Article:

Lesnic, D. orcid.org/0000-0003-3025-2770 and Bradji, A. (2025) A note on an iterative algorithm for solving an inverse problem for a fractional-order partial differential equation. Communications on Analysis and Computation. ISSN 2837-0562

https://doi.org/10.3934/cac.2025002

A note on an iterative algorithm for solving an inverse problem for a fractional-order partial differential equation

Abdallah Bradji[1] and Daniel Lesnic[2,*]

[1] *LMA Laboratory, Faculty of Sciences, Badji Mokhtar-Annaba University, Annaba 23000, Algeria*

[2] *University of Leeds, Department of Applied Mathematics, Leeds LS2 9JT, UK*

[*]Correspondence to: D.Lesnic@leeds.ac.uk

Abstract. We consider an ill-posed inverse problem for a fractional-order partial differential equation (PDE). For its solution, we establish an iterative algorithm based on a sequence of well-posed problems previously developed for classical (non-fractional) elliptic and parabolic PDEs. For exact data, we prove the convergence of the algorithm and we establish its rate of the convergence. As with any regularising algorithm, in case of noisy data the iterations have to be stopped at an appropriate threshold before the solution's instability starts to manifest.

Keywords: fractional-order partial differential equation; inverse and ill-posed problem; iterative algorithm

AMS Subject Classification: 35R11, 35R30, 35R25, 26A33, 65N12

1. Introduction

We consider an ill-posed inverse problem for a fractional-order PDE previously treated in [2, 11] in which the stability of such problem was restored using a boundary-value problem regularization method and a Fourier truncation method, respectively. However, the former requires extending the solution domain and considering a non-local boundary condition, whilst the latter one is truncating the Fourier series expansion to achieve stable approximate solutions. In this paper, we propose an iterative algorithm that preserves the original solution domain and the governing PDE, based on solving a sequence of associated well-posed problems. For the classical non-fractional order parabolic heat equation this algorithm was developed in [7, Section 2.1] and was tested numerically in [9] in the context of regularizing the exponentially ill-posed backward heat conduction problem (BHCP). Also, for the elliptic Laplace equation this algorithm was developed in [7, Section 2.2] and was tested numerically in [8] in the context of regularizing the severely ill-posed Cauchy problem. In this note, we accommodate this iterative algorithm for solving an inverse problem for the following fractional-order PDE:

$$\partial_t^\alpha u(\boldsymbol{x},t) + \Delta u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Omega_T := \Omega \times (0,T), \tag{1}$$

where $1 < \alpha < 2$, Ω is an open bounded connected subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), $T > 0$ and ∂_t^α is the Caputo derivative of order α given by

$$\partial_t^\alpha \varphi(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \varphi_{\tau\tau}(\tau) d\tau, \tag{2}$$

where Γ is the Gamma function. Equation (1) is equipped, for simplicity, with the homogeneous Dirichlet boundary conditions

$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{3}$$

Also, at $t = 0$ we supply

$$-u_t(\boldsymbol{x}, 0) = G(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{4}$$

and

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{5}$$

where the Cauchy data G and f may not be exact, i.e., it can be noisy. We mention that a modified equation to (1) given by $\partial_t^\alpha u - \Delta u(\boldsymbol{x}, t) = 0$ subjected to conditions (3)-(5) yields the more well-known time fractional diffusion-wave problem investigated elsewhere [1, 6]. As $\alpha \nearrow 2$, then the system of equations (1), (3)-(5) forms the Cauchy problem for the elliptic Laplace's equation in $(d+1)$-dimensions, whilst for $\alpha \searrow 1$ it becomes the BHCP for the heat equation with a source term given by $u_t(\boldsymbol{x}, t) + \Delta u(\boldsymbol{x}, t) = -G(\boldsymbol{x})$ for $(\boldsymbol{x}, t) \in \Omega_T$. Both these classical inverse problems for elliptic and parabolic PDEs have at most one solution, which, however, does not dependent continuously on the data (4) and (5). In the fractional case $\alpha \in (1, 2)$, the inverse problem (1), (3)-(5) is also ill-posed [2, 5]. To restore stability, we use a sequence of well-posed problems whose solutions tend to the solution of (1), (3)-(5), of the following general form:

$$\begin{cases} \partial_t^\alpha \Psi(\boldsymbol{x}, t) + \Delta\Psi(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Omega_T, \\ \Psi(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial\Omega \times (0, T), \\ -\Psi_t(\boldsymbol{x}, 0) = \zeta(\boldsymbol{x}), \quad \Psi(\boldsymbol{x}, T) = \theta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{6}$$

Using the new variable $\overline{\Psi} = \Psi_t$, the problem (6) recasts as

$$\begin{cases} \partial_t^{\alpha-1} \overline{\Psi}(\boldsymbol{x}, t) - \int_t^T \Delta\overline{\Psi}(\boldsymbol{x}, s) ds = -\Delta\theta(\boldsymbol{x}), & (\boldsymbol{x}, t) \in \Omega_T, \\ \overline{\Psi}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial\Omega \times (0, T), \\ -\overline{\Psi}(\boldsymbol{x}, 0) = \zeta(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{7}$$

Using this new form, the following stability result is proved in [2].

Lemma 1.1 (Stability estimate for problem (6), cf. [2]). *Assume that θ and $\zeta \in H^2(\Omega)$. Then any solution of the problem (6) is satisfying the following estimate:*

$$\|\partial_t^{\frac{1+\alpha}{2}} \Psi\|_{L^2(\Omega_T)} \leq C \left(\|\Delta\theta\|_{L^2(\Omega)} + \|\Delta\zeta\|_{L^2(\Omega)} \right). \tag{8}$$

The following corollary, deduced from Lemma 1.1, is useful.

Corollary 1.1 (A new $L^\infty(L^2)$–estimate for problem (6))**.** *Under the hypotheses of Lemma 1.1, the following estimate holds for any solution of problem (6):*

$$\|\Psi\|_{L^\infty(0,T;\,L^2(\Omega))} \le C\left(\|\theta\|_{H^2(\Omega)} + \|\zeta\|_{H^2(\Omega)}\right). \tag{9}$$

Proof. For $\beta := \alpha - 1 \in (0,1)$ we have that $\partial_t^{\frac{\beta}{2}+1}\Psi = \partial_t^{\frac{\beta}{2}}\Psi_t$ and from (8) we deduce that

$$\|\partial_t^{\frac{\beta}{2}}\Psi_t\|_{L^2(\Omega_T)} \le C\left(\|\Delta\theta\|_{L^2(\Omega)} + \|\Delta\zeta\|_{L^2(\Omega)}\right). \tag{10}$$

Using Fubini's theorem, the fractional Poincaré - Friedrichs inequality [3, Theorem 2.10] (which also stems from the application of [4, equation (9) and Lemma 3.1]) and the third condition in (6), inequality (10) implies that, for $\mathcal{S}(\boldsymbol{x},t) := \Psi_t(\boldsymbol{x},t) + \zeta(\boldsymbol{x})$ we have

$$\begin{aligned}\|\mathcal{S}\|^2_{L^2(\Omega_T)} &= \int_\Omega \|\mathcal{S}(\boldsymbol{x},\cdot)\|^2_{L^2(0,T)} d\boldsymbol{x} \le C(T,\alpha)\int_\Omega \|\partial_t^{\frac{\beta}{2}}\mathcal{S}(\boldsymbol{x},\cdot)\|^2_{L^2(0,T)} d\boldsymbol{x}\\ &= C(T,\alpha)\int_\Omega \|\partial_t^{\frac{\beta}{2}}\Psi_t(\boldsymbol{x},\cdot)\|^2_{L^2(0,T)} d\boldsymbol{x} \le C\left(\|\Delta\theta\|_{L^2(\Omega)} + \|\Delta\zeta\|_{L^2(\Omega)}\right)^2.\end{aligned}$$

Using the triangle inequality, this implies

$$\|\Psi_t\|_{L^2(\Omega_T)} \le C\left(\|\Delta\theta\|_{L^2(\Omega)} + \|\zeta\|_{H^2(\Omega)}\right). \tag{11}$$

Using the identity $\Psi(\cdot,t) = -\int_t^T \Psi_\tau(\cdot,\tau)d\tau + \theta(\cdot)$ (which stems from the fourth condition in (6)) and the Cauchy-Schwarz inequality yield the desired estimate (9). □

To compute the solutions of the problems (1), (3)-(5) and (6), we use the separation of variables method. Let us denote by $(\varphi_m, \mu_m)_{m\in\mathbb{N}^*}$, where $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, the solution of the eigenvalue problem

$$-\Delta\varphi(\boldsymbol{x}) = \mu\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega \quad \text{and} \quad \varphi(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\partial\Omega. \tag{12}$$

It is known that (12) has a sequence of eigenvalues $0 < \mu_1 < \ldots < \mu_m \to +\infty$, as $m\to\infty$, and that the eigenfunctions set $(\varphi_m)_{m\in\mathbb{N}^*}$ may be taken to constitute an orthonormal basis of $L^2(\Omega)$. Let us write the solution of (1), (3)-(5) as

$$u(\boldsymbol{x},t) = \sum_{m=1}^{\infty}\psi_m(t)\varphi_m(\boldsymbol{x}). \tag{13}$$

Taking the inner product of (1) with φ_m, using (13) together with the orthonormal property and, subsequently, using the conditions (4) and (5) we obtain

$$\partial_t^\alpha\psi_m(t) - \mu_m\psi_m(t) = 0, \quad t\in(0,T), \tag{14}$$

$$-\psi_m'(0) = G_m \quad \text{and} \quad \psi_m(0) = f_m, \tag{15}$$

where $G_m = (G,\varphi_m)_{L^2(\Omega)}$ and $f_m = (f,\varphi_m)_{L^2(\Omega)}$. The solution of (14)–(15) (see for instance [12]) is given by

$$\psi_m(t) = f_m E_{\alpha,1}(\mu_m t^\alpha) - G_m t E_{\alpha,2}(\mu_m t^\alpha), \quad t\in[0,T], \tag{16}$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function defined by

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)}, \quad z \in \mathbb{C}. \tag{17}$$

Gathering (13) and (16) yields the following expression for the solution of the ill-posed problem (1), (3)-(5):

$$u(\boldsymbol{x},t) = \sum_{m=1}^{\infty} \left(f_m E_{\alpha,1}(\mu_m t^\alpha) - G_m t E_{\alpha,2}(\mu_m t^\alpha)\right)\varphi_m(\boldsymbol{x}). \tag{18}$$

In a similar way, the solution of the well-posed problem (6) is expressed as follows:

$$\Psi(\boldsymbol{x},t) = \sum_{m=1}^{\infty} \overline{\psi}_m(t)\varphi_m(\boldsymbol{x}), \tag{19}$$

where $\overline{\psi}_m(t)$ is the solution of the following fractional differential problem:

$$\partial_t^\alpha \overline{\psi}_m(t) - \mu_m \overline{\psi}_m(t) = 0, \quad t \in (0,T), \tag{20}$$

$$-\overline{\psi}'_m(0) = \zeta_m \quad \text{and} \quad \overline{\psi}_m(T) = \theta_m, \tag{21}$$

where $(\zeta_m, \theta_m) = \left((\zeta,\varphi_m)_{L^2(\Omega)}, (\theta,\varphi_m)_{L^2(\Omega)}\right)$. Problem (20)–(21) has the solution

$$\overline{\psi}_m(t) = \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\theta_m - \left(\frac{tE_{\alpha,2}(\mu_m t^\alpha)E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\right)\zeta_m, \quad t \in [0,T]. \tag{22}$$

Using (9) and (19) we obtain the estimate

$$\sup_{t\in[0,T]} |\overline{\psi}_m(t)|^2 \le \sup_{t\in[0,T]} \sum_{m=1}^{+\infty} |\overline{\psi}_m(t)|^2 \le C\left(\|\theta\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)}\right). \tag{23}$$

2. Iterative algorithm

To solve the problem (1), (3)-(5), we suggest the following iterative algorithm previously developed for elliptic and parabolic non-fractional PDEs in [7, Section 2]:

- Initial guess step:

$$\begin{cases} \partial_t^\alpha \psi_m^{(0)}(t) - \mu_m \psi_m^{(0)}(t) = 0, & t \in (0,T), \\ -\left(\psi_m^{(0)}\right)'(0) = G_m, & \psi_m^{(0)}(T) = \xi_m, \end{cases} \tag{24}$$

  where $\xi \in H^2(\Omega)$ is an initial guess for the solution at $t = T$ and $\xi_m = (\xi, \varphi_m)_{L^2(\Omega)}$.
- Recurrence relation: Assuming that $\psi_m^{(n)}$ has been computed, solve for $\psi_m^{(n+1)}$ satisfying

$$\begin{cases} \partial_t^\alpha \psi_m^{(n+1)}(t) - \mu_m \psi_m^{(n+1)}(t) = 0, & t \in (0,T), \\ -\left(\psi_m^{(n+1)}\right)'(0) = G_m, & \psi_m^{(n+1)}(T) = \psi_m^{(n)}(T) - \sigma_m\left(\psi_m^{(n)}(0) - f_m\right), \end{cases} \tag{25}$$

where σ_m are positive parameters that will be chosen later and the superscript $^{(n)}$ does not denote the usual high-order ordinary derivatives of a function. Note that the condition at $t = 0$ in equation (25) remains invariant in the iterative procedure. The solutions of the problems (24) and (25) are given by

$$\psi_m^{(0)}(t) = \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\xi_m - \left(\frac{tE_{\alpha,2}(\mu_m t^\alpha)E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\right) G_m \tag{26}$$

and

$$\begin{aligned}\psi_m^{(n+1)}(t) &= \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\left(\psi_m^{(n)}(T) - \sigma_m \psi_m^{(n)}(0) + \sigma_m f_m\right) \\ &- \left(\frac{tE_{\alpha,2}(\mu_m t^\alpha)E_{\alpha,1}(\mu_m T^\alpha) - TE_{\alpha,2}(\mu_m T^\alpha)E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\right) G_m.\end{aligned} \tag{27}$$

Let us set

$$\Psi^{(0)}(\boldsymbol{x}, t) = \sum_{m=1}^{\infty} \psi_m^{(0)}(t)\varphi_m(\boldsymbol{x}) \quad \text{and} \quad \Psi^{(n+1)}(\boldsymbol{x}, t) = \sum_{m=1}^{\infty} \psi_m^{(n+1)}(t)\varphi_m(\boldsymbol{x}). \tag{28}$$

Remark 2.1 (On the sequence defined by (28)). *The function $\Psi^{(0)}$ is the solution of the following problem:*

$$\begin{cases} \partial_t^\alpha \Psi^{(0)}(\boldsymbol{x}, t) + \Delta \Psi^{(0)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Omega_T, \\ \Psi^{(0)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial\Omega \times (0, T), \\ -\Psi_t^{(0)}(\boldsymbol{x}, 0) = G(\boldsymbol{x}), \quad \Psi^{(0)}(\boldsymbol{x}, T) = \xi(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{29}$$

In the case when $\sigma_m =: \sigma$ for all $m \in \mathbb{N}^*$, the functions $\Psi^{(n+1)}$ are the solutions of the following problems:

$$\begin{cases} \partial_t^\alpha \Psi^{(n+1)}(\boldsymbol{x}, t) + \Delta \Psi^{(n+1)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \Omega_T, \\ \Psi^{(n+1)}(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial\Omega \times (0, T), \\ -\Psi_t^{(n+1)}(\boldsymbol{x}, 0) = G(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \Psi^{(n+1)}(\boldsymbol{x}, T) = \Psi^{(n)}(\boldsymbol{x}, T) - \sigma\left(\Psi^{(n)}(\boldsymbol{x}, 0) - f(\boldsymbol{x})\right), & \boldsymbol{x} \in \Omega, \end{cases} \tag{30}$$

where, for any $n \in \mathbb{N}$, $\Psi^{(n)}$ is a solution of the problem (6) with $(\zeta, \theta) = (G, \xi)$ when $n = 0$, and $\zeta = G$ and θ is depending on σ and the previous $\Psi^{(n-1)}$ when $n \in \mathbb{N}^*$.

The aim of this section is bi-fold:

- We prove the convergence of the solutions of the well-posed problems (24)–(25) towards the solution of the problem (14)–(15), see Theorem 2.1 below. This convergence requires only that the parameters $(\sigma_m)_m$ are chosen such that (32) holds (see also Remark 2.2).
- We prove that $\Psi^{(n+1)}$ given by (28) converges towards the solution u (given by (18)) of the ill-problem (1), (3)-(5), as $n \to +\infty$, using a suitable choice of σ_m such that (35) holds (see also Remark 2.2). These results are stated in Theorem 2.2. The convergence $\Psi^{(n+1)} \to u$ is proved under the assumptions (35) and (36) on, respectively, the parameter γ_m and the datum f and G.

Theorem 2.1 (Convergence of the algorithm (24)–(25) towards the solution of the ill-posed problem (14)–(15))**.** *For each $m \in \mathbb{N}^*$, let us consider the sequence of solutions $\left(\psi_m^{(n)}\right)_n$ (given by (26)–(27)) of the problems (24) and (25), and let us denote*

$$\gamma_m := 1 - \frac{\sigma_m}{E_{\alpha,1}(\mu_m T^\alpha)}. \tag{31}$$

Assume that the parameters $(\sigma_m)_m$ are chosen such that

$$\gamma_m \in (-1, 1). \tag{32}$$

Then, for any but fixed m and for any $t \in [0, T]$

$$\psi_m^{(n+1)}(t) - \psi_m(t) \to 0, \quad \text{as} \quad n \to +\infty. \tag{33}$$

In addition to this, we have

$$\psi_m^{(n+1)}(t) - \psi_m(t) = \gamma_m^{n+1} \left(\frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \xi_m - E_{\alpha,1}(\mu_m t^\alpha) f_m + \frac{T E_{\alpha,2}(\mu_m T^\alpha) E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m \right). \tag{34}$$

Theorem 2.2 (Convergence of the sequence (28) towards the solution of the ill-posed problem (1), (3)-(5))**.** *Assume that the ill-posed problem (1), (3)-(5) has the solution given by (18), and let $\left(\Psi^{(n)}\right)_{n\in\mathbb{N}}$ be the sequence given by (28). Let γ_m be given by (31) and σ_m be chosen such that*

$$\gamma_m \in \left(-\frac{1}{s}, \frac{1}{s}\right). \tag{35}$$

for some $s > 1$ independent of m. Assume also that datum f and G satisfy

$$\sum_{m=1}^{\infty} \left(E_{\alpha,1}(\mu_m T^\alpha)\right)^2 (f_m)^2 + \sum_{m=1}^{\infty} \left(T E_{\alpha,2}(\mu_m T^\alpha)\right)^2 (G_m)^2 =: \Upsilon < +\infty. \tag{36}$$

Then the following error estimate holds, for all $n \in \mathbb{N}$,

$$\sup_{t\in[0,T]} \|\Psi^{(n+1)}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \le \frac{C}{s^{2(n+1)}} \left(\|\xi\|_{L^2(\Omega)}^2 + \Upsilon \right), \tag{37}$$

where $C \ge 0$ is a constant independent of n. So, we have the convergence in the $L^2(\Omega)$–norm and uniform in time,

$$\sup_{t\in[0,T]} \|\Psi^{(n+1)}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \to 0, \quad \text{as } n \to \infty. \tag{38}$$

Remark 2.2 (Sufficient conditions for the hypotheses (32) and (35)). *First, it is worth to mention that the hypothesis (35) yields (32), i.e. the hypothesis (35) is stronger than (32). Indeed, (35) is needed to obtain (37) and subsequently the uniform convergence (38), which is based on a uniform bound (with respect to m) for the ratio γ_m^{n+1} involved in the error (34). The following sufficient conditions yield the hypotheses (32) and (35):*

- *The hypothesis (32) of Theorem 2.1 can be reached for instance when* $\sigma_m = \sigma$ *for all* $m \in \mathbb{N}^*$ *and, recalling that* $\mu_1 < \mu_m$ *for all* $m \in \mathbb{N} \setminus \{0,1\}$,

$$0 < \sigma < 2E_{\alpha,1}(\mu_1 T^\alpha). \tag{39}$$

- *The hypothesis (35) of Theorem 2.2 can be reached when*

$$\left(1 - \frac{1}{s}\right) E_{\alpha,1}(\mu_m T^\alpha) < \sigma_m < \left(1 + \frac{1}{s}\right) E_{\alpha,1}(\mu_m T^\alpha). \tag{40}$$

Lemma 2.1. *Let* $\left(\psi_m^{(n)}\right)_{n\in\mathbb{N}}$ *be given by (26)–(27), which are the solutions of the problems (24)–(25) and the parameter* γ_m *be given by (31). Denoting* $r_m^{(n)} := \psi_m^{(n)}(T) - \sigma_m \psi_m^{(n)}(0)$, *then*

$$r_m^{(0)} = \gamma_m \xi_m - \frac{\sigma_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m, \tag{41}$$

$$r_m^{(n+1)} = \gamma_m^{n+2}\xi_m + \sigma_m f_m \sum_{j=1}^{n+1} \gamma_m^j - \frac{\sigma_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m \sum_{j=0}^{n+1} \gamma_m^j \quad \text{for } n \in \mathbb{N}, \tag{42}$$

and, if (32) holds, we have for any but fixed m,

$$r_m^{(n+1)} \to E_{\alpha,1}(\mu_m T^\alpha) f_m - \sigma_m f_m - T E_{\alpha,2}(\mu_m T^\alpha) G_m, \quad \text{as} \quad n \to \infty. \tag{43}$$

Proof. The expression (41) can be deduced from (26). Besides that, (27) leads to

$$\begin{aligned} r_m^{(n+1)} &= r_m^{(n)} + \sigma_m f_m - \frac{\sigma_m}{E_{\alpha,1}(\mu_m T^\alpha)} \left(r_m^{(n)} + \sigma_m f_m\right) - \frac{\sigma_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m \\ &= \gamma_m r_m^{(n)} + \sigma_m \gamma_m f_m - \frac{\sigma_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m \\ &= \gamma_m^{n+1} r_m^{(0)} + \left(\sigma_m \gamma_m f_m - \frac{\sigma_m T E_{\alpha,2}(\mu_m T^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} G_m\right) \sum_{j=0}^{n} \gamma_m^j. \end{aligned} \tag{44}$$

Gathering (44) and (41) yields the desired expression (42). Letting $n \to \infty$ in (42) and using (32) lead to the desired limit (43). This completes the proof of Lemma 2.1. □

Proof of Theorem 2.1

Letting $n \to \infty$ in (27), and using respectively (43) and (16) imply that, as $n \to \infty$,

$$\begin{aligned} \psi_m^{(n+1)}(t) \to & \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \left(E_{\alpha,1}(\mu_m T^\alpha) f_m - T E_{\alpha,2}(\mu_m T^\alpha) G_m\right) \\ & - \left(\frac{t E_{\alpha,2}(\mu_m t^\alpha) E_{\alpha,1}(\mu_m T^\alpha) - T E_{\alpha,2}(\mu_m T^\alpha) E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)}\right) G_m \\ & = E_{\alpha,1}(\mu_m t^\alpha) f_m - t E_{\alpha,2}(\mu_m t^\alpha) G_m = \psi_m(t), \end{aligned}$$

which proves (33). To prove (34), subtract (16) from (27) and use (42) to obtain

$$\psi_m^{(n+1)}(t) - \psi_m(t) = \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \left(r_m^{(n)} + \sigma_m f_m - E_{\alpha,1}(\mu_m T^\alpha) f_m + T E_{\alpha,2}(\mu_m T^\alpha) G_m\right)$$

$$= \frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \left[\gamma_m^{n+1} \xi_m + \left(\sigma_m \sum_{j=1}^{n} \gamma_m^j + \sigma_m - E_{\alpha,1}(\mu_m T^\alpha)\right) f_m + T E_{\alpha,2}(\mu_m T^\alpha) \gamma_m^{n+1} G_m\right].$$

The second term on the right hand side of the above expression can be written as

$$\sigma_m \sum_{j=1}^{n} \gamma_m^j + \sigma_m - E_{\alpha,1}(\mu_m T^\alpha) f_m = \sigma_m \gamma_m \frac{\gamma_m^n - 1}{\gamma_m - 1} + \sigma_m - E_{\alpha,1}(\mu_m T^\alpha) f_m$$

$$= \sigma_m - E_{\alpha,1}(\mu_m T^\alpha) \gamma_m^n f_m.$$

Gathering the last two relations yields the desired expression (34). □

Proof of Theorem 2.2

Using (34), the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the fact that $\frac{E_{\alpha,1}(\mu_m t^\alpha)}{E_{\alpha,1}(\mu_m T^\alpha)} \leq 1$ and recalling that u and $\Psi^{(n+1)}$ are, respectively, the solution of the problem (1), (3)-(5) and the functions defined by (28), we have

$$\|\Psi^{(n+1)}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \gamma_m^{2(n+1)} \left(\|\xi\|_{L^2(\Omega)}^2 + \Upsilon\right). \tag{45}$$

This together with the assumption (35) give (37). This completes the proof of Theorem 2.2. □

Remark 2.3 (Advantages of assumptions (36)). *The assumption (36) is given in terms of the Mittag-Leffler function, which is "relatively" not easy to check. However, using the estimates* [11, equations (2.15)–(2.16), Lemma 2.3], *it can be shown that under the conditions*

$$\frac{1}{\alpha^2} \sum_{m=1}^{\infty} \exp\left(2\mu_m^{\frac{1}{\alpha}} T\right) |f_m|^2 < \infty \quad \text{and} \quad \frac{1}{\alpha^2} \sum_{m=1}^{\infty} \mu_m^{-\frac{2}{\alpha}} \exp\left(2\mu_m^{\frac{1}{\alpha}} T\right) |G_m|^2 < +\infty. \tag{46}$$

the hypothesis (36) of Theorem 2.2 is satisfied. The advantage of the assumption (36) is that it is a condition on the given datum f and G, whereas the assumption used in [11, equation (3.26), Theorem 3.1] *is a condition on the unknown exact solution u. The assumption (36) does not only serve to get the desired convergence results of Theorem 2.2, but it also makes the series (18) convergent. From this point, (36) is a natural assumption. In case of noisy measurements replacing the exact Cauchy data in (4) and (5), these could be first mollified, see, for instance,* [10]*, before checking directly the assumption (36), or (46).*

3. Conclusions

An ill-posed fractional partial differential problem has been considered and a sequence of well-posed problems has been established. The formulation of the sequence was based on an iterative algorithm developed in [7, Section 2] for elliptic and parabolic classical (non-fractional) PDEs.

The convergence of solutions of this sequence towards the solution of the underlying problem has been proved. Restoring the stability of the inverse and ill-posed problem using regularization by choosing an appropriate stopping iteration number similar to that derived in [7, Theorem 2 and Section 2.2] for non-fractional elliptic PDEs, when the Cauchy data (4) and (5) is noisy, is deferred to a future work. Future work will also concern the computational implementation of the developed iterative algorithm.

Acknowledgement
A. Bradji would like to acknowledge the MCS team of LAGA Laboratory in USPN (Sorbonne Paris Nord University) and the EUR (École Universitaire de Recherche de Paris Nord en Mathématiques et Informatique).

References

[1] Bradji, A. (2020) A new analysis for the convergence of the gradient discretisation method for multidimensional time fractional diffusion and diffusion-wave equations, *Comput. Math. Appl.* **79** (2), 500-520.

[2] Bradji, A. and Lesnic, D. (2024) Analysis of direct mixed and inverse Cauchy problems associated to a fractional-order partial differential equation, *Fractional Calculus and Applied Analysis*, (submitted).

[3] Ervin, V. and Rop, J.P. (2006) Variational formulation for the stationary fractional advection dispersion equation, *Numer. Meth. Partial Diff. Eqns.* **22** (3), 558-576.

[4] Jiao, F. and Zhou, Y. (2011) Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.* **62** (3), 1181-1199.

[5] Jin, B. and Rundell, W. (2015) A tutorial on inverse problems for anomalous diffusion processes, *Inverse Problems* **31** (3), Article ID 035003, (40 Pages).

[6] Kian, Y. and Yamamoto, M. (2017) On existence and uniqueness of solutions for semilinear fractional wave equations, *Fract. Calc. Appl. Anal.* **20** (1), 117-138.

[7] Kozlov, V.A. and Maz'ya, V.G. (1990) On iterative procedures for solving ill-posed boundary value problems that preserve the differential equations, *Leningrad Math. J.* **1** (5), 1207–1228.

[8] Mera, N.S., Elliott, L., Ingham, D.B. and Lesnic, D. (2000) An iterative boundary element method for the solution of a Cauchy steady state heat conduction problem, *Computer Modeling Eng. Sci.* **1**, 101–106.

[9] Mera, N.S., Elliott, L., Ingham, D.B. and Lesnic, D. (2001) An iterative boundary element method for solving the one dimensional backward heat conduction problem, *Int. J. Heat Mass Transfer* **44**, 1937–1946.

[10] Murio D. A. (2007) Stable numerical solution of a fractional-diffusion inverse heat conduction problem, *Computers & Mathematics with Applications* **53**, 1492–1501.

[11] Nguyen Huy, T., Tran Dong, X., Nguyen Anh, T. and Lesnic, D. (2018) On the Cauchy problem for a semilinear fractional elliptic equation, *Appl. Math. Lett.* **83**, 80–86.

[12] Zhao, X. and An, F. (2016) The eigenvalues and sign-changing solutions of a fractional boundary value problem, *Adv. Continuous Discrete Models* **2016**, Article ID 109, (14 pages).