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Stability of polydisc slicing

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Abstract

We prove a dimension-free stability result for polydisc slicing due to Oleszkiewicz and Pełczyński. Intriguingly, compared to the real case, there is an additional asymptotic maximizer. In addition to Fourier-analytic bounds, we crucially rely on a self-improving feature of polydisc slicing, established via probabilistic arguments.

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1 | INTRODUCTION

The study of sections of convex bodies has a long and rich history. Many results about extremal sections and their stability are known (see the recent survey [40], and the references therein). An influential result of this type is Ball's cube slicing theorem from [4], which states that the hyperplane sections of the unit volume cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$ have volume bounded between 1 and $\sqrt{2}$ (the lower bound had been known earlier and goes back to the independent works [26] of Hadwiger and [27] of Hensley). Ball's upper bound famously gave a simple counterexample to the Busemann–Petty problem in dimensions $n \geqslant 10$ (see [5, 14, 24, 32]). For many other ensuing works, see, for instance, [2, 6, 8, 10, 29–31, 37, 38, 41, 43, 44, 49], as well as the comprehensive surveys [40, 50]. Both bounds for cube slicing are sharp, the lower one uniquely attained at hyperplanes orthogonal to the vectors e_i, $1 \leqslant i \leqslant n$, the upper bound uniquely attained at hyperplanes orthogonal to the vectors $e_i \pm e_j$, $1 \leqslant i < j \leqslant n$, where $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. However, only recently quantitative stability results have been developed: for every hyperplane $a^\perp$ in $\mathbb{R}^n$ orthogonal to the unit vector a in $\mathbb{R}^n$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$, we have

$$1 + \frac{1}{54}|a - e_1|^2 \leqslant \mathrm{vol}_{n-1}([-\tfrac{1}{2}, \tfrac{1}{2}]^n \cap a^\perp) \leqslant \sqrt{2} - 6 \cdot 10^{-5}\left|a - \frac{e_1 + e_2}{\sqrt{2}}\right|, \tag{1}$$

where here and throughout this paper $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. A *local* version of the upper bound has been established by Melbourne and Roberto in [36] (with applications in information theory), while the stated lower and upper bounds are from [16] (with the numerical value of the constant in the upper bound from [22], where it is instrumental in extending Ball's cube slicing to the ℓ_p balls for $p > 10^{15}$). Distributional stability of Ball's inequality has been very recently studied in [23].

The goal of this paper is to derive a complex analogue of (1). Across the areas in convex geometry, significant efforts have been made to extend many fundamental and classical results well-known from real spaces to complex ones. For example, see [3, 7, 9, 11, 12, 18, 19, 21, 30, 31, 33–35] (sometimes complex-counterparts turn out to be "easier," e.g., [28, 39, 46], but for certain problems, on the contrary, satisfactory results have been elusive, e.g., [48]). A counterpart of Ball's cube slicing in $\mathbb{C}^n$ was discovered by Oleszkiewicz and Pełczyński in [42]. Let $\mathbb{D}$ be the unit disc in the complex plane and let

$$\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D} = \{z \in \mathbb{C}^n,\ \max_{j \leqslant n} |z_j| \leqslant 1\}$$

be the polydisc in $\mathbb{C}^n$, the complex analogue of the cube. For $z, w \in \mathbb{C}^n$, we let as usual $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ be their standard inner product. Oleszkiewicz and Pełczyński proved that for every (complex) hyperplane $a^\perp = \{z \in \mathbb{C}^n, \langle z, a\rangle = 0\}$ orthogonal to the vector a in $\mathbb{C}^n$, we have

$$1 \leqslant \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leqslant 2. \tag{2}$$

Interestingly, this is in fact formally a *generalization* of Ball's result (see Szarek's argument in [42, Remark 4.4]). The lower bound is attained uniquely at hyperplanes orthogonal to the standard basis vectors e_j, $1 \leqslant j \leqslant n$, the upper one is attained uniquely at hyperplanes orthogonal to the vectors $e_j + e^{it}e_k$, $1 \leqslant j < k \leqslant n$, $t \in \mathbb{R}$. In this setting, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the standard embedding and vol is always Lebesgue measure on the appropriate subspace whose dimension is usually indicated in the lower script (as, for instance, here $a^\perp$ becomes a subspace in $\mathbb{R}^{2n}$ of real dimension $2n-2$). Note that, in particular, $\mathrm{vol}_{2n-2}(\mathbb{D}^{n-1}) = \pi^{n-1}$ (obtained as the canonical section $\mathbb{D}^n \cap (1, 0, \ldots, 0)^\perp$), which is the normalizing factor above. Thanks to the symmetries of $\mathbb{D}^n$ under the permutations of the coordinates as well as complex rotations along axes $z \mapsto (e^{it_1}z_1, \ldots, e^{it_n}z_n)$, it suffices to consider real nonnegative vectors with say nonincreasing components. The main result of this paper is the following dimension-free stability result that refines (2). It is natural to introduce the normalized section function,

$$A_n(a) = \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp), \qquad a \in \mathbb{R}^n,$$

so that $A_n(e_1) = \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^{n-1}) = 1$.

Theorem 1. *For $n \geqslant 2$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$, we have*

$$A_n(a) \geqslant 1 + \frac{1}{8}|a - e_1|^2, \tag{3}$$

as well as

$$A_n(a) \leqslant 2 - \min\left\{10^{-40}\left|a - \frac{e_1+e_2}{\sqrt{2}}\right|, \frac{1}{76}\sum_{j=1}^n a_j^4\right\}. \tag{4}$$

We do not try to optimize the numerical values of the constants involved (for the sake of clarity). Before we move to proof, several remarks are in place.

Remark 1. In contrast to the real case, the deficit term in our upper bound (4) is more complicated and features the minimum over two quantities: the distance to the *unique* extremizer and the ℓ_4 norm of a. The latter appears to account for the fact that

$$\lim_{n\to\infty} A_n\left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right) = 2.$$

In other words, curiously, polidysc slicing admits an additional *asymptotic* (Gaussian) extremizer $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp$, $n \to \infty$. In the real case,

$$\lim_{n\to\infty} \mathrm{vol}_{n-1}\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^n \cap \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)^\perp\right) = \sqrt{\frac{6}{\pi}} < \sqrt{2}.$$

Remark 2. Up to the absolute constants, (4) is sharp, in that the asymptotic behavior of the right-hand side as a function of the quantities involved $|a - \frac{e_1+e_2}{\sqrt{n}}|$ and $\sum_{j=1}^n a_j^4$ is best possible. Indeed, for the former quantity, consider vectors $a = (\sqrt{\frac{1}{2}+\epsilon}, \sqrt{\frac{1}{2}-\epsilon}, 0, \dots, 0)$ and note that, by combining (5) and Lemma 2, we get $A_n(a) = (\frac{1}{2}+\epsilon)^{-1} = 2 - \epsilon + O(\epsilon^2)$ as $\epsilon \to 0$, while the left-hand side is $2 - \Theta(\epsilon)$. For the latter quantity, testing with $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ gives the right-hand side of the order $2 - \Theta(\frac{1}{n})$, while $A_n(a) = \frac{1}{2}\int_0^\infty (\frac{2J_1(t/\sqrt{n})}{t/\sqrt{n}})^n t\,\mathrm{d}t$ (see Subsection 3.2) which, by using the power series expansion (the definition) of the Bessel function, $\frac{2}{t}J_1(t) = 1 - \frac{t^2}{8} + \frac{t^4}{3\cdot 2^6} + O(t^6)$, $t \to 0$, leads to $A_n(a) = 2 - \Theta(\frac{1}{n})$ as well, as $n \to \infty$.

Remark 3. The presence of the asymptotic extremizer also attests to the fact that bound (4) with a better term $\sum_{j=1}^n |a_j|^p$ with some $p < 4$ in place of $\sum_{j=1}^n a_j^4$ would not hold. Indeed, for $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$, $A_n(a) = 2 - \Theta(\frac{1}{n})$, as explained in Remark 2, whereas $\sum_{j=1}^n |a_j|^p = n^{1-p/2} \gg n^{-1}$, if $p < 4$.

2 | A SKETCH OF OUR APPROACH

The lower bound is established by quantifying a simple convexity argument leading to the main term (akin to the real case, as done in [16]).

For the upper bound, we principally follow the strategy developed in [16] (see also [22, section 5]). However, the presence of the asymptotic extremizer (see Remark 1) is a new obstacle. To wit, there are several entirely different arguments, depending on the hyperplane $a^\perp$ (in what follows we always assume as in Theorem 1 that a is a unit vector with nonnegative nonincreasing components). Here is a rough roadmap.

(a) When a is *close* to the extremizer $\frac{e_1+e_2}{\sqrt{2}}$, we reapply polydisc slicing in a lower dimension to a portion of a, which yields its self-improvement and gives a quantitative deficit (this is largely inspired by a similar phenomenon for Szarek's inequality from [47] discovered in [20]). This part crucially uses probabilistic insights put forward in [15–17] and perhaps constitutes the most subtle point of the whole analysis.
(b) When a has all coordinates *well* below $\frac{1}{\sqrt{2}}$, we employ Fourier-analytic bounds and quantitative versions of the Oleszkiewicz–Pełczyński integral inequality for the Bessel function. This results in the ℓ_4 norm quantifying the improvement *near* the asymptotic extremizer.
(c) When a has one coordinate around $\frac{1}{\sqrt{2}}$ and the others small, a is neither close to the extremizer $\frac{e_1+e_2}{\sqrt{2}}$, nor the Fourier-analytic bounds are applicable. We rely on probabilistic insights again and use a Berry–Esseen type bound.
(d) When a has a coordinate *barely* above $\frac{1}{\sqrt{2}}$, we use a Lipschitz property of the normalized section function and reduce the analysis to the previous cases.
(e) When a has a coordinate *well*-above $\frac{1}{\sqrt{2}}$, we use a projection argument.

3 | ANCILLARY RESULTS AND TOOLS

As in the proof we consider several cases that require different approaches and tools, this section that includes auxiliary results is split into several subsections.

3.1 | The role of independence

Our approach, to a large extent, relies on the following probabilistic formula for the volume of sections of the polydisc, obtained in [12] by Fourier-analytic means (see also [17] for a *direct* derivation): for every $n \geqslant 1$ and every *unit* vector a in $\mathbb{R}^n$, we have

$$A_n(a) = \mathbb{E}\left|\sum_{k=1}^{n} a_k \xi_k\right|^{-2}, \tag{5}$$

where $\xi_1, \xi_2, \ldots$ are independent random vectors uniform on the unit sphere S^3 in $\mathbb{R}^4$.

To leverage independence and rotational symmetry in (5), we note the following general observation.

Lemma 2. *Let $d \geqslant 3$ and let X and Y be independent rotationally invariant random vectors in $\mathbb{R}^d$. Then*

$$\mathbb{E}|X+Y|^{2-d} = \mathbb{E}\min\{|X|^{2-d}, |Y|^{2-d}\}.$$

In particular, in $\mathbb{R}^4$,

$$\mathbb{E}|X+Y|^{-2} = \mathbb{E}\min\{|X|^{-2}, |Y|^{-2}\}. \tag{6}$$

The special case of $d = 3$ appeared as [16, Lemma 6.6], whereas for the general case we follow the argument from [17, Remark 15] (see also Corollary 17 therein).

Proof of Lemma 2. Let ξ_1, ξ_2 be independent random vectors uniform on the unit sphere S^{d-1} in $\mathbb{R}^d$. By rotational invariance, X and Y have the same distributions as $|X|\xi_1$ and $|Y|\xi_2$. Conditioning on the values of the magnitudes $|X|$ and $|Y|$, it thus suffices to show that for every $a_1, a_2 \geqslant 0$, we have

$$\mathbb{E}|a_1\xi_1 + a_2\xi_2|^{2-d} = \min\{a_1^{2-d}, a_2^{2-d}\}.$$

By homogeneity and symmetry, this will follow from the special case of $a_1 = 1, a_2 = t \in (0,1)$. By rotational invariance, we have

$$h(t) = \mathbb{E}|\xi_1 + t\xi_2|^{2-d} = \mathbb{E}|e_1 + t\xi_2|^{2-d} = \frac{1}{\mathrm{vol}_{d-1}(S^{d-1})}\int_{S^{d-1}} |e_1 + t\xi|^{2-d}\mathrm{d}\xi,$$

(in the sense of the usual Lebesgue surface integral) and our goal is to argue that this equals 1 for all $0 < t < 1$. Let $F(x) = |x|^{2-d}$. On the sphere, for every $x \in S^{d-1}$, x is the outer-normal, hence the divergence theorem yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}} F(e_1 + t\xi)\mathrm{d}\xi &= \int_{S^{d-1}} \langle \nabla F(e_1 + t\xi), \xi\rangle \mathrm{d}\xi \\ &= \int_{B_2^d} \mathrm{div}_x(\nabla F(e_1 + tx))\mathrm{d}x \\ &= t\int_{B_2^d} (\Delta F)(e_1 + tx)\mathrm{d}x = 0\end{aligned}$$

as $\Delta F = 0$ ($e_1 + tx$ never vanishes for $x \in B_2^d, 0 < t < 1$). Noting that clearly $h(0) = 1$, this finishes the proof. $\square$

3.2 | Integral inequality

Another key ingredient is the Fourier-analytic expression for the section function,

$$A_n(a) = \frac{1}{2}\int_0^\infty \left(\prod_{j=1}^n \frac{2J_1(a_j t)}{a_j t}\right) t\mathrm{d}t \tag{7}$$

(see [42, (5)]) and, crucially, the resulting upper bound obtained from Hölder's inequality with $L_{a_j^{-2}}$ norms (this idea perhaps goes back to Haagerup's work [25]): for every $n \geqslant 1$ and every *unit* vector a in $\mathbb{R}^n$, we have

$$A_n(a) \leqslant 2\prod_{k=1}^n \Psi(a_k^{-2})^{a_k^2}, \tag{8}$$

where for $s > 0$,

$$\Psi(s) = \frac{s}{4}\int_0^\infty \left|\frac{2J_1(t)}{t}\right|^s t\mathrm{d}t. \tag{9}$$

Here $J_1(t) = \frac{t}{2}\sum_{k=0}^\infty \frac{(-1)^k}{2^{2k}k!(k+1)!}t^{2k}$ is the Bessel function (of the first kind) of order 1. As $J_1(t) = O(t^{-1/2})$ as $t \to \infty$ (see, e.g., [1, 9.2.1.]), $\Psi(s)$ is finite for all $s > \frac{4}{3}$ (for $s \leqslant \frac{4}{3}$, we let $\Psi(s) = \infty$, so that (8) formally holds).

Oleszkiewicz and Pełczyński's approach crucially relies on the fact that

$$\sup_{s\geqslant 2}\Psi(s) = 1,$$

and that the supremum is attained at $s = 2$ as well as when $s \to \infty$. Implicit in their proof of this subtle claim is the following quantitative version, crucial for us.

Lemma 3. *For the special function Ψ defined in (9), we have*

$$\Psi(s) \leqslant \begin{cases} 1 - \frac{1}{12}(s-2)^2, & 2 \leqslant s \leqslant \frac{8}{3}, \\ 1 - \frac{1}{151s}, & s > \frac{8}{3}. \end{cases} \tag{10}$$

Proof. When $2 \leqslant s \leqslant \frac{8}{3}$, we have

$$\Psi(s) \leqslant \frac{s}{2}e^{-\frac{s-2}{2}},$$

as showed in [42] (*Proof of Proposition 1.1 in Case (II)*, p. 290). It remains to apply an elementary bound to $v = \frac{s}{2} - 1 \in [0, \frac{1}{3}]$,

$$(v+1)e^{-v} \leqslant (v+1)(1 - v + \tfrac{v^2}{2}) = 1 - \tfrac{v^2}{2} + \tfrac{v^3}{2} \leqslant 1 - \tfrac{v^2}{3}.$$

When $s \geqslant \frac{8}{3}$, it is showed in [42] (*Proof of Proposition 1.1 in Case (I)*, p. 288) that

$$\begin{aligned}\Psi(s) &\leqslant 1 - \frac{1}{3s} + \frac{1}{3s^2} + \frac{8s}{3s-4}(60\pi^2)^{-s/4} \\ &= 1 - \frac{1}{s}\left(\frac{1}{3} - \frac{1}{3s} - \frac{8s^2}{3s-4}(60\pi^2)^{-s/4}\right).\end{aligned}$$

It remains to note that the function in the bracket is increasing in s on $[\frac{8}{3}, \infty)$, thus it is at least its value at $s = \frac{8}{3}$, which is greater than $\frac{1}{151}$. □

3.3 | Lipschitz property of the section function and complex intersection bodies

In perfect analogy to the real case, there is a complex analogue of the classical Busemann's theorem from [13] saying that $x \mapsto \frac{|x|}{\mathrm{vol}_{n-1}(K\cap x^\perp)}$ defines a norm on $\mathbb{R}^n$, if K is a symmetric convex body in $\mathbb{R}^n$.

Theorem 4 Koldobsky–Paouris–Zymonopoulou, [34]. *Let K be a complex symmetric convex body K in $\mathbb{C}^n$, that is K is a convex body in $\mathbb{R}^{2n}$ with $e^{it}z \in K$, whenever $z \in K$, $t \in \mathbb{R}$. Then the function*

$$z \mapsto \frac{|z|}{(\mathrm{vol}_{2n-2}(K \cap z^{\perp}))^{1/2}}$$

defines a norm on $\mathbb{C}^n$.

We use this result to establish a Lipschitz property of the section function A_n.

Lemma 5. *For unit vectors a, b in $\mathbb{R}^n$, we have*

$$|A_n(a) - A_n(b)| \leqslant 4\sqrt{2}|a-b|.$$

Proof. Let $K = (\frac{1}{\pi}\mathbb{D})^n$ be the volume 1 polydisc, so that $A_n(a) = \mathrm{vol}_{2n-2}(K \cap a^{\perp})$. Then, by Theorem 4, $N(a) = |a|A_n(a)^{-1/2}$ is a norm, thus for *unit* vectors a and b, we have

$$\begin{aligned} |A_n(a) - A_n(b)| = |N(a)^{-2} - N(b)^{-2}| &= \frac{N(a)+N(b)}{N(a)^2N(b)^2}|N(a)-N(b)| \\ &\leqslant \frac{N(a)+N(b)}{N(a)^2N(b)^2}N(a-b). \end{aligned}$$

By the definition of N, the right-hand side becomes

$$A_n(a)A_n(b)\frac{A_n(a)^{-1/2} + A_n(b)^{-1/2}}{A_n(a-b)^{1/2}}|a-b|$$

and using the polydisc slicing inequalities, that is $1 \leqslant A_n(x) \leqslant 2$ for every vector x, the result follows. □

3.4 | Berry–Esseen bound

Finally, we will employ a Berry–Esseen type bound with explicit constant for random vectors in $\mathbb{R}^4$. Recently, Raič has obtained such a result for an arbitrary dimension.

Theorem 6 (Raič [45]). *Let $X_1, \dots, X_n$ be independent mean 0 random vectors in $\mathbb{R}^d$ such that $\sum_{j=1}^n X_j$ has the identity covariance matrix. Let G be a standard Gaussian random vector in $\mathbb{R}^d$. Then*

$$\sup_A \left| \mathbb{P}\left(\sum_{j=1}^n X_j \in A\right) - \mathbb{P}(G \in A)\right| \leqslant (42d^{1/4} + 16)\sum_{j=1}^n \mathbb{E}|X_j|^3,$$

where the supremum is over all Borel convex sets in $\mathbb{R}^d$.

4 | PROOF OF THEOREM 1

In this section, we will present the proof of Theorem 1. We recall that a is assumed to be a unit vector in $\mathbb{R}^n$ such that $a_1 \geqslant a_2 \geqslant \dots \geqslant a_n \geqslant 0$.

We begin with a short proof of the lower bound (3). First note that using (5) and the convexity of $(\cdot)^{-1}$ (Jensen's inequality),

$$A_n(a) = \mathbb{E}\left(\left|\sum_{k=1}^n a_k\xi_k\right|^2\right)^{-1} \geqslant \left(\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^2\right)^{-1} = \left(\sum_{k=1}^n a_k^2\right)^{-1} = 1,$$

which gives the sharp lower bound without the error term. This of course can be easily improved upon (the same idea is used in the proof of [16, Theorem 6.1]). We let $Y = 2\sum_{k<l} a_k a_l \langle \xi_k, \xi_l\rangle$ so that

$$\left|\sum_{k=1}^n a_k\xi_k\right|^2 = 1 + Y.$$

We have an elementary inequality $(1+y)^{-1} \geqslant 1 - y + \frac{3}{4}y^2 - \frac{1}{4}y^3$, $y > -1$ (after simplifications, equivalent to $\frac{1}{4}y^2(y-1)^2 \geqslant 0$). This leads to the bound

$$A_n(a) \geqslant 1 - \mathbb{E}Y + \frac{3}{4}\mathbb{E}Y^2 - \frac{1}{4}\mathbb{E}Y^3.$$

Plainly, $\mathbb{E}Y = 0$ (by symmetry). Moreover, it was shown in the course of the proof of [16, Theorem 6.1] that $\mathbb{E}Y^3 \leqslant \mathbb{E}Y^2$ (the case $d = 4$ therein) and $\mathbb{E}Y^2 \geqslant \frac{1}{4}|a - e_1|^2$. These result in (3).

We move on to the upper bound (4). Its proof requires considering multiple cases dependent on the size of the two largest coordinates of the vector a.

For the convenience of the reader, we include the following pictorial guide to the proof.

4.1 | Two largest coordinates are close to $\frac{1}{\sqrt{2}}$: Local stability via self-improvement

We set

$$\delta(a) = \left|a - \frac{e_1 + e_2}{\sqrt{2}}\right|^2 = \left(a_1 - \frac{1}{\sqrt{2}}\right)^2 + \left(a_2 - \frac{1}{\sqrt{2}}\right)^2 + a_3^2 + \dots + a_n^2$$
$$= 2 - \sqrt{2}(a_1 + a_2).$$

When $n = 2$, from Lemma 2, we have

$$A_2(a) = \min\{a_1^{-2}, a_2^{-2}\} = a_1^{-2}$$

and we check that this is at most $2 - \sqrt{\delta(a)}$.

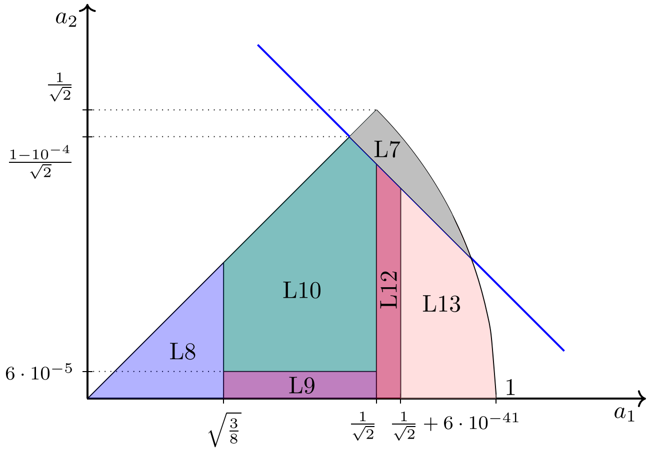

FIGURE 1 We consider six cases. The labels Lk correspond to the lemmas in which a given case is resolved. In Subsection 4.1, we explain the case where two largest coordinates are near $\frac{1}{\sqrt{2}}$, corresponding to L7 in the picture above. In Subsection 4.2, we explain the bound when all coordinates are below $\sqrt{3/8}$, that is, we cover the region L8. In Subsection 4.3, we study the case where a_1 is below $1/\sqrt{2}$, which we examine in two regimes depending on the value of a_2 corresponding to L9 and L10. We address the case when a_1 is only slightly above $\frac{1}{\sqrt{2}}$, marked as L12, in Subsection 4.4. Finally, in Subsection 4.5, we complete the picture by settling the case when a_1 is large (L13). We put these bounds together, proving the theorem, in Subsection 4.6.

One way to verify that $a_1^{-2} \leqslant 2 - \sqrt{\delta(a)}$ is to set $a_1 = \cos\theta$, $\theta \in [0, \frac{\pi}{4}]$, so that then $\delta(a) = 2 - \sqrt{2}(\cos\theta + \sin\theta) = 2 - 2\cos(\frac{\pi}{4} - \theta) = 4\sin^2(\frac{\pi}{8} - \frac{\theta}{2})$. Letting $t = \frac{\pi}{8} - \frac{\theta}{2} \in [0, \frac{\pi}{8}]$, we have $a_1 = \cos(\frac{\pi}{4} - 2t)$ and the desired inequality becomes $\frac{1}{\cos^2(\frac{\pi}{4} - 2t)} \leqslant 2(1 - \sin t)$. Moreover, $\cos^2(\frac{\pi}{4} - 2t) = \frac{1}{2}(\cos(2t) + \sin(2t))^2 = \frac{1}{2}(1 + \sin(4t))$, so it suffices to check that $(1 - \sin t)(1 + \sin(4t)) \geqslant 1$, $t \in [0, \frac{\pi}{8}]$. Note that on this interval, $\sin(4t) = 4\sin t\cos t\cos(2t) \geqslant 4\sin t\cos^2(2t) \geqslant 2\sin t$ and $(1 - \sin t)(1 + 2\sin t) = 1 + (1 - 2\sin t)\sin t \geqslant 1$, as $\sin t < \sin(\frac{\pi}{6}) = \frac{1}{2}$.

Hence, Theorem 1 holds when $n = 2$. We can assume from now on that $n \geqslant 3$.

Our goal here is to establish Theorem 1 for vectors a that are *near* the extremizer. This relies on a self-improving feature of the polydisc slicing result.

Lemma 7. *We have, $A_n(a) \leqslant 2 - \frac{1}{25}\sqrt{\delta(a)}$, provided that $\delta(a) \leqslant \frac{1}{5000}$.*

Proof. We let $X = a_1\xi_1 + a_2\xi_2$ and $Y = \sum_{j=3}^n a_j\xi_j$. Then, using (5), (6) and the concavity of $t \mapsto \min\{\alpha, t\}$, we obtain

$$A_n(a) = \mathbb{E}\min\{|X|^{-2}, |Y|^{-2}\} \leqslant \mathbb{E}_X \min\{|X|^{-2}, \mathbb{E}_Y|Y|^{-2}\}.$$

By polydisc slicing, $\mathbb{E}_Y|Y|^{-2} \leqslant \frac{2}{1-a_1^2-a_2^2}$. We thus get

$$A_n(a) \leqslant \mathbb{E}\min\left\{|X|^{-2}, \frac{2}{1 - a_1^2 - a_2^2}\right\} = \mathbb{E}|X|^{-2} - \mathbb{E}\left(|X|^{-2} - \frac{2}{1 - a_1^2 - a_2^2}\right)_+.$$

Using (6) again, we get that $\mathbb{E}|X|^{-2} = \min\{a_1^{-2}, a_2^{-2}\} = a_1^{-2}$.

It will be more convenient to work with the rotated variables

$$u_1 = \frac{a_1 + a_2}{\sqrt{2}}, \qquad u_2 = \frac{a_1 - a_2}{\sqrt{2}},$$

for which $u_1 = 1 - \frac{\delta(a)}{2} \in [1 - 10^{-4}, 1]$, $u_2 > 0$ and $u_1^2 + u_2^2 = a_1^2 + a_2^2 < 1$. Then, in terms of u_1, u_2, we have

$$\tfrac{1}{2}A_n(a) \leqslant \frac{1}{(u_1 + u_2)^2} - \mathbb{E}\left(\tfrac{1}{2}|X|^{-2} - \frac{1}{1 - u_1^2 - u_2^2}\right)_+.$$

Note also that

$$|X|^2 = a_1^2 + a_2^2 + 2a_1a_2\theta = u_1^2 + u_2^2 + (u_1^2 - u_2^2)\theta,$$

where θ is a random variable with density $\frac{2}{\pi}(1 - x^2)^{1/2}$ on $[-1, 1]$ (the distribution of $\langle \xi_1, \xi_2 \rangle$ that is the same as the one of $\langle \xi_1, e_1 \rangle$). We will use this representation in what follows.

Consider two cases:

Case 1: $u_1^2 + 9u_2^2 \geqslant 1$. We simply neglect the second term (the expectation), to obtain the upper bound of the form

$$\tfrac{1}{2}A_n(a) \leqslant \frac{1}{(u_1 + u_2)^2} \leqslant \frac{1}{\left(u_1 + \sqrt{\frac{1-u_1^2}{9}}\right)^2}.$$

Denoting for brevity $\delta = \delta(a) \in [0, \frac{1}{5000}]$ we crudely lower bound the denominator of the right-hand side,

$$u_1 + \sqrt{\frac{1 - u_1^2}{9}} = 1 - \frac{\delta}{2} + \sqrt{\frac{\delta}{18}\left(2 - \frac{\delta}{2}\right)} \geqslant 1 - \frac{\delta}{2} + \sqrt{\frac{\delta}{10}} \geqslant 1 + \frac{1}{2}\sqrt{\frac{\delta}{10}}.$$

Therefore,

$$A_n(a) \leqslant 2\left(1 + \frac{1}{2}\sqrt{\frac{\delta}{10}}\right)^{-2} \leqslant 2\left(1 - \frac{1}{2}\sqrt{\frac{\delta}{10}}\right) = 2 - \sqrt{\frac{\delta(a)}{10}},$$

where we used that $(1 + x)^{-2} \leqslant 1 - x$ holds for $x \in [0, \frac{1}{2}]$.

Case 2: $u_1^2 + 9u_2^2 \leqslant 1$. We use a more refined lower bound on the expectation, namely

$$\mathbb{E}\left(\tfrac{1}{2}|X|^{-2} - \frac{1}{1 - u_1^2 - u_2^2}\right)_+ \geqslant \mathbb{E}\left[\left(\tfrac{1}{2}|X|^{-2} - \frac{1}{1 - u_1^2 - u_2^2}\right)\mathbf{1}_{\left\{\frac{1}{2}|X|^{-2} \geqslant \frac{2}{1-u_1^2-u_2^2}\right\}}\right]$$

$$\geqslant \frac{1}{1-u_1^2-u_2^2}\mathbb{E}\left[\mathbf{1}_{\left\{\frac{1}{2}|X|^{-2}\geqslant\frac{2}{1-u_1^2-u_2^2}\right\}}\right]$$
$$= \frac{1}{1-u_1^2-u_2^2}\mathbb{P}\left(|X|^2 \leqslant \frac{1-u_1^2-u_2^2}{4}\right).$$

Recalling that $|X|^2 = u_1^2+u_2^2+(u_1^2-u_2^2)\theta$, the condition $|X|^2 \leqslant \frac{1-u_1^2-u_2^2}{4}$ becomes $\theta \leqslant \frac{1-5(u_1^2+u_2^2)}{4(u_1^2-u_2^2)} = -1+\theta_0$ with $\theta_0 = \frac{1-u_1^2-9u_2^2}{4(u_1^2-u_2^2)}$. Note that by our assumption $0<\theta_0$ and that $\theta_0<1$. Indeed, as $u_1>u_2$ and $5(u_1^2+u_2^2)\geqslant 5u_1^2 = 5(1-\delta/2)^2 \geqslant 5(1-10^{-4})^2 > 1$ we get that $-1+\theta_0 < 0$ and the claim follows.

Therefore, using that $\theta_0<1$ we estimate the probability of the event $|X|^2 \leqslant \frac{1-u_1^2-u_2^2}{4}$ by

$$\mathbb{P}(\theta \leqslant -1+\theta_0) = \frac{2}{\pi}\int_{-1}^{-1+\theta_0}\sqrt{1-x^2}\mathrm{d}x = \frac{2}{\pi}\int_0^{\theta_0}\sqrt{x(2-x)}\mathrm{d}x$$
$$\geqslant \frac{2}{\pi}\int_0^{\theta_0}\sqrt{x}\mathrm{d}x = \frac{4}{3\pi}\theta_0^{3/2}.$$

Putting this together and using the fact that $1-u_1^2-u_2^2 \leqslant 1-u_1^2$ and $u_1^2-u_2^2\leqslant 1$, we get

$$\frac{1}{2}A_n(a) \leqslant \frac{1}{(u_1+u_2)^2} - \frac{1}{1-u_1^2-u_2^2}\mathbb{P}\left(|X|^2\leqslant\frac{1-u_1^2-u_2^2}{4}\right)$$
$$\leqslant \frac{1}{(u_1+u_2)^2} - \frac{1}{1-u_1^2-u_2^2}\cdot\frac{4}{3\pi}\left(\frac{1-u_1^2-9u_2^2}{4(u_1^2-u_2^2)}\right)^{3/2}$$
$$\leqslant \frac{1}{(u_1+u_2)^2} - \frac{1}{6\pi}\frac{(1-u_1^2-9u_2^2)^{3/2}}{1-u_1^2}. \tag{11}$$

We claim that the right-hand side as a function of u_2 is decreasing. Indeed, its derivative equals

$$-2(u_1+u_2)^{-3} + \frac{9}{2\pi}\frac{u_2(1-u_1^2-9u_2^2)^{1/2}}{1-u_1^2} \leqslant -2(u_1+u_2)^{-3} + \frac{9}{2\pi}\frac{u_2}{\sqrt{1-u_1^2}}.$$

As $1-u_1^2 \geqslant 9u_2^2$, the second term is at most $\frac{3}{2\pi} < \frac{1}{2}$. Crudely, $u_1+u_2 = a_1\sqrt{2} < \sqrt{2}$, so the first term is at most $-2\sqrt{2}^{-3} = -\frac{1}{\sqrt{2}}$ and hence the derivative is negative. Setting $u_2=0$ in (11) thus gives

$$\frac{1}{2}A_n(a) \leqslant \frac{1}{u_1^2} - \frac{1}{6\pi}\sqrt{1-u_1^2} = \left(1-\frac{\delta}{2}\right)^{-2} - \frac{1}{6\pi}\sqrt{\frac{\delta}{2}\left(2-\frac{\delta}{2}\right)}$$
$$\leqslant 1+2\delta - \frac{1}{6\pi}\sqrt{1-\frac{1}{2}\cdot 10^{-4}}\sqrt{\delta},$$

where we have used $(1-x/2)^{-2} \leqslant 1+2x$, $0 \leqslant x \leqslant \frac{1}{2}$. As $\delta \leqslant \sqrt{\frac{1}{5000}}\sqrt{\delta}$, the right-hand side is at most

$$1+\left(\frac{2}{\sqrt{5000}}-\frac{1}{6\pi}\sqrt{1-\frac{1}{2}\cdot 10^{-4}}\right)\sqrt{\delta}<1-\frac{\sqrt{\delta}}{50}. \qquad \square$$

We note for future reference that the complementary case to the one considered in Lemma 7 is

$$\delta(a) \geqslant \frac{1}{5000}. \tag{12}$$

As $a_2 \leqslant \frac{a_1+a_2}{2} = \frac{1-\delta(a)/2}{\sqrt{2}}$, this in particular implies that a_2 is bounded away from $\frac{1}{\sqrt{2}}$,

$$a_2 \leqslant \frac{1-10^{-4}}{\sqrt{2}}. \tag{13}$$

4.2 | All weights are small

When all weights are small and bounded away from $\frac{1}{\sqrt{2}}$, we can rely on the Fourier analytic bound (8) because Lemma 3 guarantees savings across all weights. This case results with the term $\|a\|_4^4$ in (4) that quantifies the distance to the *asymptotic extremizer* $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$, $n \to \infty$.

Lemma 8. *We have, $A_n(a) \leqslant 2\exp\{-\frac{1}{151}\|a\|_4^4\}$, provided that $a_1 \leqslant \sqrt{\frac{3}{8}}$.*

Proof. By the assumption, $a_k^{-2} \geqslant \frac{8}{3}$ for all k, thus, using (8) and (10),

$$A_n(a) \leqslant 2\prod_{k=1}^{n}\Psi(a_k^{-2})^{a_k^2} \leqslant 2\prod_{k=1}^{n}\left(1-\tfrac{1}{151}a_k^2\right)^{a_k^2} \leqslant 2\exp\left\{-\tfrac{1}{151}\sum_{k=1}^{n}a_k^4\right\}. \qquad \square$$

4.3 | Largest weight is moderately below $\frac{1}{\sqrt{2}}$

Suppose that $a_1 = \frac{1}{\sqrt{2}}$. Then $\Psi(a_1^{-2}) = 1$ and the Fourier-analytic bound in the proof of Lemma 8 only gives that $A_n(a) \leqslant 2\exp\{-\frac{1}{151}\sum_{k=2}^{n}a_k^4\}$. When a_2 is bounded away from 0, this allows to conclude that $A_n(a)$ is bounded away from 2. Otherwise, we use the Gaussian approximation for $\sum_{k=2}^{n}a_k\xi_k$. A toy case illustrating why this works is the vector $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2(n-1)}}, \dots, \frac{1}{\sqrt{2(n-1)}})$ for large n. Then, if G denotes a standard Gaussian random vector in $\mathbb{R}^4$ independent of the ξ_j, the central limit theorem suggests that $A_n(a)$ is well-approximated by

$$\mathbb{E}\left|\frac{1}{\sqrt{2}}\xi_1+\frac{1}{\sqrt{2}}\frac{G}{2}\right|^{-2} = 2(1-e^{-2})$$

(for a computation of this expectation, see (14) below). Of course, to make this heuristics quantitative, we shall use a Berry–Esseen type bound, Raič's Theorem 6.

Thus, we brake the analysis now into two further subcases.

4.3.1 | Second largest weight is small

Lemma 9. *We have, $A_n(a) \leqslant 2 - 10^{-5}$, provided that $\sqrt{\frac{3}{8}} \leqslant a_1 \leqslant \frac{1}{\sqrt{2}}$ and $a_2 \leqslant 6 \cdot 10^{-5}$.*

Proof. We let $Y = \sum_{j=2}^{n} a_j \xi_j$ and observe that, by (5) and (6),

$$A_n(a) = \mathbb{E}\left|a_1\xi_1 + Y\right|^{-2} = \mathbb{E}\min\left\{a_1^{-2}, |Y|^{-2}\right\} = \int_0^{a_1^{-2}} \mathbb{P}\left(|Y|^{-2} > t\right)\mathrm{d}t.$$

Note that Y has covariance matrix $\frac{1-a_1^2}{4}\mathrm{Id}$. Therefore, using the Berry–Esseen bound from Theorem 6 (applied to $d = 4$ and $X_j = \frac{2}{\sqrt{1-a_1^2}} a_j\xi_j$, $j = 2, \ldots, n$),

$$\mathbb{P}\left(|Y|^{-2} > t\right) \leqslant \mathbb{P}\left(\left(\sqrt{\frac{1-a_1^2}{4}}|G|\right)^{-2} > t\right) + (42\sqrt{2} + 16)\sum_{j=2}^{n} \mathbb{E}\left|\frac{2}{\sqrt{1-a_1^2}} a_j\xi_j\right|^3,$$

where G denotes a standard Gaussian random vector in $\mathbb{R}^4$. As $|G|^2$ has density $\frac{x}{4}e^{-x/2}$, $x > 0$ ($\chi^2(4)$ distribution), we obtain

$$\begin{aligned}
\int_0^{a_1^{-2}} \mathbb{P}\left(\left(\sqrt{\frac{1-a_1^2}{4}}|G|\right)^{-2} > t\right)\mathrm{d}t &= \mathbb{E}\min\left\{a_1^{-2}, \left(\sqrt{\frac{1-a_1^2}{4}}|G|\right)^{-2}\right\} \\
&= \int_0^{\infty} \min\left\{a_1^{-2}, \frac{4}{1-a_1^2}\frac{1}{x}\right\}\frac{x}{4}e^{-x/2}\mathrm{d}x \\
&= \frac{1}{a_1^2}\left(1 - e^{-\frac{2a_1^2}{1-a_1^2}}\right).
\end{aligned} \tag{14}$$

Moreover, plainly,

$$\sum_{j=2}^{n} \mathbb{E}\left|\frac{2}{\sqrt{1-a_1^2}} a_j\xi_j\right|^3 = \frac{8}{(1-a_1^2)^{3/2}}\sum_{j=2}^{n} a_j^3 \leqslant \frac{8}{(1-a_1^2)^{3/2}} a_2 \sum_{j=2}^{n} a_j^2 = \frac{8a_2}{\sqrt{1-a_1^2}}.$$

Putting these together yields,

$$A_n(a) \leqslant \frac{1}{a_1^2}\left(1 - e^{-\frac{2a_1^2}{1-a_1^2}}\right) + \frac{8(42\sqrt{2}+16)a_2}{a_1^2\sqrt{1-a_1^2}}.$$

It can be checked that the first term is a decreasing function of a_1^2. Consequently, using $\frac{3}{8} \leqslant a_1^2 \leqslant \frac{1}{2}$ and $a_2 \leqslant 6 \cdot 10^{-5}$, we get

$$A_n(a) \leqslant \frac{8}{3}\left(1 - e^{-\frac{6}{5}}\right) + \frac{8(42\sqrt{2}+16)\cdot 6 \cdot 10^{-5}}{\frac{3}{8}\sqrt{\frac{1}{2}}} < 2 - 10^{-5}.$$

□

4.3.2 | Second largest weight is bounded away from 0

The goal here is to treat the case when a_2 is not *too* small.

Note that in the following lemma instead of assuming that (13) holds, we assume slightly less, that is, that $a_2 \leqslant \frac{1-10^{-5}}{\sqrt{2}}$. We will use this in Subsection 4.4.

Lemma 10. *We have, $A_n(a) \leqslant 2 - 10^{-19}$, provided that $\sqrt{\frac{3}{8}} \leqslant a_1 \leqslant \frac{1}{\sqrt{2}}$ and $6 \cdot 10^{-5} \leqslant a_2 \leqslant \frac{1-10^{-5}}{\sqrt{2}}$.*

Proof. Note that $\Psi(a_k^{-2}) \leqslant 1$ for each k, as guaranteed by (10) as $a_k^{-2} \geqslant 2$ for each k. Using this (for all k except $k = 2$) in conjunction with (5) gives

$$A_n(a) \leqslant 2\prod_{k=1}^{n} \Psi(a_k^{-2})^{a_k^2} \leqslant 2\Psi(a_2^{-2})^{a_2^2}.$$

Furthermore, again by (10),

$$\begin{aligned}\Psi(a_2^{-2}) &\leqslant 1 - \min\left\{\tfrac{1}{151}a_2^2,\ \tfrac{1}{12}(a_2^{-2}-2)^2\right\} \leqslant 1 - \min\left\{\tfrac{36}{151}10^{-10},\ \tfrac{1}{3}((1-10^{-5})^{-2}-1)^2\right\} \\ &= 1 - \frac{36}{151}\cdot 10^{-10}.\end{aligned}$$

Thus,

$$A_n(a) \leqslant 2\left(1 - \frac{36}{151}\cdot 10^{-10}\right)^{a_2^2} \leqslant 2\left(1 - \frac{36}{151}\cdot 10^{-10}a_2^2\right) < 2 - 10^{-19}.$$

□

Putting Lemmas 9 and 10 together yields the following corollary, needed in the sequel.

Corollary 11. *We have, $A_n(a) \leqslant 2 - 10^{-19}$, provided that $\sqrt{\frac{3}{8}} \leqslant a_1 \leqslant \frac{1}{\sqrt{2}}$ and $a_2 \leqslant \frac{1-10^{-5}}{\sqrt{2}}$.*

4.4 | Largest weight is moderately above $\frac{1}{\sqrt{2}}$

Lemma 12. *We have, $A_n(a) \leqslant 2 - 10^{-20}$, provided that $\frac{1}{\sqrt{2}} < a_1 \leqslant \frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}$ and (13).*

Proof. We consider the following modification of a, the vector

$$b = \left(\frac{1}{\sqrt{2}}, \sqrt{a_1^2 + a_2^2 - \frac{1}{2}}, a_3, \dots, a_n \right).$$

This is a unit vector with $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$ and

$$b_2^2 \leqslant \left(\frac{1}{\sqrt{2}} + 6 \cdot 10^{-41} \right)^2 + \left(\frac{1 - 10^{-4}}{\sqrt{2}} \right)^2 - \frac{1}{2} < \left(\frac{1 - 10^{-5}}{\sqrt{2}} \right)^2.$$

By Lemma 5 and Corollary 11 applied to b, we get

$$A_n(a) \leqslant A_n(b) + 4\sqrt{2}|a - b| \leqslant 2 - 10^{-19} + 8|a - b|.$$

As $\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 = \frac{a_1^2 - \frac{1}{2}}{\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} + a_2} \leqslant \sqrt{a_1^2 - \frac{1}{2}}$, we have

$$|a - b|^2 = \left(a_1 - \frac{1}{\sqrt{2}} \right)^2 + \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 \right)^2 \leqslant 2a_1 \left(a_1 - \frac{1}{\sqrt{2}} \right) < 10^{-40}$$

and, consequently,

$$A_n(a) \leqslant 2 - 10^{-19} + 8 \cdot 10^{-20} < 2 - 10^{-20}.$$

□

4.5 | Largest weight is bounded below away from $\frac{1}{\sqrt{2}}$

Lemma 13. *We have, $A_n(a) \leqslant 2 - 12\sqrt{2} \cdot 10^{-41}$, provided that $a_1 \geqslant \frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}$.*

Proof. Combining (5) and (6) applied to $X = a_1 \xi_1$ gives

$$A_n(a) \leqslant a_1^{-2} \leqslant 2(1 + 6\sqrt{2} \cdot 10^{-41})^{-2} \leqslant 2(1 - 6\sqrt{2} \cdot 10^{-41}),$$

where we used that $(1 + x)^{-2} \leqslant 1 - x$ for $x \leqslant \frac{1}{2}$.

□

4.6 | Putting things together

Proof of Theorem 1. Let us summarize what we proved. Without loss of generality, we assume that a is a unit vector such that $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \geqslant a_n \geqslant 0$. We considered several cases depending on the values of a_1 and a_2, which we illustrated on Figure 1 and which we discussed in Lemmas 7, 8, 9, 10, 12, 13. Putting them together, we get that

$$A_n(a) \leqslant 2 - \min\left(\frac{1}{25}\sqrt{\delta(a)}, \frac{2}{151}\|a\|_4^4, 10^{-5}, 10^{-19}, 10^{-20}, 12\sqrt{2}\cdot 10^{-41}\right)$$

Recall that $\delta(a) = |a - \frac{e_1+e_2}{\sqrt{2}}|^2$ is assumed to be at most $\frac{1}{5000}$ (Hence, $\frac{1}{\sqrt{2}}10^2\sqrt{\delta} < 1$). Therefore, we may rewrite this as

$$\begin{aligned} A_n(a) &\leqslant 2 - \min\left(\min\{\frac{1}{25}, \frac{1}{\sqrt{2}}10^{-3}, \frac{1}{\sqrt{2}}10^{-17}, \frac{1}{\sqrt{2}}10^{-18}, 12\cdot 10^{-39}\}\sqrt{\delta(a)}, \frac{1}{76}\|a\|_4^4\right) \\ &\leqslant 2 - \min\left(\frac{6}{5}10^{-40}\sqrt{\delta(a)}, \frac{1}{76}\|a\|_4^4\right), \end{aligned}$$

which finishes the proof. □

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