ANNALES
HENRI LEBESGUE

TIMOTHY GOWERS

KATARZYNA WYCZESANY

A COUNTEREXAMPLE TO A STRENGTHENING OF A QUESTION OF V. D. MILMAN

UN CONTRE-EXEMPLE À UNE VERSION RENFORCÉE D'UNE QUESTION DE V. D. MILMAN

ABSTRACT. — Let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^n$ and let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space. A subspace $Y \subset X$ is *strongly α-Euclidean* if there is a constant t such that $t|y| \leqslant \|y\| \leqslant \alpha t|y|$ for every $y \in Y$, and say that it is *strongly α-complemented* if $\|P_Y\| \leqslant \alpha$, where P_Y is the orthogonal projection from X to Y and $\|P_Y\|$ denotes the operator norm of P_Y with respect to the norm on X. We give an example of a normed space X of arbitrarily high dimension that is strongly 2-Euclidean but contains no 2-dimensional subspace that is both strongly $(1+\epsilon)$-Euclidean and strongly $(1+\epsilon)$-complemented, where $\epsilon > 0$ is an absolute constant. This property is closely related to an old question of Vitali Milman. The example is probabilistic in nature.

RÉSUMÉ. — Soit $|\cdot|$ la norme euclidienne standard sur $\mathbb{R}^n$ et soit $X = (\mathbb{R}^n, \|\cdot\|)$ un espace normé. Un sous-espace $Y \subset X$ est *fortement α-euclidien* s'il existe une constante t telle que $t|y| \leqslant \|y\| \leqslant \alpha t|y|$ pour tout $y \in Y$, et *fortement α-complémenté* si $\|P_Y\| \leqslant \alpha$, où P_Y est la projection orthogonale de X sur Y et $\|P_Y\|$ désigne la norme d'opérateur de P_Y par

rapport à la norme sur X. Nous donnons un exemple d'un espace normé X avec une dimension arbitrairement grande qui est fortement 2-euclidien mais ne contient pas de sous-espace à 2 dimensions qui soit à la fois fortement $(1 + \epsilon)$-euclidien et fortement $(1 + \epsilon)$-complémenté, où $\epsilon > 0$ est une constante absolue. Cette propriété est étroitement liée à une vieille question de Vitali Milman. L'exemple est de nature probabiliste.

1. Introduction

A famous theorem of Dvoretzky [Dvo61] which, in particular, solved an important conjecture posed by Grothendieck [Gro56], asserts that for every positive integer k and every $\epsilon > 0$ there exists a positive integer $n = n(\epsilon, k)$ such that every normed space of dimension at least n has a subspace Y of dimension k such that $d(Y, \ell_2^k) \leqslant 1 + \epsilon$, where d is the Banach–Mazur distance given by

$$d(X, W) = \inf \left\{ \|T\| \cdot \left\| T^{-1} \right\| \,\middle|\, T : X \to W \text{ is a linear isomorphism} \right\}.$$

This number has an intuitive geometric meaning: if $X = (\mathbb{R}^n, \| \cdot \|_X)$ and $W = (\mathbb{R}^n, \| \cdot \|_W)$ are normed spaces with $d(X, W) \leqslant d$, and K_X and K_W are the unit balls of X and W, respectively, then there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $K_X \subset AK_W \subset dK_X$.

A highly influential second proof of Dvoretzky's theorem was given by Vitali Milman [Mil71], which exploited measure concentration and led to many other arguments based on the same fundamental idea. In his paper, Milman established a sharp dependence of n on k. More precisely, he showed that $k \geqslant c\epsilon^2 \log n$, and demonstrated that in fact the conclusion of Dvoretzky's theorem holds for a random subspace (of appropriate dimension) with probability close to 1. For a comprehensive exposition we refer the reader to [AAGM15, Chapter 5].

Let us call an n-dimensional normed space X *C-Euclidean* if $d(X, \ell_2^n) \leqslant C$. A fairly straightforward use of Milman's method yields the following statement.

THEOREM 1.1 (V. D. Milman [Mil71]). — *For every $C > 1$ and every $\epsilon > 0$ there exists $c > 0$ such that for every $n \in \mathbb{N}$, every n-dimensional C-Euclidean normed space X has a subspace Y of dimension at least cn that is $(1 + \epsilon)$-Euclidean.*

In other words, under the additional hypothesis that X is C-Euclidean, one can obtain a linear dependence between the dimension of Y and the dimension of X.

There is a vast literature on finding "nice" subspaces of normed spaces under various conditions (see for example [AAGM15, BL89, FLM77, GM01, Lin92, LS08, Mil92, Sch13, STJ80]), but most of this literature pays little attention to how those subspaces sit in the main space. In particular, a desirable property for a subspace $Y \subset X$ is that it should be *complemented*. In an infinite-dimensional context, one says that Y is complemented if $Y = PX$ for a continuous projection P on X. In a finite-dimensional context, we need a more quantitative definition: Y is said to be α-complemented if $Y = PX$ for a projection P of operator norm at most α. In geometric terms this means that there exists a projection $P : X \to Y$ such that $PK_X \subset \alpha K_X \cap Y$.

There are several open problems about the existence of complemented subspaces. For example, it is not known whether there is a constant C such that for every k there exists n_0 such that for every $n \geqslant n_0$, every n-dimensional normed space has a C-complemented subspace of dimension at least k and codimension at least k. (For a partial result in this direction, see [STJ09].)

In this paper we consider the following question of Milman. To the best of our knowledge it has not appeared in print, except in a related paper of ours [GW22]. However, it has been promoted by Milman for many years in private correspondence with several mathematicians.

QUESTION 1.2. — *Let $k \in \mathbb{N}$, let $C \in \mathbb{R}$, and let $\epsilon > 0$. Does there exist $n \in \mathbb{N}$ such that every C-Euclidean normed space X of dimension at least n has a k-dimensional subspace Y that is $(1 + \epsilon)$-Euclidean and $(1 + \epsilon)$-complemented?*

We do not answer the question, but we give a negative answer to a question that is sufficiently close to Milman's to suggest very strongly that Milman's question has a negative answer.

To describe our result, we introduce two further definitions. We shall write $|\cdot|$ for the standard Euclidean norm on $\mathbb{R}^n$.

DEFINITION 1.3. — *A normed space $X = (\mathbb{R}^n, \|\cdot\|)$ is called* strongly α-Euclidean *if there is a constant $t > 0$ such that $t|x| \leqslant \|x\| \leqslant \alpha t|x|$ for every $x \in X$.*

DEFINITION 1.4. — *A subspace Y of $X = (\mathbb{R}^n, \|\cdot\|)$ is called* strongly α-complemented *if the orthogonal projection $P_Y : X \to Y$ has norm at most α, i.e. $\|P_Y x\| \leqslant \alpha \|x\|$ for all $x \in X$.*

Note that the first definition is stronger than being α-Euclidean, because instead of asking for any linear map T such that $|Tx| \leqslant \|x\| \leqslant \alpha |Tx|$, we ask for T to be a multiple of the identity. In other words, we require the unit ball K_X to satisfy $tB \subset K_X \subset \alpha t B$, where B is the Euclidean unit ball and not merely an ellipsoid. Also, the second definition is stronger than simply being α-complemented because we require the projection to be orthogonal with respect to the standard inner product on $\mathbb{R}^n$.

These are natural strengthenings to consider, in the light of the fact that Milman's proof of Theorem 1.1 begins by observing that without loss of generality X is strongly C-Euclidean and then proceeds to find a strongly $(1 + \epsilon)$-Euclidean subspace. Thus, one would expect Question 1.2 to have a positive answer if and only if the following question also has a positive answer.

QUESTION 1.5. — *Let $k \in \mathbb{N}$, let $C \in \mathbb{R}$, and let $\epsilon > 0$. Does there exist $n \in \mathbb{N}$ such that every strongly C-Euclidean normed space X of dimension at least n has a k-dimensional subspace Y that is strongly $(1 + \epsilon)$-Euclidean and strongly $(1 + \epsilon)$-complemented?*

Our main theorem is an example that shows that the answer to Question 1.5 is negative.

THEOREM 1.6. — *There exist constants $\epsilon > 0$ and $C \in \mathbb{R}$ such that for all sufficiently large $n \in \mathbb{N}$ there is an n-dimensional normed space that is C-Euclidean*

but contains no 2-dimensional subspace that is both strongly $(1 + \epsilon)$-Euclidean and strongly $(1 + \epsilon)$-complemented.

The rest of the paper is devoted to a proof of the above theorem. In the next section we will give some preliminary definitions and lemmas. We define the normed space in Section 3. In the final two sections we first outline the proof and then give it in detail.

2. Good vectors

We start with a definition that allows us to reformulate in a convenient way the condition that Y is strongly $(1 + \epsilon)$-Euclidean and strongly $(1 + \epsilon)$-complemented.

We write $\langle \cdot, \cdot \rangle$ for the standard inner product.

DEFINITION 2.1. — *Let $X = (\mathbb{R}^n, \| \cdot \|)$ be a normed space and let $x \in X$. We say that x is ϵ-good if*

$$\langle x, z \rangle \leqslant (1 + \epsilon) \frac{\|z\|}{\|x\|} |x|^2,$$

for every vector $z \in \mathbb{R}^n$.

Note that x is ϵ-good if and only if λx is ϵ-good for every $\lambda \neq 0$.

To see what this definition means geometrically, consider the orthogonal projection P_x on to the 1-dimensional subspace of $\mathbb{R}^n$ generated by x. Writing x' for the normalized vector $x/|x|$, this has the formula

$$P_x z = \langle x', z \rangle x'.$$

Hence, the operator norm of P_x (as a map from X to X) is the maximum of the quantity

$$\frac{\langle x', z \rangle \|x'\|}{\|z\|} = \frac{\langle x, z \rangle \|x\|}{|x|^2 \|z\|}$$

over all non-zero $z \in \mathbb{R}^n$. It follows that x is ϵ-good if and only if P_x has operator norm at most $1 + \epsilon$. This is also equivalent to saying that the subspace $\mathrm{Span}(x)$ is strongly $(1 + \epsilon)$-complemented, and hence that

$$P_x(K_X) \subset (1 + \epsilon) K_X \cap \mathrm{Span}(x).$$

We now show that a subspace Y of a space X is strongly $(1 + \epsilon)$-Euclidean and strongly $(1 + \epsilon)$-complemented for some small ϵ if and only if every $y \in Y$ is δ-good for some small δ.

LEMMA 2.2. — *Let $X = (\mathbb{R}^n, \| \cdot \|)$ be a normed space and let $Y \subset X$ be a subspace.*

   (1) *If Y is strongly $(1 + \epsilon)$-complemented and strongly $(1 + \epsilon)$-Euclidean, then every $y \in Y$ is $(2\epsilon + \epsilon^2)$-good.*
   (2) *If $\epsilon \leqslant 1/36\pi^2$ and every point in Y is ϵ-good, then Y is strongly $(1 + \epsilon)$-complemented and strongly $(1 + 3\pi\sqrt{\epsilon})$-Euclidean.*

Before we prove the statement, note that this characterization reduces Question 1.5 to the following question.

QUESTION 2.3. — *Let $\epsilon > 0$, $C \geqslant 1$ and $k \in \mathbb{N}$. Does there exist n such that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$ such that $|x| \leqslant \|x\| \leqslant C|x|$ for every $x \in \mathbb{R}^n$, then the space $(\mathbb{R}^n, \|\cdot\|)$ has a subspace Y of dimension k such that every $y \in Y$ is ϵ-good?*

Proof of Lemma 2.2. — Let P_Y be the orthogonal projection onto Y. If Y is strongly $(1 + \epsilon)$-Euclidean and strongly $(1 + \epsilon)$-complemented, then $\|P_Y x\| \leqslant (1+\epsilon)\|x\|$ for every $x \in X$ and there exists $\lambda \in \mathbb{R}$ such that $\lambda|y| \leqslant \|y\| \leqslant (1+\epsilon)\lambda|y|$ for every $y \in Y$. From this it follows that for every $y \in Y$ and every $x \in X$ we have

$$\langle y, x \rangle = \langle y, P_Y x \rangle \leqslant |y| \, |P_Y x| \leqslant |y| \frac{1}{\lambda} \|P_Y x\| \leqslant (1+\epsilon)\lambda \frac{|y|^2}{\|y\|} \frac{1}{\lambda} (1+\epsilon)\|x\|$$

$$= (1+\epsilon)^2 \frac{\|x\|}{\|y\|} |y|^2,$$

which implies that every point y in Y is $(2\epsilon + \epsilon^2)$-good, as claimed.

Conversely, assume that every point in Y is ϵ-good, so that for every $y \in Y$ and every $x \in X$ we have the inequality

$$\langle y, x \rangle \leqslant (1+\epsilon) \frac{\|x\|}{\|y\|} |y|^2.$$

Choose $x \in X$. Then $P_Y x \in Y$, so

$$|P_Y x|^2 = \langle P_Y x, P_Y x \rangle = \langle P_Y x, x \rangle \leqslant (1+\epsilon) \frac{\|x\|}{\|P_Y x\|} |P_Y x|^2,$$

and therefore $\|P_Y x\| \leqslant (1+\epsilon)\|x\|$. It follows that Y is strongly $(1+\epsilon)$-complemented.

Now assume for a contradiction that the subspace Y is not strongly $(1 + a)$-Euclidean with $0 < a$. In particular, this means that we can find two unit vectors $y, w \in Y$ such that $\|y\| = \|w\|(1 + a)$. Without loss of generality we may assume that $a \leqslant 1/2$.

Let us consider a sequence of unit vectors $w = x_0, x_1, \ldots, x_{m-1}, x_m = y$ that are equally spaced along the shortest arc that joins w to y (which is unique, since w cannot equal $-y$). By the pigeonhole principle there exists i such that

$$\|x_i\|(1+a)^{1/m} \leqslant \|x_{i+1}\|.$$

We shall choose m to ensure that x_i is a witness for x_{i+1} not being ϵ-good. Indeed, if we assume that m is at least $3\pi^2/a$ then since the angle between x_i and x_{i+1} is at most π/m we get that

$$\langle x_{i+1}, x_i \rangle \frac{\|x_{i+1}\|}{\|x_i\| \, |x_{i+1}|^2} \geqslant \cos\left(\angle x_i x_{i+1}\right)(1+a)^{1/m} \geqslant \left(1 - \frac{\pi^2}{2m^2}\right)\left(1 + \frac{a}{m} - \frac{a^2}{2m}\right)$$

$$\geqslant 1 + \frac{a}{m} - \frac{a^2 m^2 + \pi^2 m + \pi^2 a}{2m^3} \geqslant 1 + \frac{a}{2m}.$$

Here we used the fact that for $0 < \gamma < 1$ and $a > 0$ we have that $(1+a)^\gamma \geqslant 1 + \gamma a - \frac{\gamma(1-\gamma)}{2}a^2 \geqslant 1 + \gamma a - \frac{\gamma a^2}{2}$, and the assumptions that $0 < a \leqslant 1/2$ and $m \geqslant 3\pi^2/a$.

It follows that the point x_{i+1} is not $\frac{a}{2m}$-good. Therefore, if every point is ϵ-good, we must have that $\frac{a}{\lceil 6\pi^2/a \rceil} \leqslant \epsilon$, which implies that $a \leqslant 3\pi\sqrt{\epsilon}$. Thus, we find that Y is strongly $(1 + 3\pi\sqrt{\epsilon})$-Euclidean, which completes the proof of Lemma 2.2. $\square$

Next, we give an equivalent condition for a point to be ϵ-good for some small ϵ. Before we state the result, let us recall that a *support functional* of a norm $\|\cdot\|$ at x is any non-zero linear functional f such that for every y with $\|y\| \leqslant \|x\|$ we have $f(y) \leqslant f(x)$. Geometrically, if τ is the value of a support functional f at x such that $\|x\| = 1$, then $H = f^{-1}(\tau)$ is a hyperplane that is tangent to the unit ball of the norm $\|\cdot\|$ at x. Note that if the norm is differentiable, then writing $g(x)$ for $\|x\|$, we have that any multiple of $g'(x)$ is a support functional at x.

In the next proposition, we shall use the standard identification of $\mathbb{R}^n$ with its dual. That is, we identify a vector z with the linear functional $y \mapsto \langle y, z \rangle$.

PROPOSITION 2.4. — *Let $(X, \|\cdot\|)$ be a normed space and suppose that $|x| \leqslant \|x\| \leqslant C|x|$ for every $x \in X$. For every $\delta > 0$ there exists $\epsilon > 0$ such that if $x \in X$ is any ϵ-good point, then there exist y, z such that $|x| = |y|$, $|x - y| < \delta|x|$, z is a support functional at y and $|y - z| < \delta|x|$. Conversely, for every $\epsilon > 0$ there exists $\delta > 0$ such that if there exist y, z such that $|y - z| < \delta|x|$, z is a support functional at y and $|x - y| < \delta|x|$, then x is an ϵ-good point.*

Proof. — We shall do the second part first. Let $0 < \epsilon \leqslant 1$ and suppose that there exist y, z such that z is a support functional for y, and $|y - x|$ and $|z - y|$ are both at most $\delta|x|$.

Now let $w \in X$. Then

$$\langle w, x \rangle = \langle w, z \rangle + \langle w, y - z \rangle + \langle w, x - y \rangle \leqslant \langle w, z \rangle + 2\delta|w||x|.$$

But z is a support functional for y, so

$$\langle w, z \rangle \leqslant \|w\| \, \|z\|^* = \|w\| \frac{\langle y, z \rangle}{\|y\|}$$

We also have that

$$\|y\| \geqslant \|x\| - C|x - y| \geqslant \|x\| - C\delta|x| \geqslant (1 - C\delta)\|x\|.$$

Finally, since $|x - z| \leqslant 2\delta|x|$ we have that $|z| \leqslant (1 + 2\delta)|x|$, so

$$\langle y, z \rangle = \langle x, z \rangle + \langle y - x, z \rangle \leqslant |x||z| + \delta|x||z| \leqslant (1 + \delta)(1 + 2\delta)|x|^2.$$

Putting all this together, we find that

$$\langle w, x \rangle \leqslant \frac{(1 + \delta)(1 + 2\delta)}{1 - C\delta} \frac{\|w\|}{\|x\|}|x|^2 + 2\delta|w||x| \leqslant \left(\frac{(1 + \delta)(1 + 2\delta)}{1 - C\delta} + 2C\delta \right) \frac{\|w\|}{\|x\|}|x|^2.$$

It can be checked that if we set $\delta = \epsilon/5C$, then the factor in brackets is at most $1 + \epsilon$.

For the other direction, assume that for all y such that $|y| = |x|$ and $|x - y| < \delta|x|$ we have that $|y - z| > \delta|x|$, where z is the support functional at y, chosen such that $|z| = |y|$.

We can assume that $|x| = 1$ and that for every unit vector y with $|y - x| < \delta$, we have that $|y - z| \geqslant \delta$. It follows that

$$\langle y, z \rangle = 1 - |y - z|^2/2 \leqslant 1 - \delta^2/2$$

and therefore that

$$|z - \langle y, z \rangle y|^2 = 1 - \langle y, z \rangle^2 \geqslant \delta^2 - \delta^4/4,$$

which implies that the component of z orthogonal to y has size at least $\sqrt{3}\delta/2 \geqslant \delta/2$.

It follows that for any $\gamma < \delta$ we can find a path on the unit sphere that starts at x and ends at a point at distance at least γ from x such that the norm $\| \cdot \|$ decreases at a rate of at least $\delta/2$ along the path. This gives us a unit vector $\bar{y}$ such that $|\bar{y} - x| \leqslant \gamma$ and

$$\|\bar{y}\| \leqslant \|x\| - \gamma\delta/2 \leqslant \|x\|(1 - \gamma\delta/2C).$$

It follows that $\langle x, \bar{y} \rangle > 1 - \gamma^2/2$, so

$$\langle x, \bar{y} \rangle > \frac{(1 - \gamma^2/2)}{(1 - \gamma\delta/2C)} \frac{\|\bar{y}\|}{\|x\|} |x|^2.$$

Setting $\gamma = \delta/2C$, we deduce that x is not $\delta^2/8C^2$-good. $\qquad\square$

3. Definition of the norm and an important observation

The norm has a fairly simple definition. Let P be a random orthogonal projection of rank $n/2$ and let $A = I + P$. Then we define

$$(3.1) \qquad \|x\| = \langle x, Ax \rangle^{1/2} + \eta n^{-1/2}\|x\|_1,$$

where $\eta > 0$ is an absolute constant to be chosen later. (Note that $|x| \leqslant \|x\| \leqslant (\sqrt{2} + \eta)|x|$, so as long as $\eta \leqslant 2 - \sqrt{2}$, this norm is strongly 2-Euclidean.) The first part of this norm is a weighted ℓ_2 norm with respect to a random orthonormal basis, where half the weights are 2 and half are 1, and the second is a multiple of the standard ℓ_1 norm. Our aim now is to prove that with probability greater than zero (and in fact close to 1) there is no 2D subspace that consists entirely of ϵ-good points, for some absolute constant $\epsilon > 0$. That is, we will prove Theorem 1.6 and therefore give a negative answer to Question 1.5.

The next lemma tells us what the support functionals are at a vector x. Let us use the notation $\text{sign}(t)$ for the multivalued function from $\mathbb{R}$ to $\mathbb{R}$ that takes t to 1 if $t > 0$, to -1 if $t < 0$, and to any element of $[-1, 1]$ if $t = 0$. Then if $x \in \mathbb{R}^n$ we write $\text{sign}(x)$ for the result of applying the multivalued function sign pointwise. Let us also write $\|x\|_A$ for $\langle x, Ax \rangle^{1/2}$.

LEMMA 3.1. — *The support functionals at x are multiples of $\frac{Ax}{\|x\|_A} + \eta n^{-1/2} \text{sign}(x)$.*

Proof. — Given y such that $\|y\| \leqslant \|x\|$ we must show that $\langle \frac{Ax}{\|x\|_A} + \eta n^{-1/2} \operatorname{sign}(x), y \rangle$ can be bounded above by $\langle \frac{Ax}{\|x\|_A} + \eta n^{-1/2} \operatorname{sign}(x), x \rangle$. Indeed, we have that

$$\left\langle \frac{Ax}{\|x\|_A} + \eta n^{-1/2} \operatorname{sign}(x), y \right\rangle = \frac{\langle Ax, y \rangle}{\|x\|_A} + \eta n^{-1/2} \langle \operatorname{sign}(x), y \rangle \leqslant \|y\|_A + \eta n^{-1/2} \|y\|_1$$
$$= \|y\|$$

with equality if $y = x$, which finishes the proof. $\qquad\square$

COROLLARY 3.2. — *Let Q be such that $P + Q = I$, let ϵ be sufficiently small and let Y be a subspace that consists entirely of ϵ-good points. Then for every unit vector $x \in Y$ there exists a unit vector y with $|x - y| \leqslant \delta$ and a value of $\operatorname{sign}(y)$ such that*

$$d\left(\eta n^{-1/2} \operatorname{sign}(y), PY + QY \right) \leqslant 7\delta,$$

where δ tends to zero with ϵ.

Proof. — By Lemma 3.1 and Proposition 2.4 we find that if a unit vector x is ϵ-good, then there exists a unit vector y, a value of $\operatorname{sign}(y)$, and a scalar μ such that $|y - x| < \delta$ and

$$\left| \mu \left(\frac{Ay}{\|y\|_A} + \eta n^{-1/2} \operatorname{sign}(y) \right) - y \right| < \delta/3,$$

where $\delta > 0$ tends to zero with ϵ. Since $|Ay| \leqslant \sqrt{2}\|y\|_A$ and $n^{-1/2}|\operatorname{sign}(y)| \leqslant 1$, our assumption that $\sqrt{2} + \eta \leqslant 2$ implies that $\mu \geqslant (1 - \delta/3)/2$. For sufficiently small ϵ, we therefore have that $\mu \geqslant 1/3$.

Recalling that $A = I + P$ and rearranging, we obtain the inequality

$$(3.2) \qquad \left(\frac{1}{\mu} - \frac{2}{\|y\|_A} \right) Py + \left(\frac{1}{\mu} - \frac{1}{\|y\|_A} \right) Qy \approx_\delta \eta n^{-1/2} \operatorname{sign}(y),$$

where we write $u \approx_\delta v$ to mean that $|u - v| \leqslant \delta$. (We shall use this convenient notation throughout the rest of this paper.)

Finally, noting that and that $\max(|\frac{1}{\mu} - \frac{2}{\|y\|_A}|, |\frac{1}{\mu} - \frac{1}{\|y\|_A}|) \leqslant 3$ and that $|Py - Px| \leqslant |y - x| \leqslant \delta$ and the same for $|Qy - Qx|$, we get that

$$\left(\frac{1}{\mu} - \frac{2}{\|y\|_A} \right) Px + \left(\frac{1}{\mu} - \frac{1}{\|y\|_A} \right) Qx \approx_{7\delta} \eta n^{-1/2} \operatorname{sign}(y).$$

Therefore, we get that for every $x \in Y$ there is a unit vector y with $|x - y| \leqslant \delta$ and such that $d(\eta n^{-1/2} \operatorname{sign}(y), PY + QY) \leqslant 7\delta$, which finishes the proof. $\qquad\square$

Corollary 3.2 tells us, in particular, that if we have a 2-dimensional subspace Y that consists entirely of ϵ-good points, then every unit vector $x \in Y$ is close to a vector y such that $n^{-1/2} \operatorname{sign}(y)$ is close to the subspace $PY + QY$, which has dimension at most 4. This is the main observation we shall use to obtain a contradiction. (In fact, what we shall use is the slightly weaker approximation (3.2).)

4. Outline of the proof and some technical lemmas

Let us call a non-zero vector a *sign vector* if all its coordinates have the same absolute value. As before, we write X for $(\mathbb{R}^n, \|\cdot\|)$, though sometimes we abuse notation and use X to refer simply to the vector space $\mathbb{R}^n$.

In order to show that the norm defined in (3.1) indeed constitutes a counterexample to Question 1.5, that is, that there is no two-dimensional subspace of $(\mathbb{R}^n, \|\cdot\|)$ that consists entirely of ϵ-good points, we shall obtain a contradiction using more precise versions of the following statements.

(1) Every ϵ-good point is close to PX or QX.
(2) With high probability, no point that is close to PX or QX can be approximated by a point with only a few distinct coordinates.
(3) If Y is a 2-dimensional subspace that consists entirely of ϵ-good points, then for every $x \in Y$ there exists x' close to x such that $\mathrm{sign}(x')$ is close to the subspace $PY + QY$.
(4) Using the first two statements, we deduce that the vectors $\mathrm{sign}(x')$ are not approximately contained in a 4-dimensional subspace.

Corollary 3.2 is our precise version of Statement 3. Let us now prove Statement 1, which is also fairly simple.

LEMMA 4.1. — *Let x be an ϵ-good vector in $(X, \|\cdot\|)$, such that $|x| = 1$. Then either $d(x, PX) \leqslant 3\delta + 2\eta$ or $d(x, QX) \leqslant 3\delta + 2\eta$, where δ is given by Proposition 2.4.*

Proof. — From the beginning of the proof of Corollary 3.2 we obtain a unit vector y such that $|x - y| \leqslant \delta$ and such that

$$\frac{Ay}{\|y\|_A} + \eta n^{-1/2} \mathrm{sign}(y) \approx_\delta \lambda y,$$

where we have written λ for $1/\mu$ and used the fact that $\mu \geqslant 1/3$.

We have that

$$d(x, PX) \leqslant d(x, y) + d(y, PX) \leqslant \delta + |y - Py| = \delta + |Qy|$$

and similarly for $d(x, QX)$. Hence, our goal is to bound $\min\{|Py|, |Qy|\}$.

But $|Ay| \geqslant \|y\|_A$, and $\eta n^{-1/2} |\mathrm{sign}(y)| \leqslant \eta$, so $\lambda \geqslant 1 - \eta - \delta$ and

$$|Ay - \lambda\|y\|_A y| < (\delta + \eta)\|y\|_A \leqslant \sqrt{2}(\delta + \eta).$$

Thus, y is an approximate eigenvector of A and it remains to prove that an approximate eigenvector of A must be close to an eigenvector. (This is of course false for general linear maps.)

Since $P + Q = I$, we have $y = Py + Qy$ and $Ay = 2Py + Qy$, so if ν is any scalar, then

$$|Ay - \nu y|^2 = (2 - \nu)^2 |Py|^2 + (1 - \nu)^2 |Qy|^2.$$

Writing $2 - \nu = a + 1/2$ and $1 - \nu = a - 1/2$, one can rewrite the right-hand side as

$$\left(a + \frac{|Py|^2 - |Qy|^2}{2}\right)^2 + \frac{1}{4}\left(1 - \left(|Py|^2 - |Qy|^2\right)^2\right),$$

from which we see that if $|Ay - \nu y|^2 \leqslant \tau$ then

$$\left(|Py|^2 - |Qy|^2\right)^2 \geqslant 1 - 4\tau,$$

which implies that either $|Py|^2 \leqslant 2\tau$ or $|Qy|^2 \leqslant 2\tau$. In our case, we may set $\tau = 2(\delta + \eta)^2$, so $\min\{|Py|, |Qy|\} \leqslant 2(\delta + \eta)$, which gives us the bound stated. $\qquad\square$

Next, we formulate and prove a suitable version of Statement 2. We begin with a crude upper bound for the volume of the γ-expansion of the unit sphere of a subspace of dimension cn. (Much more accurate estimates exist, but for us a simple argument suffices.)

LEMMA 4.2. — *Let $m = cn$ and let $\gamma > 0$. Assume that $2^{n+1}\gamma \geqslant 1$. Then the probability that a random unit vector has distance at most γ from a given subspace $Y \subset X$ of dimension m is at most $24^n \gamma^{n-m}$.*

Proof. — A spherical cap in S^{n-1} of Euclidean radius 2γ has volume at most $(4\gamma)^{n-1} \leqslant 8^n \gamma^n = (8\gamma)^n$. By standard estimates, we can also find a γ-net of Y of cardinality at most $(3/\gamma)^m$. But every point of S^{n-1} that is within γ of Y is within 2γ of a point in the γ-net, and from this the result follows. $\qquad\square$

LEMMA 4.3. — *Let k be a positive integer, let $c, \gamma > 0$, and let Y be a random subspace of ℓ_2^n of dimension m. Then the probability that $Y_\gamma = \{x \in \ell_2^n : d(x, Y) \leqslant \gamma\}$ contains a unit vector x with at most k distinct coordinates is at most $(3/\gamma)^k (48k)^n \gamma^{n-m}$.*

Proof. — The number of partitions of $\{1, 2, \ldots, n\}$ into k sets is at most k^n, and for each partition $E_1, \ldots, E_k$ the set of vectors that are constant on each E_i is a k-dimensional subspace, so there is a γ-net of the unit sphere of this subspace of size at most $(3/\gamma)^k$.

The probability that Y_γ contains a vector with at most k distinct coordinates is at most the probability that $Y_{2\gamma}$ contains a point in one of these γ-nets, which is at most

$$(3/\gamma)^k (24k)^n (2\gamma)^{n-m} \leqslant (3/\gamma)^k (48k)^n \gamma^{n-m}$$

by Lemma 4.2 and a union bound. $\qquad\square$

We present one more technical lemma that is similar to Lemma 4.3, and which will be an important part of the argument. Again we make no attempt to optimize bounds.

LEMMA 4.4. — *Let Z be a random subspace of ℓ_2^n of dimension m. Then the probability that Z_γ contains a unit vector with support size at most r is at most $288^n \gamma^{n-m-r}$.*

Proof. — The number of sets of size at most r is $\binom{n}{r} \leqslant 2^n$. For each such set E the size of a γ-net of the unit sphere of the space of vectors supported on E is at most $(3/\gamma)^r$, and for each point in such a net the probability that it is in $Z_{2\gamma}$ is at most $24^n (2\gamma)^{n-m}$. Therefore, the probability we wish to bound is at most

$$2^n (3/\gamma)^r 24^n (2\gamma)^{n-m} \leqslant 288^n \gamma^{n-m-r},$$

which proves the lemma. $\qquad\square$

The key point we shall need from the above lemma is that for any $c > 0$ there exists $\gamma > 0$ such that if $n - m - r \geqslant cn$, then the probability that Z_γ contains a point with small support is small. In particular, we have the following statement.

COROLLARY 4.5. — *Let $\gamma = 2^{-37}$. Let $X = \ell_2^n$ with $n \geqslant 2$, and let $P : X \to X$ be a random orthogonal projection of rank $n/2$ and let $Q = I - P$. Then the probability that either $PX_{2\gamma}$ or $QX_{2\gamma}$ contains a unit vector of support size at most $n/4$ is at most $(\frac{2}{3})^n$.*

Proof. — Applying Lemma 4.4, we find that the probability that $PX_{2\gamma}$ contains a vector of support size at most $n/4$ is at most $288^n (2\gamma)^{n/4} = (288/512)^n$. The same is true of $QX_{2\gamma}$ and the result follows with room to spare. $\qquad\square$

For the remainder of the paper, we shall assume that P has been chosen in such a way that neither $PX_{2\gamma}$ nor $QX_{2\gamma}$ contains a vector of support size at most $n/4$, and neither PX_γ nor QX_γ contains a vector with at most five distinct coordinates. By Lemma 4.3 and Corollary 4.5 such a P exists.

5. The set of signs cannot be squeezed into a 4-dimensional subspace

Before we move to the heart of the argument, which will be a precise version of Statement 4, let us remark that as we move forward we shall be dealing with many parameters. Since we do not wish to choose them straight away we make sure that it is easy to keep track of all the dependencies by stating them clearly and giving each one a label.

If Y is a 2-dimensional subspace that consists entirely of ϵ-good points, then the equation (3.2) from the proof of Corollary 3.2 gives us for each unit vector $x \in Y$ a unit vector y with $|x - y| \leqslant \delta$ and coefficients α_y and β_y such that

$$(5.1) \qquad n^{-1/2} \operatorname{sign}(y) \approx_{\delta/\eta} \alpha_y Py + \beta_y Qy,$$

where $\alpha_y = \eta^{-1}(\lambda - 2/\|y\|_A)$ and $\beta_y = \eta^{-1}(\lambda - 1/\|y\|_A)$.

Recall also from the beginning of the proof of Lemma 4.1 that we also have the equivalent formula

$$\eta n^{-1/2} \operatorname{sign}(y) \approx_\delta \lambda y - \frac{Ay}{\|y\|_A},$$

from which it follows that $|\lambda| \in [1 - \delta - \eta, \sqrt{2} + \delta + \eta]$, since $\|y\|_A \leqslant |Ay| \leqslant \sqrt{2}\|y\|_A$. Therefore, provided that

$$(5.2) \qquad \delta + \eta \leqslant 2 - \sqrt{2}$$

it follows that $\lambda \in [\sqrt{2} - 1, 2]$. In particular, it follows that $|\alpha_y|$ and $|\beta_y|$ are at most $2\eta^{-1}$. This bound will be important later. Suppose now that

$$(5.3) \qquad 3\delta + 2\eta \leqslant \gamma.$$

If $x \in Y$ is a unit vector, then by Lemma 4.1, either $d(x, PX)$ or $d(x, QX)$ is at most $3\delta + 2\eta$. Therefore, for every $y \in Y_\gamma$, either $d(y, PX)$ or $d(y, QX)$ is at most

$3\delta + 2\eta + \gamma \leqslant 2\gamma$, so by the assumption made at the end of the previous section, y has support size at least $n/4$. That is, every vector in Y_γ has support size at least $n/4$. We shall use this property frequently in the rest of the section.

We will now use the fact that the subspace Y is 2-dimensional to get a convenient re-parametrization of the unit sphere of Y. Indeed, there exist non-negative real numbers $r_1, \ldots, r_n$ and phases $\phi_1, \ldots, \phi_n \in [0, 2\pi)$ such that each unit vector in Y is equal to

$$x = x(\theta) = \Big(r_1 \sin(\theta + \phi_1), \ldots, r_n \sin(\theta + \phi_n) \Big)$$

for some $\theta \in [0, 2\pi)$. Note that by looking at $\mathbb{E}_\theta\, |x(\theta)|^2$ we find that $\sum_i r_i^2 = 2$.

LEMMA 5.1. — *Let $\delta, \xi > 0$ x and y be two vectors in $\mathbb{R}^n$ such that $|x - y| \leqslant \delta$. Then the number of i such that $|x_i| \geqslant \xi n^{-1/2}$ and $\mathrm{sign}(x_i) \neq \mathrm{sign}(y_i)$ is at most $\xi^{-2}\delta^2 n$.*

Proof. — For each such i we have that $|x_i - y_i|^2 \geqslant \xi^2 n^{-1}$, and our hypothesis is that $\sum_i |x_i - y_i|^2 \leqslant \delta^2$. $\square$

Let $\alpha > 0$ be a constant to be chosen later, let $E = \{i : r_i \geqslant \alpha n^{-1/2}\}$, and write P_E for the coordinate projection onto E. It will also be convenient to write E_0 for $\{1, 2, \ldots, n\} \setminus E$, which we think of as the set of coordinates where Y almost vanishes.

LEMMA 5.2. — *Let $0 < c, \xi < 1$ and let θ be chosen uniformly at random from $[0, 2\pi)$. Then with probability at least $1 - \xi/\alpha c$, the number of $i \in E$ such that $|x(\theta)_i| < \xi n^{-1/2}$ is less than $c|E|$.*

Proof. — We will use Markov's inequality to upper bound the probability of the complementary event. In order to estimate the expected number of $i \in E$ such that $|x(\theta)_i| < \xi n^{-1/2}$ recall that since $r_i \geqslant \alpha n^{-1/2}$ for every $i \in E$, and $x(\theta)_i = r_i \sin(\theta + \phi_i)$, we have that for each $i \in E$,

$$\mathbb{P}\Big[|x(\theta)_i| < \xi n^{-1/2}\Big] \leqslant \mathbb{P}\Big[|\sin(\theta + \phi_i)| < \xi/\alpha\Big].$$

Further, note that $|\sin(\theta + \phi)|$ on the interval $[0, 2\pi)$ has the same distribution as $\sin \theta$ on the interval $[0, \pi/2)$. But if θ belongs to that interval, then $\sin \theta \geqslant 2\theta/\pi$ with equality only at 0 and $\pi/2$, so $\sin \theta < \xi/\alpha$ only if $\theta < \pi\xi/2\alpha$, which is true with probability less than ξ/α. Therefore, the expected number of $i \in E$ such that $|x(\theta)_i| < \xi n^{-1/2}$ is less than $\xi|E|/\alpha$.

It follows that

$$\mathbb{P}\Big[\big| \big\{i \in E : |x(\theta)_i| < \xi n^{-1/2}\big\}\big| \geqslant c|E|\Big] \leqslant \frac{\xi|E|\alpha^{-1}}{c|E|} = \frac{\xi}{\alpha c},$$

which, by taking the complement, completes the proof. $\square$

We will now introduce a notion of a "typical" θ (and $x(\theta)$).

DEFINITION 5.3. — *For choices of ξ and c that we shall make later, we call θ and $x(\theta)$* typical *if*

   (i) *the number of $i \in E$ such that $|x(\theta)_i| < \xi n^{-1/2}$ is less than $c|E|$ (that is, the conclusion of Lemma 5.2 holds), and*

(ii) *there is no $i \in E$ with $\theta + \phi_i \in \{0, \pi\}$.*

Note that the second condition, which is there for convenience, holds with probability 1 and ensures that $\mathrm{sign}(x(\theta))_i \in \{-1, 1\}$ for every $i \in E$. Later we shall want to be sure that typical vectors exist, for which Lemma 5.2 tells us that a sufficient condition is the inequality

$$(5.4) \qquad\qquad\qquad \xi < \alpha c.$$

Let us now define Σ to be the set of all vectors of the form $n^{-1/2} P_E \, \mathrm{sign}(x(\theta))$ such that $x(\theta)$ is a typical element of the unit sphere of Y. Let $\beta > 0$ be another parameter to be chosen later, and let $V \subset \Sigma$ be the maximal centrally symmetric β-separated subset of Σ. Assume that V consists of $2k$ vectors (that is k antipodal pairs) and note that V is a β-net of Σ.

We will now show, assuming that

$$(5.5) \qquad\qquad\qquad \beta \leqslant 1,$$

that the set V is non-empty.

LEMMA 5.4. — *Let Y be a 2-dimensional subspace that consists entirely of ϵ-good points and let $\beta \leqslant 1$. Then every maximal centrally symmetric β-separated subset V of*

$$V \subset \Sigma = \left\{ n^{-1/2} P_E \, \mathrm{sign}(x(\theta)) : \ x(\theta) \text{ is a typical unit vector in } Y \right\},$$

has cardinality at least 2.

Proof. — Since Y is a subspace that consists entirely of ϵ-good points we have that $Y \subset PX_\gamma$ or $Y \subset QX_\gamma$ and we chose P in such a way that both $PX_{2\gamma}$ and $QX_{2\gamma}$ do not contain a vector of support size at most $n/4$ (recall the last paragraph in Section 4). It follows that every vector in Y, and even in Y_γ, has support size at least $n/4$. Recall that, for any vector, we denote its set of "large" coordinates by $E = \{i : r_i \geqslant \alpha n^{-1/2}\}$ and note that for every $y \in Y$ we have $|y - P_E y| < \alpha$ and, clearly, $P_E y$ has support size $|E|$. Assuming that

$$(5.6) \qquad\qquad\qquad \alpha \leqslant \gamma,$$

it follows that for every $y \in Y$ the set of large coordinates E has cardinality at least $n/4$.

Thus, Σ consists of vectors whose norm is at least $n^{-1/2}|E|^{1/2} \geqslant 1/2$ (and at most 1), so as long as $\beta \leqslant 1$, which we assumed, we can choose any typical vector $x(\theta)$, and let $V = \{\pm n^{-1/2} P_E \, \mathrm{sign}(x(\theta))\}$, and thereby obtain a β-separated subset of Σ. This proves the result. $\qquad\square$

Since V is centrally symmetric it has even cardinality. Let this cardinality be $2k$. We now consider three cases.

5.1. Case 1: $k = 1$.

Let $\zeta = \xi/\alpha c$ and let $V = \{v, -v\}$. Since θ is typical with probability at least $1 - \zeta$, every closed interval of length greater than $2\pi\zeta$ contains a typical θ. If

$n^{-1/2}P_E \operatorname{sign}(x(\theta)) \approx_\beta v$, then $n^{-1/2}P_E \operatorname{sign}(-x(\theta)) \approx_\beta -v$, so there is at least one θ such that $n^{-1/2}P_E \operatorname{sign}(x(\theta)) \approx_\beta v$ and one such that $n^{-1/2}P_E \operatorname{sign}(x(\theta)) \approx_\beta -v$.

Let A be the set of typical $x(\theta)$ such that $n^{-1/2}P_E \operatorname{sign}(x(\theta)) \approx_\beta v$ and note that $-A$ is the set of typical $x(\theta)$ such that $n^{-1/2}P_E \operatorname{sign}(x(\theta)) \approx_\beta -v$. Note also that every typical $x(\theta)$ belongs to A or $-A$, and that therefore neither A nor $-A$ is empty. Writing B for the set of points at distance at most $\pi\zeta$ from A, we also have that B and $-B$ are closed and that $B \cup -B$ is the entire unit circle of Y. To see the last assertion, let $x(\theta_0)$ be a point in the unit circle of Y. Then the closed interval of length $2\pi\zeta$ centred at θ_0 contains a typical point θ, so $x(\theta)$ belongs to $A \cup (-A)$ and therefore $x(\theta_0) \in B \cup -B$, as the distance from $x(\theta_0)$ to $x(\theta)$ is at most $\pi\zeta$. Since the unit circle is connected, $B \cap -B$ is non-empty, from which it follows that there exist typical unit vectors $x, y \in Y$ such that $|x - y| \leqslant 2\pi\zeta$ and such that $n^{-1/2}P_E \operatorname{sign}(x) \approx_\beta v$, and $n^{-1/2}P_E \operatorname{sign}(y) \approx_\beta -v$.

It follows that $n^{-1/2}P_E \operatorname{sign}(x)$ differs from v in at most $\beta^2 n/4$ coordinates, and $n^{-1/2}P_E \operatorname{sign}(y)$ differs from $-v$ in at most $\beta^2 n/4$ coordinates. Therefore, $P_E \operatorname{sign}(x)$ and $P_E \operatorname{sign}(y)$ are equal in at most $\beta^2 n/2$ coordinates. Moreover, by Lemma 5.1, the number of i for which $\operatorname{sign}(x_i) \neq \operatorname{sign}(y_i)$ and $|x_i| \geqslant \rho n^{-1/2}$ is at most $\rho^{-2}(2\pi\zeta)^2 n = (\frac{2\pi\xi}{\alpha c\rho})^2 n$.

From these two facts it follows that the number of coordinates $i \in E$ for which $|x_i| \geqslant \rho n^{-1/2}$ is at most $((\frac{2\pi\xi}{\alpha c\rho})^2 + \frac{\beta^2}{2})n$. Therefore, we find that x has distance at most ρ from a vector of support size at most $((\frac{2\pi\xi}{\alpha c\rho})^2 + \frac{\beta^2}{2})n$.

If we choose parameters in such a way that

$$(5.7) \qquad\qquad\qquad \rho \leqslant \gamma$$

and

$$(5.8) \qquad\qquad\qquad \left(\frac{2\pi\xi}{\alpha c\rho}\right)^2 + \frac{\beta^2}{2} < 1/4$$

we obtain a contradiction with the fact that Y_γ does not contain a vector of support size less than $n/4$.

5.2. Case 2: $2 \leqslant k \leqslant 4$

Let $V = \{\pm v_1, \dots, \pm v_k\}$. Since each v_i is of the form $n^{-1/2}P_E \operatorname{sign}(x(\theta))$ for some typical vector $x(\theta) \in Y$, it takes values $\pm n^{-1/2}$ in E.

We now show that either this case can be reduced to the case $k = 1$ with β replaced by 48β or there is a subset V' of V consisting of at least two antipodal pairs such that V' is a 3κ-separated κ-net of Σ and $\beta \leqslant \kappa \leqslant 16\beta$.

Since V is a β-net of Σ, then if it is 3β separated then we are done. If not, we can find $i \neq j$ such that $|v_i - v_j| \leqslant 3\beta$. Then we can remove $\pm v_j$ from V and we will still have a 4β-net. Similarly, if $V' = V \setminus \{\pm v_j\}$ is 12β-separated we are done, but if it contains two distinct elements v_i, v_j such that $|v_i - v_j| \leqslant 12\beta$, then again we can remove $\pm v_j$ and we will still have a 16β-net. Finally, if there are two distinct elements v_i, v_j with $|v_i - v_j| \leqslant 48\beta$, then we may remove $\pm v_j$ and end up with V' of

the form $\{v, -v\}$ and we are back in case $k = 1$. However, now β is replaced by 48β, which we must allow for when choosing our parameters, so we need to strengthen condition (5.5) to the condition

$$(5.9) \qquad\qquad\qquad \beta \leqslant 1/48.$$

If the process stops before we reach $k = 1$, then we have a 3κ-separated κ-net $V' = \{\pm v_1, \ldots, \pm v_m\}$ of Σ such that $2 \leqslant m \leqslant 4$ and $\kappa \leqslant 16\beta$ as claimed.

Recall that every interval of length greater than $2\pi\zeta$ contains a typical θ, and hence by a connectedness argument similar to the one used for the case $k = 1$ there must exist θ, ϕ, and $v_i \neq \pm v_j$ such that

$$|\theta - \phi| \leqslant 2\pi\zeta,$$

$$\left| n^{-1/2} P_E \operatorname{sign}(x(\theta)) - v_i \right| \leqslant \kappa,$$

and

$$\left| n^{-1/2} P_E \operatorname{sign}(x(\phi)) - v_j \right| \leqslant \kappa.$$

Since $|v_i \pm v_j| \geqslant 3\kappa$, it follows that $n^{-1/2}|P_E \operatorname{sign}(x(\theta)) - P_E \operatorname{sign}(x(\phi))|$ and $n^{-1/2}|P_E \operatorname{sign}(x(\theta)) + P_E \operatorname{sign}(x(\phi))|$ are both at least κ. Now recall equation (5.1) saying that for every $x \in Y$ we can find y with $|x - y| \leqslant \delta$ such that

$$n^{-1/2} \operatorname{sign}(y) \approx_{\delta/\eta} \alpha_y P y + \beta_y Q y$$

for some constants α_y and β_y. Let us choose $y(\theta)$ and $y(\phi)$ that have this relationship with $x(\theta)$ and $x(\phi)$, respectively.

By Lemma 5.1 and the assumption that x is typical, the number of coordinates in E such that $\operatorname{sign}(x(\theta))$ and $\operatorname{sign}(y(\theta))$ differ is at most $c|E| + \xi^{-2}\delta^2 n \leqslant (c + \xi^{-2}\delta^2)n$, so

$$n^{-1/2}\left| P_E \operatorname{sign}(x(\theta)) - P_E \operatorname{sign}(y(\theta)) \right| \leqslant 2\left(c + \xi^{-2}\delta^2\right)^{1/2},$$

and similarly for ϕ. If we choose parameters in such a way that

$$(5.10) \qquad\qquad\qquad c + \xi^{-2}\delta^2 \leqslant \beta^4/256,$$

it follows that these distances are both at most $\beta^2/8$.

Let us write $\alpha(\theta)$ instead of $\alpha_{y(\theta)}$, and similarly for ϕ. Then

$$n^{-1/2} \operatorname{sign}(y(\theta)) \approx_{\delta/\eta} \alpha(\theta) P y(\theta) + \beta(\theta) Q y(\theta),$$

and

$$n^{-1/2} \operatorname{sign}(y(\phi)) \approx_{\delta/\eta} \alpha(\phi) P y(\phi) + \beta(\phi) Q y(\phi).$$

Also, since $|\theta - \phi| \leqslant 2\pi\zeta$, we have $|x(\theta) - x(\phi)| \leqslant 2\pi\zeta$ (because $|x(\theta) - x(\phi)| < |\theta - \phi|$), and therefore $|y(\theta) - y(\phi)| \leqslant 2\pi\zeta + 2\delta$. It follows that

$$\left| \alpha(\phi) P y(\phi) + \beta(\phi) Q y(\phi) - (\alpha(\phi) P y(\theta) + \beta(\phi) Q y(\theta)) \right| \leqslant 2(\pi\zeta + \delta)(|\alpha(\phi)| + |\beta(\phi)|).$$

Now recall that $|\alpha(\phi)|$ and $|\beta(\phi)|$ are both at most $2\eta^{-1}$. We therefore obtain the approximation

$$n^{-1/2} \operatorname{sign}(y(\phi)) \approx_\sigma \alpha(\phi) P y(\theta) + \beta(\phi) Q y(\theta),$$

where $\sigma = \delta/\eta + 8(\pi\zeta + \delta)/\eta = (8\pi\zeta + 9\delta)/\eta$.

It is convenient to encapsulate our knowledge so far as an approximate matrix equation

$$\begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \alpha(\phi) & \beta(\phi) \end{pmatrix} \begin{pmatrix} Py(\theta) \\ Qy(\theta) \end{pmatrix} \approx_\sigma n^{-1/2} \begin{pmatrix} \operatorname{sign}(y(\theta)) \\ \operatorname{sign}(y(\phi)) \end{pmatrix}$$

where by $\approx_\sigma$ in this context we mean that the approximation holds coordinatewise.

The rough idea of what we shall now do is as follows. Because $\operatorname{sign}(y(\theta))$ and $\operatorname{sign}(y(\phi))$ are not roughly proportional to each other, the matrix $\left(\begin{smallmatrix} \alpha(\theta) & \beta(\theta) \\ \alpha(\phi) & \beta(\phi) \end{smallmatrix}\right)$ is well-invertible, which allows us to deduce from the approximate matrix equation that $Py(\theta)$ and $Qy(\theta)$ can both be approximated by linear combinations of $n^{-1/2} \operatorname{sign}(y(\theta))$ and $n^{-1/2} \operatorname{sign}(y(\phi))$, with coefficients that are not too large. Therefore, $y(\theta)$ can as well, which implies that $x(\theta)$ can be. But $\operatorname{sign}(y(\theta))$ and $\operatorname{sign}(y(\phi))$ take at most two values each on almost all of E, and $x(\theta)$ is small outside E, so $x(\theta)$ can be approximated by a vector whose coordinates have at most five distinct values. Then we can obtain a contradiction from Lemma 4.3.

To carry out this argument we begin by making precise the statement that $\operatorname{sign}(y(\theta))$ and $\operatorname{sign}(y(\phi))$ are not roughly proportional.

LEMMA 5.5. — *Let u and v be vectors in $\mathbb{R}^n$ that take values $\pm n^{-1/2}$ in a set E of size m. Suppose that there are r values in E with $u_i = v_i$ and s values with $u_i \neq v_i$. Then for every $\lambda \in \mathbb{R}$, $|u - \lambda v| \geqslant 2(rs/mn)^{1/2}$.*

Proof. — We have that

$$n|u - \lambda v|^2 \geqslant r(1-\lambda)^2 + s(1+\lambda)^2$$
$$= \left(1 + \lambda^2\right) m + 2\lambda(s - r).$$

This is minimized when $\lambda = (r - s)/m$, and the minimum works out to be $4rs/m$. The lemma follows on dividing both sides by n and taking the square root. $\qquad\square$

We showed earlier that the distance between $n^{-1/2} P_E \operatorname{sign}(x(\theta))$ and $\pm n^{-1/2} P_E \operatorname{sign}(x(\phi))$ is at least κ, which by assumption is at least β. It follows further that the conditions of Lemma 5.5 apply to $n^{-1/2} \operatorname{sign}(x(\theta))$ and $n^{-1/2} \operatorname{sign}(x(\phi))$ with both r and s at least $\beta^2 n/4$. Therefore, using the trivial bound $|E| \leqslant n$, we deduce that

$$n^{-1/2} \Big| \operatorname{sign}(x(\theta)) - \lambda \operatorname{sign}(x(\phi)) \Big| \geqslant \beta^2/2.$$

Therefore, using the fact that

$$n^{-1/2} \Big| \operatorname{sign}(x(\theta)) - \operatorname{sign}(y(\theta)) \Big| \leqslant \beta^2/8$$

and

$$n^{-1/2} \Big| \operatorname{sign}(x(\phi)) - \operatorname{sign}(y(\phi)) \Big| \leqslant \beta^2/8,$$

we find that

$$n^{-1/2} \Big| \operatorname{sign}(y(\theta)) - \lambda \operatorname{sign}(y(\phi)) \Big| \geqslant \frac{\beta^2}{2} - \frac{\beta^2}{8}(|\lambda| + 1) \geqslant \beta^2/8$$

when $|\lambda| \leqslant 2$. In case $|\lambda| \geqslant 2$ we can instead use the bound

$$\left(1 + \lambda^2\right) m + 2\lambda(s - r) \geqslant (|\lambda| - 1)^2 m$$

to deduce that

$$n^{-1/2}\Big|\operatorname{sign}(x(\theta)) - \lambda\operatorname{sign}(x(\phi))\Big| \geqslant \frac{\beta^2}{2}(|\lambda| - 1),$$

from which it follows that

$$n^{-1/2}\Big|\operatorname{sign}(y(\theta)) - \lambda\operatorname{sign}(y(\phi))\Big| \geqslant \frac{\beta^2}{2}(|\lambda| - 1) - \frac{\beta^2}{8}(|\lambda| + 1),$$

which is again at least $\beta^2/8$.

Now that we have shown in a precise sense that $n^{-1/2}\operatorname{sign}(y(\theta))$ and $n^{-1/2}\operatorname{sign}(y(\phi))$ are not approximately proportional to each other, we turn to deducing that the matrix $\left(\begin{smallmatrix} \alpha(\theta) & \beta(\theta) \\ \alpha(\phi) & \beta(\phi) \end{smallmatrix}\right)$ is well-invertible, by which we simply mean that its determinant is not too small.

LEMMA 5.6. — *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a 2×2 real matrix, let x and y be vectors in a Euclidean space such that $\langle x, y\rangle = 0$, and let*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then there exists λ such that $|u - \lambda v| \leqslant \frac{|x|\,|y|\,|\det(A)|}{|v|}$

Proof. — Consider first the case where x and y are unit vectors. Then

$$|u - \lambda v|^2 = (a - \lambda c)^2 + (b - \lambda d)^2$$
$$= \left(c^2 + d^2\right)\lambda^2 - 2(ac + bd)\lambda + a^2 + b^2.$$

This is minimized when $\lambda = \frac{ac+bd}{c^2+d^2}$, and the minimum is

$$a^2 + b^2 - \frac{(ac + bd)^2}{c^2 + d^2} = \frac{(ad - bc)^2}{c^2 + d^2}\ ,$$

which proves the result.

In the general case, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a|x| & b|y| \\ c|x| & d|y| \end{pmatrix}\begin{pmatrix} x/|x| \\ y/|y| \end{pmatrix}\ .$$

Using the case for unit vectors, we deduce that there exists λ such that

$$|u - \lambda v|^2 \leqslant \frac{|x|^2|y|^2\det(A)^2}{c^2|x|^2 + d^2|y|^2},$$

and again the result is proved. □

Let us now apply this lemma with $A = \left(\begin{smallmatrix} \alpha(\theta) & \beta(\theta) \\ \alpha(\phi) & \beta(\phi) \end{smallmatrix}\right)$, $x = Py(\theta)$ and $y = Qy(\theta)$. Let $\left(\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right) = A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and let $u = n^{-1/2}\operatorname{sign}(y(\theta))$, $v = n^{-1/2}\operatorname{sign}(y(\phi))$. The approximate matrix equation proved earlier states that $|u - u'| \leqslant \sigma$ and $|v - v'| \leqslant \sigma$. It follows from the lemma that there exists λ such that

$$\frac{|Py(\theta)|\,|Qy(\theta)|\,|\det(A)|}{|v'|} \geqslant |u' - \lambda v'| \geqslant |u - \lambda v| - (1 + |\lambda|)\sigma.$$

But we have shown that $n^{-1/2}|\operatorname{sign}(y(\theta)) - \lambda \operatorname{sign}(y(\phi))| \geqslant \beta^2/8$ for every λ. Since we also know that $|Py(\theta)| \leqslant 1$ and $|Qy(\theta)| \leqslant 1$, it follows that

$$|\det(A)| \geqslant \left(\beta^2/8 - (1 + |\lambda|)\sigma\right) |v'|.$$

From the proof of the last lemma it follows that the minimum distance is achieved when

$$\lambda = \frac{\alpha(\phi)\alpha(\theta)|x|^2 + \beta(\phi)\beta(\theta)|y|^2}{|v'|^2}.$$

Recall also from the beginning of the section that $\alpha(\phi), \alpha(\theta), \beta(\phi), \beta(\theta) \leqslant 2\eta^{-1}$. Since $|v' - v| \leqslant \sigma$, $v = n^{-1/2}\operatorname{sign}(y(\phi))$, and every $y(\phi)$ has support size at least $n/4$, we get that $|v'| \geqslant 1/2 - \sigma$.

Hence

$$|\lambda| \leqslant \frac{4}{\eta^2(1/2 - \sigma)^2}.$$

Assuming that

(5.11) $\sigma \leqslant \beta^2\eta^2/2^{10}$

we may deduce that $\sigma(1 + |\lambda|) \leqslant \beta^2/16$ and hence that $|\det A| \geqslant (1/2 - \sigma)\beta^2/16 \geqslant \beta^2/64$.

Let us rewrite the approximate matrix equation as

$$\begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \alpha(\phi) & \beta(\phi) \end{pmatrix} \begin{pmatrix} Py(\theta) \\ Qy(\theta) \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix} \approx_\sigma n^{-1/2} \begin{pmatrix} \operatorname{sign}(y(\theta)) \\ \operatorname{sign}(y(\phi)) \end{pmatrix}.$$

Then

$$\begin{pmatrix} Py(\theta) \\ Qy(\theta) \end{pmatrix} = \det(A)^{-1} \begin{pmatrix} \beta(\phi) & -\beta(\theta) \\ -\alpha(\phi) & \alpha(\theta) \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Since the coefficients of A have absolute value at most $2\eta^{-1}$, it follows that both $Py(\theta)$ and $Qy(\theta)$ are linear combinations of u' and v' with coefficients of absolute value at most $128\eta^{-1}\beta^{-2}$. Using again the fact that $|u' - n^{-1/2}\operatorname{sign}(y(\theta))|$ and $|v' - n^{-1/2}\operatorname{sign}(y(\phi))|$ are both at most σ, it follows that both $Py(\theta)$ and $Qy(\theta)$ can be approximated to within $256\eta^{-1}\beta^{-2}\sigma$ by the corresponding linear combinations of $n^{-1/2}\operatorname{sign}(y(\theta))$ and $n^{-1/2}\operatorname{sign}(y(\phi))$, and hence that $y(\theta)$ can be approximated to within $512\eta^{-1}\beta^{-2}\sigma$ by a linear combination of $n^{-1/2}\operatorname{sign}(y(\theta))$ and $n^{-1/2}\operatorname{sign}(y(\phi))$ with coefficients of absolute value at most $256\eta^{-1}\beta^{-2}$.

Now recall that

$$n^{-1/2}\left|P_E\operatorname{sign}(x(\theta)) - P_E\operatorname{sign}(y(\theta))\right| \leqslant 2\left(c + \xi^{-2}\delta^2\right)^{1/2},$$

and similarly for ϕ, which implies in particular that $n^{-1/2}P_E\operatorname{sign}(y(\theta))$ and $n^{-1/2}P_E\operatorname{sign}(y(\phi))$ can be approximated to within $2(c + \xi^{-2}\delta^2)^{1/2}$ by vectors whose coordinates take just the values $\pm n^{-1/2}$ on E. This is because $x(\theta)$ and $x(\phi)$ are typical, which implies, by the second condition in the definition of "typical", that all coordinates of $P_E\operatorname{sign}(x(\theta))$ and $P_E\operatorname{sign}(x(\phi))$ are ± 1.

Putting together the bounds obtained in the last two paragraphs we get that $P_E y(\theta)$ can be approximated by a linear combination of $n^{-1/2}P_E\operatorname{sign}(x(\theta))$ and $n^{-1/2}P_E\operatorname{sign}(x(\phi))$ to within $512\eta^{-1}\beta^{-2}\sigma + 1024\eta^{-1}\beta^{-2}(c + \xi^{-2}\delta^2)^{1/2}$. Thus, $P_E y(\theta)$ can be approximated by a vector with at most four distinct coordinates.

This in turn implies that $P_E x(\theta)$ can be approximated to within $\delta + 512\eta^{-1}\beta^{-2}\sigma + 1024\eta^{-1}\beta^{-2}(c + \xi^{-2}\delta^2)^{1/2}$ by such a vector. But $|x(\theta) - P_E x(\theta)| \leqslant \alpha$, so we end up with the conclusion that $x(\theta)$ can be approximated to within $\alpha + \delta + 512\eta^{-1}\beta^{-2}\sigma + 1024\eta^{-1}\beta^{-2}(c + \xi^{-2}\delta^2)^{1/2}$ by a vector that takes at most five distinct values (the fifth value being zero). But $x(\theta) \in Y$, so it is an ϵ-good point, which implies by Lemma 4.1 that $d(x(\theta), PX) \leqslant 3\delta + 2\eta$ or $d(x(\theta), QX) \leqslant 3\delta + 2\eta$. Provided we have chosen our parameters in such a way that

$$(5.12) \qquad \alpha + 4\delta + 2\eta + 512\eta^{-1}\beta^{-2}\sigma + 1024\eta^{-1}\beta^{-2}\left(c + \xi^{-2}\delta^2\right)^{1/2} \leqslant \gamma,$$

this contradicts the fact that P was chosen to ensure that neither PX_γ nor QX_γ contains a vector with at most five distinct coordinates (see the end of Section 4 where the choice of P was described).

5.3. Case 3: $k \geqslant 5$.

We begin with a simple lemma to estimate how well we can simultaneously approximate k orthonormal vectors by a $(k-1)$-dimensional subspace.

LEMMA 5.7. — *Let W be a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ and let $u_1, \ldots, u_k$ be an orthonormal sequence. Then there exists i such that $d(u_i, W) \geqslant k^{-1/2}$.*

Proof. — Without loss of generality $n = k$. Now let v be a unit vector orthogonal to W. Then the orthogonal projection P_W to W is given by the formula $P_W(x) = x - \langle x, v \rangle v$, from which it follows that $d(x, W) = |\langle x, v \rangle|$. But $\sum_{i=1}^{k} \langle u_i, v \rangle^2 = 1$, so there must exist i such that $|\langle u_i, v \rangle| \geqslant k^{-1/2}$, which proves the lemma. $\square$

Now, with the help of the assumption that $k \geqslant 5$, we prove that we cannot find a 4-dimensional subspace that approximately contains all the vectors in Σ. For convenience, let us reorder the coordinates in such a way that $E = \{1, 2, \ldots, m\}$ and $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_m < 2\pi$. Then for every typical vector $x(\theta)$, the set of i such that $x(\theta)_i > 0$ is an interval mod m.

LEMMA 5.8. — *Let W be a 4-dimensional subspace of X. Then Σ contains a vector u such that $d(u, W) \geqslant \beta/2\sqrt{5}$.*

Proof. — Let $0 \leqslant \theta_1 < \cdots < \theta_5 < \pi$ be such that the points $u_j = n^{-1/2} P_E \operatorname{sign}(x(\theta_j)) \in \Sigma$ and that together with the points $-u_j$ form a β-separated subset of Σ. For each $j \in \{1, 2, \ldots, 5\}$ let $[a_j, b_j]$ be the interval mod m of i such that $(u_j)_i = n^{-1/2}$.

Note that $a_1 \geqslant \cdots \geqslant a_5$ and $b_1 \geqslant \cdots \geqslant b_5$, where here we refer to the cyclic ordering on the integers mod m. It follows that for each j, $(u_{j+1} - u_j)_i = 2n^{-1/2}$ on the interval $[a_{j+1}, a_j)$, $-2n^{-1/2}$ on the interval $(b_{j+1}, b_j]$, and zero everywhere else. In particular, the vectors of the form $u_{j+1} - u_j$ for $j = 1, 2, 3, 4$, together with the vector $u_1 + u_5$, are orthogonal.

Since we have that $|u_i \pm u_j| \geqslant \beta$ for every $i \neq j$, setting $v_j = \frac{u_{j+1} - u_j}{|u_{j+1} - u_j|}$ for $j = 1, \ldots, 4$ and $v_5 = \frac{u_1 + u_5}{|u_1 + u_5|}$, we may deduce from Lemma 5.7 that $d(v_j, W) \geqslant 1/\sqrt{5}$ for some j. If $j = 1, \ldots, 4$ then this implies that $d(u_{j+1} - u_j, W) \geqslant \beta/\sqrt{5}$, which implies

that either $d(u_{j+1}, W) \geqslant \beta/2\sqrt{5}$ or $d(u_j, W) \geqslant \beta/2\sqrt{5}$. Similarly, if $j = 5$ we get that either $d(u_1, W) \geqslant \beta/2\sqrt{5}$ or $d(u_5, W) \geqslant \beta/2\sqrt{5}$. This proves the Lemma 5.8. $\quad\square$

Let $u = P_E \operatorname{sign}(x(\theta))$ be given by Lemma 5.8. Recall that (with parameters that satisfy condition (5.12)) we have

$$n^{-1/2}\big|P_E \operatorname{sign}(x(\theta)) - P_E \operatorname{sign}(y(\theta))\big| \leqslant \beta^2/8.$$

Recall also equation (5.1), which reads

$$n^{-1/2} \operatorname{sign}(y(\theta)) \approx_{\delta/\eta} \alpha(\theta) P y(\theta) + \beta(\theta) Q y(\theta)$$

for some coefficients $\alpha(\theta), \beta(\theta)$ that have absolute values at most $2\eta^{-1}$. We also know that $|y(\theta) - x(\theta)| \leqslant \delta$. It follows that

$$n^{-1/2} P_E \operatorname{sign}(x(\theta)) \approx_{\beta^2/8 + \delta/\eta + 4\delta/\eta} \alpha(\theta) P_E P x(\theta) + \beta(\theta) P_E Q x(\theta).$$

It follows that the distance from $n^{-1/2} P_E \operatorname{sign}(x(\theta))$ to the subspace $P_E(PY + QY)$, which has dimension at most 4, is at most $\beta^2/8 + 5\delta/\eta$. If we pick parameters in such a way that

$$(5.13) \qquad\qquad \frac{\beta^2}{8} + \frac{5\delta}{\eta} < \frac{\beta}{2\sqrt{5}},$$

then this contradicts Lemma 5.8.

5.4. Choosing parameters

We conclude by showing that there exists a choice of parameters which fulfils all the conditions. We are not optimizing this choice.

First, recall that we have already chosen γ in the Corollary 4.5 to be 2^{-37}. Further we see that $\beta = 2^{-6}$ satisfies (5.5) and (5.9). We can further choose $\eta = 2^{-40}$, $\alpha = 2^{-40}$, $\rho = 2^{-40}$ and then $c = 2^{-205}$, $\xi = 2^{-403}$ and finally $\delta = 2^{-506}$. It is easy to check that these parameters meet all the conditions.

This finishes the proof that the 2-Euclidean norm defined in (3.1), for $n \geqslant 35$ (which comes from our choice of γ and the condition in Lemma 4.2 and the fact that we use a 2γ-expansion), contains no 2-dimensional subspace which is both strongly $(1 + \epsilon)$-complemented and strongly $(1 + \epsilon)$-Euclidean with $\epsilon = \delta^2/8C^2 = 2^{-1017}$. (the dependence of ϵ on δ was established in the proof of Lemma 2.4).

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Timothy GOWERS
Collège de France,
11 Pl. Marcelin Berthelot,
75231 Paris (France)
and the University of Cambridge,

Department of Pure Mathematics
and Mathematical Statistics,
Wilberforce Road, Cambridge
CB3 0WB (United Kingdom)
wtg10@dpmms.cam.ac.uk

Katarzyna WYCZESANY
Carnegie Mellon University,
Department of Mathematical Sciences,
5000 Forbes Ave, Pittsburgh,
PA 15213 (United States)
kwycz@cmu.edu