) have been applied to $T\bar{T}$ -perturbed models. This includes the thermodynamic Bethe ansatz (TBA) [3–8], and, most recently, the form factor program [9–13]. If we put our focus on the scattering theory of these models, and on the traditional pathway to studying IQFTs, it is natural to also consider the effect of a $T\bar{T}$ perturbation and its generalisations on IQFTs in the presence of integrable boundaries. The study of this problem was initiated in [14] (see also the more recent study [15]) as we discuss later.

It has been known for a long time that in the presence of a boundary, a new set of functions $R_a(\theta)$ come to play a prominent role in the formulation of the scattering theory of the model. The S -matrix remains unchanged but scattering processes off the boundary now need to be accounted for, while retaining integrability. In this context, the function $R_a(\theta)$ is the reflection amplitude off the boundary. Unitarity and crossing relations lead to the constraints [16]:

$$R_a(\theta)R_a(-\theta) = 1 \quad \text{and} \quad R_a(\theta)R_{\bar{a}}(\theta + i\pi) = S_{aa}(2\theta), \quad (2)$$

where \bar{a} is the particle conjugate to a . As we can see, reflection amplitudes are related to the scattering phase. Thus, when this changes, like in the presence of irrelevant perturbations, we expect $R_a(\theta)$ to also change. In the presence of stable bound states [17, 18] there are additional requirements for the functions $R_a(\theta)$ in the form of boundary bootstrap equations. These take the form:

$$R_a(\theta + i\eta_{ac}^b)R_b(\theta + i\eta_{bc}^a)S_{ab}(\theta + i\eta_{bc}^a + i\eta_{ac}^b) = R_c(\theta), \quad (3)$$

where η_{ab}^c are values related to the position of the poles of the scattering matrix. If the S -matrix $S_{ab}(\theta)$ has a pole at $\theta = iu_{ab}^c$ corresponding to the formation of the bound state c in the process $a + b \mapsto c$, then $\eta_{ab}^c := \pi - u_{ab}^c$. Systematic solutions to these equations have been famously constructed for affine Toda field theories [19–21].

This program can be further extended by considering the possibility of a “dynamical” boundary, namely a boundary which can be excited to a different state by particle collision. This is associated with a pole of the reflection amplitudes themselves. This possibility was first put forward in [22]. Notably, it is fully compatible with integrability. If we label the type of boundary by capital letters, then we can see this as the process $a + A \mapsto B$. In this case, boundary reflection amplitudes acquire an extra index $R_a^A(\theta)$ and there are additional boundary bootstrap equations

$$R_a^A(\theta) = R_a^B(\theta)S_{ab}(\theta + i\eta_{aA}^B)S_{ab}(\theta - i\eta_{aA}^B), \quad (4)$$

where $\theta = i\eta_{aA}^B$ is a pole of the amplitude $R_a^A(\theta)$. The solutions to these equations in Toda field theories were studied in great detail in [23].

Once solutions to these equations have been found, they can be employed as input data in the study of the thermodynamic properties of massive boundary IQFTs as done in [24,25]. They can also be employed in the context of computing correlation functions and their building blocks (form factors). This may be done either by employing the boundary state as proposed in [16] or by developing a form factor program for boundary IQFTs, as done in [26] and employed for example in [27–33].

This paper is organised as follows: In Section 2 we review the construction of deformed reflection amplitudes, starting from (1). This overlaps with the work [14] but is presented here in the more restrictive setting of IQFT. In Section 3 we review the boundary form factor program, focusing only on one-particle form factors, particularly the so-called minimal part. In Section 4 we introduce the set of reflection amplitudes of the sinh-Gordon theory and discuss the special case of Dirichlet boundary conditions. In Section 5 we show that the minimal form factor admits a representation of the $\text{T}\bar{\text{T}}$ type. This representation is functionally similar to the results of [13]. In Section 6 we discuss the extension of our construction to more general boundary conditions. We conclude in Section 7.

2 Reflection Amplitudes and Irrelevant Perturbations

Consider for simplicity a theory with no bound states. Let $R_a(\theta)$ be a reflection amplitude which preserves integrability and has no pole leading to excited boundary states. In this case the only relevant equations for $R_a(\theta)$ are (2). We will now promote $R_a(\theta)$ to $R_a^\alpha(\theta)$ to denote the deformed solution to equations (2) corresponding to the deformed S -matrix (1). We expect that

$$R_a^\alpha(\theta) = R_a(\theta)\Lambda_a^\alpha(\theta), \quad (5)$$

for some function $\Lambda_a^\alpha(\theta)$ which satisfies

$$\Lambda_a^\alpha(\theta)\Lambda_a^\alpha(-\theta) = 1 \quad \text{and} \quad \Lambda_a^\alpha(\theta)\Lambda_a^\alpha(\theta + i\pi) = \Phi_{aa}^\alpha(2\theta). \quad (6)$$

It is very easy to see that these equations are solved by

$$\Lambda_a^\alpha(\theta) = \sqrt{\Phi_{aa}^\alpha(2\theta)}. \quad (7)$$

This is the standard type of solution, namely a $2\pi i$ periodic, odd function of 2θ and it agrees with the boundary scattering factor found in [14]. This gives the universal change of the reflection amplitudes in boundary IQFTs after a $\text{T}\bar{\text{T}}$ perturbation. The solutions for $\Lambda_a^\alpha(\theta)$ can however be more general than this. While the factor $\sqrt{\Phi_{aa}^\alpha(2\theta)}$ needs to be there, any function of the type $\sinh(k\theta)$ with k odd can be added to the exponent, providing a new solution to (2). In general we have

$$\Lambda_a^\alpha(\theta) = \sqrt{\Phi_{aa}^\alpha(2\theta)} \exp \left[-i \sum_{k \in \mathbb{Z}} \gamma_k m^{2(2k+1)} \sinh((2k+1)\theta) \right]. \quad (8)$$

The fact that the modification is not unique for a given S -matrix is expected since in general there are multiple boundary conditions allowed for one single scattering amplitude¹. From here onwards we will consider the simplest solution only (with $\gamma_i = 0$). The question of how the reflection amplitudes are deformed under irrelevant perturbations has already been discussed in the literature a few years ago [14] and then employed to develop a generalised boundary thermodynamic Bethe ansatz. Here we proceed instead to discuss the form factor program in the presence of boundaries.

3 Boundary Form Factor Program and Minimal Form Factor

The boundary form factor program was proposed in [26] and subsequently applied to several models and fields [27–33]. The main idea of the program is to address the computation of correlation functions in the presence of a boundary. There are two viewpoints we may take. If the boundary is located at the origin of time it can be represented by a boundary state in the Ghoshal-Zamolodchikov sense [35]. In this case matrix elements of local fields may be computed in terms of the matrix elements obtained in the absence of a boundary, assuming these are known via the standard form factor program [36, 37]. This can be achieved by expanding the boundary state in terms of bilinears of the Zamolodchikov-Faddeev algebra. It is also possible to think of the boundary as located in space, say at the origin. In this case, one can derive a set of modified form factor equations for the matrix elements of local fields which now must take into account scattering processes off the boundary. These equations were presented in [26]. For our purposes, we are only interested in the equations for the one-particle form factors of a local field \mathcal{O} , namely:

$$F_a^{\mathcal{O}}(\theta) = R_a(\theta)F_a^{\mathcal{O}}(-\theta) \quad \text{and} \quad F_a^{\mathcal{O}}(\theta) = R_a(i\pi - \theta)F_a^{\mathcal{O}}(2\pi i - \theta), \quad (9)$$

where

$$F_a^{\mathcal{O}}(\theta) := \langle 0 | \mathcal{O}(0) | \theta \rangle_a, \quad (10)$$

with $|\theta\rangle_a$ an in-state containing a single particle of species a and $|0\rangle$ the vacuum state. Due to breaking of translation invariance, the one-particle form factor is rapidity dependent, even for spinless fields. Thus, it is the simplest non-trivial form factor that may be computed and a building block for higher particle form factors.

The solution procedure for equations (9) was presented in [26] and follows the usual construction. Starting with an integral representation for $R_a(\theta)$, a corresponding integral representation of $F_a^{\mathcal{O}}(\theta)$ can be found. The minimal solution to the equations, without poles in the physical strip, is entirely determined by $R_a(\theta)$ whereas the pole structure of $F_a^{\mathcal{O}}(\theta)$ is related to the operator \mathcal{O} .

Let $r_a^{\min}(\theta)$ be a minimal solution of (9). What would be its deformed version in the presence of a generalised $\overline{\text{T}\overline{\text{T}}}$ perturbation? Following [10–12] we observe that the equations (9) are linear and factorised. Therefore, we expect that the modified minimal form factor to be of the form

$$r_a^{\min}(\theta; \boldsymbol{\alpha}) = r_a^{\min}(\theta)\varphi_a^{\boldsymbol{\alpha}}(\theta), \quad (11)$$

¹Many interesting examples are known. The simplest cases are the Ising field theory, where a family of distinct boundary conditions exist parametrised by the boundary magnetic field [16]. Similarly, the sinh-Gordon model admits a two-parameter family of solutions, as found in [34]. We will discuss these two models in Sections 4-6.

with

$$\varphi_a^\alpha(\theta) = \Lambda_a^\alpha(\theta)\varphi_a^\alpha(-\theta) \quad \text{and} \quad \Lambda_a^\alpha(\theta)\varphi_a^\alpha(\theta) = \varphi_a^\alpha(2\pi i - \theta), \quad (12)$$

which is solved by

$$\log \varphi_a^\alpha(\theta) = \frac{2\theta - i\pi}{2\pi} i \log \Lambda_a^\alpha(\theta) = \frac{2\theta - i\pi}{4\pi} i \log \Phi_{aa}^\alpha(2\theta). \quad (13)$$

However, there is a larger family of solutions. Indeed, the exponent above, can be modified by a sum of $\cosh(k\theta)$ functions with $k \in \mathbb{Z}$, and still satisfy all requirements. We therefore find that the most general minimal solution to (9) is

$$r_a^{\min}(\theta; \boldsymbol{\alpha}, \boldsymbol{\beta}) = r_a^{\min}(\theta)\varphi_a^\alpha(\theta)C_a^\beta(\theta), \quad (14)$$

with $\varphi_a^\alpha(\theta)$ given by (13) and

$$\log C_a^\beta(\theta) = \sum_{s \in \mathcal{S}'} \beta_s m_a^{2s} \cosh(s\theta). \quad (15)$$

Thus, the solution is parametrised by parameters $\boldsymbol{\alpha}$, which are determined by the deformation of the S -matrix, and $\boldsymbol{\beta}$ which can in principle be freely chosen.

The presence of free parameters in the minimal form factor is an issue that we also encountered when considering theories without boundaries [9–12] and whose meaning, for the time being, is not fully understood. Traditionally, we would expect the minimal form factor to be entirely fixed by analyticity and asymptotics requirements. Indeed, this is the case for standard IQFTs, as we shall see in the next section. However, for models perturbed by a finite number of irrelevant perturbations finding the natural choice of parameters $\boldsymbol{\beta}$ remains difficult. Progress in this direction will be reported soon [38].

4 The sinh-Gordon Model with Dirichlet Boundary Conditions

The sinh-Gordon model with Dirichlet boundary conditions was one of the examples considered in [26] and later in [28]. This particular choice of boundary conditions has the advantage that the minimal form factor $r_a^{\min}(\theta)$ coincides with the one-particle form factor, that is, there are no additional poles to be included, which makes calculations particularly simple. In this case non-vanishing form factors associated with odd particle numbers can be identified as corresponding to the operator $\partial_x \phi$, where ϕ is the sinh-Gordon field. More generally, as reported in [34], there is a two-parameter family of solutions for the reflection amplitudes of the sinh-Gordon model. They can be written in terms of fundamental blocks $(x)_\theta$, $[x]_\theta$ as

$$R(\theta, E, F) = \left(\frac{1}{2}\right)_\theta \left(\frac{2+B}{4}\right)_\theta \left(1 - \frac{B}{4}\right)_\theta \left[\frac{E-1}{2}\right]_\theta \left[\frac{F-1}{2}\right]_\theta, \quad (16)$$

where²

$$(x)_\theta := \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)}, \quad [x]_\theta = -(x)_\theta(1-x)_\theta = \frac{\tanh \frac{1}{2}(\theta + i\pi x)}{\tanh \frac{1}{2}(\theta - i\pi x)}. \quad (17)$$

²Note that there is a minus sign difference between the definitions of $[x]_\theta$ in [26] and [13]. Here we are using the same definitions as in [13]. Compared to [34] the blocks $(x)_\theta$ differ by a factor 1/2 in the definition of x .

The sinh-Gordon two-body scattering matrix is simply

$$S(\theta) = \left[-\frac{B}{2} \right]_{\theta} = \frac{\sinh \theta - i \sin \frac{\pi B}{2}}{\sinh \theta + i \sin \frac{\pi B}{2}}, \quad (18)$$

with $B \in [0, 2]$ a coupling constant [39–41]. The simplest version of (16) is obtained by removing the F -dependent factor and by setting $E = 0$, while introducing an overall minus sign (this is due to the particular definition of our $[x]_{\theta}$ symbol, as explained in footnote 2). This corresponds to Dirichlet boundary conditions that fix the boundary field to 0. In that special case, the amplitude (16) reduces to

$$R(\theta) = - \left(-\frac{1}{2} \right)_{\theta} \left(\frac{2+B}{4} \right)_{\theta} \left(1 - \frac{B}{4} \right)_{\theta}. \quad (19)$$

This choice also cancels out the pole of the reflection amplitude at $\theta = \frac{i\pi}{2}$ that is present in (16) due to the block $(\frac{1}{2})_{\theta}$. A special property of this amplitude is that for $B = 0$ it reduces to $R(\theta) = 1$ which corresponds to a free boson solution (the sinh-Gordon S -matrix reduces to 1 for $B = 0$). The minimal form factor solution corresponding to this free boson case is proportional to $\sinh \theta$. In [26], the minimal form factor solution corresponding to (19) was given as

$$r^{\min}(\theta) = \frac{\sinh \theta}{i + \sinh \theta} u(\vartheta, B) \quad (20)$$

with

$$u(\vartheta, B) = \exp \left[-2 \int_0^{\infty} \frac{dx}{x} \left(\cos \frac{\vartheta x}{\pi} - 1 \right) \frac{\cosh \frac{x}{2}}{\sinh^2 x} \left(\sinh \frac{x B}{4} + \sinh \left(1 - \frac{B}{2} \right) \frac{x}{2} + \sinh \frac{x}{2} \right) \right], \quad (21)$$

where $\vartheta = \frac{i\pi}{2} - \theta$. The normalisation is chosen so that $u(0, B) = 1$. We have that

$$u(\vartheta, 0) = \exp \left[-2 \int_0^{\infty} \frac{dx}{x} \left(\cos \frac{\vartheta x}{\pi} - 1 \right) \frac{\cosh \frac{x}{2}}{\sinh^2 x} \left(2 \sinh \frac{x}{2} \right) \right] = -\frac{i}{2} (i + \sinh \theta), \quad (22)$$

so that for $B = 0$ we recover the free boson solution $r^{\min}(\theta) = -\frac{i}{2} \sinh \theta$. For our purposes however, it is interesting to emphasise the connection with free fermions instead. The sinh-Gordon model is a fermionic theory, in the sense that $S(\theta = 0) = -1$ and if we factor out this -1 from the S -matrix $[-B/2]_{\theta}$ what remains can be seen as a CDD factor. This means that the sinh-Gordon S -matrix is of the type (1) with a CDD factor given by a sum over all odd integers and coefficients $m^{2s} \alpha_s$ which are functions of B . The precise formulae were discussed in [13]. Hence, according to our derivation in Section 3, the minimal form factor (21) should also admit a representation of the type (14) with (13). We will now show that this is indeed the case.

5 A New Minimal Form Factor Representation

Let

$$\omega(\vartheta, B) := \log u(\vartheta, B). \quad (23)$$

then, the derivative w.r.t. ϑ is,

$$\omega'(\vartheta, B) = h(\vartheta, B) + h(\vartheta, 2 - B) + g(\vartheta), \quad (24)$$

with

$$h(\vartheta, B) = \frac{2}{\pi} \int_0^\infty dx \frac{\sin \frac{\vartheta x}{\pi} \cosh \frac{x}{2} \sinh \frac{x B}{4}}{\sinh^2 x} \quad \text{and} \quad g(\vartheta) = \frac{1}{\pi} \int_0^\infty dx \frac{\sin \frac{\vartheta x}{\pi}}{\sinh x}. \quad (25)$$

We have that $g(\vartheta)$ can be easily integrated to $g(\vartheta) = \frac{1}{2} \tanh \frac{\vartheta}{2}$, while $h(\vartheta, B)$ can be computed using contour integration, along the same lines of the computations presented in [13]. For example, we have the integral

$$I(a, b) = \int_{-\infty}^\infty dx \frac{e^{(b+ia)x}}{\sinh^2 x} = i\pi(b+ia) \frac{1 + e^{i\pi(b+ia)}}{1 - e^{i\pi(b+ia)}}, \quad (26)$$

from where it follows

$$\begin{aligned} h(\vartheta, B) &= \frac{1}{8\pi i} \left[I\left(\frac{\vartheta}{\pi}, \frac{2+B}{4}\right) + I\left(\frac{\vartheta}{\pi}, \frac{-2+B}{4}\right) - I\left(\frac{\vartheta}{\pi}, \frac{2-B}{4}\right) - I\left(\frac{\vartheta}{\pi}, -\frac{2+B}{4}\right) \right. \\ &\quad \left. - I\left(-\frac{\vartheta}{\pi}, \frac{2+B}{4}\right) - I\left(-\frac{\vartheta}{\pi}, \frac{-2+B}{4}\right) + I\left(-\frac{\vartheta}{\pi}, \frac{2-B}{4}\right) + I\left(-\frac{\vartheta}{\pi}, -\frac{2+B}{4}\right) \right] \\ &= \frac{4\vartheta \sin \frac{\pi B}{2} - 4\pi \sin \frac{\pi B}{4} \sinh \vartheta + \pi B \sinh(2\vartheta)}{4\pi(\cosh(2\vartheta) + \cos \frac{\pi B}{2})}. \end{aligned} \quad (27)$$

Integrating gives

$$\begin{aligned} \omega(\vartheta, B) &= \log a(B) + \log \cosh \frac{\vartheta}{2} - \frac{i\vartheta}{2\pi} \log \left[\frac{\sin \frac{B\pi}{2} + i \sinh(2\vartheta)}{\sin \frac{B\pi}{2} - i \sinh(2\vartheta)} \right] \\ &\quad + \frac{1}{2} \log \left[\cos \frac{B\pi}{4} + \cosh \vartheta \right] + \frac{1}{4} \log \left[\frac{\cosh \vartheta + \sin \frac{B\pi}{4}}{\cosh \vartheta - \sin \frac{B\pi}{4}} \right] - \frac{B}{8} \log \left[\frac{\cosh(2\vartheta) - \cos \frac{\pi B}{4}}{\cosh(2\vartheta) + \cos \frac{\pi B}{4}} \right] \\ &\quad + \frac{i}{4\pi} \left(\text{Li}_2(-ie^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(ie^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-ie^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(ie^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right) \\ &\quad - \frac{i}{4\pi} \left(\text{Li}_2(-e^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(e^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-e^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(e^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right). \end{aligned} \quad (28)$$

Here $\log a(B)$ is an integration constant which can be fixed by asymptotic requirements. Requiring that $\omega(0, B) = 0$ and after some simplifications, we obtain

$$\begin{aligned} \log a(B) &= -\frac{1}{2} \log \left(\frac{\sin \frac{\pi(2+B)}{8}}{\sin \frac{\pi(2-B)}{8}} \right) - \log \left(2 \cos \frac{B\pi}{8} \right) + \frac{B}{4} \log \tan \frac{B\pi}{4} \\ &\quad + \frac{i}{2\pi} \left(\text{Li}_2(-e^{\frac{i\pi B}{2}}) - \text{Li}_2(e^{\frac{i\pi B}{2}}) \right) - \frac{i\pi(B-1)}{8}. \end{aligned} \quad (29)$$

It is interesting to consider the various contributions to (28):

- The contribution

$$\log \cosh \frac{\vartheta}{2} = \log \cos \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right), \quad (30)$$

is such that when taking exponential of $\omega(\vartheta, B)$ it gives a factor $\cosh \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right)$ in the minimal form factor that combines with the prefactor $\frac{\sinh \theta}{\sinh \theta + i}$ in (20) to give

$$r_{\text{fixed}}(\theta) = -\frac{i}{2} \frac{\sinh \theta}{\cosh \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right)}, \quad (31)$$

which is the minimal form factor corresponding to the Ising model with reflection amplitude

$$R^0(\theta) = \left(-\frac{1}{2} \right)_\theta. \quad (32)$$

This is known as the fixed boundary condition of the Ising model, and corresponds to the limit of infinite boundary magnetic field, as discussed in [16, 26] (see also Section 6 for further discussion).

- The contribution

$$-\frac{i\vartheta}{2\pi} \log \left[\frac{\sin \frac{B\pi}{2} + i \sinh(2\vartheta)}{\sin \frac{B\pi}{2} - i \sinh(2\vartheta)} \right] = \frac{2\theta - i\pi}{4\pi} i \log(-S(2\theta)) = \frac{2\theta - i\pi}{4\pi} i \log \Phi^{\text{shG}}(2\theta), \quad (33)$$

where $\Phi^{\text{shG}}(\theta)$ is minus the scattering matrix of the sinh-Gordon model, which can be seen as a CDD factor. Hence, sinh-Gordon emerges as a perturbation of the Ising field theory.

- The remaining terms in (28) add up to an even function of θ which admits a formal expansion as a sum of $\cosh(s\theta)$ functions with s integer, both odd and even. This is similar to the computations presented in the Appendix of [13].

In summary, the minimal form factor $r^{\min}(\theta)$ introduced in (20) can be rewritten as

$$r^{\min}(\theta) = a(B) r_{\text{fixed}}(\theta) e^{\frac{2\theta - i\pi}{4\pi} i \log(-S(2\theta))} C^\beta(\theta), \quad (34)$$

with $r_{\text{fixed}}(\theta)$ given by (31), $S(\theta)$ the sinh-Gordon S -matrix and $C^\beta(\theta)$ given by

$$\begin{aligned} & \log C^\beta(\theta) \\ &= \frac{1}{2} \log \left[\cos \frac{B\pi}{4} + \cosh \vartheta \right] + \frac{1}{4} \log \left[\frac{\cosh \vartheta + \sin \frac{B\pi}{4}}{\cosh \vartheta - \sin \frac{B\pi}{4}} \right] - \frac{B}{8} \log \left[\frac{\cosh(2\vartheta) - \cos \frac{\pi B}{4}}{\cosh(2\vartheta) + \cos \frac{\pi B}{4}} \right] \\ &+ \frac{i}{4\pi} \left(\text{Li}_2(-ie^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(ie^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-ie^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(ie^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right) \\ &- \frac{i}{4\pi} \left(\text{Li}_2(-e^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(e^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-e^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(e^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right), \quad (35) \end{aligned}$$

with $\vartheta = \frac{i\pi}{2} - \theta$ as before. As indicated by (15), this function admits a formal expansion in terms of $\cosh(\ell\theta)$ functions. We can show that

$$\begin{aligned} \log C^\beta(\theta) &= -\frac{1}{2} \log 2 - \sum_{\ell=1}^{\infty} \frac{1}{\ell} \cos \left(\frac{B}{4} \ell \pi \right) \cosh(\ell\theta) \\ &- \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \sin \left(\frac{B}{4} (2\ell+1)\pi \right) \cosh((2\ell+1)\theta) \\ &+ \frac{B}{2} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \cos \left(\frac{B}{4} (2\ell+1)\pi \right) \cosh(2(2\ell+1)\theta) \\ &- \sum_{\ell=0}^{\infty} \frac{1}{\pi(2\ell+1)^2} \sin \left(\frac{B}{2} (2\ell+1)\pi \right) \cosh(2(2\ell+1)\theta). \quad (36) \end{aligned}$$

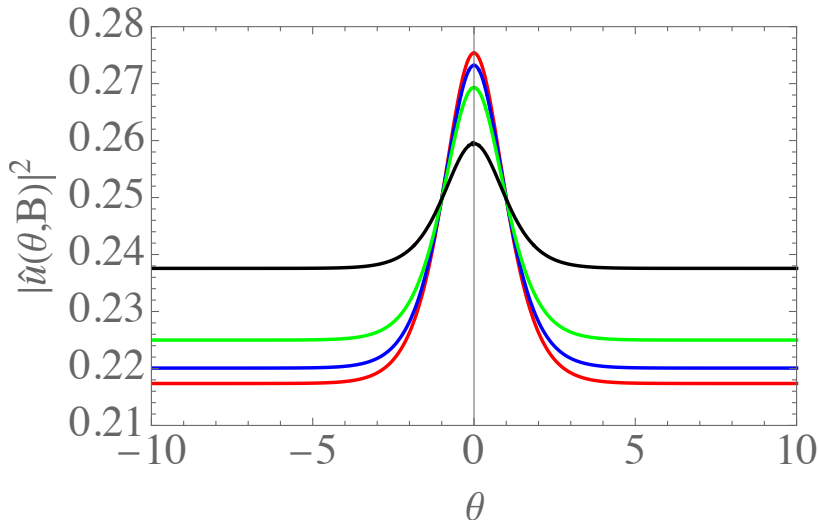


Figure 1: The absolute value squared of the function $\hat{u}(\theta, B) := \frac{u(\vartheta, B)}{i + \sinh \theta}$ with $u(\vartheta, B) = \exp \omega(\vartheta, B)$ evaluated numerically from (28). The colours correspond to different values of B : $B = 1$ (red), $B = 0.7$ (blue), $B = 0.5$ (green) and $B = 0.2$ (black).

Comparing to (15) we identify the coefficients (we take the mass scale $m = 1$)

$$\begin{aligned}
 \beta_0 &= -\frac{1}{2} \log(2), \\
 \beta_{2\ell+1} &= -\frac{\cos\left(\frac{B}{4}(2\ell+1)\pi\right) + (-1)^\ell \sin\left(\frac{B}{4}(2\ell+1)\pi\right)}{2\ell+1}, \\
 \beta_{4\ell} &= -\frac{\cos(B\ell\pi)}{4\ell}, \\
 \beta_{4\ell+2} &= -\frac{\cos\left(\frac{B}{2}(2\ell+1)\pi\right) + B \cos\left(\frac{B}{4}(2\ell+1)\pi\right)}{2(2\ell+1)} - \frac{1}{\pi(2\ell+1)^2} \sin\left(\frac{B}{2}(2\ell+1)\pi\right).
 \end{aligned} \tag{37}$$

6 More General Boundary Conditions

We have just seen that the simplest boundary condition/reflection amplitude (19) in the sinh-Gordon theory admits a new interpretation. It can be seen, at the level of the minimal form factor at least, as resulting from irrelevantly perturbing the boundary Ising model with fixed boundary conditions. The fixed boundary condition corresponds to taking the magnetic field h , which parametrizes all Ising boundary conditions, to infinity. Indeed, the most general reflection amplitude in the Ising model can be written as [16]

$$R_x(\theta) = -[x]_\theta \left(-\frac{1}{2}\right)_\theta, \tag{38}$$

where x is related to the magnetic field h as $\sin(\pi x) = 1 - \frac{h^2}{2m^2}$. There are two simple boundary conditions known as “free” and “fixed”. They correspond to

- $h = 0$ ($x = \frac{1}{2}$) with reflection amplitude

$$R_{\frac{1}{2}}(\theta) = \left(\frac{1}{2}\right)_\theta, \quad (39)$$

This is the *free boundary condition*. As in the general case, this reflection amplitude has a pole at $\theta = \frac{i\pi}{2}$. This pole is dynamical, i.e. it changes position when changing the value of h .

- $h \rightarrow \infty$ ($x \rightarrow i\infty - \frac{\pi}{2}$), with reflection amplitude

$$R_{i\infty - \frac{\pi}{2}}(\theta) = \left(-\frac{1}{2}\right)_\theta \quad (40)$$

this is called *fixed boundary condition* and it is the simplest configuration. It corresponds to moving the pole away from the physical strip. Notice that in this case the factor $-[x]_\theta \rightarrow 1$ while in the free case it is non trivial.

It is not difficult to generalise the construction of the minimal form factor (34) to the case of generic reflection amplitudes (16). One viewpoint is to consider the Ising field theory with generic boundary conditions itself as a “perturbation” of the Ising field theory with fixed boundary conditions. Again, this is meant in the sense of how we compute the minimal form factor. This viewpoint allows us to both generalise the Ising and the sinh-Gordon results, to more general boundary conditions. We discuss this below.

6.1 A $\overline{\text{T}\overline{\text{T}}}$ Picture of the Boundary Ising Model with Generic Boundaries

Let $r_x(\theta)$ be the minimal form factor of the Ising field theory with generic boundary conditions, corresponding to the reflection amplitude (38). The minimal form factor should be a modified version of the solution for fixed boundary conditions such that

$$r_x(\theta) = r_{\text{fixed}}(\theta)\varphi_x(\theta), \quad (41)$$

which implies

$$\varphi_x(\theta) = -[x]_\theta\varphi_x(-\theta), \quad -[x]_\theta\varphi_x(\theta) = \varphi_x(2\pi i - \theta). \quad (42)$$

The task is now to compute the new function $\varphi_x(\theta)$. Employing the standard integral representations that can be found in many places, such as [26], we have that

$$-[x]_\theta = (x)_\theta(1-x)_\theta = \exp\left\{2\int_0^\infty \frac{dt \sinh \frac{t\theta}{i\pi}}{t \sinh^2 t} (\sinh tx + \sinh t(1-x))\right\} \quad (43)$$

so we can write

$$\varphi_x(\vartheta) = \exp\left\{2\int_0^\infty \frac{dt (\sinh tx + \sinh t(1-x)) \cosh \frac{t}{2}}{t \sinh^2 t} \left(1 - \cos \frac{t\vartheta}{\pi}\right)\right\} \quad (44)$$

where we again use the variable $\vartheta = \frac{i\pi}{2} - \theta$. As before we take the logarithmic derivative

$$\begin{aligned} \frac{d}{d\vartheta} \log \varphi_x(\vartheta) &= \frac{2}{\pi} \int_0^\infty \frac{dt (\sinh tx + \sinh t(1-x)) \cosh \frac{t}{2}}{t \sinh^2 t} \sin \frac{t\vartheta}{i\pi} \\ &= h(\vartheta, 4x) + h(\vartheta, 4-4x), \end{aligned} \quad (45)$$

where $h(\vartheta, x)$ is the same function defined earlier in (25). We can therefore use the same formula (27) to write

$$h(\vartheta, 4x) = \frac{\vartheta \sin 2\pi x - \pi \sin \pi x \sinh \vartheta + \pi x \sinh(2\vartheta)}{\pi(\cosh(2\vartheta) + \cos 2\pi x)}. \quad (46)$$

Until now everything is pretty much the same as in previous sections. However, the sum (45) simplifies greatly, so that after integration, we have simply

$$\log \varphi_x(\vartheta) = \int d\vartheta [h(\vartheta, 4x) + h(\vartheta, 4 - 4x)] = \log(\cosh \vartheta + \sin \pi x) + c. \quad (47)$$

The constant is easily fixed to $c = -\log(1 + \sin \pi x)$ so as to ensure that $\log \varphi_x(0) = 0$. Therefore, writing everything back in terms of θ we get

$$\varphi_x(\vartheta) = \frac{\sin \pi x - i \sinh \theta}{1 + \sin \pi x} = \frac{\sin \pi x - i \sinh \theta}{\sin \pi x + i \sinh \theta} \frac{\sin \pi x + i \sinh \theta}{1 + \sin \pi x}, \quad (48)$$

or

$$\log \varphi_x(\vartheta) = \log(-[x]_\theta) + \log(\sin \pi x + i \sinh \theta) - \log(1 + \sin \pi x). \quad (49)$$

This gives the generic factor that has to be added any time the boundary condition is changed. In particular, for $x = \frac{1}{2}$ we can obtain the modification of the minimal form factor with fixed boundary conditions that corresponds to free boundary conditions in the Ising case. Since the factor $\varphi_x(\vartheta)$ accounts for the contribution to the minimal form factor of a generic square block $-[x]_\theta$ in the reflection amplitude, it can also be adapted to deal with the blocks $[\frac{E-1}{2}]_\theta$ and $[\frac{E-1}{2}]_\theta$ in the sinh-Gordon amplitude (16).

7 Conclusion and Outlook

In this paper we have studied boundary IQFTs perturbed by $\overline{\text{T}\overline{\text{T}}}$ and higher spin irrelevant operators. We discussed how the deformation of the two-body scattering matrix (1) propagates to a deformation of the reflection amplitudes off the boundary (5)-(8) and how these give rise to a deformation of the one-particle minimal form factor (11)-(15). While the deformation of the reflection amplitudes had already been discussed in [14, 15], this work initiates the study of form factors of irrelevantly perturbed boundary theories.

We find that the form factor deformation is very similar to the bulk case. A further analogy is that this deformation suggests a factorised minimal form factor structure that is also reproduced for more standard boundary IQFTs. We show this to be the case for the sinh-Gordon model with Dirichlet boundary conditions, which we discuss in detail here. Through a computation which is analogous to that presented in [13] we show that the boundary one-particle minimal form factor admits a new representation which forgoes integrals or infinite products, is very explicit and numerically efficient. In this representation, the boundary sinh-Gordon model with Dirichlet boundary condition can be interpreted as the Ising field theory with fixed boundary conditions in the presence of infinitely many irrelevant perturbations with specific coupling constants which are functions of the sinh-Gordon coupling B . The effect of adding irrelevant boundary operators in the Ising and sine-Gordon models was studied in [42]. Indeed, similar to our case and to the results of [14, 15], it was shown that such perturbations induce deformations of the reflection amplitudes which modify the UV properties of the theory. A similar conclusion

has also been reached in the context of gravity, where it has been shown that $T\bar{T}$ deformation of 2D conformal field theory can be seen as coming from a modification of the boundary conditions in a 3D (Chern-Simons) gravity theory [43, 44].

Our construction easily generalises to other models and boundary conditions. More importantly, it should now be possible to progress to constructing non-minimal form factor solutions, as done for the bulk case in [9, 10]. We hope to return to this problem in the near future.

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