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Minimal Models RG flows: non-invertible symmetries & non-perturbative description

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In this letter we continue the investigation of RG flows between minimal models that are protected by non-invertible symmetries. RG flows leaving unbroken a subcategory of non-invertible symmetries are associated with anomaly-matching conditions that we employ systematically to map the space of flows between Virasoro Minimal models beyond the $\mathbb{Z}_2$-symmetric proposed recently in the literature. We introduce a family of non-linear integral equations that appear to encode the exact finite-size, ground-state energies of these flows, including non-integrable cases, such as the recently proposed $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$. Our family of NLIEs encompasses and generalises the integrable flows known in the literature: $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(1,2)}$ and $\phi_{(2,1)}$. This work uncovers a new interplay between exact solvability and non-invertible symmetries. Furthermore, our non-perturbative description provides a non-trivial test for all the flows conjectured by anomaly matching conditions, but so far not-observed by other means.

Introduction. The systematic identification of Renormalization Group (RG) flows between quantum field theories is a paramount problem in theoretical physics. Global symmetries are central to this quest, providing non-perturbative constraints on the RG flows between the ultra-violet (UV) and infra-red (IR) fixed points. Symmetries dictate the structure of the interaction generated along the flows. By matching their anomalies, we can put strong constraints on the IR theory. In recent years, building on the seminal paper [1], a profound effort has been devoted to exploring generalizations to the usual notion of global symmetries, such as *higher-form*, *non-invertible*, or more general *higher-categorical* symmetries, extending the usual *group-like* structure of symmetry generators to the more general algebraic structure of fusion higher-categories (for recent reviews see [2, 3]). In the context of two-dimensional conformal field theories (CFT), non-invertible symmetries are ubiquitous [4–6]: it is well known that topological line operators, acting as generators of 0-form symmetries, do not form generically a group but a fusion category. In the special case of Rational 2d CFTs with diagonal modular invariance, i.e. the Virasoro Minimal Models $\mathcal{M}(p,q)$, the set of topological line operators is known to coincide with the finitely many Verlinde line defect, forming a *fusion modular category* [4, 7]. From this perspective, the study of RG flows from a minimal model provides a unique arena where we have a complete understanding of the full set of *categorical* symmetries of the UV theory, and it has indeed recently received considerable attention [8–11]. This approach was first undertaken in [4] and, more recently, in [12], where the authors predict infinitely many new RG flows between minimal models: $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$, preserving a special A_{q-1} fusion category containing the standard $\mathbb{Z}_2$ symmetry. Moreover, some special deformations of minimal models are *integrable*. Integrability imposes that the S-Matrix of the deformed theory is factorized and satisfies the Yang-Baxter equation along the entire flow [13–15]. Integrable flows allow for an exact, non-perturbative description through the Thermodynamic Bethe Ansatz (TBA) equations [16] – equivalently, a Non-Linear Integral Equation (NLIE) [17, 18] – encoding the exact energy spectrum along the whole flow.

Here, we extend the investigation of RG flows between minimal models predicted by anomaly-matching conditions associated with non-invertible symmetries. We also present evidence that the ground-state energy of all these RG flows – not just the integrable ones – admits an explicit NLIE description. We base this statement on the observation that a three-parameter family of NLIE encodes non-trivial features of the RG flows predicted by anomaly-matching conditions. In particular, for multi-operator deformations – which is the case for most of the RG flows we looked at – the scaling function obtained from the NLIEs shows clear signs of multi-scale behaviour in the UV, with exponents agreeing with the operators predicted by the anomaly-matching conditions. Additionally, for the deformations triggered by the $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$ and $\phi_{(1,2)}$ operators, the kernels of the NLIEs reduce to the known ones [19–24]. These are substantial evidence supporting the interpretation of the NLIEs as a universal description for RG flows between minimal models. Definitive evidence will come from accurate numerical investigation and by comparison against Conformal Perturbation Theory [16] or Hamiltonian Truncation [25]. We will embark on this project in the near future. The NLIEs are a potential non-perturbative description for all the flows $\mathcal{M}(kq+I,q) \xrightarrow{\phi_{(1,2k+1)}} \mathcal{M}(kq-I,q)$ conjectured in [12], and provide an explicit proof of their existence. They greatly expand the class of flows that can be studied non-perturbatively, even in the absence of a known integrable structure.

Minimal models RG flows. Consider a UV fixed point described by a Virasoro minimal model $\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p,q)$. Basic notions about Virasoro Minimal Models, fixing the conventions used in this letter, may be found in the Supplemental Material in Appendix A. In this letter, we are interested in studying deformations of the UV theory by one of its relevant ($h_{(r,s)} < 1$) primary fields:

$$\mathcal{M}(p,q) + g_{(r,s)} \int \mathrm{d}x\, \phi_{(r,s)}\,. \tag{1}$$

The IR fixed point at the end of this RG flow may be either *gapped* or *gapless*. The former case is typical for generic deformations, having an IR described by a Topological Quantum Field Theory (TQFT). We will not consider these in our analysis, albeit they can be studied with similar techniques to those discussed here [4, 12, 26–29]. We refer the reader to the discussion at the end of this letter for comments on this matter. In the latter case, when the IR theory is gapless, we assume here it may be described by another minimal model itself :

$$\mathcal{T}_{\rm UV} \quad = \quad \mathcal{M}(p,q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p',q') \quad = \quad \mathcal{T}_{\rm IR}\,.$$

An important constraint on $\mathcal{T}_{\rm IR}$ is given by the $c_{\rm eff}$-theorem: along RG flows between $\mathcal{PT}$-symmetric non-unitary CFT, the effective central charge:

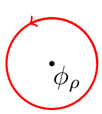
$$c_{\rm eff}(p,q) = 1 - \frac{6}{pq} \qquad (2)$$

is monotonically decreasing [30]. Equation (2) reduces to the usual Zamolodchikov c-theorem [31] for the case of unitary CFT. We will assume that $\mathcal{PT}$-symmetry is always preserved along our flows, as tested by now in all the examples considered in the literature [9–11, 32, 33], where it has been observed that CFT transition happens precisely at the spontaneous $\mathcal{PT}$ breaking locus. Furthermore, $\mathcal{PT}$-symmetry guarantees the reality of the energy spectrum at finite volume (and therefore of conformal dimensions) along the entire flow down to the IR CFT, as is the case for the non-unitary minimal models. Stringent constraints follow from the non-invertible symmetry lines of the minimal models. We will describe a very general strategy we plan to employ also for more general UV fixed points in future work.
Whenever, for any state on the cylinder $|\Phi\rangle$, the line $\mathcal{L}_\sigma$ commutes with the deformation triggering the RG flow:

$$\left[\mathcal{L}_\sigma, \phi_{(r,s)}\right]|\Phi\rangle = 0\,, \qquad (3)$$

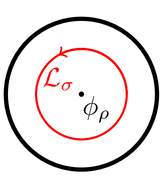
than the line operator $\mathcal{L}_\sigma$ is unbroken by the deformation. The maximal subset $\{\mathcal{L}_\sigma\}^{(r,s)}_{\rm UV} \subset \mathcal{V}_{(p,q)}$ of Verlinde lines commuting with the deformation that are closed under fusion generates the symmetry that is preserved along the RG flow. Using the fusion rules and Verlinde line action on the primary fields:

$$\mathcal{L}_\sigma\,|\phi_\rho\rangle = $$



$$= \frac{S_{\sigma\rho}}{S_{0\rho}}\,|\phi_\rho\rangle\,, \qquad (4)$$

(3) is turned into a trigonometric equation for the label σ, at fixed (r,s). In particular, it implies that the quantum dimension of the preserved lines is an RG flow invariant:



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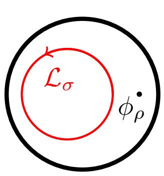

(5)

as well as the spin content of the defect Hilbert spaces associated with the preserved lines $\mathcal{H}_{\mathcal{L}_\sigma}$[1]. These two pieces of categorical data that are RG-invariant quantities are to be considered as 't Hooft anomaly matching conditions in the realm of fusion categories. To explore the possible $\mathcal{T}_{\rm IR}$ we proceed as follows:

1. Given $\mathcal{T}_{\rm UV}$, for any relevant primary field $\phi_{(r,s)}$ of $\mathcal{T}_{\rm UV}$, we compute the fusion subcategory[2] of Verlinde lines $\{\mathcal{L}_\sigma\}^{(r,s)}_{\rm UV}$ commuting with the perturbation via eq. (3).
2. We generate a list of the possible minimal models $\mathcal{T}_{\rm IR}$ that satisfy $f(\mathcal{T}_{\rm UV}) > f(\mathcal{T}_{\rm IR})$. This list is always finite.
3. We select among the $\mathcal{T}_{\rm IR}$ determined above, only the minimal models containing a fusion subcategory $\{\mathcal{L}_\rho\}_{\rm IR}$ of Verlinde lines coinciding with $\{\mathcal{L}_\sigma\}^{(r,s)}_{\rm UV}$. This means that all the quantum dimensions, fusion rules, and spins in the defect Hilbert spaces in these two subcategories coincide $\{\mathcal{L}_\sigma\}^{(r,s)}_{\rm UV}$.

This, for any given (p,q) produces a list of candidate flows

$$\{\mathcal{M}(p,q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p',q')\}\,, \qquad (6)$$

fulfilling all the anomaly-matching conditions by construction. This procedure shall be regarded as exclusive rather than inclusive, meaning that fulfilling the anomalies does not automatically guarantee that the flow between the two minimal models will exist dynamically.

Generically more than a single relevant operator may trigger the same flow in eq. (6). If the set of operators that triggers the flow also preserves the same fusion subcategory of topological lines, then along the flow all such operators will be dynamically generated. Then, the critical point in the IR will be hit only by fine-tuning a combination of the UV deformations, we refer to the supplemental materials for a more detailed illustration of this. Lastly, the existence of a gapless flow triggered by a given relevant operator does not exclude the existence of gapped phases for other generic choices of critical coupling. For example, the $\phi_{(1,3)}$ perturbation of the tricritical Ising model $\mathcal{M}(5,4)$ flows to either a gapped phase or to $\mathcal{M}(4,3)$ depending on the sign of the perturbation. One can also determine what is the entering direction of the flow in the IR. Indeed, given that the topological lines are preserved along the flow, one can determine which irrelevant operators of $\mathcal{T}_{\rm IR}$, commute with the same subcategory of the UV theory. It may happen that no such operators in the IR exist. In this case, the flow either enters along the $T\bar{T}$ direction[3] or along

[1] More precisely, the defect spin content in the IR is a subset of the one in the UV due to possible massive decouplings.

[2] Note that this implies the closure of this category under fusion.

[3] This is for instance the case for the flow between Ising model and the Yang-Lee model ($\mathcal{M}(2,5)$), or the integrable flow $\mathcal{M}(4,5) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(3,4)$. See Appendix B for more details and examples.

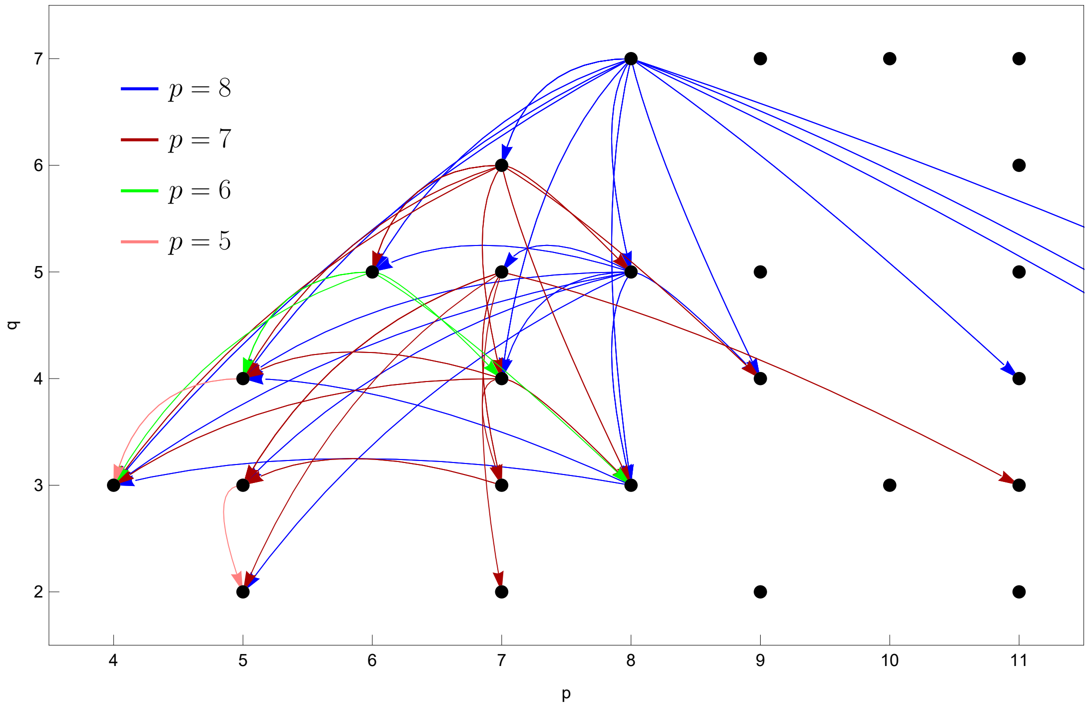

FIG. 1: Flows between minimal models $\mathcal{T}_{\rm UV} = \mathcal{M}(p \leq 8, q)$. A detailed description is reported in Appendix B.

the least irrelevant descendant of a (relevant) primary that has commutations relations with the preserved topological lines and RG invariants consistent with the UV data. From this perspective, the non-invertible symmetries also provide a strong organizing principle to organize the effective field theory (or conformal perturbation theory) expansion around both the UV and IR. The procedure is easily automatized and implemented in `Mathematica`. In Figure 1 we report the outcome up to $p = 8$, but the procedure can be easily extended to any value of p. Among others, we reproduce all the flows conjectured in [12], corresponding to the $\mathbb{Z}_2$ symmetric flows, as well as the $\phi_{(1,2)}$, and $\phi_{(2,1)}$ flows known to be integrable [24, 34], but that do no belonging to that family. In addition, we find new spurious flows that do not belong to the families above, but that are associated with anomaly-matching. An example is the flow $\mathcal{M}(7,5) \xrightarrow{\phi_{(2,3)}} \mathcal{M}(11,3)$ discussed in the supplemental materials, together with a detailed discussion up to $p = 7$. An interesting case is the one of the integrable flows with $\phi_{(1,5)}$, while $\phi_{(1,3)}$ could a priory be dynamically generated along the flow, the solutions of the NLIEs suggest that this is not the case. We plan to study in detail the relation between integrability and non-invertible symmetries somewhere else.

Description via NLIE Since the seminal article [35], it was shown that certain special perturbations of minimal models could be described as *quantum reductions* of integrable quantum field theories. Specifically, perturbations controlled by the relevant field $\phi_{(1,3)}$ are obtained as quantum reductions of the sine-Gordon (sG) model [36–38], while perturbations by the fields $\phi_{(1,5)}$, $\phi_{(2,1)}$ and $\phi_{(1,2)}$ arise from the quantum reduction of the "Zhiber-Mikhailov-Shabat" (ZMS) model [34, 39–41].

Thanks to their integrability, it has been possible to derive a Non-Linear Integral Equation (NLIE) that *non-perturbatively* encodes the energies $E_{\rm s}(R)$ of any state s on a cylinder of radius R, which acts as an RG parameter. As functions of $r = Rm$, with m being the mass scale of the system, the energies interpolate between the UV regime

$$E_{\rm s}(R) \xrightarrow{r\to 0} -\frac{\pi(c_{\rm UV} - 24h_{\rm s})}{6R} ,$$

the usual Casimir behaviour [42], and the IR one

$$E_{\rm s}(R) \xrightarrow{r\to\infty} {\rm N}_{\rm s} \in \mathbb{Z}_{\geq 0} .$$

In [19, 22, 23] it was shown that the integrable structure of sG could be equally well employed to encode *massless flows* interpolating between successive unitary minimal models

$$\{\mathcal{M}(p+1,p) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(p,p-1)\} .$$

Soon it became clear that this description could also address flows between $\mathcal{M}(p,q)$ and $\mathcal{M}(2p-q,q)$ along $\phi_{(1,3)}$ [20, 21] and, using the integrable structure of ZMS, massless flows between $\mathcal{M}(2p+I,p)$ and $\mathcal{M}(2p-I,p)$ along $\phi_{(1,5)}$ and between $\mathcal{M}(2p-I,p)$ and $\mathcal{M}(2p-I,p-I)$ along $\phi_{(2,1)}$ [24].

One of the main results of this letter is that the NLIEs encoding the finite size spectrum of massless flows between minimal models can be extended – at the very least on a qualitative level – beyond the $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$ and $\phi_{(1,2)}$ cases, to the whole family of flows predicted by anomaly matching conditions associated to non-invertible symmetries.

The structure of the "massless NLIEs" is the same as for the known cases [19, 24]: one first computes the solutions $f_{\rm R}(\theta)$ and $f_{\rm L}(\theta)$ to the following coupled NLIE system

$$\begin{aligned} f_{\rm R}(\theta) &= {\rm i}\alpha' - {\rm i}\frac{r}{2}e^{\theta} - \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_{\rm s}^{\sigma}} d\theta' \left[\phi(\theta-\theta')L_{\rm R}^{-\sigma}(\theta') + \chi(\theta-\theta')L_{\rm L}^{\sigma}(\theta')\right] , \\ f_{\rm L}(\theta) &= -{\rm i}\alpha' - {\rm i}\frac{r}{2}e^{\theta} + \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_{\rm s}^{\sigma}} d\theta' \left[\phi(\theta-\theta')L_{\rm L}^{\sigma}(\theta') + \chi(\theta-\theta')L_{\rm R}^{-\sigma}(\theta')\right] , \end{aligned} \tag{7}$$

where $L^{\pm}_{\rm \genfrac{}{}{0pt}{}{R}{L}}(\theta) = \log\left[1+\exp\left(\pm f_{\rm \genfrac{}{}{0pt}{}{R}{L}}(\theta)\right)\right]$. Then, the *scaling function* $f_s(r) = 6RE_{\rm s}(R)/\pi$ is determined as

$$f_s(r) = \sum_{\sigma=\pm}\frac{3{\rm i}r\sigma}{2\pi^2}\int_{\mathcal{C}_{\rm s}^{\sigma}} d\theta \left[e^{-\theta}L_{\rm L}^{\sigma}(\theta) - e^{\theta}L_{\rm R}^{-\sigma}(\theta)\right] . \tag{8}$$

In these equations, the parameter α' is known as *twist*. The kernels $\phi(\theta)$ and $\chi(\theta)$ identify the specific theory, while the contours $\mathcal{C}_{\rm s}^{\pm}$ determine the state. In particular, the ground state is obtained with the choice $\mathcal{C}_{\rm s}^{\pm} = \mathbb{R} \pm {\rm i}\eta$, with $\eta \gtrsim 0$.

The flows described in this letter correspond to the following choice of kernels

$$\begin{aligned} \phi(\theta) &= -\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{{\rm i}\theta\omega} \frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{2\xi-\kappa}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)} , \\ \chi(\theta) &= -\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{{\rm i}\theta\omega} \frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{\kappa-2}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)} , \end{aligned} \tag{9}$$

where $\kappa > 2$ and $\xi > 1$, making the Fourier image integrable[4] on $\mathbb{R}$. The physical parameters of the UV and IR CFTs are determined as follows

$$\begin{aligned} c_{\rm eff}^{\rm UV}(p,q) &\equiv 1 - \frac{6}{pq} = 1 - 3\left(\frac{\alpha'}{\pi}\right)^2 \frac{(\xi-1)^2}{\xi(\xi+1)} , \\ c_{\rm eff}^{\rm IR}(p',q') &\equiv 1 - \frac{6}{p'q'} = 1 - 3\left(\frac{\alpha'}{\pi}\right)^2 \frac{\xi-1}{\xi} , \\ h_{(r,s)} &\equiv \frac{(pr-qs)^2-(p-q)^2}{4pq} = 1 - \frac{1}{z_{(r,s)}}\frac{\kappa}{\xi+1} . \end{aligned} \tag{10}$$

with $h_{(r,s)}$ being the conformal dimension of the perturbing field $\phi_{(r,s)}$ in the UV and $z_{(r,s)} = 1, 2$ depending on whether the field $\phi_{(r,s)}$ is even or odd under the natural $\mathbb{Z}_2$ symmetry in the UV[5]. Fixing these three physical parameters, i.e. choosing a UV starting CFT, together with an outgoing direction, and a target IR CFT uniquely fixes the form of the NLIEs (7). Consequently, any additional information extracted from (8) can be considered a non-trivial prediction. One quantity that can be analytically computed is the conformal dimension of the operator that attracts the flow in the IR CFT:

$$h_{(r',s')} = 1 + \frac{1}{z_{(r',s')}}\frac{\kappa}{\xi-1} . \tag{11}$$

The request that this conformal dimension appears, as it should, in the Kač table of the IR minimal model $\mathcal{M}(p',q')$ enforces a constraint on the allowed values of the integers p, q, p', q', r, s, r', and s'.

$$\frac{p(r+1)-q(s-1)}{p'(r'+1)-q'(s'-1)} = -\frac{p'(r'-1)-q'(s'+1)}{p(r-1)-q(s+1)} , \tag{12}$$

where we assumed that[6] $z_{(r,s)} = z_{(r',s')}$. While we could not find the most general solution to the above Diophantine equation, we can verify that the special family of solutions that corresponds to the flows discovered in [12]

$$\{\mathcal{M}_{(\mu p+I,p)} \xrightarrow{\phi_{(1,2\mu+1)}} \mathcal{M}_{(\mu p-I,p)}\} , \tag{13}$$

solve all the constraints with 2μ and $\mu p - I$ being positive integers. This family includes the familiar $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$ and $\phi_{(1,2)}$ flows[7]. In the supplementary material, we show how the NLIEs (7) reduce the known integrable cases for $\mu = 1/2, 1, 2$. where $p'/2 \le p \le p'-2$. In the ancillary Mathematica notebook is included a routine that determines all the flows allowed by the above constraint. Further restrictions can be imposed on the solutions using the non-invertible symmetry matching.

Numerical analysis and conformal perturbation theory. Extracting analytically any further non-trivial prediction from NLIEs of the form (7, 8) is a notoriously arduous task. We can make some headway by studying them numerically. In particular, we can compare the behaviour of the scaling function for large and small values of r to the predicted behaviour

[4] The limit case $\kappa = 2$ yields the $\phi_{(1,3)}$ massless flows [19–21]. This limit is non-trivial and is discussed in details in the supplementary material.

[5] Specifically, $z_{(r,s)} = (3-(-1)^{p(r-1)-q(s-1)})/2$.

[6] This is reasonable as we expect the parity of the field under $\mathbb{Z}_2$ symmetry is preserved along the flow.

[7] The case $\phi_{(2,1)}$ displayed in eq. (3.9) in [24] is recovered from (13) by setting $\mu = 1/2$, re-defining $p = 2P - J$ and $I = J/2$ and finally swapping the indices of the minimal models and of the primary fields.

of the ground-state energy along the flow (6). Contrary to the well-known integrable cases, we expect the general flow to be a *multi-field deformation* of the UV CFT, with the IR theory only arising upon fine-tuning of the critical coupling of the various deforming fields, e.g. in the flow $\mathcal{M}(7,2) \to \mathcal{M}(5,2)$, where both UV fields $\phi_{(1,2)}$ and $\phi_{(1,3)}$ were seen to contribute by using a Hamiltonian truncation method [32, 33]. Indeed, all relevant operators allowed by the preserved generalized symmetries will contribute to the flow, in agreement with the standard Wilsonian RG lore. For a flow triggered by a number M of relevant UV fields $\{\phi_{(r_i,s_i)}\}_{i=1}^{M}$, the expected small r behaviour of the scaling function (8) is

$$\begin{aligned} f(r) &\overset{r\to 0}{=} \frac{3r^2/(4\pi)}{\sin\left(\frac{\pi\kappa}{\xi+1}\right)} + \sum_{\{l_i\}=0}^{\infty} a_{l_1,\dots,l_M} r^{\sum_{i=1}^{M} l_i y_{(r_i,s_i)}}\,, \\ y_{(r,s)} &= 2z_{(r,s)}(1-h_{(r,s)})\,, \\ a_{0,0,\dots,0} &= f(p,q) = 1-\frac{6}{pq}\,. \end{aligned} \tag{14}$$

Here the coefficients $a_{l_1,\dots,l_M}$ are proportional to the correlation functions of the perturbing fields on the vacuum (see supplementary material for details). While the expansion (14) is expected to have a finite radius of convergence [16, 43], the situation in the IR is much less under control. There, the Conformal Perturbation Theory (CPT) expansion

$$\begin{aligned} f(r) \overset{r\to\infty}{=} f(p',q') + \sum_{l=1}^{\infty}\left(a'_l r^{l y_{(r',s')}} + b'_l r^{-2l}\right) + \\ +(\text{further contributions})\,, \end{aligned} \tag{15}$$

is asymptotic, and there is very little [44] control over the omitted "further contributions". We performed a numerical analysis of the NLIEs (7) for several cases and found that, in all of them, the scaling function (8) agrees perfectly with the expected behaviours (14, 15). While the parameters (10), are built in the kernel by construction, the agreement with the multiple sum for small r shall be regarded as a highly non-trivial check that our data passes with flying colours. In principle, further support can come from comparing the first few coefficients of the expansions with the estimates coming from CPT. We will report on this in a future publication. Figure 2 reports the numerical results for the flow $\mathcal{M}(10,3) \to \mathcal{M}(8,3)$, triggered by $\phi_{(1,7)}$, first proposed in [45], which has recently received a lot of attention [9, 11, 46]. The fit that includes contributions from all the perturbing fields, is numerically favoured, independently agreeing with the results obtained in [9] by employing Hamiltonian truncation and CPT methods.

Outlook. In this paper, we studied RG flows between generic minimal models. Many flows can be conjectured by the matching of the global symmetries. For these flows, we propose an NLIE description encoding the ground state energy non-perturbatively. It would be interesting to confront our ground state energy with the results that can be independently obtained by Conformal Perturbation Theory and Hamiltonian Truncation. While here we focussed on gapless RG flows between Minimal Models, our methods extend to the ones to gapped phases. In this case, the anomalies of non-invertible symmetries predict a non-trivial structure of the vacua of the TQFT and particle-soliton degeneracies [26–29]. A direct application would be to check whether the RG flows between QCD_2 theories proposed in [47] may be obtained via matching of the anomalies associated with lines of the coset models in QCD_2 as initiated recently in [28].
Our NLIEs also admit a simple extension to the massive version, similar to what happens for the $\phi_{(1,3)}$ and $\phi_{(1,5)}$, $\phi_{(1,2)}$ cases. For these integrable massive flows, the NLIE describes the ground state energy of, respectively the sG and ZMS theories. In general, we expect the massive version of our equations to be related to the ground state of a (time-like) Liouville CFT deformed by several vertex operators. This perspective suggests the possibility of studying the one-point functions of these theories using the reflection relations proposed in [48–50]. We plan to follow this path in the near future.

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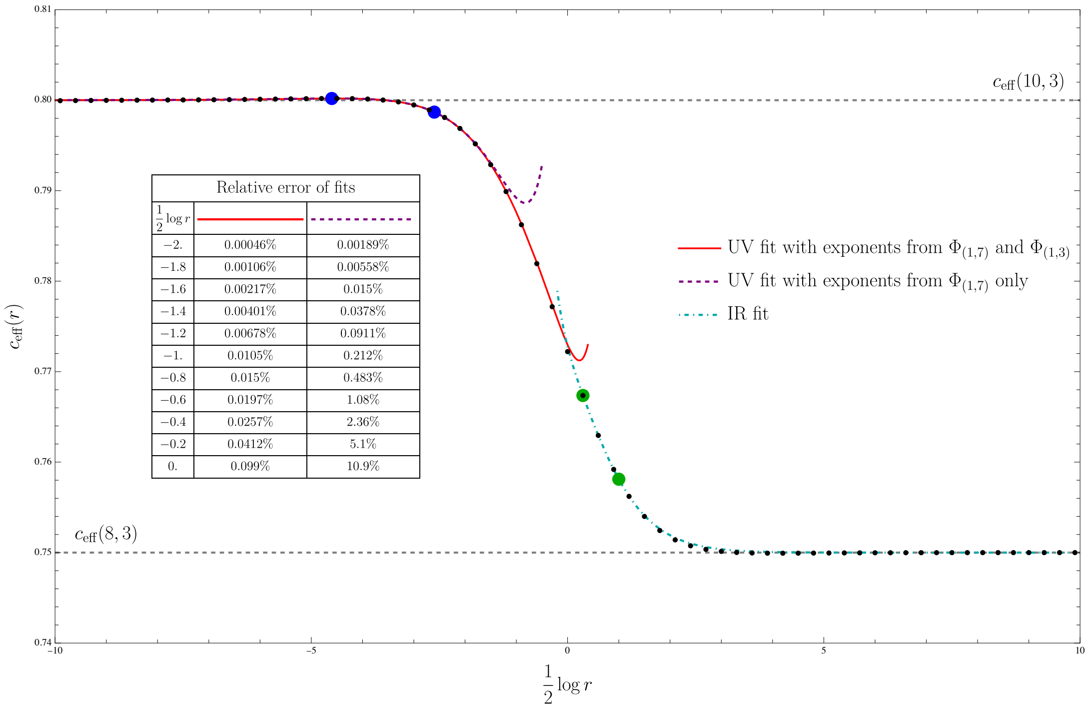

$\frac{1}{2}\log r$	UV fit ($\Phi_{(1,7)}$ and $\Phi_{(1,3)}$, red line)	UV fit ($\Phi_{(1,7)}$ only, purple dashed line)
−2.	0.00046%	0.00189%
−1.8	0.00106%	0.00558%
−1.6	0.00217%	0.015%
−1.4	0.00401%	0.0378%
−1.2	0.00678%	0.0911%
−1.	0.0105%	0.212%
−0.8	0.015%	0.483%
−0.6	0.0197%	1.08%
−0.4	0.0257%	2.36%
−0.2	0.0412%	5.1%
0.	0.099%	10.9%

FIG. 2: Plot of the ground state scaling function (8) for parameters (α', ξ, κ) supposed to correspond to the flow $\mathcal{M}(10,3) \to \mathcal{M}(8,3)$, triggered by $\phi_{(1,7)}$ in the UV. We fitted the data against both UV and IR CPT predictions, using only the points between the pairs of larger blue and green dots, respectively. We see that the fit performed including the contributions of all perturbing fields (red line) – here $y_{(1,5)} = 3y_{(1,7)}$ – performs much better than the fit for a single field perturbation (purple line). In the table are collected the relative errors for the two fits on points that were not used for the fit.

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Appendix A: Primer on minimal models.

In this section we collect some basic notions on minimal models and we fix the conventions used in this letter. Virasoro Minimal Models $\mathcal{M}(p,q)$, with $p>q$ are 2d rational CFTs enjoying diagonal Modular Invariance. Their central charge is:

$$c(p,q)=1-6\frac{(p-q)^2}{pq}\,, \tag{16}$$

they consist of $(p-1)(q-1)/2$ primaries $\phi_{(r,s)}$ in a fundamental domain $r=1,\cdots,p-1\,,s=1,\cdots,q-1$ with $sp+rq<pq$, having conformal weights:

$$h_{(r,s)}=h_{(p-r,q-s)}=\frac{(pr-qs)^2-(p-q)^2}{4pq}\,. \tag{17}$$

For simplicity, we will often denote the primaries just by ϕ_i, with $\phi_0=\phi_{(1,1)}$.

The primary fields form the fusion ring:

$$[\phi_\rho]\times[\phi_\sigma]=\sum_\kappa N^{\kappa}_{\rho\sigma}\,[\phi_\kappa] \tag{18}$$

where the fusion coefficients $N^\kappa_{\rho\sigma}$ are given in terms of the modular S-matrix $S_{r\rho}$ as:

$$N^{\kappa}_{\rho\sigma}=\sum_\lambda\frac{S_{\rho\lambda}\,S_{\sigma\lambda}S_{\kappa\lambda}}{S_{0\lambda}} \tag{19}$$

$$S_{(r,s),(\rho,\sigma)}=(-1)^{1+s\rho+r\sigma}\sqrt{\frac{8}{pq}}\sin\left(\pi\frac{p}{q}r\rho\right)\,\sin\left(\pi\frac{q}{p}s\sigma\right) \tag{20}$$

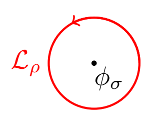
The topological Verlinde line operators, are in one-to-one correspondence with the primary fields $\mathcal{L}_{(r,s)}$ (that we will also equivalently denote as $\mathcal{L}_\rho$). They satisfy the same fusion ring as the primary fields:

$$\mathcal{L}_\rho\times\mathcal{L}_\sigma=\sum_\lambda N^\lambda_{\rho\sigma}\,\mathcal{L}_\lambda\,. \tag{21}$$

We denote the fusion modular category generated by the Verlinde lines by $\mathcal{V}_{(p,q)}$.

Let us denote $\phi_\rho\,|0\rangle=|\phi_\rho\rangle$ a a state on the cylinder. Then, the action of a Verlinde line on a primary field is given by:

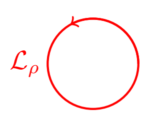
$$\mathcal{L}_\rho\,|\phi_\sigma\rangle=$$



$$=\frac{S_{\rho\sigma}}{S_{0\sigma}}\,|\phi_\sigma\rangle\,. \tag{22}$$

The action of a Verlinde line on a field, should be thought as generating the Ward Identity for the discrete symmetries associated to the topological line. The vacuum expectation value of a Verlinde line on the cylinder is known as *Quantum dimension*:

$$\langle\mathcal{L}_\rho\rangle=\langle 0|\,\mathcal{L}_\rho\,|0\rangle=$$



$$=d_\rho\,. \tag{23}$$

Whenever the symmetry generated by $\mathcal{L}_\rho$ is an invertible and non-anomalous group-like element, then $d_\rho=1$, but it is positive an irratinal number for an non-invertible element of the fusion category, and it can be dressed by a phase whenever the symmetry has a 't Hooft anomaly.

The torus partition function with the insertion of a Verlinde line operator $\mathcal{L}$ along the spatial cycle is after a modular transformation, the partition function over the defect Hilbert space $\mathcal{H}_\mathcal{L}$ of local operators living at the endpoint of $\mathcal{L}$. It is given explicitly by a twisted trace over the Virasoro characters:

$$Z_{\mathcal{L}_\lambda}(\tau,\overline{\tau})=\sum_{\rho,\sigma}N^\lambda_{\rho\sigma}\,\chi_\rho(\tau)\,\overline{\chi}_\sigma(\overline{\tau})\,. \tag{24}$$

From (24) we read off the spin content of the defect Hilbert space by evaluating

$$s_\lambda=(h_\rho-h_\sigma)\mod\mathbb{Z} \tag{25}$$

over non-zero fusion coefficients. Spins being non-semi integers signal the presence of a 't Hooft anomaly of the global symmetry generated by $\mathcal{L}_\lambda$

Appendix B: Detailed description of RG from $\mathcal{M}_{(p,q)}$ with $p, q \leq 7$

In this appendix we provide a detailed discussion of all the flows with $\max(p, q) \leq 7$ that can be predicted based on generalized symmetries constraints. We indicate the identity line $\mathcal{L}_{(1,1)} := \mathbb{1}$

Flows with $p \leq 5$.

The only flows here are

- $\boldsymbol{\mathcal{M}(5,3) \xrightarrow{\phi_{(2,1)}} \mathcal{M}(5,2)}$. The relevant $\left(h_{(1,2)} = \frac{3}{4}\right)$ deformation preserves the non-unitary Fibonacci category[8] generated by the non-invertible element $\mathcal{L}_{(1,3)}$, with

$$d_{(1,3)} = \frac{1}{2}\left(1 - \sqrt{5}\right) , \qquad s_{(1,3)} = \mathbb{Z} \pm \left\{0, \frac{1}{5}\right\} , \qquad \mathcal{L}_{(1,3)} \times \mathcal{L}_{(1,3)} = \mathbb{1} + \mathcal{L}_{(1,3)} . \tag{26}$$

  $\mathcal{L}_{(1,3)}$ of $\mathcal{M}_{3,5}$ flows to the unique non-trivial line $\mathcal{L}_{(1,2)}$ of the Lee-Yang model $\mathcal{M}(5,2)$ that famously satisfies the anomalous fusion category above. More interesting is the fate of the $\phi_{(2,1)}$ primary field that enters in the IR in the $T\overline{T}$-direction[9] This deformation was known to be integrable from [24].

- $\boldsymbol{\mathcal{M}(5,4) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(4,3)}$. This is gives the famous integrable Zamolodchikov flow preserving the entire Ising fusion category (Tambara-Yamagami TY_2 category) generated by the Kramers-Wannier duality defect $\mathcal{L}_{\phi_{(3,1)}} := \mathcal{N}$ and the invertible $\mathbb{Z}_2$-line $\mathcal{L}_{(2,1)} := \eta$. This flow is studied in [4], and we will not repeat the discussion here. We want to emphasize, that, in principle, there could be a putative flow $\mathcal{M}(5,4) \xrightarrow{\phi_{(1,2)}} \mathcal{M}(4,3)$ only preserving the invertible $\mathbb{Z}_2$-line η but not the Kramers-Wannier duality defect $\mathcal{N}$. Such a flow could only happen if the Kramers-Wannier duality is restored by symmetry enhancement in the IR, a very unlikely instance, given that, in both cases the IR is controlled by the $\bar{T}T$ operator that preserves the KW deformation, rather than breaking it. Henceforth, in the $\phi_{(1,3)}$ flow no operators may be generated, and the flow enters purely along the $\bar{T}T$ direction.

Flows with $p = 6$.

- $\boldsymbol{\mathcal{M}(6,5) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(5,4)}$. This is the integrable flow between tetracritical Ising and tricritical Ising, this is discussed in details in [4]. The flow preserves the non-invertible lines of $\{\mathcal{L}_{2,1}, \mathcal{L}_{3,1}, \mathcal{L}_{4,1}\}$ flowing to the $\{\mathcal{L}_{1,2}, \mathcal{L}_{1,3}, \mathcal{L}_{3,1}\}$ lines in $\mathcal{M}(5,4)$ respectively, having spin content:

$$s_{(2,1)} = \mathbb{Z} \pm \left\{0\,, \frac{1}{10}\,, \frac{2}{5}\,, \frac{1}{2}\right\} , \quad s_{(3,1)} = \mathbb{Z} \pm \left\{0\,, \frac{2}{5}\right\} , \quad s_{(4,1)} = \mathbb{Z} \pm \left\{0\,, \frac{1}{2}\right\} . \tag{27}$$

  The flow enters along the $\phi_{3,1}$ direction in the IR

- $\boldsymbol{\mathcal{M}(6,5) \xrightarrow{\phi_{(2,3)}} \mathcal{M}(7,4)}$. The flow is allowed by the effective central charge theorem as $f\,(7,4) = \frac{11}{14}$, $f\,(6,5) = \frac{5}{6}$. The $\phi_{(2,3)}$ deformation preserves the non-anomalous $\mathbb{Z}_2$ generated by the $\mathcal{L}_{(4,1)}$ line, but breaks the KW duality defects. In the IR, the flow enters along the direction of the second descendant of $\phi_{(1,2)}$.

- $\boldsymbol{\mathcal{M}(6,5) \xrightarrow{\phi_{(2,3)}} \mathcal{M}(8,3)}$. With the same field, there could also be a flow to this minimal model preserving the same symmetry line. It could enter along the first descendent of $\phi_{(1,5)}$. This would be consistent with the flow $\mathcal{M}(7,4) \to \mathcal{M}(8,3)$ also predicted below.

Both the last two flows are not anomalous, and we do not have any means to exclude them so far. An hint on this matter comes from our NLIE for which the scaling function in the UV does not exhibit agreement with a deformation triggered by $\phi_{(2,3)}$.

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[8] There are two fibonacci Categories with fusion rules $W^2 = 1 + W$, distinguished by their anomalies (F-symbols), or equivalently, by the defect Hilbert Space spins [4]

[9] We thank Roberto Tateo for clarifications on this point.

Flows with $p = 7$.

- $\mathcal{M}(7,3) \xrightarrow{\phi_{(1,5)}} \mathcal{M}(5,3)$. This flow is triggered by the most relevant field $\phi_{(1,5)}$ preserving the anomalous $\mathbb{Z}_2$ line $\mathcal{L}_{(2,1)}$ [51]:

$$\mathcal{L}_{(2,1)} \times \mathcal{L}_{(2,1)} = \mathbb{1}\,, \quad d_{(2,1)} = -1\,, \quad s_{(2,1)} = \mathbb{Z} \pm \frac{1}{4} \tag{28}$$

Along the flow also $\phi_{(1,3)}$ could be dynamically generated given that it commutes with $\mathcal{L}_{(2,1)}$ as well. In the IR, the flow enters along the $T\bar{T}$ direction, with contribution also from the descendents of $\phi_{(1,2)}$ of the IR model. Yet, it has been observed that $\phi_{(1,3)}$ does not contribute, and $\phi_{(1,5)}$ is a single-field integrable deformation. Indeed, integrability protects further the flow, by generating this operator. We plan to go back to this point somewhere else.

- $\mathcal{M}(7,4) \xrightarrow{\phi_{(2,1)}} \mathcal{M}(7,3)$. It preserves the category generated by $\{\mathbb{1}, \mathcal{L}_{(1,3)}, \mathcal{L}_{(1,5)}\}$ that is an $SU(2)_4$ fusion ring:

$$\begin{gathered}\mathcal{L}_{(1,3)} \times \mathcal{L}_{(1,3)} = \mathbb{1} + \mathcal{L}_{(1,3)} + \mathcal{L}_{(1,5)}\,, \qquad \mathcal{L}_{(1,5)} \times \mathcal{L}_{(1,5)} = \mathbb{1} + \mathcal{L}_{(1,3)} \\ \mathcal{L}_{(1,3)} \times \mathcal{L}_{(1,5)} = \mathcal{L}_{(1,5)} \times \mathcal{L}_{(1,3)} = \mathcal{L}_{(1,3)} + \mathcal{L}_{(1,5)}\,,\end{gathered} \tag{29}$$

and anomalies:

$$\begin{aligned} d_{(1,3)} &= -\frac{1}{2}\csc\left(\frac{3\pi}{14}\right)\,, \qquad & s_{(1,3)} &= \mathbb{Z} \pm \left\{0\,, \frac{1}{7}\,, \frac{2}{7}\right\}\,, \\ d_{(1,5)} &= 2\sin\left(\frac{\pi}{14}\right)\,, \qquad & s_{(1,5)} &= \mathbb{Z} \pm \left\{0\,, \frac{2}{7}\,, \frac{5}{7}\right\}\,. \end{aligned} \tag{30}$$

This flow is known to be integrable from [24]. No other operators commute with this fusion category, and no other operator (apart from its descendants) may be generated along the flow. The flow enters in $\mathcal{M}(3,7)$ along the $\phi_{(1,2)}$. The line operators $\mathcal{L}_{(1,3)}, \mathcal{L}_{(1,5)}$ flow to the same line operators of $\mathcal{M}(7,3)$.

- $\mathcal{M}(7,4) \xrightarrow{\phi_{(1,4)}} \mathcal{M}(5,4)$. The flow is generated by $\phi_{(1,4)}$ and preserves only the invertible non-anomalous $\mathbb{Z}_2$ line $\mathcal{L}_{(3,1)}$ and generates also the relevant operator $\phi_{(1,2)}$, that is also dynamically generated. The flow enters along the $\phi_{(1,4)}$ direction. It naturally implies all the flows from $\mathcal{M}(5,4)$.

- $\mathcal{M}(7,4) \xrightarrow{\phi_{(1,4)}} \mathcal{M}(8,3)$. This flows is generated by $\phi_{(1,4)}$ and $\phi_{(1,2)}$ with the former being the most relevant. It preserves the same It enters along the $\phi_{(2,1)}$ direction.

The difference in the last two flow is given by the relative critical coupling between the two fields.

- $\mathcal{M}(7,5) \xrightarrow{\phi_{(2,1)}} \mathcal{M}(7,2)$. This is triggered by the relevant field $\phi_{(2,1)}$ and preserves the $SU(2)_4$ category generated by the same lines as in $\mathcal{M}(7,4) \to \mathcal{M}(7,3)$ but with:

$$\begin{aligned} d_{(1,3)} &= \sin\left(\frac{\pi}{7}\right)\sec\left(\frac{3\pi}{14}\right)\,, \qquad & s_{(1,3)} &= \mathbb{Z} \pm \left\{0\,, \frac{1}{7}\,, \frac{3}{7}\right\}\,, \\ d_{(1,5)} &= -2\sin\left(\frac{3\pi}{14}\right)\,, \qquad & s_{(1,5)} &= \mathbb{Z} \pm \left\{0\,, \frac{1}{7}\,, \frac{2}{7}\right\}\,. \end{aligned} \tag{31}$$

It enters along the $T\bar{T}$ direction.

- $\mathcal{M}(7,5) \xrightarrow{\phi_{(1,2)}} \mathcal{M}(5,2)$. It preserves the anomalous Fibonacci category generated by $\mathcal{L}_{(3,1)}$ as in (26). It enters in $\mathcal{M}(5,2)$ along the $T\bar{T}$ direction. No other relevant operators may be generated along the flow.

- $\mathcal{M}(7,5) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(5,3)$. The flow triggered by $\phi_{(1,3)}$ preserves the category $\{\mathbb{1}\,, \mathcal{L}_{(2,1)}\,, \mathcal{L}_{(3,1)}\,, \mathcal{L}_{(4,1)}\}$, satisfying the fusion ring:

$$\begin{aligned} &\mathcal{L}_{(2,1)} \times \mathcal{L}_{(2,1)} = \mathbb{1} + \mathcal{L}_{(3,1)}\,, && \mathcal{L}_{(2,1)} \times \mathcal{L}_{(3,1)} = \mathcal{L}_{(2,1)} + \mathcal{L}_{(4,1)}\,, && \mathcal{L}_{(2,1)} \times \mathcal{L}_{(4,1)} = \mathcal{L}_{(3,1)} \\ &\mathcal{L}_{(3,1)} \times \mathcal{L}_{(2,1)} = \mathcal{L}_{(2,1)} + \mathcal{L}_{(4,1)}\,, && \mathcal{L}_{(3,1)} \times \mathcal{L}_{(3,1)} = \mathbb{1} + \mathcal{L}_{(3,1)}\,, && \mathcal{L}_{(3,1)} \times \mathcal{L}_{(4,1)} = \mathcal{L}_{(2,1)} \\ &\mathcal{L}_{(4,1)} \times \mathcal{L}_{(2,1)} = \mathcal{L}_{(3,1)}\,, && \mathcal{L}_{(4,1)} \times \mathcal{L}_{(3,1)} = \mathcal{L}_{(2,1)}\,, && \mathcal{L}_{(4,1)} \times \mathcal{L}_{(4,1)} = \mathbb{1}\,, \end{aligned} \tag{32}$$

and RG invariants:

$$\begin{aligned} &d_{(2,1)} = \frac{-1}{2}(1+\sqrt{5})\,, && d_{(3,1)} = \frac{1}{2}(1+\sqrt{5})\,, && d_{(4,1)} = -1\,, \\ &s_{(2,1)} = \mathbb{Z} \pm \left\{\frac{1}{20}, \frac{1}{4}, \frac{9}{20}\right\}\,, && s_{(3,1)} = \mathbb{Z} \pm \left\{0, \frac{1}{5}\right\}\,, && s_{(4,1)} = \mathbb{Z} \pm \frac{1}{4}\,. \end{aligned} \tag{33}$$

The flow enters along the $T\bar{T}$ direction in $\mathcal{M}(5,3)$. Further primaries cannot be generated along the $\phi_{(1,3)}$ flow. $\phi_{(1,2)}$ would only preserve $\mathcal{L}_{(3,1)}$, and the relevant fields $\phi_{(2,2)}, \phi_{(2,4)}$ would only preserve $\mathcal{L}_{(4,1)}$ and are therefore not allowed. Yet, one cannot exclude in principle to have a less protected flow, even though this would be quite unnatural, as the symmetries broken by $\phi_{(1,2)}$ or $\phi_{(2,2)}, \phi_{(2,4)}$, must be somehow restored at the critical IR point. For this reason, we believe that the correct flow is the one discussed above, and indeed it is the one that is known to be integrable.

- $\boldsymbol{\mathcal{M}(7,5) \xrightarrow{\phi_{(2,2)}} \mathcal{M}(11,3)}$. This flow along may generate also the less relevant $\phi_{(2,4)}$ preserves the anomalous $\mathbb{Z}_2$ symmetry generated by $\mathcal{L}_{(4,1)}$, with

$$d_{(4,1)} = -1\,, \qquad s_{(4,1)} = \mathbb{Z} \pm \frac{1}{4}\,. \tag{34}$$

It enters in the IR along the direction of $\phi_{(2,2)}$ which preserves $\mathcal{L}_{(2,1)}$ (being the image of $\mathcal{L}_{(4,1)}$ of the UV). We checked that this flow is reproduced by solution of the NLIEs.

- $\boldsymbol{\mathcal{M}(7,5) \xrightarrow{\phi_{(2,2)}} \mathcal{M}(7,3)}$. This is another flow triggered by $\phi_{(2,4)}$ and $\phi_{(2,2)}$. It enters in $\mathcal{M}(11,3)$ along the $\phi_{(2,1)}$ direction. Note that this is allowed by the Effective Central charge theorem given that $f(11,3) = \frac{9}{11} < f(7,5) = \frac{29}{35}$. It is consistent with the flow $\mathcal{M}(11,3) \xrightarrow{\phi_{(1,7)}} \mathcal{M}(7,3)$ that is also predicted the non-invertible symmetries.

- $\boldsymbol{\mathcal{M}(7,6) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(6,5)}$. This is one the flow of the Ising series. It is integrable with deformation $\phi_{(1,3)}$ that preserves the whole subcategory $\left\{\mathcal{L}_{(1,1)}, \mathcal{L}_{(2,1)}, \cdots, \mathcal{L}_{(5,1)}\right\}$. For the sake of clarity, let us denote $\mathcal{L}_{(k,1)}$ as (k). Then the fusion ring is:

$$\begin{aligned} &(2)\times(2) = \mathbb{1} + (3)\,, && (2)\times(3) = (2)+(4)\,, && (2)\times(4) = (3)+(5)\,, && (2)\times(5) = (4) \\ &(3)\times(2) = (2)+(4)\,, && (3)\times(3) = \mathbb{1}+(3)+(5)\,, && (3)\times(4) = (2)+(4)\,, && (3)\times(5) = (3) \\ &(4)\times(2) = (3)+(5)\,, && (4)\times(3) = (2)+(4)\,, && (4)\times(4) = \mathbb{1}+(3)\,, && (4)\times(5) = (2) \\ &(5)\times(2) = (4)\,, && (5)\times(3) = (3)\,, && (5)\times(4) = (2)\,, && (5)\times(5) = \mathbb{1}\,, \end{aligned} \tag{35}$$

with anomalies:

$$\begin{aligned} &d_{(2,1)} = \sqrt{3}\,, && d_{(3,1)} = 2\,, && d_{(4,1)} = \sqrt{3}\,, && d_{(5,1)} = 1 \\ &s_{(2,1)} = \mathbb{Z} \pm \left\{\frac{1}{24}, \frac{1}{8}, \frac{3}{8}, \frac{11}{24}\right\}\,, && s_{(3,1)} = \mathbb{Z} \pm \left\{0, \frac{1}{3}, \frac{1}{2}\right\}\,, && s_{(4,1)} = s_{(2,1)}\,, && s_{(5,1)} = \mathbb{Z} \pm \left\{0, \frac{1}{2}\right\} \end{aligned} \tag{36}$$

It is integrable with $\phi_{(1,3)}$. Deformations by $\phi_{(2,1)}, \phi_{(3,2)}, \phi_{(3,3)}$ would break part of this global symmetry. As argued above, it is very unlikely that these flow would have $\mathcal{M}(6,5)$ as an IR fixed point given that the broken symmetries would need to be restored at the end of the flow. Henceforth, we expect this flow to be triggered by the integrable deformation $\phi_{(1,3)}$ with no other operators contributing. It enters along the $\phi_{(1,3)}$ direction being the only irrelevant operator of $\mathcal{M}(5,6)$ having the correct commutation relations with the lines $\left\{\mathbb{1}, \mathcal{L}_{(1,2)}, \mathcal{L}_{(1,3)}, \mathcal{L}_{(1,4)}, \mathcal{L}_{(4,1)}\right\}$ having the corresponding quantum dimensions and defect Hilbert space spin content, and being the respective flow of the preserved lines in the UV.

- $\boldsymbol{\mathcal{M}(7,6) \xrightarrow{\phi_{(3,2)}} \mathcal{M}(9,4)}$. This flow triggered by the deformation $\phi_{(3,2)}$ preserves the non anomalous $\mathbb{Z}_2$ line $\mathcal{L}_{(1,5)}$. The operator $\phi_{(3,3)}$ is generated along the flow and it enters along the $\phi_{(2,1)}$ direction.

- $\boldsymbol{\mathcal{M}(7,6) \xrightarrow{\phi_{(3,2)}} \mathcal{M}(7,4), \mathcal{M}(8,3), \mathcal{M}(8,5)}$. The flows are triggered by the same deformation $\phi_{(3,2)}$ allowing also for $\phi_{(2,2)}$ and enters along the $\phi_{(2,3)}$ direction in $\mathcal{M}(7,4)$. They are all "long" flows implied by the series of short flows $\mathcal{M}(9,4) \xrightarrow{\phi_{(1,5)}} \mathcal{M}(7,4) \xrightarrow{\phi_{(1,4)}} \mathcal{M}(8,3)$, and $\mathcal{M}(8,5) \xrightarrow{\phi_{(1,2)}} \mathcal{M}(8,3)$. They are distinguished by the values of critical couplings. Given that the symmetry is in this cases non-anomalous, there is no a priori reason to exclude these two flows but no strong indication of their existance can be put foward.

Appendix C: Recovering the known, integrable cases

The NLIEs (7) with kernels (9) are conjectured to encode the spectrum of a very wide class of massless flows. Amongst these – in particular, in the class of Nakayama's flows (13) – are the more familiar $\phi_{(1,3)}$, $\phi_{(1,5)}$, and $\phi_{(1,2)}$ flows, studied since the '90s in the literature [19–24]. It is straightforward to check that the choice $\kappa = 3$ yields the same kernels considered in [24]. Indeed it is not difficult to see that the parameter μ in (13) is related to κ as $\mu = \kappa/z_{(1,2\mu+1)} - 1$. Since $z_{(1,2\mu+1)}$ can be either 1 or 2 for $\mathbb{Z}_2$ even or odd operators, respectively, we have the two options $\mu = 2$ or $\mu = 1/2$, corresponding, respectively, to $\phi_{(1,5)}$ and $\phi_{(1,2)}$ deformations.

Recovering $\phi_{(1,3)}$ is slightly more subtle[10]. In fact, one would need to set $\kappa = 2$ in the expressions of (9), which is not immediately possible, since for $\kappa = 2$ the Fourier image of $\phi(\theta)$ is not integrable on $\mathbb{R}$. We need to proceed more carefully. We notice that

$$\lim_{\kappa\to 2} \hat{\phi}(\omega) = 2\hat{\phi}_{\mathrm{Z}}(\omega) - 1 \,, \qquad \lim_{\kappa\to 2} \hat{\chi}(\omega) = 2\hat{\chi}_{\mathrm{Z}}(\omega) \,, \tag{37}$$

where $\hat{\phi}(\omega)$ and $\hat{\chi}(\omega)$ are the Fourier images of the kernels (9). The Fourier images $\hat{\phi}_{\mathrm{Z}}(\omega)$ and $\hat{\chi}_{\mathrm{Z}}(\omega)$ are those used in [20, 21, 52] to describe the $\phi_{(1,3)}$ massless flows. Then we find

$$\lim_{\kappa\to 2} \phi(\theta) = 2\phi_{\mathrm{Z}}(\theta) - \delta(\theta) \,, \qquad \lim_{\kappa\to 2} \chi(\theta) = 2\chi_{\mathrm{Z}}(\theta) \,. \tag{38}$$

At the level of the NLIEs (7), we take the limit $\kappa \to 2$ and find

$$\begin{aligned} f_{\mathrm{R,Z}}(\theta) &= \mathrm{i}\alpha' - \mathrm{i}\frac{r}{2}e^{\theta} - 2\sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta' \left[\phi_{\mathrm{Z}}(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta') + \chi_{\mathrm{Z}}(\theta-\theta')L^{\sigma}_{\mathrm{L,Z}}(\theta')\right] + \\ &\quad + \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta'\, \delta(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta') \,, \\ f_{\mathrm{L,Z}}(\theta) &= -\mathrm{i}\alpha' - \mathrm{i}\frac{r}{2}e^{\theta} + 2\sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta' \left[\phi(\theta-\theta')L^{\sigma}_{\mathrm{L,Z}}(\theta') + \chi(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta')\right] + \\ &\quad - \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta'\, \delta(\theta-\theta')L^{\sigma}_{\mathrm{L,Z}}(\theta') \,. \end{aligned} \tag{39}$$

Now, from the definition of the functions $L^{\pm}$ we have the identity

$$\sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta'\, \delta(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta') = -\int_{-\infty}^{\infty} d\theta'\, \delta(\theta-\theta') f_{\mathrm{R,Z}}(\theta') = -f_{\mathrm{R,Z}}(\theta) \,, \tag{40}$$

where we used the Cauchy theorem. A similar manipulation can be performed for the term with subscript L. Now, a slight shuffling of the furniture in (39), yields

$$\begin{aligned} f_{\mathrm{R,Z}}(\theta) &= \mathrm{i}\frac{\alpha'}{2} - \mathrm{i}\frac{r}{4}e^{\theta} - \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta' \left[\phi_{\mathrm{Z}}(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta') + \chi_{\mathrm{Z}}(\theta-\theta')L^{\sigma}_{\mathrm{L,Z}}(\theta')\right] \,, \\ f_{\mathrm{L,Z}}(\theta) &= -\mathrm{i}\frac{\alpha'}{2} - \mathrm{i}\frac{r}{4}e^{\theta} + \sum_{\sigma=\pm}\sigma \int_{\mathcal{C}_s^\sigma} d\theta' \left[\phi(\theta-\theta')L^{\sigma}_{\mathrm{L,Z}}(\theta') + \chi(\theta-\theta')L^{-\sigma}_{\mathrm{R,Z}}(\theta')\right] \,, \end{aligned} \tag{41}$$

which coincides precisely with the equations in [20, 21, 52], provided that we rescale α' and r by a factor 2.

[10] We thank Roberto Tateo for suggesting this calculation.

$$\begin{array}{ccccc}
\mathcal{M}(13,4) & \xrightarrow{\phi_{(1,7)}} & \mathcal{M}(11,4) & \xrightarrow{\phi_{(1,6)}} & \mathcal{M}(9,4) \\
 & & & & \big\downarrow \phi_{(1,5)} \\
\mathcal{M}(4,3) & \xleftarrow{\phi_{(1,3)}} & \mathcal{M}(5,4) & \xleftarrow{\phi_{(1,4)}} & \mathcal{M}(7,4)
\end{array}$$

FIG. 3: Chain of flows from $\mathcal{M}(13,4)$ to Ising. We only report in the figure the most relevant deformation triggering the flow.

Appendix D: Chain of flows from $\mathcal{M}(14,3)$

We analyse here the chain reported of 3, starting with $\mathcal{M}(13,4)$ and landing ultimately on the Ising model $\mathcal{M}(4,3)$. Using the procedure discussed in the main text we can split this flow in

- $\mathcal{M}(13,4) \rightarrow \mathcal{M}(11,4)$. The most relevant operator triggering the flow is $\phi_{(1,7)}$ preserving the TY_2 fusion category generated by the identity line, the invertible non-anomalous $\mathbb{Z}_2$ line $\mathcal{L}_{(3,1)}$ and $\mathcal{L}_{(2,1)}$ with $d_{(2,1)} = \sqrt{2}$. The spin content of the defect Hilbert space is:

$$s_{(2,1)} = \mathbb{Z} \pm \left\{ \frac{1}{16}, \frac{7}{16} \right\}, \qquad s_{(3,1)} = \mathbb{Z} \pm \left\{ 0, \frac{1}{2} \right\} \tag{42}$$

  In the IR the flow enters in the $\phi_{(1,7)}$ direction in $\mathcal{M}(11,4)$. Furthermore the same category of lines is preserved by the entire tower $\{\phi_{(1,3)}, \phi_{(1,5)}, \phi_{(1,7)}\}$, that are therefore dynamically generated along the $\phi_{(1,7)}$ flow. The operators $\phi_{(1,2)}, \phi_{(1,4)}, \phi_{(1,6)}$ commute only with the $\mathcal{L}_{(1,3)}$, so they cannot be generated by $\phi_{(1,7)}$ flow.

- $\mathcal{M}(11,4) \rightarrow \mathcal{M}(9,4)$. This flow is interesting because it explicitly break the Kramers-Wannier duality defect. It was also observed by Nakayama. The perturbations are $\phi_{(1,2)}, \phi_{(1,4)}, \phi_{(1,6)}$, in the IR $\phi_{(1,6)}$ arrives on $\phi_{(1,6)}$.

- $\mathcal{M}(9,4) \rightarrow \mathcal{M}(7,4)$. Both UV and IR have anomalous TY_2 category having quantum dimensions $(1, -\sqrt{2}, 1)$. This subalgebra is preserved by $\phi_{(1,5)}, \phi_{(1,3)}$ in the UV, with the former being the most relevant. They enter the IR in the $\phi_{(1,5)}, \phi_{(1,3)}$ directions. The flow is integrable, and single field with $\phi_{(1,5)}$.

- $\mathcal{M}(7,4) \rightarrow \mathcal{M}(5,4)$. Same feature as before: they break KW duality. As in $\mathcal{M}(7,4)$ is anomalous, and in $\mathcal{M}(5,4)$ is not. The most relevant operator in the UV is $\phi_{(1,4)}$, while $\phi_{(1,2)}$ also preserves $\mathbb{Z}_2$. They enter the IR along the $\phi_{(1,2)}$ direction.

- $\mathcal{M}(5,4) \rightarrow \mathcal{M}(4,3)$. This is the famous Zamolodchikov tri-critical Ising to Ising flow. The only allowed perturbing operator in the UV is $\phi_{(1,3)}$, and the flows enters the IR along the $T\bar{T}$ direction.

We numerically analysed the flows belonging to the chain in Figure 3. We fitted the numerical data against the expected UV and IR behaviours (14, 15). We found excellent agreement with the predicted behaviour in all cases except the IR of $\mathcal{M}(7,4) \rightarrow \mathcal{M}(5,4)$, in which the numerical data was too unstable to be reliable. More quantitatively, the goodness-of-fit data is collected in Table I. Note that all fits were performed with a CPT series truncated at 10 terms (bulk energy coefficient excluded).

Flow	UV χ^2_{red}	UV d.o.f.	IR χ^2_{red}	IR d.o.f.
$\mathcal{M}(13,4) \rightarrow \mathcal{M}(11,4)$	0.935364	39	1.69241	59
$\mathcal{M}(11,4) \rightarrow \mathcal{M}(9,4)$	1.47083	59	0.988184	58
$\mathcal{M}(9,4) \rightarrow \mathcal{M}(7,4)$	1.02055	58	1.02304	58
$\mathcal{M}(7,4) \rightarrow \mathcal{M}(5,4)$	5.21225	51	56439.4	38
$\mathcal{M}(5,4) \rightarrow \mathcal{M}(4,3)$	1.16287	30	0.842416	48

TABLE I: Quantitative, goodness-of-fit data of the behaviours (14, 15) for the numerical data obtained by standard iteration of the NLIEs (7) for the chain of flows in Figure 3.

The results are represented graphically in Figure 4.

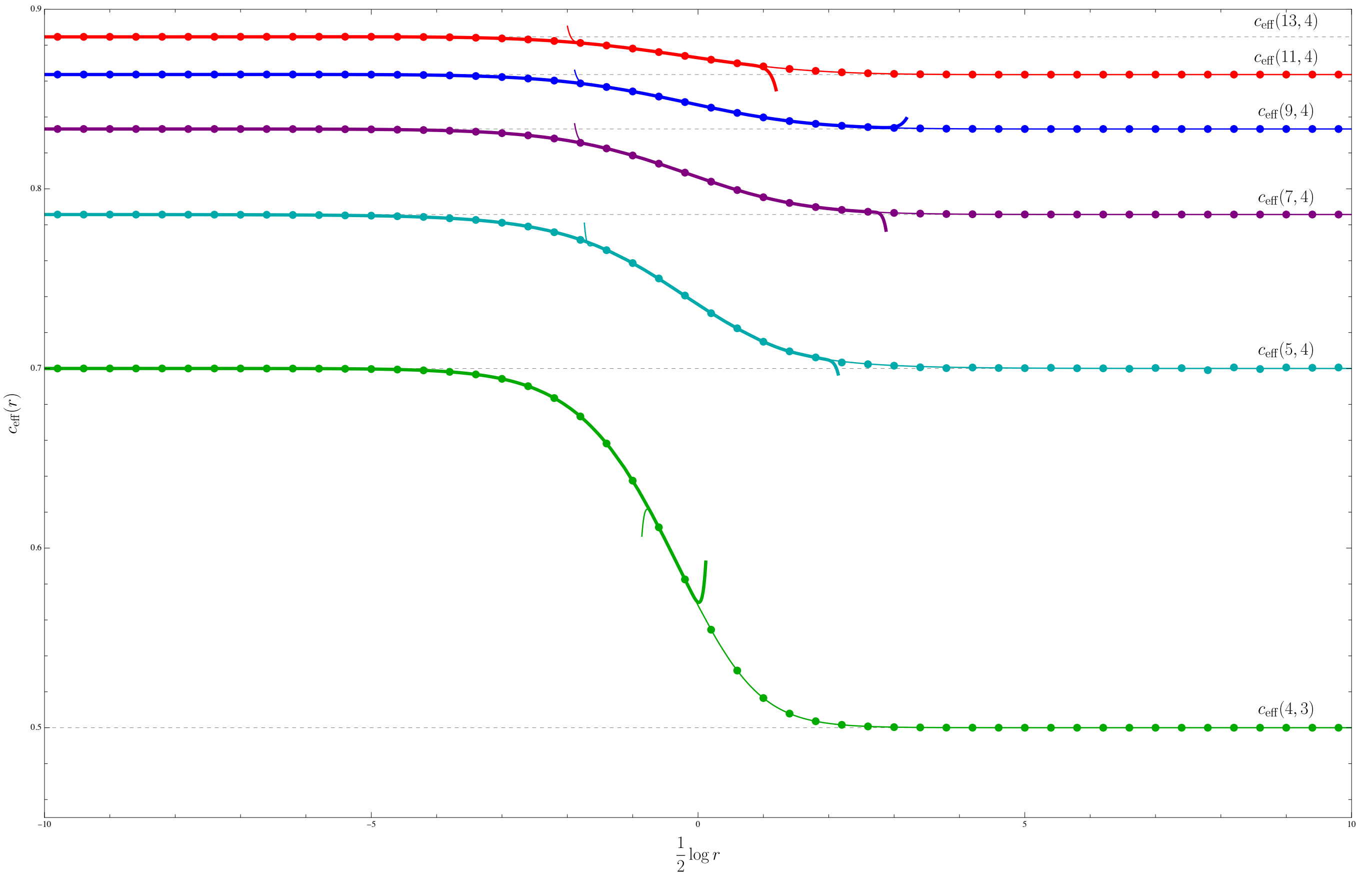

FIG. 4: Plot of the scaling functions $f(r)$ for the chain of flows in Figure 3. The dots represent numerical data obtained by standard iteration of the NLIEs (7). The thick and thin lines were obtained by fitting the numerical data against, respectively, the expected UV and IR behaviours (14, 15).