Article:

Martin, P. orcid.org/0000-0002-8141-9465, Rowell, E.C. and Torzewska, F. (2025) Classification of charge-conserving loop braid representations. Journal of Algebra, 666. pp. 878-931. ISSN 0021-8693

https://doi.org/10.1016/j.jalgebra.2024.12.011

Classification of charge-conserving loop braid representations

Paul Martin, Eric C. Rowell, and Fiona Torzewska

July 24, 2024

Contents

Abstract

Here a loop braid representation is a monoidal functor F from the loop braid category L to a suitable target category, and is N-charge-conserving if the target is the category Match^N of charge-conserving matrices (specifically Match^N is the same rank-N charge-conserving monoidal subcategory of the monoidal category Mat used to classify braid representations in [27]) with F strict, and surjective on $\mathbb{N}$, the object monoid. We classify and construct all such representations. In particular we prove that representations at given N fall into varieties indexed by a set in bijection with the set of pairs of plane partitions of total degree N.

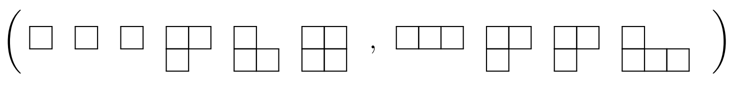
1 Introduction

Viewing the braid group B_n as a group of motions of n points in the 2-disk leads to vast generalisations when pondered in 3 spatial dimensions, including motions of links in the 3-ball [9]. The simplest of these is the loop braid group LB_n: motions of n unlinked, oriented circles [10, 17, 9, 42]. The representation theory of LB_n is (despite much intriguing progress - see for example [5, 43, 34, 8, 11, 20]) largely unknown, and the aim of a systematic study of extending braid representations to LB_n inspired [12] in which a loop braid group version of the Hecke algebra was discovered. This revealed a surprise: there exists a 4×4 non-group-type (see [20]) Yang-Baxter operator R that admits a lift yielding a local representation of LB_n. Is this R an isolated example, or does it fit into a larger family? An appropriate context for answering this question is suggested by a salient feature of this R: it is *charge conserving*[1] in the sense of [27] (also described below).

In this article we classify charge conserving loop braid representations. A preliminary step is to classify charge conserving braid representations, which was carried out in [27]. Our results can be interpreted as a classification of monoidal functors from the Loop braid category L to the category of charge-conserving matrices Match^N that are surjective on objects, and strict. This L is the diagonal category made up of loop braid groups LB_n, exactly paralleling the relationship between MacLane's braid category [25] and the Artin braid groups [2].

Just as the braid groups and the Yang-Baxter equation manifest as key components of several areas of mathematics and physics, so the loop braid groups are key to applications that require a higher-dimensional generalisation. Their study is thus partially motivated by various such applications. One is the aim of formulating a notion of higher quantum group (cf. e.g. [22, 8]). Another is the aim of determining statistics of loop-like excitations (see e.g. [32]) in 3D topological phases of matter (see e.g. [23, 4]), which in turn has applications to topological quantum computation, see e.g. [31, 36]. More precisely, monoidal local loop braid representations, such as those we study here, are used to connect models of 3D topological phases to quantum computing models [16][33, Sec.4.2][45, Sec.7.1]. Another example is construction of solutions to the tetrahedron equation [21, 13].

The result (Theorem 7.3) may be summarised as follows.
The set of all varieties of charge-conserving loop braid representations may be indexed by the set of 'signed multisets' of compositions where each composition has at most two parts. A *signed multiset* is an ordered pair of multisets. For example

$\lambda =$ 

(To connect with the corresponding index set for braid representations one should think of two-coloured compositions with each part having a different colour, so that the colouring is forced and hence need not be explicitly recorded.)
We will show explicitly
(I) how to construct a variety of representations from each such index; and
(II) that every charge-conserving representation arises this way.

[1] The idea for the term charge-conserving comes from the XXZ spin-chain setting - cf. e.g. [6, Ch.8 *et seq*] - hence also 'spin-chain representation', but the spin-chain context makes less sense for loop braid.

As discussed in detail in §2, category L is monoidally generated by two kinds of exchange of pairs of loops, a non-braiding exchange denoted s and a braiding exchange denoted σ. Thus we can give a solution, a monoidal functor F, by giving the pair $(F(s), F(\sigma))$.

As alluded to above, the classification of braid representations in [27] actually progressed serendipitously from the aim of a systematic study of extensions of braid representations to loop braids (in Damiani et al [12]). So in the present paper the original aim is realised.
(To unpack this background a bit: rather than *extension*, this can be seen as 'merging' braid and symmetric group representations — which raises the question of how to bring their separate universes (spaces on which they act) together. Each has its own up-to-isomorphism freedom. So one idea is to rigidify one or both of them when bringing them together. The idea of rigidification on the braid side set up some choices and a direction of travel which, so far, ends with charge-conservation... which then turns out to facilitate a complete classification in this setting!)
Given the route that led to braid-representation classification, it is natural to make loop braids one of the next structures to be studied using the charge-conserving machinery (and its broad underlying philosophy of paying active attention to the target category as well as the source).

A rough 'route map' for the present paper is provided by the stages of the braid representation classification in [27]. In particular then, one would start with a physical realisation of loop braids, lifting the 'Lizzy category' from [27]. For the sake of brevity (and given that the strong parallel is stretched by the absence of a 4d laboratory) we have the option to jump this and pass to the next stage: a presentation - see §3. But see §2 (supported by §C) for a workable heuristic. The target category is recalled in §4, where the further properties of Match categories that we shall need (cf. [27, §3]) are obtained. In §5.1 (and §7.1) we prove the key Lemmas determining the form of solutions in low rank. In §6 we introduce the combinatorial structures that we shall need, adapting those developed in [27, §5] in light of §5.1. And in §7 we prove the classification Theorem.

Recipe in brief

We now *outline* our combinatorial parameterisation of isomorphism classes (under the group of symmetries as in [27]) of functors $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$.

As mentioned above, a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ is determined by a pair of charge-conserving $N^2 \times N^2$ matrices, $(\mathsf{F}(s), \mathsf{F}(\sigma))$. We denote these here by $S = \mathsf{F}(s)$ and $R = \mathsf{F}(\sigma)$. As with all charge conserving matrices, R may be encoded as a sequence $R \leftrightarrow (a_1, a_2, \dots, a_N, A(1,2), \dots, A(N-1, N))$ where $a_i \in \mathbb{C}^*$ and the $A(i,j)$ are 2×2 matrices; similarly $S \leftrightarrow (b_1, \dots, b_N, B(1,2), \dots, B(N-1,N))$. (See §3.1 for details.)

Let $J_N^\pm$ denote the set of signed multisets of two-part composition diagrams of total degree N. (By results of Sagan [37], these are in bijection with pairs of plane partitions.) For example the following signed multisets are in $J_3^\pm$, with diagrams before the comma having a $+$, and diagrams after a $-$. (Full details are in §6.3.)

$$(\square^3,),\ (\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}^1,),\ (\square\square^1, \square^1),\ (, \square^1 \square\square^1)$$

Within each sign, we observe the following convention for ordering diagrams: first in ascending total size, secondly, for diagrams of equal total size, in ascending order of the second part of each composition. Given an element of $J_N^\pm$, we refer to compositions as nations, labelling nations as $n_1, n_2, \dots$, and within a nation n_t we refer to each part of the composition as a county, labelling the top part of the composition diagram $s_{t,1}$ and the second part $s_{t,2}$.

Now to an element $\lambda \in J_N^\pm$ we construct the operators R and S as follows. First, label the boxes in λ with the numbers $\{1, \dots, N\}$ in order with the first numbers going left to right in the first county of n_1, then the second county and so on with $n_2, n_3, \dots$. The following is an example with $N = 11$:

$$\lambda \rightsquigarrow (\boxed{1}\boxed{2}\quad \boxed{3}\boxed{4}\quad \begin{smallmatrix}\boxed{5}\\ \boxed{6}\end{smallmatrix}, \ \boxed{7}\ \ \boxed{8}\ \ \boxed{9}\ \ \boxed{10}\boxed{11}\ ) \tag{1}$$

Now, for each nation n_t we assign parameters α_t to county $s_{t,1}$ and β_t to county $s_{t,2}$ (if it is non-empty) such that $\alpha_t + \beta_t \neq 0$. Next, for each pair of distinct nations n_r, n_t we assign two non-zero parameters $\mu_{r,t}, C_{r,t}$.

Firstly, if i resides in county $s_{t,1}$ then $a_i = \alpha_t$ whereas if i resides in county $s_{t,2}$ then $a_i = \beta_t$. If $\mathrm{sgn}(n_t) = +$ then $b_i = 1$ (resp. $b_i = -1$) if i resides in $s_{t,1}$ (resp. in $s_{t,2}$). The sign of b_i is opposite to this in case $\mathrm{sgn}(n_t) = -$.

Consider each pair of individuals $i < j \in \{1, \dots, N\}$.

1. If $i \in n_r$ and $j \in n_t$ with $r \neq t$ then $A(i,j) = \begin{pmatrix} 0 & \mu_{r,t}/C_{r,t} \\ \mu_{r,t}C_{r,t} & 0 \end{pmatrix}$, and $B(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
2. If i and j are in the same nation n_t but different counties $s_{t,x}$ and $s_{t,y}$ (note $x < y$ by construction), then $A(i,j) = \begin{pmatrix} \alpha_t + \beta_t & \alpha_t \\ -\beta_t & 0 \end{pmatrix}$ and $B(i,j) = \mathrm{sgn}\,(n_t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
3. If i and j are both in the first, (respectively second), county in n_t then $A(i,j) = \mathrm{sgn}\,(n_t) \begin{pmatrix} \alpha_t & 0 \\ 0 & \alpha_t \end{pmatrix}$, (respectively $A(i,j) = \begin{pmatrix} \beta_t & 0 \\ 0 & \beta_t \end{pmatrix}$) and $B(i,j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Our results imply that

1. This construction of (R, S) does provide a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ and
2. For any such functor, the corresponding pair (R, S) may be transformed into an equivalent pair (R', S') of the above form by means of two basic symmetries: simultaneous local basis permutations and/or simultaneous conjugation by a diagonal matrix X.

In [20] we used the nomenclature, adapted from [1], *loop braided vector spaces* (LBVSs) for a general triple (V, R, S) where V is an N-dimensional vector space and (R, S) defines a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Mat}^N$. Our main result can then be phrased as a classification of charge conserving LBVSs.

Acknowledgements. We thank Emmanuel Wagner, Celeste Damiani, Sinead Lyle, Joao Faria Martins and Alex Bullivant for useful conversations. PPM thanks EPSRC for support under Programme Grant EP/W007509/1. PPM thanks Paula Martin for useful conversations, and Leonid Bogachev for a reference on asymptotics of plane partitions. The research of ECR was partially supported by USA NSF grants DMS-2000331 and DMS-2205962. FT thanks support from the EPSRC under grant EP/S017216/1.

2 Some basics of loop braids

Mac Lane's monoidal braid category B [25, Sec.XI.4] has object monoid generated by a single object - a single strand of hair, or a single point from which this hair is extruded. The category is then generated by an elementary braid in $\mathsf{B}(2,2)$ and its inverse. The monoidal category L has object monoid generated by a single circle or loop; and the category is generated by *two* 'exchanges' in $\mathsf{L}(2,2)$. In [27] we used the 'Lizzy category' to give a geometric framework for B. This section aims merely to visualise the two generators of L in an analogous way.[2] (As such the section can optionally be skipped. For representation theory we can rely on the presentation given in §3.)

For each $n \in \mathbb{N}$, let C_n be a configuration of n unlinked oriented circles in a box in $\mathbb{R}^3$. We will fix C_n so the i-th loop is a circle of small radius in the xy-plane centred at $(i, 0, 0)$. (Up to isomorphism it will not matter precisely which configuration we take for C_n.) The loop braid group LB_n is a 'motion group' for C_n. See for example Dahm [9], Goldsmith [17], Lin [24], Fenn–Rimany-Rourke [15], Baez-Crans-Wise [4], Brendle-Hatcher [7], Damiani [10] and references therein. Or see Appendix C.

In the spirit of the braid category B the groups LB_n form a natural diagonal subcategory of a motion groupoid (informally speaking, $\mathrm{Mot}_{\mathbb{R}^3}$ as in [42], except we should stress compact support — in [42] it is proved that a loop braid group is a motion group in a box B^3 with fixed boundary).

[2] As opposed to the hybrid combinatorial visualisations of [4] borrowed from virtual braids, for example.

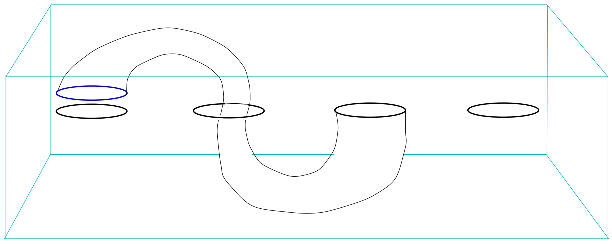


Figure 1: Schematic overlay visualisation of a loop motion.

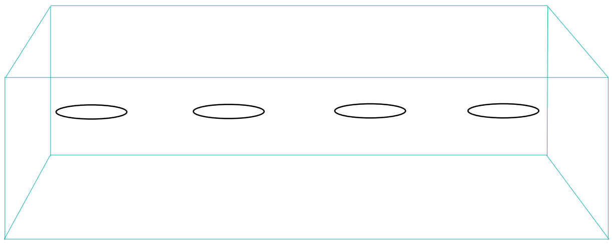


Figure 2: Initial configuration of loops in a box (C_n with $n = 4$). Also this is the overlay visualisation of a static motion, representative of the identity in LB_4.

The monoidal structure is indicated by placing one row of circles following another, to make a longer row: $C_n \sqcup C_m = C_{n+m}$. See Fig.7.

The braid category B is the diagonal groupoid whose groups of morphisms are the ordinary braid groups. In this context the braid groups have various relevant realisations (for representation theory it is convenient to work with efficient presentations, but for intuition and application geometric realisations are more useful). In [27] a realisation as hair-braiding is used. In this case one may consider a square (or other topological disk) that is a fixed-height section through the hair, thus cutting the hair at n points in the square. An initial configuration, P_n say, places the n points at regular intervals in the square. In our loop-braid case the square section through a 3d braid-laboratory is replaced by a box (it would be a section through a 4d loop-braid laboratory), and the points by circles, as in Fig.2.

In the braid case one can either think of the braiding as taking place over time (particles move in the square and world-lines braid), or acting by physical continuity on the hair above the point of action - between this point and the fixed scalp as it were.

In the former perception, we can visualise the braiding by showing all the locations visited by each point - still just drawn in the square. An 'overlay' visualisation. This has the drawback that the same location can be visited at different times (indeed a simple static point is an extreme form of this), but the merit is that the visualisation remains in 2d. The drawback can be minimised

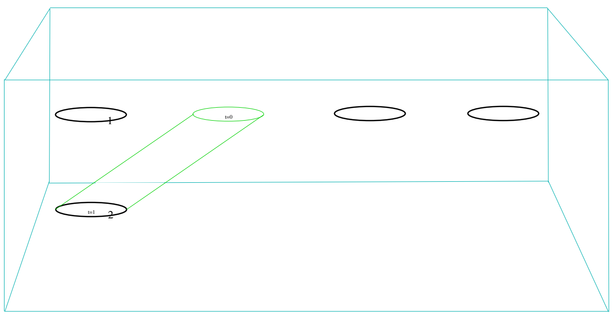


Figure 3: Overlay view of simple non-diagonal motion, moving loop-2 under loop-1.

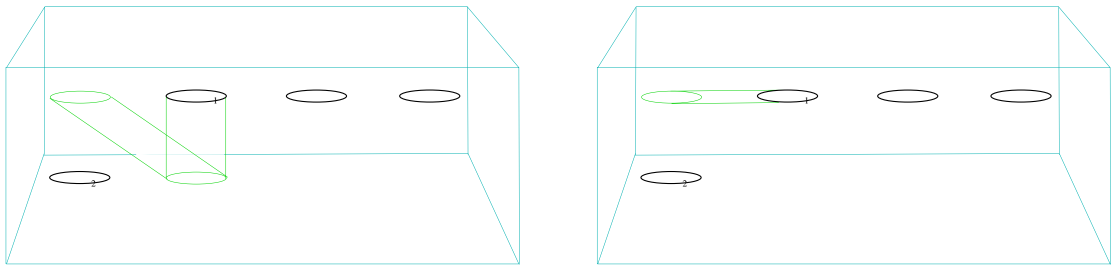


Figure 4: Equivalent motions, moving loop-1 over, indirectly or directly.

by only drawing 'simple' changes [3] in a single instance [30]. For this it is convenient momentarily to break out of the diagonal groupoid with object set $\{P_n\}_n$ and into the more general, in the sense of considering partial braidings, passing to configurations different from P_n. Collectively the braidings and partial braidings are 'motions'. (Depending on the realisation, a motion may be an object set trajectory, or a path in a space of homeomorphisms of the ambient space that restricts to this trajectory. Here we focus on describing the object set trajectory.) The morphisms in the category (the elements in the groups) are certain equivalence classes of motions - see below.

In the loop braid case we thus have overlay visualisations in 3d. We can further use artists-impression to indicate these 3d objects on the 2d page, finally giving representations like those in Fig.3 and Fig.4. Here the colour-code is green for the initial and intermediate points, and black for the final points. Note in this visualisation that the 'world-line' of a static point is simply a point.

Two 'motions' between the same initial and final configurations (configurations a and b, say) are equivalent if one can be homotopically deformed into the other in the box. This is a process that is not made easy to track by the overlay picture! (Cf. Fig.4.) But this will not be an issue for us here. Note that the specific figure illustrates one analogue here of the triangle equivalence of polygonal knots. By another such analogue, the image of a circle at some intermediate point in a motion need not be a simple translate of the circle. Examples of this follow shortly. (However see (2.1).)

Consider also the the sequence in Fig.5. This sequence is a complete one, in the sense that we finally return, set-wise, to the starting configuration. Similarly Fig.6. Note in general that two motions are composable if the tail of one is the source of the next. A visualisation of the composition would amount to overlaying and replacing all now-intermediate black to green (but we will not need this).

(**2.1**) It is a useful fact that for each loop i every equivalence class of motions contains a representative in which loop i is at most translated during the motion (i.e. circularity, attitude and size are preserved). N.B. this is certainly not true for any *pair* of loops.

(**2.2**) Here the loop braid category L is the 'category of loop braid motions' (strictly speaking a subcategory of the category of all loop motions), as follows.
Let $\varsigma \in \mathsf{L}(2,2)$ denote the class of motions in which two loops are exchanged by one 'passing through' another, as in Fig.6 (ignoring the extra two loops).
Let $\varrho \in \mathsf{L}(2,2)$ denote the class where the two loops pass around each other, as in Fig.5.
Then L is the category generated by these motions and inverses. (This does not include motions in which a single loop undergoes an orientation-reversing flip.) The monoidal composition is as indicated in Fig.7.

We will not need to dwell on this beautiful but technical construction in order to do representation theory, because we have the following isomorphism (3.2). Indeed the main reason for recalling aspects of the construction of L here is to grasp why the presentation given by (3.2) takes the form that it does.

[3]Here we will leave the simple-change notion entirely informal and example-based. Note that the separation between two 'particles' in some intermediate moment can be small, so simpleness is not necessarily enough to render distinct paths from a to b homotopic in the relevant space.

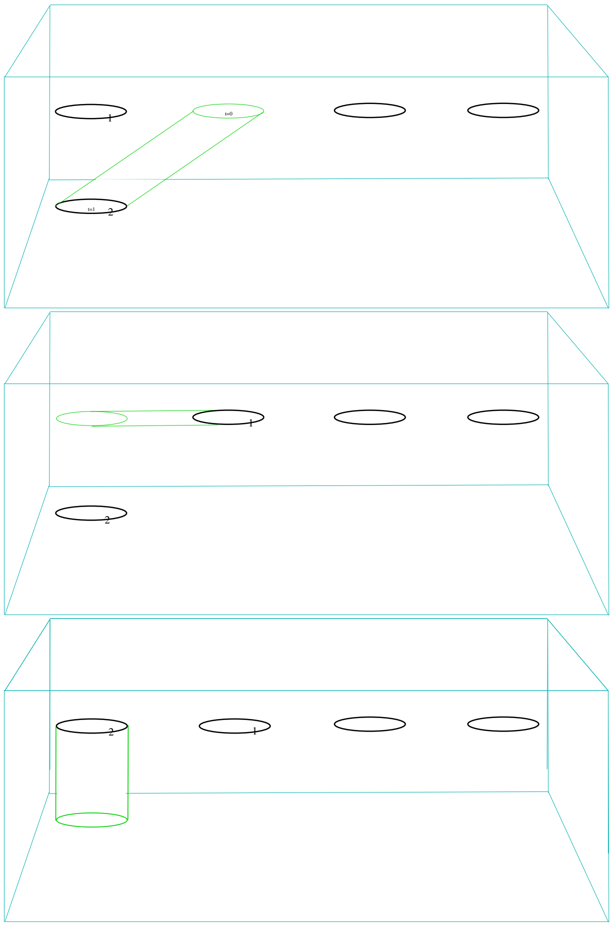


Figure 5: Sequence exchanging loop-1 and loop-2. The sequence runs top to bottom: (a) Move loop-2 down. (b) Move loop-1 over. (c) Move loop-2 up.

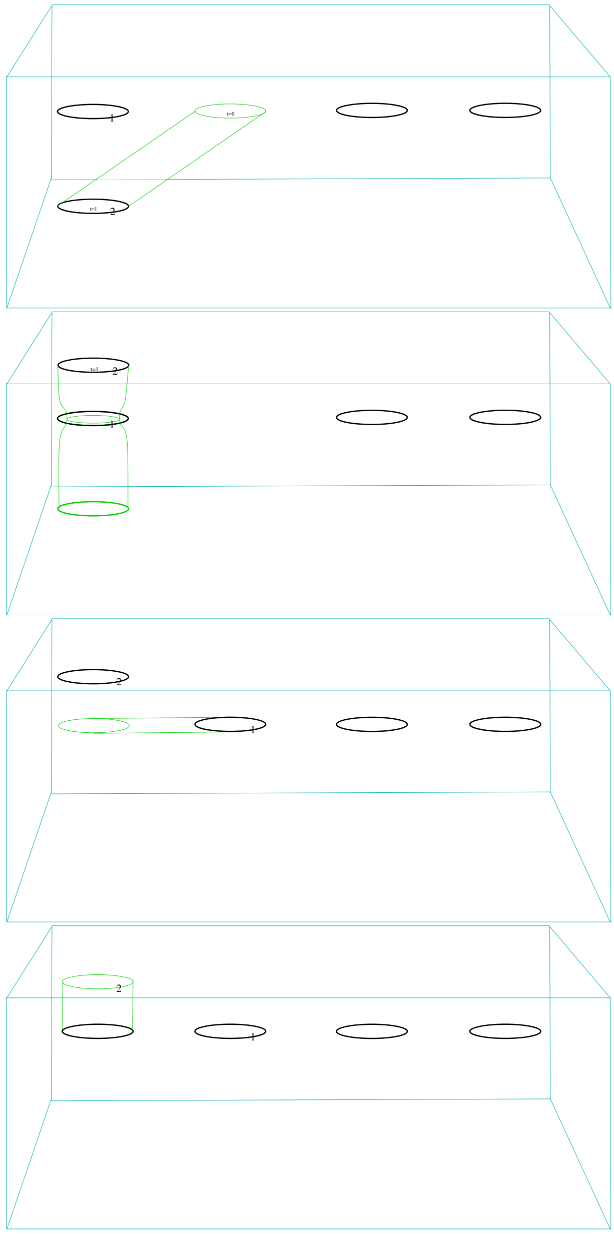


Figure 6: Sequence exchanging loop-1 and loop-2: (a) Move loop-2 under loop-1. (b) Move loop-2 up through loop-1. (c) Move loop-1 over. (d) Move loop-2 down.

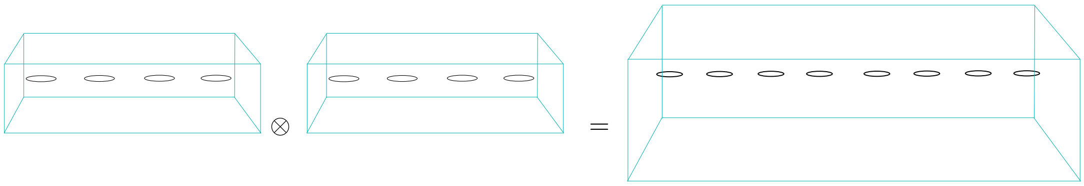


Figure 7: Schematic for monoidal composition.

3 The setup, and the story so far

We study *natural* monoidal functors $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$, that is monoidal functors such that $\mathsf{F}(1) = 1$. A strict natural monoidal functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ is called N-*charge-conserving*. Next we explain how to work with L; explain key properties of Match^N; and hence explain how to give a functor F.

A presentation for the loop braid category as a strict monoidal category may be given as follows.

(**3.1**) The category L' is the strict monoidal (diagonal, groupoid) category with object monoid the natural numbers, and two generating morphisms (and inverses) both in $\mathsf{L}'(2, 2)$, call them σ and s, obeying

$$s^2 = 1 \otimes 1 \tag{2}$$

where (as a morphism) 1 denotes the unit morphism in rank one;

$$s_1 s_2 s_1 = s_2 s_1 s_2 \tag{3}$$

where $s_1 = s \otimes 1$ and $s_2 = 1 \otimes s$,

$$\text{(I) } \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad \text{(II) } \sigma_1\sigma_2 s_1 = s_2\sigma_1\sigma_2, \qquad \text{(III) } \sigma_1 s_2 s_1 = s_2 s_1 \sigma_2. \tag{4}$$

Note that because of (2), relation (4)(III) is equivalent to $s_1 s_2 \sigma_1 = \sigma_2 s_1 s_2$. On the other hand the 'reverse' of (4)(II) is not imposed.

(**3.2**) The map on generators of L' to L given by $\sigma \mapsto \varsigma$ and $s \mapsto \varrho$ (recall (2.2)) extends to an isomorphism. (For the σ/ς motion it makes a difference which loop is which, leading to the asymmetry of (4)(II), so later we will be careful with conventions.)

(**3.3**) By (3.2) we may simply consider the representation theory of L'. Or equivalently, we have a representation of L provided that the images of ς and ϱ obey identities corresponding to the presentation of L' above. In practice we will simply identify the two categories henceforward.

Just as a strict monoidal functor $F : \mathsf{B} \to \mathsf{Match}^N$ (or to any target) can be given by giving the image of the elementary braid $\sigma \in \mathsf{B}(2, 2)$, so a functor $F : \mathsf{L} \to \mathsf{Match}^N$ can be given by giving the pair

$$F_* = (F(s), F(\sigma)) \tag{5}$$

(here $*$ is some label for the representation). We discuss this in more detail in C.2. The pairs that follow in §5.1 *et seq* thus give functors in this way.

(**3.4**) An initial organisational scheme for solutions is provided by the classification on restriction to B, as in [27], so we recall this briefly in §3.2.

The foundational aspect of this is the Match^N category itself, which we recall briefly in §3.1.

3.1 The story so far: Match^N categories

Here Mat is the monoidal category of matrices over a given commutative ring k. We take $k = \mathbb{C}$. (See (4.6) below for monoidal product conventions.) For $N \in \mathbb{N}$ let $\underline{N} := \{1, 2, ..., N\}$.

A *natural* monoidal category is a strict monoidal category with object monoid freely generated by a single object. Recall from [27] that Mat^N denotes the natural monoidal subcategory of Mat generated (in the obvious sense) by the object N in Mat (renamed object 1 in Mat^N, with $1 \otimes 1 = 1 + 1$).

(**3.5**) The index set for rows of a matrix in $\mathsf{Mat}^N(n, m)$ is the set of words in $\underline{N}^n$, and similarly for columns. We sometimes write $|ijk\rangle$ to emphasise that the word ijk is being used as a column/row index.

A matrix $M \in \mathsf{Mat}^N(n, n)$ is *charge conserving* if $M_{w,w'} = \langle w|M|w'\rangle \neq 0$ implies that w is a perm of w'. That is $w = \sigma w'$ for some $\sigma \in \Sigma_n$, where symmetric group Σ_n acts by place permutation.

(**3.6**) The subset of Mat^N of charge conserving (cc) matrices forms a monoidal subcategory (see for example [27, Lem.3.7I]) denoted Match^N.

(**3.7**) **Lemma**. [27, Lem.3.7III] For each $M \leq N$ and each injective function $\psi : \underline{M} \to \underline{N}$ there is a monoidal functor $f^\psi : \mathsf{Match}^N \to \mathsf{Match}^M$ given on morphisms by $f^\psi(a)_{vw} = a_{\psi v, \psi w}$ ($a \in \mathsf{Match}^N(m,m)$ say).
In particular we have a group action of the symmetric group Σ_N on Match^N. □

Example. Consider the nontrivial bijection $\omega : \underline{2} \to \underline{2}$. On $\mathsf{Match}^2(2,2)$ this gives

$$\begin{pmatrix} a_1 &&& \\ & a & b & \\ & c & d & \\ &&& a_2 \end{pmatrix} \overset{f^\omega}{\mapsto} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \begin{pmatrix} a_1 &&& \\ & a & b & \\ & c & d & \\ &&& a_2 \end{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$= \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} \begin{pmatrix} a_1 &&& \\ & a & b & \\ & c & d & \\ &&& a_2 \end{pmatrix} \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} = \begin{pmatrix} a_2 &&& \\ & d & c & \\ & b & a & \\ &&& a_1 \end{pmatrix} \tag{6}$$

(**3.8**) Let C be a natural category, as in [27]. For $\varsigma \in C(2,2)$ its image under a functor $F : C \to \mathsf{Match}^N$ with $F(1) = 1$ will be some $R \in \mathsf{Match}^N(2,2)$. Let K_N denote the (simplicially directed) complete graph, as in [27]. The (possibly) nonzero entries in R are in correspondence with assignments of a scalar $a_i = a_i(R)$ to each vertex i of K_N and a matrix

$$\begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = \begin{pmatrix} a_{ij}(R) & b_{ij}(R) \\ c_{ij}(R) & d_{ij}(R) \end{pmatrix}$$

to each directed edge. Altogether for $N = 2$ we have

$$R = \begin{pmatrix} a_1(R) &&& \\ & a_{12}(R) & b_{12}(R) & \\ & c_{12}(R) & d_{12}(R) & \\ &&& a_2(R) \end{pmatrix} \tag{7}$$

(**3.9**) Recall from (5) that to give a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ it is enough to give the images $\mathsf{F}(s)$ and $\mathsf{F}(\sigma)$. Here we assume the object map $\mathsf{F}(1) = 1$, so $\mathsf{F}(s)$ and $\mathsf{F}(\sigma) \in \mathsf{Match}^N(2,2)$.

For $N = 2$ giving $M \in \mathsf{Match}^2(2,2)$ is easy to do explicitly, as in (6) or (7), but for general N it can be helpful to view elements of $\mathsf{Match}^N(2,2)$ geometrically, using the complete graph K_N, as in [27]. The point is that the non-zero elements of $M \in \mathsf{Match}^N(2,2)$ break up into 1×1 and 2×2 blocks, with the former $a_i = \langle ii|M|ii\rangle$ indexed by vertices i and the latter $A(i,j)$ (the submatrix $\left(\begin{smallmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{smallmatrix}\right)$ associated to $|ij\rangle$ and $|ji\rangle$) by edges (i,j) of K_N.

Alternatively we may encode a fixed M as a list of the scalars followed by a list of the matrices in a suitable order. For our pair of matrices $R = \mathsf{F}(\sigma), S = \mathsf{F}(s)$, thus giving a functor, we will use:

$$\underline{\alpha}(R) = (a_1, a_2, ..., a_N, A(1,2), A(1,3), A(2,3), ..., A(N-1,N)) \tag{8}$$

$$\underline{\alpha}(S) = (b_1, b_2, ..., b_N, B(1,2), B(1,3), B(2,3), ..., B(N-1,N))$$

That is, to each edge (i,j) of K_N we will associate two matrices, giving $A(i,j)$ and $B(i,j)$, and so on, giving F this way.

3.2 The story so far: classification of braid representations

The classification of braid representations we need can be given in rank N indexing either by the set $\mathfrak{S}_N$ of two-coloured multitableaux; or, working up to the Σ_N symmetry, by the set $\mathfrak{T}_N$ of multisets of two-coloured compositions, of total degree N [27].

In rank $N = 2$ for example these multisets may be written

For each index λ there is an orbit of varieties of solutions. A variety is obtained (complete with named variables) by numbering the boxes of λ up to order within rows. (We will recall them explicitly shortly.) The size of this variety depends on whether one wants to give representations up to 'gauge' isomorphism (i.e. a set of representations that are a transversal of isomorphism classes) or *all* representations. The former is natural for B itself, but since extension to L will restrict the symmetry of isomorphism it is natural here to consider the latter.

We will retain most of the convenient language of [27]: An individual composition in a multiset is a *nation*; and a part in a composition is a *county*. In [27] these partitions depend initially on 4 cases: $+, -, 0$ and $/$, with $+$ and $-$ further refined into $\mathsf{a}, \underline{\mathsf{a}}, \mathsf{f}$ and $\underline{\mathsf{f}}$ as defined below in (3.10).

The rank $N = 2$ and $N = 3$ cases will be particularly important for us. In rank $N = 2$ the six varieties have convenient special names. Case □□ — a single county in a single nation — gives rise to the trivial or 0-variety of solutions. Case □ □ is the /-variety. Case  is the a-variety and (depending on box numbering) $\underline{\mathsf{a}}$-variety. Case  yields the f-variety and $\underline{\mathsf{f}}$-variety.

(**3.10**) We now recall explicitly the classification for $N = 2$ of all $R \in \mathsf{Match}^N(2,2)$ which satisfy the relation (4 II), i.e. the complete set from [27, Prop.3.21] of cc braid representations in rank $N = 2$. We have six families of solutions given here. In each of the following families all variables must be non-zero. Organisationally we also insist that in the last four cases $\alpha + \beta \neq 0$ since the subset $\alpha + \beta = 0$ is included in the $F_/$ family of solutions; and similarly in F_{f} and $F_{\underline{\mathsf{f}}}$ cases $\alpha \neq \beta$ as this is included in the cases F_{a} and $F_{\underline{\mathsf{a}}}$ respectively.

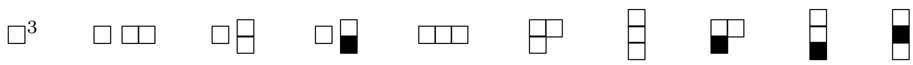
$$F_0 = \begin{pmatrix} \alpha &&& \\ & \alpha && \\ && \alpha & \\ &&& \alpha \end{pmatrix}, \qquad F_/ = \begin{pmatrix} \alpha &&& \\ && \gamma\chi & \\ & \frac{\gamma}{\chi} && \\ &&& \beta \end{pmatrix}, \qquad F_{\mathsf{f}} = \begin{pmatrix} \alpha &&& \\ & \alpha+\beta & \chi & \\ & -\frac{\alpha\beta}{\chi} && \\ &&& \alpha \end{pmatrix}$$

$$F_{\mathsf{a}} = \begin{pmatrix} \alpha &&& \\ & \alpha+\beta & \chi & \\ & -\frac{\alpha\beta}{\chi} && \\ &&& \beta \end{pmatrix}, \quad F_{\underline{\mathsf{f}}} = \begin{pmatrix} \alpha &&& \\ && \chi & \\ & -\frac{\alpha\beta}{\chi} & \alpha+\beta & \\ &&& \alpha \end{pmatrix}, \quad F_{\underline{\mathsf{a}}} = \begin{pmatrix} \alpha &&& \\ && \chi & \\ & -\frac{\alpha\beta}{\chi} & \alpha+\beta & \\ &&& \beta \end{pmatrix} \tag{9}$$

The parameter χ, where it appears, is here 'unphysical', i.e. it can be changed by X-symmetry (discussed in §4.1) and does not affect the spectrum. In contrast for $F_/$ say, α, β, γ all affect the spectrum. To match the 'gauge choice' made in [27] we should set $\chi = \gamma$ in $F_/$ (and γ here is μ there); and $\chi = \beta$ in other cases.

The aim of Section 5.1 then is to establish which of the above braid solutions extend to loop braid solutions. Precisely, given an $F(\sigma)$ of one of the above types, when does there exist an $F(s)$ such that all (2), (3), and (4) are satisfied?

(**3.11**) In case $N = 3$ we have (from [27, Prop.5.1]) the braid multisets



corresponding to the braid solutions as recalled in (A.1). There are also , , and , treated in [27] by using the additional flip symmetry.

The main point in extending the $N = 3$ braid solutions to loop braid is going to be to note from Lemma 5.2 that extension at each edge is, up to sign, either by an identity matrix (extending type 0) — so that the vertex eigenvalues at each end are forced the same; or by a signed permutation/P matrix where the vertex signs are either forced different (type a) or nominally free (type /); with no extension for type f. We will thus see the following rules:
1. In the same county vertices must have the same sign;
2. In different counties in the same nation the signs are different (hence there are at most two

counties in a nation);
And a 'non-rule': Vertex signs are not constrained between different nations.

In §5.1 we obtain the $N = 2$ Lemma. In §6 we will then implement the rules to obtain all solutions in all ranks.

4 Preparations for calculus in Match categories

Both to prove the main Theorem, and the key Lemma, we will need to be able to compute in Match categories. Here we develop the required machinery.

(**4.1**) Let $\mathscr{M}$ be a monoidal presentation for a strict monoidal, indeed natural, category C. We say a relation has width n if it is in $hom_C(n, n)$. For example the relation (4(II)) above has width 3. We say the presentation $\mathscr{M}$ has width w if w is the maximum width among relations in $\mathscr{M}$. For example the presentation in (2-4) has width 3.

(**4.2**) Let C be presented by $\mathscr{M}$. We may give a monoidal functor $\mathsf{F} : C \to D$ by giving the images of the generators. A function F from generators to D gives a functor F provided that the images of the relations hold. The image of each relation $a = b$ say can be checked by checking $F(A) = 0$ where the 'anomaly' $F(A) = F(a) - F(b) \in D(w, w)$ (we assume D linear).

As noted in (3.5), a basis element of the space acted on by $\mathsf{Match}^N(w, w)$ is $|i_1 i_2..i_w\rangle$, $i_j \in \{1, 2, .., N\}$. By the cc property every $\mathsf{F}(A)|i_1 i_2..i_w\rangle = \sum_{\sigma\in\Sigma_w} k_\sigma(\mathsf{F}(A))|\sigma i_1 i_2..i_w\rangle$ for some scalars k_σ, indeed for every $M \in \mathsf{Match}^N(w, w)$, $M|i_1 i_2..i_w\rangle = \sum_{\sigma\in\Sigma_w} k_\sigma(M)|\sigma i_1 i_2..i_w\rangle$, i.e. the $\mathsf{F}(A)$ action on $\underline{N}^w$ breaks into Σ_w orbits. Thus:

(**4.3**) **Lemma**. The image of a relation of width w is verified if and only if it is verified on each subspace $\{\sigma|i_1 i_2..i_w\rangle \mid \sigma \in \Sigma_w\}$, the subspace of the subset $\{i_1, i_2, .., i_w\} \subset \underline{N}$. We note that this is the same as verification on $\mathsf{Match}^w(w, w)$ up to relabelling, *for each subset*. I.e. the same as to say that F restricts to a functor $\mathsf{F}' : \mathsf{L} \to \mathsf{Match}^w$ on each subset. (But note also that the various subsets interlock, and every restriction must hold.) □

(**4.4**) **Remark**. Note that the monoidal presentation (2-4) has width 3. Thus a pair F_* as in (5) induces a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ if anomalies vanish in every restriction to $\{i, j, k\} \subseteq \underline{N}$.

Specifically for $R_1 R_2 R'_1 = R'_2 R_1 R_2$ say — with $\underline{\alpha}(R) = (A_1, A_2, .., A_N, \left(\begin{smallmatrix} A_{12} & B_{12} \\ C_{12} & D_{12}\end{smallmatrix}\right), \left(\begin{smallmatrix} A_{13} & B_{13} \\ C_{13} & D_{13}\end{smallmatrix}\right), ..)$ (as in 8), $\underline{\alpha}(R') = (a_1, a_2, .., a_N, \left(\begin{smallmatrix} a_{12} & b_{12} \\ c_{12} & d_{12}\end{smallmatrix}\right), ..)$ say — every term in the anomaly acting on $|ijk\rangle$ is a cubic with indices in i, j, k. E.g. $A_{12}A_{23}a_{12}$, $A_{13}B_{12}c_{12}$ when $ijk = 123$.

4.1 X-symmetry

Let C be a natural category (a strict monoidal category with object monoid freely generated by a single object, denoted 1). The proof of the X-Lemma in [27] observes that if R is the image of a generic element in $C(2, 2)$ under a functor $F : C \to \mathsf{Match}^N$ — thus with elements $a_{ij} = a_{ij}(R)$ and so on — then the *braid anomaly*

$$A_R \;:=\; (R \otimes 1_N)(1_N \otimes R)(R \otimes 1_N) - (1_N \otimes R)(R \otimes 1_N)(1_N \otimes R)$$

has entries that are cubics in the various indeterminate entries in R; but in particular in each entry we have one of the following (writing b_{ij} for $b_{ij}(R)$ and so on):
b_{ij} (or c_{ij}) appears as an overall factor;
b_{ij} and c_{ij} only appear in the form $b_{ij}c_{ij}$.

Each of the three factors in a term in the cubic come, note, from one of the three factors in a term in A_R. Now suppose we have a second element S in $C(2, 2)$, with entries $a_{ij}(S)$ and so on. Then in A_{RRS}, say, one of the factors in each term in a cubic will now comes from S, so the cubics are modified by some b_{ij} becoming $b_{ij}(S)$ and so on. We observe that simultaneous conjugation by an invertible diagonal matrix still preserves equalities for such cubics. The general X-Lemma follows immediately from this.

(**4.5**) **Lemma**. Let $X \in \mathsf{Mat}^N(2,2)$ be any invertible diagonal matrix (hence $X \in \mathsf{Match}^N(2,2)$) and $F : \{\sigma, s\} \to \mathsf{Match}^N(2,2)$ be any pair. If F induces a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ then so does F^X where $F^X(\sigma) = XF(\sigma)X^{-1}$ and $F^X(s) = XF(s)X^{-1}$. □

A more explicit proof is given in §B.

This construction gives an action on the set of all functors, denoted $(\mathsf{L}, \mathsf{Match}^N)$ (see also (7.4)), of the abelian group $\Delta^N := \mathsf{Mat}^N_\Delta(2,2)$ of invertible diagonal matrices.

This action together with the action of Σ_N (the bijections from (3.7)) generate a group of symmetries of $(\mathsf{L}, \mathsf{Match}^N)$, that we can call 'core symmetries'.

4.2 Ket calculus: conventions

(**4.6**) Recall (see e.g. [27]) the convention that the Mat^N (and hence Match^N) categories use the Kronecker product in the Ab convention. The Ab convention is as indicated by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \end{pmatrix} = \begin{pmatrix} ae & be & af & bf \\ ce & de & cf & df \end{pmatrix}, \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_1 \\ a_1 b_2 \\ a_2 b_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(In fact either convention is fine, but we need to fix one. Note that *Maxima* and MAPLE use the aB convention by default.) This means in particular that the ordered basis 1,2 of $\mathbb{C}^2$ (specifically we might take $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, although even this is a choice) passes, for $(\mathbb{C}^2)^2$, to order 11,21,12,22 with earlier indices changing more quickly. That is $|21\rangle = |2\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Thus without further adulteration

$$\begin{aligned}
\begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} |21\rangle &= \begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ c \\ 0 \end{pmatrix} = a|21\rangle + c|12\rangle \\
\begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} |12\rangle &= \begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ d \\ 0 \end{pmatrix} = b|21\rangle + d|12\rangle \\
\langle 21| \begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix} = \begin{pmatrix} 0 & a & b & 0 \end{pmatrix} = a\langle 21| + b\langle 12|
\end{aligned} \tag{10}$$

For $M, N \in \mathsf{Match}^N(1,1)$ we have $M \otimes N|ij\rangle = M|i\rangle \otimes N|j\rangle$ and so on. [4]

4.3 Back to generalities

(**4.7**) Let 1_N denote the identity matrix in $\mathsf{Match}^N(1,1)$. Given $R, S \in \mathsf{Match}^N(2,2)$ then define

$$A_{RRS} = (R \otimes 1_N)(1_N \otimes R)(S \otimes 1_N) - (1_N \otimes S)(R \otimes 1_N)(1_N \otimes R) \quad \in \mathsf{Match}^N(3,3) \tag{11}$$

In particular this yields the *braid anomaly* matrix $A_R = A_{RRR}$ by substituting $S = R$.

In general for $R \in \mathsf{Match}^N(2,2)$ we will write $R_1 = R \otimes 1_N$ and $R_2 = 1_N \otimes R$. So then $A_{RRR} = R_1 R_2 R_1 - R_2 R_1 R_2$.

In light of (4) it will be useful to have facility with computing this kind of operator. One approach is to resolve charge-conserving matrices as sums of monomial matrices, as in §4.4. Another is to use a ket calculus as in §B.

[4] In [27] the notation $|\underline{ij}\rangle$ is used, and different conventions are adopted. The choice of conventions is, in any case, essentially unimportant there. But here we must be careful.

4.4 Calculus via sums of monomial matrices

(**4.8**) Supposing that $N \in \mathbb{N}$ is fixed, let $\mathsf{P} = \mathsf{P}_{(N)}$ denote the simple permutation matrix in $\mathsf{Match}^N(2,2)$, given by $\mathsf{P}|ij\rangle = |ji\rangle$ for all i,j; and 1_N the identity matrix in $\mathsf{Match}^N(1,1)$. Indeed for given N we may simply write P for $\mathsf{P}_{(N)}$ and $\mathbf{1}$ for 1_N.

Note that one way to resolve $R \in \mathsf{Match}^N(2,2)$ into a sum of two monomial matrices is

$$R \;=\; \Delta(R)\,(1_N \otimes 1_N) \;+\; D(R)\,\mathsf{P} \tag{12}$$

where the diagonal matrix $\Delta(R)$ has zeros in the ii positions and the diagonal matrix $D(R)$ does not. Thus for example with $N = 2$:

$$\Delta(R) = \begin{pmatrix} 0 &&& \\ & a_{12}(R) && \\ && d_{12}(R) & \\ &&& 0 \end{pmatrix} \quad \text{and} \quad D(R) = \begin{pmatrix} a_1(R) &&& \\ & b_{12}(R) && \\ && c_{12}(R) & \\ &&& a_2(R) \end{pmatrix} \tag{13}$$

so

$$R = \begin{pmatrix} 0 &&& \\ & a_{12}(R) && \\ && d_{12}(R) & \\ &&& 0 \end{pmatrix} 1_4 + \begin{pmatrix} a_1(R) &&& \\ & b_{12}(R) && \\ && c_{12}(R) & \\ &&& a_2(R) \end{pmatrix} \mathsf{P} \tag{14}$$

(**4.9**) Consider the case $R = D(R)P_N$. A useful observation is

$$D(R)\,\mathsf{P} \;=\; \mathsf{P}\,D^{-}(R) \tag{15}$$

where $D^{-}(R)$ is given by swapping the 12,12 and 21,21 entries and so on, so here b_{ij} and c_{ij}.

So for example in the simple monomial case, with $\mathcal{R} = D(r)\mathsf{P}$ and $\mathcal{R}' = D(R)\mathsf{P}$, say, then

$$\mathcal{R}_1\mathcal{R}'_2 \;=\; ((D(r)\mathsf{P}) \otimes 1_N)\;\; (1_N \otimes (D(R)\mathsf{P})) \tag{16}$$

can be 'straightened' by promoting the second $D(R)$. Of course the tensor product affects the promotion.

For example with $N = 2$, and naming variables by: $\mathcal{R} = \begin{pmatrix} a_1 &&& \\ && b_{12} & \\ & c_{12} && \\ &&& a_2 \end{pmatrix}$, $\mathcal{R}' = \begin{pmatrix} A_1 &&& \\ && B_{12} & \\ & C_{12} && \\ &&& A_2 \end{pmatrix}$, $\mathcal{R}'' = \begin{pmatrix} \alpha_1 &&& \\ && \beta_{12} & \\ & \gamma_{12} && \\ &&& \alpha_2 \end{pmatrix}$, the product in (16) becomes

$$\mathcal{R}_1\mathcal{R}'_2 = \begin{pmatrix} a_1 &&&&&&& \\ & b_{12} &&&&&& \\ && c_{12} &&&&& \\ &&& a_2 &&&& \\ &&&& a_1 &&& \\ &&&&& b_{12} && \\ &&&&&& c_{12} & \\ &&&&&&& a_2 \end{pmatrix} P_1 \begin{pmatrix} A_1 &&&&&&& \\ & A_1 &&&&&& \\ && B_{12} &&&&& \\ &&& B_{12} &&&& \\ &&&& C_{12} &&& \\ &&&&& C_{12} && \\ &&&&&& A_2 & \\ &&&&&&& A_2 \end{pmatrix} P_2$$

$$= \begin{pmatrix} a_1A_1 &&&&&&& \\ & b_{12}B_{12} &&&&&& \\ && c_{12}A_1 &&&&& \\ &&& a_2B_{12} &&&& \\ &&&& a_1C_{12} &&& \\ &&&&& b_{12}A_2 && \\ &&&&&& c_{12}C_{12} & \\ &&&&&&& a_2A_2 \end{pmatrix} P_1P_2 \tag{17}$$

Thus

$$\mathcal{R}_1\mathcal{R}'_2\mathcal{R}''_1 \;=\; ((D(r)\mathsf{P}) \otimes 1_N)\;\; (1_N \otimes (D(R)\mathsf{P}))\;\; ((D(\rho)\mathsf{P}) \otimes 1_N)$$

(note again the chosen variable names) can be 'straightened' by

$$\begin{pmatrix} a_1A_1 &&&&&&& \\ & b_{12}B_{12} &&&&&& \\ && c_{12}A_1 &&&&& \\ &&& a_2B_{12} &&&& \\ &&&& a_1C_{12} &&& \\ &&&&& b_{12}A_2 && \\ &&&&&& c_{12}C_{12} & \\ &&&&&&& a_2A_2 \end{pmatrix} P_1P_2 \begin{pmatrix} \alpha_1 &&&&&&& \\ & \beta_{12} &&&&&& \\ && \gamma_{12} &&&&& \\ &&& \alpha_2 &&&& \\ &&&& \alpha_1 &&& \\ &&&&& \beta_{12} && \\ &&&&&& \gamma_{12} & \\ &&&&&&& \alpha_2 \end{pmatrix} P_1$$

$$= \begin{pmatrix} a_1A_1\alpha_1 &&&&&&& \\ & b_{12}B_{12}\alpha_1 &&&&&& \\ && c_{12}A_1\beta_{12} &&&&& \\ &&& a_2B_{12}\beta_{12} &&&& \\ &&&& a_1C_{12}\gamma_{12} &&& \\ &&&&& b_{12}A_2\gamma_{12} && \\ &&&&&& c_{12}C_{12}\alpha_2 & \\ &&&&&&& a_2A_2\alpha_2 \end{pmatrix} P_1P_2P_1$$

Meanwhile

$$\mathcal{R}_2''\mathcal{R}_1\mathcal{R}_2' \;=\; (1_N \otimes (D(\rho)\mathsf{P})) \;\; ((D(r)\mathsf{P}) \otimes 1_N) \;\; (1_N \otimes (D(R)\mathsf{P})) \;\; =$$

$$\begin{pmatrix} \alpha_1 &&&&&&& \\ & \alpha_1 &&&&&& \\ && \beta_{12} &&&&& \\ &&& \beta_{12} &&&& \\ &&&& \gamma_{12} &&& \\ &&&&& \gamma_{12} && \\ &&&&&& \alpha_2 & \\ &&&&&&& \alpha_2 \end{pmatrix} P_2 \begin{pmatrix} a_1A_1 &&&&&&& \\ & b_{12}B_{12} &&&&&& \\ && c_{12}A_1 &&&&& \\ &&& a_2B_{12} &&&& \\ &&&& a_1C_{12} &&& \\ &&&&& b_{12}A_2 && \\ &&&&&& c_{12}C_{12} & \\ &&&&&&& a_2A_2 \end{pmatrix} P_1P_2$$

$$= \begin{pmatrix} \alpha_1a_1A_1 &&&&&&& \\ & \alpha_1b_{12}B_{12} &&&&&& \\ && \beta_{12}a_1C_{12} &&&&& \\ &&& \beta_{12}b_{12}A_2 &&&& \\ &&&& \gamma_{12}c_{12}A_1 &&& \\ &&&&& \gamma_{12}a_2B_{12} && \\ &&&&&& \alpha_2c_{12}C_{12} & \\ &&&&&&& \alpha_2a_2A_2 \end{pmatrix} P_2P_1P_2$$

From this elementary warm-up exercise we observe immediately that the permutation factors in $\mathcal{R}_1\mathcal{R}_2'\mathcal{R}_1''$ and $\mathcal{R}_2''\mathcal{R}_1\mathcal{R}_2'$ agree: $\mathsf{P}_1\mathsf{P}_2\mathsf{P}_1 = \mathsf{P}_2\mathsf{P}_1\mathsf{P}_2$, so that for example, we have the following.

Lemma 4.10. *Fix $N = 2$ and $\mathsf{P} = \mathsf{P}_{(N=2)}$. For all $\mathcal{R} = D(\mathcal{R})\ \mathsf{P}$ and $R'' = \Delta(R'')1_4 + D(R'')\mathsf{P}$ (including singular)*

$$\mathcal{R}_1\ \mathcal{R}_2\ R_1'' \;=\; R_2''\ \mathcal{R}_1\ \mathcal{R}_2 \tag{18}$$

Proof. First consider the case $R'' = \mathcal{R}'' = D(R'')\mathsf{P}$. Compare the diagonal factors in the evaluations of $\mathcal{R}_1\mathcal{R}_2'\mathcal{R}_1''$ and $\mathcal{R}_2''\mathcal{R}_1\mathcal{R}_2'$ above and note that in the present case $\mathcal{R} = \mathcal{R}'$ so $a_1 = A_1$, $c_{12} = C_{12}$, so $a_1C_{12} = a_1c_{12} = A_1c_{12}$ and so on. Thus (18) holds in this case.

Now suppose we turn on the diagonal terms in (14) for R''. Thus $R'' = \Delta'' + Q''$, say. Of course $\mathcal{R}_1\mathcal{R}_2Q_1'' = Q_2''\mathcal{R}_1\mathcal{R}_2$ from the established case of (18) so the full version requires $\mathcal{R}_1\mathcal{R}_2\Delta_1'' - \Delta_2''\mathcal{R}_1\mathcal{R}_2$ to vanish. Again by the established part of the Lemma we have $\mathcal{R}_1\mathcal{R}_2\Delta_1''\mathsf{P}_1 = \Delta_2''\mathsf{P}_2\mathcal{R}_1\mathcal{R}_2$, so $\mathcal{R}_1\mathcal{R}_2\Delta_1'' = \Delta_2''\mathsf{P}_2\mathcal{R}_1\mathcal{R}_2\mathsf{P}_1 = \Delta_2''\mathsf{P}_2\mathsf{P}_2\mathcal{R}_1\mathcal{R}_2$ (using established special case $\mathcal{R}_1\mathcal{R}_2\mathsf{P}_1 = \mathsf{P}_2\mathcal{R}_1\mathcal{R}_2$ at the last step) as required. □

Thus in particular we have, for each $\mathcal{R} = D(\mathcal{R})\mathsf{P}$, a solution to the braid relation by putting $R'' = \mathcal{R}$, but we also have solutions to the mixed relations for any R'' of the more general form.

5 Rank $N = 2$ loop braid solutions

We will prove here that there are three kinds of solutions in rank $N = 2$, given in Lem.5.2.

(**5.1**) A couple of organisational principles are convenient to have in mind for spin-chain braid representations $F : \mathsf{B} \to \mathsf{Match}^N$. These are derived more or less directly from [27] (and familiarity with this paper will significantly help the reader here).

(P.I) Each braid representation in rank N restricts to a representation in lower rank by taking a subset of indices $\{1, 2, ..., N\}$. Thus in particular each representation with $N > 2$ restricts to a collection of $N = 2$ representations (just as complete graph K_N restricts to a collection of K_2s). And these $N = 2$ representations fall into one of six types: 0, /, a, $\underline{\mathsf{a}}$, f, $\underline{\mathsf{f}}$ (recalled explicitly in (9)).

(P.II) Noting that each $N = 2$ braid representation is given by $F(\sigma)$ parameterised by

$$F(\sigma) = \begin{pmatrix} a_1 &&& \\ & a_{12} & b_{12} & \\ & c_{12} & d_{12} & \\ &&& a_2 \end{pmatrix}$$

we note (a) that either $F(\sigma)$ is a scalar multiple of the identity, or at least one of a_{12}, d_{12} vanishes; (b) that a_1, a_2 are eigenvalues, and $a_{12} + d_{12}$ is the sum of the 'middle two' eigenvalues.
(P.III) Restricted to the generator s our $F(s)$ must give a braid representation that is also a representation of the permutation category — of symmetric groups. In particular $s^2 = 1$ so the Jordan form of $F(s)$ is diagonal. For $F(s)$ with $N = 2$ its middle 2×2 has eigenvalues $+1, +1$ or $-1, -1$, or eigenvalues $1, -1$. Thus either it is up-to-sign the identity matrix — whereupon we have $F(s) = \pm 1_4$, type 0 by the classification; or else its diagonal entries obey $a_{12} + d_{12} = 0$ and hence (cf. (P.II)) $a_{12} = d_{12} = 0$ — type /.
(P.IV) By the X-Lemma (see (4.5)) each X-orbit of solutions has a representative where the nonzero off-diagonal elements of $F(s)$ are all 1.

5.1 Rank $N = 2$ loop braid solutions: main Lemma

Here we give a complete set of solutions in rank $N = 2$:

Lemma 5.2. *If $(F(s), F(\sigma))$ gives a loop braid functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^2$ then it is one of the following (organised according to the braid representation type of $F(\sigma)$).*

(I) If $R = F(\sigma)$ is of type a or type $\underline{\mathsf{a}}$ then $F(s)$ must take the form $S = \begin{pmatrix} 1 & & & \\ & & 1/c & \\ & c & & \\ & & & -1 \end{pmatrix} = \Sigma\mathsf{P}$ where $\Sigma = \begin{pmatrix} 1 & & & \\ & 1/c & & \\ & & c & \\ & & & -1 \end{pmatrix}$ for some $c \neq 0$, up to overall sign. Then for each specific such S, i.e. each c, we get a type a solution if and only if for some A_1, A_2 we have $R = \begin{pmatrix} A_1 & & & \\ & A_1{+}A_2 & A_1/c & \\ & -cA_2 & 0 & \\ & & & A_2 \end{pmatrix}$. And similarly in type $\underline{\mathsf{a}}$. Note this means that the braid "gauge" parameter χ in (9) is locked to $\chi = A_1/c$. In particular if we want again to be free to choose χ then $c = A_1/\chi$ is forced; and if we want to chose c (to set $c = 1$ say, see later), for given A_1, then χ is not *free. Note $A_1, A_2 \neq 0$ by invertability and $A_1 + A_2 \neq 0$ in type a or $\underline{\mathsf{a}}$. Note, in this case $SR = \begin{pmatrix} A_1 & & & \\ & -A_2 & & \\ & c(A_1{+}A_2) & A_1 & \\ & & & -A_2 \end{pmatrix}$.*

Applying f^σ from (6) we get $f^\sigma(S) = \begin{pmatrix} -1 & & & \\ & & c & \\ & 1/c & & \\ & & & 1 \end{pmatrix}$ and $f^\sigma(R) = \begin{pmatrix} A_2 & & & \\ & 0 & -cA_2 & \\ & A_1/c & A_1{+}A_2 & \\ & & & A_1 \end{pmatrix}$. We can bring this S back into the previous form by applying the homomorphism given by $S \rightsquigarrow -S$ and taking a new c given by $c \rightsquigarrow -1/c$. Since (almost) all A_1, A_2 give solutions, interchanging them takes us to a point on the same variety, then giving $R \rightsquigarrow \begin{pmatrix} A_1 & & & \\ & 0 & A_1/c & \\ & -cA_2 & A_1{+}A_2 & \\ & & & A_2 \end{pmatrix}$, which, with the restored S, is thus a viable parameterisation for type $\underline{\mathsf{a}}$.
(II) If $R = F(\sigma)$ is of type f or $\underline{\mathsf{f}}$ there is no solution.
(III) If $R = F(\sigma)$ is of type 0 then $S = \pm 1_4$, and there is no further constraint on $F(\sigma)$.
(IV) If $R = F(\sigma)$ is of type / then $S = \Sigma\mathsf{P}$ where $\Sigma = \begin{pmatrix} 1 & & & \\ & 1/c & & \\ & & c & \\ & & & \pm 1 \end{pmatrix}$ for some $c \neq 0$, up to overall sign, and then $R = \begin{pmatrix} A_1 & & & \\ & & \mu/Cc & \\ & \mu Cc & & \\ & & & A_2 \end{pmatrix}$ with the four further variables A_1, A_2, C, μ non-zero but otherwise free. Note these variables are independent of c. To match (9) we put $\mu Cc = \gamma/\chi$ and $\mu/Cc = \gamma\chi$, so $\mu/\gamma = \gamma/\mu = Cc\chi$ so $\mu = \gamma$ and $Cc = 1/\chi$. The parameterisation has been chosen so that c again captures the X-symmetry, while A_1, A_2, C, μ are 'physical/non-gauge' in the sense that (unlike c) they do *affect operator spectrum. In particular $SR = \begin{pmatrix} A_1 & & & \\ & \mu C & & \\ & & \mu/C & \\ & & & A_2 \end{pmatrix}$.*

Proof. Observe that all possibilities for $F(\sigma)$ are considered, by (9), so it remains to verify the extensions $F(s)$ in each case and determine any further constraints on $F(\sigma)$. First consider the general form of $F(s) = S$. The braid classification as in (9) applies to $F(s)$, and (5.1)(P.III), so S is type 0 or /. In cases (I), (II), (IV) $R_1 \neq R_2$, so from the $R_1 S_2 S_1 = S_2 S_1 R_2$ identity (4)(III) we have that S cannot be the unit matrix (or minus) here. So by (5.1)(P.III) we have that $S = \Sigma\mathsf{P}$ where $\Sigma = \begin{pmatrix} 1&&& \\ &1/c&& \\ &&c& \\ &&&\pm1 \end{pmatrix}$ for some c and some sign, up to an overall sign. Conversely in case (III) S must be type 0.

Cases (I) and (II):

Here we consider $F(\sigma)$ of form $R = \Delta + \mathcal{R} = \Delta + D\mathsf{P}$ where $\Delta = \begin{pmatrix} 0&&& \\ &a&& \\ &&d& \\ &&&0 \end{pmatrix}$ and

$$D\mathsf{P} = \begin{pmatrix} A_1&&& \\ &&B& \\ &C&& \\ &&&A_2 \end{pmatrix} = \begin{pmatrix} A_1&&& \\ &B&& \\ &&C& \\ &&&A_2 \end{pmatrix} \begin{pmatrix} 1&&& \\ &&1& \\ &1&& \\ &&&1 \end{pmatrix}$$

for given $, A_1, A_2$ and suitable a, d, B, C. This is the most general form, but we know in particular from (5.1)(II) that $ad = 0$. For this case of R, then, the RRS anomaly here takes the form

$$(\Delta + \mathcal{R})_1\ (\Delta + \mathcal{R})_2\ S_1\ -\ S_2\ (\Delta + \mathcal{R})_1\ (\Delta + \mathcal{R})_2 \tag{19}$$

Expanding (19) we get

$$\underbrace{\mathcal{R}_1\mathcal{R}_2S_1 - S_2\mathcal{R}_1\mathcal{R}_2}_{=0} + \mathcal{R}_1\Delta_2S_1 - S_2\mathcal{R}_1\Delta_2 + \Delta_1\mathcal{R}_2S_1 - S_2\Delta_1\mathcal{R}_2 + \underbrace{\Delta_1\Delta_2S_1}_{=0} - \underbrace{S_2\Delta_1\Delta_2}_{=0}$$

where the first cancellation follows from 4.10; and the $\Delta_1\Delta_2 = 0$ because by [27] in type a or f we have $\Delta = \begin{pmatrix} 0&&& \\ &a&& \\ &&0& \\ &&&0 \end{pmatrix}$ (or indeed $\Delta = \begin{pmatrix} 0&&& \\ &0&& \\ &&d& \\ &&&0 \end{pmatrix}$ for $\underline{\mathsf{a}}$ or $\underline{\mathsf{f}}$ which cases we can treat by symmetry).

Keeping to the $d = 0$ case for now, the first term we need to compute is thus

$$\mathcal{R}_1\Delta_2S_1 = D_1\mathsf{P}_1\Delta_2\Sigma_1\mathsf{P}_1 = D_1\mathsf{P}_1 \begin{pmatrix} 0&&&&&&& \\ &0&&&&&& \\ &&a&&&&& \\ &&&a&&&& \\ &&&&0&&& \\ &&&&&0&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \begin{pmatrix} 1&&&&&&& \\ &1/c&&&&&& \\ &&c&&&&& \\ &&&\pm1&&&& \\ &&&&1&&& \\ &&&&&1/c&& \\ &&&&&&c& \\ &&&&&&&\pm1 \end{pmatrix} \mathsf{P}_1 = D_1 \begin{pmatrix} 0&&&&&&& \\ &ca&&&&&& \\ &&0&&&&& \\ &&&\pm a&&&& \\ &&&&0&&& \\ &&&&&0&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \mathsf{P}_1^2$$

while the second is

$$S_2\mathcal{R}_1\Delta_2 = \Sigma_2\mathsf{P}_2D_1\mathsf{P}_1\Delta_2 = \begin{pmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1/c&&&&& \\ &&&1/c&&&& \\ &&&&c&&& \\ &&&&&c&& \\ &&&&&&\pm1& \\ &&&&&&&\pm1 \end{pmatrix} \mathsf{P}_2 \begin{pmatrix} A_1&&&&&&& \\ &B&&&&&& \\ &&C&&&&& \\ &&&A_2&&&& \\ &&&&A_1&&& \\ &&&&&B&& \\ &&&&&&C& \\ &&&&&&&A_2 \end{pmatrix} \mathsf{P}_1 \begin{pmatrix} 0&&&&&&& \\ &0&&&&&& \\ &&a&&&&& \\ &&&a&&&& \\ &&&&0&&& \\ &&&&&0&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1/c&&&&& \\ &&&1/c&&&& \\ &&&&c&&& \\ &&&&&c&& \\ &&&&&&\pm1& \\ &&&&&&&\pm1 \end{pmatrix} \mathsf{P}_2 \begin{pmatrix} 0&&&&&&& \\ &Ba&&&&&& \\ &&0&&&&& \\ &&&A_2a&&&& \\ &&&&0&&& \\ &&&&&0&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \mathsf{P}_1 = \begin{pmatrix} 0&&&&&&& \\ &Ba&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&0&&& \\ &&&&&cA_2a&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \mathsf{P}_2\mathsf{P}_1$$

then

$$\Delta_1\mathcal{R}_2S_1 = \begin{pmatrix} 0&&&&&&& \\ &a&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&0&&& \\ &&&&&a&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \begin{pmatrix} A_1&&&&&&& \\ &A_1&&&&&& \\ &&B&&&&& \\ &&&B&&&& \\ &&&&C&&& \\ &&&&&C&& \\ &&&&&&A_2& \\ &&&&&&&A_2 \end{pmatrix} \mathsf{P}_2 \begin{pmatrix} 1&&&&&&& \\ &1/c&&&&&& \\ &&c&&&&& \\ &&&\pm1&&&& \\ &&&&1&&& \\ &&&&&1/c&& \\ &&&&&&c& \\ &&&&&&&\pm1 \end{pmatrix} \mathsf{P}_1$$

$$= \begin{pmatrix} 0&&&&&&& \\ &aA_1/c&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&0&&& \\ &&&&&aC(\pm1)&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \mathsf{P}_2\mathsf{P}_1$$

and

$$S_2\Delta_1\mathcal{R}_2 = \Sigma_2\mathsf{P}_2 \begin{pmatrix} 0&&&&&&& \\ &a&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&0&&& \\ &&&&&a&& \\ &&&&&&0& \\ &&&&&&&0 \end{pmatrix} \begin{pmatrix} A_1&&&&&&& \\ &A_1&&&&&& \\ &&B&&&&& \\ &&&B&&&& \\ &&&&C&&& \\ &&&&&C&& \\ &&&&&&A_2& \\ &&&&&&&A_2 \end{pmatrix} \mathsf{P}_2$$

$$=\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1/c &&&&& \\ &&& 1/c &&&& \\ &&&& c &&& \\ &&&&& c && \\ &&&&&& \pm1 & \\ &&&&&&& \pm1 \end{pmatrix} \mathsf{P}_2 \begin{pmatrix} 0 &&&&&&& \\ & aA_1 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& aC && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix} \mathsf{P}_2 = \begin{pmatrix} 0 &&&&&&& \\ & aA_1 &&&&&& \\ && 0 &&&&& \\ &&& aC/c &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}$$

Observe from the P factors that $\mathcal{R}_1\Delta_2 S_1$ must cancel $-S_2\Delta_1\mathcal{R}_2$; and $-S_2\mathcal{R}_1\Delta_2$ must cancel $\Delta_1\mathcal{R}_2 S_1$. Necessary and sufficient in both cases are:

$$Bca = aA_1, \qquad \pm aA_2 = aC/c \tag{20}$$

Here $a \neq 0$ and so $B = A_1/c$ and $C = \pm cA_2$, so $BC = \pm A_1A_2$. Since BC is determined by A_1, A_2 we are in type a not f, so $BC = -A_1A_2$ so the relative sign in S is forced – as given in the Lemma; and $a = A_1 + A_2$ and R is as given. This concludes the proof of (I) and (II).

Note that if we want to fix $c = 1$ in case (I) (using the X-symmetry as in Lemma 4.5) then (having also fixed that $\langle 11|S|11\rangle = 1$ and determined then that $\langle 22|S|22\rangle = -1$) we also fix $B = A_1$ and $C = -A_2$. The only way to vary these latter two is to vary c in S.

Case (III):

Notice first that in each case, (2), (3), and (4) are clearly satisfied. With $F(\sigma)$ of the form $F(\sigma) = \alpha 1_4$, writing the ansatz

$$F(s) = \begin{pmatrix} a_1 &&& \\ & a_{12} & b_{12} & \\ & c_{12} & d_{12} & \\ &&& a_2 \end{pmatrix} \tag{21}$$

then (4 II) gives $F(s) \otimes 1_2 = 1_2 \otimes F(s)$, hence

$$\begin{pmatrix} a_1 &&&&&&& \\ & a_{12} & b_{12} &&&&& \\ & c_{12} & d_{12} &&&&& \\ &&& a_2 &&&& \\ &&&& a_1 &&& \\ &&&&& a_{12} & b_{12} & \\ &&&&& c_{12} & d_{12} & \\ &&&&&&& a_2 \end{pmatrix} = \begin{pmatrix} a_1 &&&&&&& \\ & a_1 &&&&&& \\ && a_{12} && b_{12} &&& \\ &&& a_{12} && b_{12} && \\ && c_{12} && d_{12} &&& \\ &&& c_{12} && d_{12} && \\ &&&&&& a_2 & \\ &&&&&&& a_2 \end{pmatrix}$$

We immediately read off $a_1 = a_{12} = d_{12} = a_2$ and $b_{12} = 0 = c_{12}$. With $s^2 = 1$, this gives the result.

Case (IV):

From the first paragraph, any $F(s)$ exist extending the / variety, they are also in the / variety. Now, with

$$F(s) = \mathfrak{d}\mathsf{P} = \begin{pmatrix} a_1 &&& \\ & b_{12} && \\ && c_{12} & \\ &&& a_2 \end{pmatrix} \begin{pmatrix} 1 &&& \\ && 1 & \\ & 1 && \\ &&& 1 \end{pmatrix},$$

(4 II) becomes

$$(D\mathsf{P} \otimes 1_2)(1_2 \otimes D\mathsf{P})(\mathfrak{d}\mathsf{P} \otimes 1_2) - (1_2 \otimes \mathfrak{d}\mathsf{P})(D\mathsf{P} \otimes 1_2)(1_2 \otimes D\mathsf{P}) = 0$$

which, by Lemma 4.10 is satisfied for any matrix $\mathfrak{d}$. Using Lemma4.10 again, (4 III) adds no additional constraints, thus using (5.1) we have all solutions. □

6 Constructions for the main Theorem

Our task here is first to give certain combinatorial sets $\mathbb{S}_N$ and $J_N^{\pm} \hookrightarrow \mathbb{S}_N$ for each rank N; second to give a parameter space for each $\lambda \in \mathbb{S}_N$; and third to explain how each element and point in parameter space yields a decoration of the complete graph K_N, that encodes a pair of matrices in $\mathsf{Match}^N(2,2)$, and hence formally a monoidal functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ Finally we must prove that this pair indeed gives a functor, and that all such functors arise this way.

Again following [27], we can enumerate solutions in two ways. One is to give all solution varieties; and the other is to give a transversal up to the Σ_N symmetries manifested by Lem.3.7. (Recall, e.g. from [27], that the absolute notion of isomorphism is less straightforward for monoidal-category representation theory than for Artinian representation theory. Our objective here is not to give a transversal with respect to some ultimate notion of isomorphism, but to understand all representations, taking advantage of the isomorphisms that serve this practical end.)

6.1 Index sets for enumerating solution varieties

Here we construct the index sets $\mathbb{S}_N$, and corresponding parameter spaces (we match to actual solutions in §6.2).

(**6.1**) Recall from [27] that for braid representations in rank-N we start with a partition of the vertices of K_N, calling the parts 'nations'. We then partition each nation further into 'counties'. In [27] the counties are ordered in each nation and partitioned into two subsets. The set of such structures is the set $\mathfrak{S}_N$ (whose elements are visualised in [27] as collections of two-coloured composition tableaux).

In our loop-braid case we find that, in order for a braid representation to extend to loop-braid, we must restrict to at most two counties per nation. The counties in each nation are ordered, so this amounts to the choice of a (first county) subset for each nation. (We then colour the counties from two colours, but in our case the colours are forced.)

Next comes the new ingredient that is not merely a restriction, which is that a subset of the nations is chosen (those that will be associated to +, i.e. +1 eigenvalue of S for individuals in the first county).

The above features characterise the loop-braid index set $\mathbb{S}_N$. It is useful also to give a more formal construction for $\mathbb{S}_N$ as follows.

We continue to use notation from [27]. And we add a few more devices. In particular for S a set, Pascal(S) denotes the set of partitions of S into an ordered pair of parts (thus Pascal(S) is in bijection with the power set $\mathcal{P}(S)$). Example:

$$\text{Pascal}(\{x,y\}) = \{(\{x,y\},\emptyset),\ (\{x\},\{y\}),\ (\{y\},\{x\}),\ (\emptyset,\{x,y\})\}$$

And Pascal$'(S)$ the subset of these in which the first part is not empty. Further, given an indexed set of sets $\{S_i\}_{i\in I}$ that are disjoint we write $\prod_{i\in I} S_i$ for the set whose elements are sets made by selecting one element from each S_i.

Given a partition p of $\underline{N}=\{1,\dots,N\}$ let us write

$$Q_p = \prod_{q\in p} \text{Pascal}'(q)$$

(thus the set given by the choices of a non-empty subset p_{i1} of each part p_i, whose elements are the collections of pairs (p_{i1},p_{i2}) where $p_{i2}=p_i\setminus p_{i1}$). Examples:

$$Q_{\{\{1,2\}\}} = \bigcup_{r\in\text{Pascal}'(\{1,2\})} \{\{r\}\} = \{\ \{(\{1,2\},\emptyset)\},\ \{(\{1\},\{2\})\},\ \{(\{2\},\{1\})\}\ \} = \left\{\boxed{1}\boxed{2},\ \begin{array}{c}\boxed{1}\\\boxed{2}\end{array},\ \begin{array}{c}\boxed{2}\\\boxed{1}\end{array}\right\}$$

(hopefully a multi tableau visualisation is easier on the eye - here specifically the multi-tableaux consist only of single tableau); and for a case requiring a proper multi-tableau:

$$Q_{\{\{1\},\{2\}\}} = \text{Pascal}'(\{1\})\prod\text{Pascal}'(\{2\}) = \{\{(\{1\},\emptyset),(\{2\},\emptyset)\}\} = \{\boxed{1}\ \ \boxed{2}\}\,.$$

Altogether then we have a set

$$\mathrm{S}'_N = \bigcup_{p\in\mathsf{P}(N)} Q_p$$

where $\mathsf{P}(N)$ is the set of partitions of $\underline{N}$. It is convenient to draw nations and their counties - ordered two-part set partitions - as composition tableaux (cf. e.g. [38]). And collections of nations as collections of tableaux. So for example

$$\mathrm{S}'_2 = Q_{\{\{1,2\}\}}\bigcup Q_{\{\{1\},\{2\}\}} = \left\{\boxed{1}\boxed{2},\ \begin{array}{c}\boxed{1}\\\boxed{2}\end{array},\ \begin{array}{c}\boxed{2}\\\boxed{1}\end{array}\right\}\cup\{\boxed{1}\ \ \boxed{2}\} = \left\{\boxed{1}\boxed{2},\ \begin{array}{c}\boxed{1}\\\boxed{2}\end{array},\ \begin{array}{c}\boxed{2}\\\boxed{1}\end{array},\ \ \boxed{1}\ \ \boxed{2}\right\}$$

(note the last set contains only a single composite element, drawn as a sequence of two nations, although this drawn order has no significance). For an example of an element in larger rank we have

$$\begin{array}{ll}\boxed{6}&\!\!\boxed{8}\\\boxed{1}&\end{array}\ \ \begin{array}{ll}\boxed{9}&\!\!\boxed{X}\\\boxed{3}&\end{array}\ \ \boxed{2}\boxed{4}\ \ \begin{array}{c}\boxed{5}\\\boxed{7}\end{array}\ \ \in \mathrm{S}'_{10} \tag{22}$$

This raises the question of how to order the nations in drawing such a picture. This is unimportant here but useful later. Later we order with smaller nations first; and then larger first counties first.

(**6.2**) Finally

$$\mathbb{S}_N = \bigcup_{z \in \mathbb{S}'_N} \text{Pascal}(z)$$

Example:

$$\mathbb{S}_2 = \left\{ (\boxed{1}\boxed{2},)\,,\; (,\boxed{1}\boxed{2})\,,\; \left(\begin{array}{c}\boxed{1}\\\boxed{2}\end{array},\right),\; \left(,\begin{array}{c}\boxed{1}\\\boxed{2}\end{array}\right),\; \left(\begin{array}{c}\boxed{2}\\\boxed{1}\end{array},\right),\; \left(,\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}\right),\; (\boxed{1}\;\boxed{2},),\; (\boxed{1},\boxed{2}),\; (\boxed{2},\boxed{1}),\; (,\boxed{1}\;\boxed{2}) \right\}$$

(**6.3**) The symmetric group Σ_N acts on $\lambda \in \mathbb{S}_N$ by application to the individuals in the counties.

The *shape* of an element $\lambda \in \mathbb{S}_N$ is the diagram obtained by ignoring the entries in the boxes. Note however that these combinatorial objects are multisets rather than sets. (As we will see, they are both beautiful and useful. We study them in §6.3.)

It will be evident that a transversal of the orbits of the Σ_N action is described by the set of shapes. (In §6.4 we give a way to convert the set of shapes into a 'standard' transversal in $\mathbb{S}_N$.)

(**6.4**) We fix the ground field $\mathbb{C}$. Associated to each $\lambda \in \mathbb{S}_N$ there is a 'type-space', call it $\mathbb{C}^\lambda$. There is a non-zero parameter for each county - we require $\alpha_s + \beta_s \neq 0$ for the two parameters in the same nation; and a pair of non-zero parameters for each pair of nations.

Let $\mathbb{S}_N^{\mathbb{C}}$ denote the set whose elements are pairs $(\lambda, \underline{x})$ where $\lambda \in \mathbb{S}_N$ and $\underline{x}$ is a point in $\mathbb{C}^\lambda$.

6.2 Recipe constructing (all) representations

Here we give a construction for varieties of cc loop braid representations (Theorem 7.3 will show this is gives them all). The varieties are indexed by $\mathbb{S}_N$, and $\mathbb{S}_N^{\mathbb{C}}$ describes the parameter space of each variety.

(**6.5**) It will be helpful to give names for the individual parameters in $\mathbb{C}^\lambda$. To this end we can order the nations (i.e. composition tableaux) in $\lambda \in \mathbb{S}_N$, $n_1, n_2, ..., n_m$, as follows — order first into the ordered pair; then within each component using the natural order on nation sizes; and then at fixed size using the natural order on second-county sizes. Finally the repeats of a given shape are ordered child-first (i.e. nation with lowest numbered resident first).

Now for each nation n_s fix the two non-zero parameter names α_s, β_s, associated to the first (upper) and second county respectively. (Recall we require in addition that $\alpha_s + \beta_s \neq 0$.) For each pair of nations n_s, n_t with $s < t$ fix the two non-zero parameter names $\mu_{s,t}$ and $C_{s,t}$.

(**6.6**) Next we give a construction for each $N \in \mathbb{N}$ of a function

$$\begin{aligned} \mathsf{R}^\circ : \mathbb{S}_N^{\mathbb{C}} &\to \mathsf{Match}^N(2,2) \times \mathsf{Match}^N(2,2) && (23) \\ (\lambda, \underline{\mathsf{x}}) &\mapsto \qquad (S, R) && (24) \end{aligned}$$

We will show that this gives all charge conserving loop braid representations.

(**6.7**) For $(\lambda, \underline{\mathsf{x}}) \in \mathbb{S}_N^{\mathbb{C}}$ we encode R in $\mathsf{R}^\circ(\lambda, \underline{\mathsf{x}})$ by $\underline{\alpha}(R) = (a_1, \ldots, a_N, A(1,2), \ldots, A(N-1,N))$ as in (8); and S by $\underline{\alpha}(S) = (b_1, \ldots, b_N, B(1,2), \ldots, B(N-1,N))$. To give a solution we give the scalars $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ and matrices $A(i,j)$, $B(i,j)$ for all $i < j$. These depend on the relationship between the counties/nations that individuals $i < j$ reside in. Specifically:

Consider each individual i. Let n_s be the nation that i resides in. If i is in the top county in n_s then $a_i = \alpha_s$. If i is in the other county of n_s then $a_i = \beta_s$. If n_s comes from the first (resp. second) part of the pair λ then the 'sign' of n_s is $\text{sgn}(n_s) = +$ (resp. $-$). If the sign is $+$ and i is in the top county then $b_i = 1$; while if the sign of n_s is $+$ and i is in the second county then $b_i = -1$. If the sign of n_s is $-$ then the cases are reversed.

Consider each pair of individuals $i < j$.
(We proceed as in Lem.5.2 but choosing the gauge/X-symmetry parameters $c_{s,t} = 1$, adopting the principle that off-diagonal elements of S are gauged to 1.)

1. If $i \in n_s$ and $j \in n_t$ with $s \neq t$ then $A(i,j) = \begin{pmatrix} 0 & \mu_{s,t}/C_{s,t} \\ \mu_{s,t}C_{s,t} & 0 \end{pmatrix}$, and $B(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ — here if C_{st} arises with $s > t$ it is to be understood as $1/C_{ts}$.

2. If i and j are in the same nation n_s but different counties with i in the top county then [5] $A(i,j) = \begin{pmatrix} \alpha_s + \beta_s & \operatorname{sgn}(n_s)\alpha_s \\ -\operatorname{sgn}(n_s)\beta_s & 0 \end{pmatrix}$ and $B(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If j is in the top county then $A(i,j) = \begin{pmatrix} 0 & -\operatorname{sgn}(n_s)\beta_s \\ \operatorname{sgn}(n_s)\alpha_s & \alpha_s + \beta_s \end{pmatrix}$ and $B(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. If i and j are both in the top, respectively bottom, county in n_s then $A(i,j) = \begin{pmatrix} \alpha_s & 0 \\ 0 & \alpha_s \end{pmatrix}$, respectively $A(i,j) = \begin{pmatrix} \beta_s & 0 \\ 0 & \beta_s \end{pmatrix}$; and $B(i,j) = \operatorname{sgn}(n_s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(**6.8**) Example. For $\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \emptyset\right)$ we have

$$\underline{\alpha}(R) = \left(\alpha_1, \alpha_1, \beta_1, \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right)\right)$$

$$\underline{\alpha}(S) = \left(1, 1, -1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

(RRR, SSS, RSS, RRS all hold and, as desired, SRR does not - these are somewhat large but routine calculations; we omit the details, but see [28]).

(**6.9**) Example. For $\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \emptyset\right)$ we have

$$\underline{\alpha}(R) = \left(\alpha_1, \beta_1, \alpha_1, \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\beta_1 \\ \alpha_1 & \alpha_1+\beta_1 \end{smallmatrix}\right)\right)$$

$$\underline{\alpha}(S) = \left(1, -1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

Observe that this is also (6.8) with $f^{(23)}$ with $(23) \in \Sigma_3$ applied, cf. (6). So again RRR, SSS, RSS, RRS all hold and, as desired, SRR does not, without further checking.

(**6.10**) Non-Example. For $\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \emptyset\right)$ we *do not* have a solution taking

$$\underline{\alpha}(R) = (\alpha_1, \beta_1, \alpha_1, \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right))$$

$$\underline{\alpha}(S) = \left(1, -1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}\right)$$

(which are riffs on a first draft recipe, but both sign versions are checked as RRR:True; SSS:True; RSS:False! in [28]).

(**6.11**) Example. For $\left(\emptyset, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right)$ we have:

$$\underline{\alpha}(R) = \left(\alpha_1, \alpha_1, \beta_1, \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & -\alpha_1 \\ \beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & -\alpha_1 \\ \beta_1 & 0 \end{smallmatrix}\right)\right)$$

$$\underline{\alpha}(S) = \left(-1, -1, 1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

[5] Alternative version: $A(i,j) = \begin{pmatrix} \alpha_s + \beta_s & \alpha_s \\ -\beta_s & 0 \end{pmatrix}$ and $B(i,j) = \operatorname{sgn}(n_s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. — note this works but diverges from our gauge choice.

(**6.12**) Cautionary Example. For $\left(\emptyset, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$ we *do not* have

$$\underline{\alpha}(R) = (\alpha_1, \alpha_1, \beta_1, \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right))$$

$$\underline{\alpha}(S) = \left(-1, -1, 1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

(RRR, SSS, RSS, all True; but RRS false with these signs - see [28]. Of course it will work with all off-diagonals $-$ in S, since this is just $-S$ from (6.8), but this does not adhere to our gauge choice).

(**6.13**) Example. For $\lambda = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right)$ we have

$$\underline{\alpha}(R) = (\alpha_1, \beta_1, \alpha_2, \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mu_{12}/C_{12} \\ \mu_{12}C_{12} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mu_{12}/C_{12} \\ \mu_{12}C_{12} & 0 \end{smallmatrix}\right))$$

$$\underline{\alpha}(S) = \left(1, -1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

(RRR, SSS, RSS, RRS all True).

(**6.14**) For $\left(\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array}\right)$ we have:

$$\underline{\alpha}(R) = (\alpha_2, \beta_1, \alpha_1, \left(\begin{smallmatrix} 0 & \mu_{21}/C_{21} \\ \mu_{21}C_{21} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mu_{21}/C_{21} \\ \mu_{21}C_{21} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\beta_1 \\ \alpha_1 & \alpha_1+\beta_1 \end{smallmatrix}\right))$$

$$\underline{\alpha}(S) = (-1, -1, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$$

so $\mathsf{R}^\circ((13).\lambda) = f^{(13)}\mathsf{R}^\circ(\lambda)$, so this works. Meanwhile for $\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline\end{array}\right)$ we have:

$$\underline{\alpha}(R) = (\alpha_2, \beta_1, \alpha_1, \left(\begin{smallmatrix} 0 & \mu_{12}/C_{12} \\ \mu_{12}C_{12} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_1+\beta_1 & \alpha_1 \\ -\beta_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mu_{21}/C_{21} \\ \mu_{21}C_{21} & 0 \end{smallmatrix}\right))$$

$$\underline{\alpha}(S) = (1, -1, -1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$$

so $\mathsf{R}^\circ((23).\lambda) = f^{(23)}\mathsf{R}^\circ(\lambda)$.

(**6.15**) Note from the construction of $\mathbb{S}_N$ that counties are unordered sets, thus permuting the indices within counties does not change the element of $\mathbb{S}_N$. It is straightforward to see from the recipe that the map R° is well defined with respect to such a permutation.

6.3 Index sets for Σ_N-orbits of solution varieties

Here we give the construction of the set $J_N^{\pm}$ (the set of shapes from (6.3)). There is an injection of $J_N^{\pm}$ into $\mathbb{S}_N$, thus there are again varieties of pairs $(S, R) \in \mathsf{Match}^N(2,2) \times \mathsf{Match}^N(2,2)$ associated to each member of $J_N^{\pm}$. We will show in Theorem 7.3 that, up to Σ_N symmetry, this subset is sufficient to give all charge conserving loop braid representations.

We start by codifying the set of shapes.

(**6.16**) Recall that a multiset on a set S is a function $f : S \to \mathbb{N}_0$ (assigning a multiplicity to each element); and that if $d : S \to \mathbb{N}$ is a degree function on S, then $J_N(S)$ is the set of multisets f that have total degree $\sum_{s \in S} f(s)d(s) = N$.

A *signed* multiset of degree N is equivalent to an ordered pair of multisets of total degree N (the first multiset gives the multiplicities of objects signed +; and the second signed -). Let us write $J_{P,M}(S) = J_P(S) \times J_M(S)$ for the set of pairs of multisets as indicated. So the set of pairs whose total degree is N is $J_N^{\pm}(S) = \sqcup_P J_{P,N-P}(S)$.

(**6.17**) As we will see, our indexing combinatorial objects at rank N are signed multisets of compositions into at-most two parts of total degree N.

Write Λ^2 for the set of at-most two-part compositions — equivalently the set

$$\mathbb{N}\times\mathbb{N}_0 = \{(1,0),\ (2,0),(1,1),\ (3,0),(2,1),(1,2),\ (4,0),(3,1),(2,2),(1,3),\ ...\} \tag{25}$$

$$= \{\square,\ \ydiagram{2},\ \ydiagram{1,1},\ \ydiagram{3},\ \ydiagram{2,1},\ \ydiagram{1,2},\ \ydiagram{4},...\}$$

Note that the number of elements of $\Lambda^2 \cong \mathbb{N}\times\mathbb{N}_0$ of degree N is N; and note the total order indicated by (25). Here write just J_N for $J_N(\mathbb{N}\times\mathbb{N}_0)$. The ordinary multisets are $J_1 = \{\square^1\}$;

$$J_2 = \{\square^2, \ydiagram{2}^1, \ydiagram{1,1}^1\}; \qquad\qquad J_3 = \{\ \square^3,\ \square^1\ydiagram{2}^1,\ \square^1\ydiagram{1,1}^1,\ \ydiagram{3}^1,\ \ydiagram{2,1}^1,\ \ydiagram{1,2}^1\ \}$$

$$J_4 = \{\square^4, \square^2\ydiagram{2}^1, \square^2\ydiagram{1,1}^1, \square^1\ydiagram{3}^1, \square^1\ydiagram{2,1}^1, \square^1\ydiagram{1,2}^1, \ydiagram{2}^2, \ydiagram{2}^1\ydiagram{1,1}^1, \ydiagram{1,1}^2, \ydiagram{4}^1, \ydiagram{3,1}^1, (2,2)^1, (1,3)^1\}$$

and so on, writing λ^t to indicate for a given function that $f(\lambda) = t$, with other multiplicities 0.

(**6.18**) For the signed versions, $J_1^\pm = J_{1,0} \cup J_{0,1}$ where: $J_{1,0} = \{(\square^1,)\}$ and $J_{0,1} = \{(, \square^1)\}$. Then $J_2^\pm = J_{2,0} \cup J_{1,1} \cup J_{0,2}$ where

$$J_{2,0} = \{(\square^2,), (\ydiagram{2}^1,), (\ydiagram{1,1}^1,)\}, \qquad J_{1,1} = \{(\square^1, \square^1)\}, \qquad J_{0,2} = \{(, \square^2), (, \ydiagram{2}^1), (, \ydiagram{1,1}^1)\}$$

Next $J_3^\pm = J_{3,0} \cup J_{2,1} \cup J_{1,2} \cup J_{0,3}$ with

$$J_{3,0} = \{(\square^3,), (\square^1\ydiagram{2}^1,), (\square^1\ydiagram{1,1}^1,), (\ydiagram{3}^1,), (\ydiagram{2,1}^1,), (\ydiagram{1,2}^1,)\},$$

$$J_{2,1} = \{(\square^2, \square^1), (\ydiagram{2}^1, \square^1), (\ydiagram{1,1}^1, \square^1)\}, \qquad J_{1,2} = \{(\square^1, \square^2), (\square^1, \ydiagram{2}^1), (\square^1, \ydiagram{1,1}^1)\},$$

$$J_{0,3} = \{(, \square^3), (, \square^1\ydiagram{2}^1), (, \square^1\ydiagram{1,1}^1), (, \ydiagram{3}^1), (, \ydiagram{2,1}^1), (, \ydiagram{1,2}^1)\}.$$

Next $J_4^\pm = J_{4,0} \cup J_{3,1} \cup J_{2,2} \cup J_{1,3} \cup J_{0,4}$ with

$$J_{4,0} = \{(\square^4,), (\square^2\ydiagram{2}^1,), (\square^2\ydiagram{1,1}^1,), (\square^1\ydiagram{3}^1,), (\square^1\ydiagram{2,1}^1,), (\square^1\ydiagram{1,2}^1,), (\ydiagram{2}^2,), (\ydiagram{2}^1\ydiagram{1,1}^1,), ...\}$$

and so on.

6.4 Recipe constructing all representations (up to symmetry)

Here we give an injection of $J_N^\pm$ into $\mathbb{S}_N$. We thus give, via R°, a construction for varieties of cc loop braid representations up to Σ_N symmetry (Theorem 7.3 will show this is gives them all).

(**6.19**) Recall that a multiset $f : S \to \mathbb{N}_0$ can be considered as a set but where entries can be repeated, or indeed missing (in this perception the repeats must somehow be given individuality from each other, for example if $f(s) = n$ then we have n copies of s individuated by $(s,1)$, $(s,2), \dots, (s,n)$ perhaps). Given an order on the underlying set S we can write the set form of f as a sequence. Individual terms that are repeated elements then receive 'individuality' from their identical siblings by their order in the sequence.

(**6.20**) Following on from (6.19), an ordered pair of multisets can then be seen as an ordered pair of sequences (or, say, as a sequence where each same-type run is partitioned into two, but we will adopt the former organisation).

In this way $\lambda \in J_N^\pm$ gives an ordered set of nations (i.e. compositions) $n_1, \dots, n_m$ — ordering first into the ordered pair $(+,-)$; then using the natural ascending order on nation sizes; and then at fixed size using the total order on $\mathbb{N}\times\mathbb{N}_0$ from (25) i.e., in ascending order by the size of the second county). Finally the repeats of a given nation are of course nominally indistinguishable, so simply ordered as written.

(**6.21**) Example: Suppose $f \in J_6$ is given by $f(\square\square) = 2$ and $f(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = 1$ and others zero; and $g \in J_5$ by $g(\square) = 3$ and $g(\square\square) = 1$ and others zero. Then $\lambda = (f, g) \in J^{\pm}_{11}$ can be represented as

$$\lambda = \; (\square\square \;\; \square\square \;\; \begin{smallmatrix}\square\\\square\end{smallmatrix}, \; \square \; \square \; \square \; \square\square) \tag{26}$$

Thus here, in this pair-of-sequences form, $n_1 = \square\square$, $n_2 = \square\square$, $n_3 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, $n_4 = \square$, $\dots$, $n_7 = \square\square$.

(**6.22**) Given $\lambda \in J^{\pm}_N$ ordered as above we obtain a composition tableaux in $\mathbb{S}_N$ by filling in the numbers in order, with the first $|n_1|$ numbers going into n_1 from left to right and then from top to bottom. From the example above we see that (26) then becomes:

$$\lambda \rightsquigarrow (\boxed{1}\boxed{2} \;\; \boxed{3}\boxed{4} \;\; \begin{smallmatrix}\boxed{5}\\\boxed{6}\end{smallmatrix}, \; \boxed{7} \; \boxed{8} \; \boxed{9} \; \boxed{10}\boxed{11} \;) \tag{27}$$

This gives an injection of $J^{\pm}_N$ into $\mathbb{S}_N$, so we may identify $\lambda \in J^{\pm}_N$ with the unique order preserving composition tableaux obtained as above, and abuse notation. Let $\mathbb{J}^{\pm}_N$ denote the subset of $\mathbb{S}^{\mathbb{C}}_N$ coming from the image of $J^{\pm}_N$, i.e., pairs $(\lambda, \underline{x})$ with $\lambda \in J^{\pm}_N$.

Now we may apply the recipe R° (6.6) to $\mathbb{J}^{\pm}_N$ to obtain a solution. We will see that, up to X-symmetry and Σ_N symmetry, *every* rank N solution (R, S) is obtained from an element of $\mathbb{J}^{\pm}_N$.

(**6.23**) See D for examples.

7 Main Theorem

7.1 Prelude to the main Theorem: a key Lemma

(**7.1**) **Lemma**. Let $F : \{\sigma, s\} \to \mathsf{Match}^N(2,2)$ be any pair as in (5). Then F induces a functor $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ if and only if every restriction to rank 2 is a functor (i.e. belongs to the list in 5.2); and the restriction to B is a functor (i.e. every restriction to rank-3 belongs to the list in A.1 up to symmetry - where we here use Lemma 4.3 to pass to rank-3).

Proof. By Lemma 4.3 (and Remark 4.4) it is enough to verify in case $N = 3$. Thus it is enough to consider all extensions of the set of functors $\mathsf{F} : \mathsf{B} \to \mathsf{Match}^3$. These are reproduced in (A.1). A nominal superset of these is given by extending in all nominally possible ways, according to 5.2, at each edge. The complete enumeration of possibilities proceeds as in (A.3). This finite set of groupings of cases is then verified by routine if lengthy direct calculation (or see e.g. (7.2)). □

(**7.2**) *Proof.* (Alternate) We may verify explicitly by checking that all cubics in rank 3 obtained from (33) with $z = Z$, and with $\zeta = Z$ (i.e. for (3)(II) and (III)) are satisfied for each matrix satisfying the assumptions of the Lemma. It is assumed that each F is a representation of L for every restriction to rank 2, thus cubics containing a repeated index, e.g. $|iij\rangle$, are assumed to be satisfied, and we only need to check equations in the permutation orbit of (34)-(38).

Recall that, as explained in 3.9, rank 3 solutions are represented by the complete graph K_3 with rank 2 solutions attached to each edge. We organise our sequence of checks by the number of restrictions of $S = F(s)$ to rank 2 solutions that are / edges. The corresponding edges in $R = F(\sigma)$ are then of type / or a, with various conditions as in (A.1). We start with the case that no edges are /, then the case all edges are /. We then divide the remaining cases into the relations (3)(III) and (3)(III), which are each subdivided by the number of / edges. In each case we show that the permutation orbits of each of (34)-(38) are always satisfied.

Suppose first that all restrictions of S and R to rank 2 solutions are in the zero varieties. Then all off diagonal entries are zero, it is thus immediate that all terms in the orbits of (35)-(38) go to zero. For (34), notice that $\delta_{ij} D_{jk} d_{ij} = d_{jk} D_{ij} \delta_{jk}$, since, by considering the Lemma 5.2, each matrix is a multiple of the identity matrix.

Now suppose all rank 2 restrictions are of / type, then all diagonal entries are 0 and all equations trivialise.

We now consider the relation (3)(III), so let $Z = \zeta$ in (34)-(38), in particular we use capital letters to refer to elements of S and lower case to elements of R. It is immediate that (37) is trivial after this substitution. We then further replace lower case elements with greek letters to avoid confusion between this c and c denoting the gauge parameter in Lemma 5.2.

Now suppose that S restricts to two solutions in the / variety, and one in the zero variety. By Σ symmetry, we may assume that the solution in the zero variety lies on the 12 edge. First notice that $D_{ij}D_{jk} = 0$ for all distinct i, j, k, since the ij or the jk edge is in the / variety. Thus, looking at (34) we must have that, for all distinct i, j, k,

$$B_{ij}D_{ik}c_{ij} = b_{jk}D_{ik}C_{jk}.$$

If ik is a / edge, $D_{ik} = 0$. Otherwise ik is the 0 edge, i.e. $ik \in \{1, 2\}$, and in both cases the condition becomes $B_{13}\gamma_{13} = \gamma_{23}B_{23}$. Let us now explain our notational convention: we use the parameter labels from Lemma 5.2 for each rank 2 matrix, and, where necessary, add a subscript to indicate the edge. Looking at (A.1), we have two cases. The first is that the 13 and 23 edges in R are a-type, in which case all A_i parameters from Lemma 5.2 are locked equal across the edges by (A.1), and the condition becomes $1/c_{13}(-c_{13}A_1) = (-c_{23}A_1)1/c_{23}$. The second case is that the 13 and 23 edges in R are /-type, in which case the μ and C parameters from Lemma 5.2 are locked equal across the edges by (A.1), and the condition becomes $1/c_{13}(\mu Cc_{13}) = (\mu Cc_{23})1/c_{23}$.

Using again that $D_{ij}D_{jk} = 0$, (35) becomes the condition $B_{ij}D_{ik}a_{ij} = d_{jk}B_{ij}D_{ik}$. If ij is zero edge we are done, so suppose not. This gives $D_{ik}a_{ij} = d_{jk}D_{ik}$. Both sides go to zero if ik is a /, so suppose not. This means $i, k \in \{1, 2\}$, and in both cases we have $a_{13} = a_{23}$, which is again true by observing that / solutions are locked equal.

The arguments for (36) and (38) are similar.

We now consider the relation (3)(II). We show that all cubics (35)-(38) are satisfied for the case $z = Z$, i.e. capital letters now refer to elements of R, and greek to elements of S. Note we have already done the case that all restrictions of S to rank 2 solutions are in the zero variety. Also (38) becomes trivial.

Suppose S restricts to two solutions in the / variety, and one in the zero variety. As above, we may assume that this zero solution is on the 12 edge. We first prove that (34) is always satisfied.

Suppose first that ik is the zero edge, thus $\delta_{ij} = \delta_{jk} = 0$. Then (34) becomes $\beta_{ij}D_{ik}C_{ij} = B_{jk}D_{ik}\gamma_{jk}$. Since ik is the zero edge, $D_{ik} \neq 0$, thus it is sufficient to observe that $\beta_{ij}C_{ij} = B_{jk}\gamma_{jk}$ becomes $\beta_{ij}C_{ij} = C_{kj}\beta_{kj}$ as $j > i, k$ and this is satisfied by the locking together of pairs of / and a type solutions.

Now suppose jk is the zero edge, thus $\gamma_{jk} = \delta_{ij} = 0$, and (34) becomes $\beta_{ij}D_{ik}C_{ij} = D_{jk}D_{ij}\delta_{jk}$. If restrictions of R to the ij and ik solutions are in the / variety, both sides become 0. For the a restrictions there are two cases, the first is $\gamma_{13}A_{23}B_{13} = D_{12}A_{13}\delta_{12}$, for which we have either $A_{23} = A_{13} = 0$, or $c_{13}(A_1 + A_2)A_1/c_{13} = A_1(A_1 + A_2)1$. These correspond to the two cases for the fourth triangle in (A.3). The other case is similar.

Finally for the case ij is the zero edge, then $C_{ij} = 0$, and, since $k = 3$, $D_{jk} = D_{ik} = 0$ and all terms in (34) go to zero. The arguments for (35),(36) and (37) are similar.

Finally we consider the case that all edges of S are of / type, and R has two edges of / type and one of a type. We will assume the a type is the edge 12. Notice first that (38) becomes trivial.

First consider (34). This becomes $\beta_{ij}D_{ik}C_{ij} = B_{jk}D_{ik}\gamma_{jk}$ Now either D_{ik} is zero, or ik is the a edge, and we have $1/c_{ij}(A_1 + A_2)(-c_{ij}\mu C) = \mu Cc_{jk}(A_1 + A_2)1/c_{jk}$. Equations (35) and (35) all go to zero by noting all terms contain either of diagonal elements of S, or pairs of diagonal elements from distinct edges of R. Equation (37) becomes $\beta_{ij}B_{ik}D_{jk} = D_{jk}B_{ij}\beta_{ik}$, and we have either $D_{jk} = 0$, or jk is the 12 edge and the condition becomes $\gamma_{ji}C_{ki} = C_{ji}\gamma_{ki}$ which follows from the locking together of pairs of / edges in (A.3). □

7.2 The main Theorem: statement and proof

Theorem 7.3. *(A)*
(I) The construction $\mathsf{R}^\circ(\lambda, \underline{\mathsf{x}})$ *gives a charge conserving monoidal functor* $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ *for every*

$(\lambda, \underline{x}) \in \mathbb{S}_N^{\mathbb{C}}$.
(II) Every such functor F *is in the orbit of some* $\mathsf{R}^\circ(\lambda, \underline{x})$ *under the X-symmetry (of 4.5).*
(B)
(I) The construction $\mathsf{R}^\circ(\lambda, \underline{x})$ *gives a monoidal functor* $\mathsf{F} : \mathsf{L} \to \mathsf{Match}^N$ *for every* $(\lambda, \underline{x}) \in \mathbb{J}_N^{\pm}$.
(II) Every such functor is in the orbit of some $\mathsf{R}^\circ(\lambda, \underline{x})$ *under the* Σ_N *and X-symmetries (of 3.7 and 4.5 respectively).*

Proof. (A,B) (I) Observe that the construction R° yields *by* construction a solution on every ij subspace (compare with Lemma 5.2); and a braid representation (compare with the recipe R in [27]). Thus it yields a solution by Lem.7.1.
(A,B) (II) See §7.3 and §7.4.

7.3 (AII) Proof

(**7.4**) As in [27] we write $\mathfrak{Func}(\mathsf{L}, \mathsf{Match}^N)$ for the category of (monoidal) functors. Let us write simply $(\mathsf{L}, \mathsf{Match}^N)$ for the object set. Consider an arbitrary such functor F, and hence the pair $S = \mathsf{F}(s)$, $R = \mathsf{F}(\sigma)$. We will determine the $(\lambda, \underline{x}) \in \mathbb{S}_N^{\mathbb{C}}$ such that $\mathsf{R}^\circ(\lambda, \underline{x}) \equiv \mathsf{F}$ up to gauge choice. We do this below by interrogating R, S for a suitable λ; and then further for a suitable $\underline{x}$. (This parallels the B case, where we interrogate R for a suitable $\lambda \in \mathfrak{S}_N$ and so on.

(**7.5**) Recall from (6.1) (or [27, (4.4)]) that $\mathfrak{S}_N$ is the set of all two-coloured multi-tableau on N boxes. Recall that $\mathsf{R} : \mathfrak{S}_N \to \mathsf{Match}^N(2,2)$ is the recipe constructing varieties of braid representations from $\mathfrak{S}_N$.

Let $R = \mathsf{F}(\sigma)$ be any braid representation. Observe from [27, (4.7) and §6.3] that the element of $\mathfrak{S}_N$ associated to R is given by $(\pi_{\mathsf{p}}(R), \pi_{\mathsf{q}}(R), \pi_\rho(R), \pi_{\mathsf{s}}(R))$. All component functions are as defined in [27], which we recall for the reader's convenience: $\pi_{\mathsf{p}}(R)$ is the partition of $\underline{N}$ into nations, $\pi_{\mathsf{q}}(R)$ further partitions the nations into counties, $\pi_\rho(R)$ orders the counties within each nation and nations within each multiset, and $\pi_{\mathsf{s}}(R)$ assigns one of two colours to each county. These are determined from $\underline{\alpha}(R) = (a_1, \ldots, a_N, \ldots, A(i,j), \ldots)$ using the forms in (3.10) as follows: The π_{p} (nation) partition are the equivalence classes from the relation $i \sim j$ if $A(i,j)$ is of type a, f or 0. The county partition π_{q} is a refinement by $i \sim j$ if $A(i,j)$ is of 0 type. The ordering π_ρ is the ordering on the counties/nations induced by the natural ordering on $\underline{N}$ by comparing the minimal individual label within each county/nation. Finally, within each nation the π_{s} bi-colouring of counties has i and j of the same color if i and j are in the same county, or in the same nation with $A(i,j)$ of type $\underline{\mathsf{a}}$ or $\underline{\mathsf{f}}$, with the first county always fixed to be the 'empty' colour.

Let us now write $\underline{\pi}(R) = (\pi_{\mathsf{p}}(R), \pi_{\mathsf{q}}(R), \pi_\rho(R), \pi_{\mathsf{s}}(R))$ for this element. That is, $\mathsf{R}(\underline{\pi}(R))$ is the variety containing R up to gauge (in this way it was shown that R constructs all representations).

We will modify the pseudo-inverse function $\underline{\pi}$ for our case.

(**7.6**) In our case firstly note that since, by Lem.5.2(II), there are no f edges (still in the sense of restricting to the braid solution part) the 'colour' must differ between *every* pair of counties in each nation of $\underline{\pi}(R)$. There are two colours, so this forces that there are at most two counties in each nation, always of different colour. - So, note, we do not need to further record colours since $\pi_{\mathsf{s}}(R)$ is uniquely determined.

This $\underline{\pi}(R)$ can be realised as a set of composition tableaux — tableaux where order in a row does not matter (so we can use natural ascending order for free), and order between rows does matter. For examples:

$$\begin{array}{|c|c|}\hline 9 & X \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 6 & 8 \\ \hline 1 & 2 & 4 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 5 \\ \hline 7 \\ \hline\end{array} \tag{28}$$

$$\begin{array}{|c|c|}\hline 9 & X \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 6 & 8 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 5 \\ \hline 2 & 4 & 7 \\ \hline\end{array}$$

In such representations as these the nations also appear *arranged* as a sequence rather than just a set, but so far this is arbitrary, cf. (6.19).

(**7.7**) Recall from (6.2) that $\mathbb{S}_N$ denotes the set of partitions of such sets into an ordered pair of subsets, restricting to the case of compositions with at most two parts (two parts of different colours, in the original $\mathfrak{S}_N$ idiom).

(**7.8**) We define $\pi_{\mathsf{r}}(R,S)$ as the partition of $\underline{\pi}(R)$ into an ordered pair of subsets, with nation s in the first part if $\langle ii|S|ii\rangle = +1$ for i the individual in the top left box of nation s.

(**7.9**) For example perhaps:

$$\pi_{\mathsf{r}}(R,S) = \left(\begin{array}{|c|}\hline 5\\ \hline 7\\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 9 & X\\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|}\hline 6 & 8 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & 4\\ \hline \end{array} \right) \tag{29}$$

For an example of large enough rank to be generic (but still neat to typeset!) we use $\{a,b,c,...\}$ instead of $\{1,2,3,...\}$:

$$\left(\begin{array}{c} f \end{array} \quad \begin{array}{c} x \end{array} \quad \begin{array}{c} z \end{array} \quad \begin{array}{cc} e & y\\ b & \end{array} \quad \begin{array}{cc} r & \\ u & v \end{array} \quad \begin{array}{cc} k & p\\ d & l \end{array} \; , \; \begin{array}{ccc} s & t & w \end{array} \quad \begin{array}{cc} c & i\\ j & \end{array} \quad \begin{array}{cc} n & o\\ h & \end{array} \quad \begin{array}{ccc} q & & \\ a & g & m \end{array} \right)$$

(**7.10**) Having interrogated R, S for a $\lambda \in \mathbb{S}_N$, we now interrogate for an $\underline{\mathsf{x}} \in \mathbb{C}^{\lambda}$.

Associated to each nation n_s, say, of $\boldsymbol{\alpha} = \underline{\pi}(R)$ there are α_s and β_s values that can be read off from R — inspecting $\langle ii|R|ii\rangle$ gives either α_s or β_s for i's nation, depending on whether i is in the first or second row. Observe that i, i' in the same row have a type-0 edge between them here, so the procedure is well-defined without specifying i further.

Associated to each pair n_s, n_t of nations, with $s < t$, there is a μ_{st} parameter value, given by square root of product of off-diagonals in R between any i in s and j in t. Observe again well-definedness.

Continuing with this n_s, n_t, we can also read off the value of $C_{st}c_{ij}$. This does depend on i, j, but the value of c_{ij} is determined from the ij off-diagonals in S, i.e. $\langle ij|S|ji\rangle$, so we can determine C_{st}. Note that we can apply X-symmetry to F to make $c_{ij} = 1$ in all cases.

To be precise, in case $i < j$,

$$c_{ij} = \langle ii|S|ii\rangle\langle ji|S|ij\rangle, \;\; C_{st} = \frac{\langle ii|S|ii\rangle\langle ji|R|ij\rangle}{c_{ij}\mu_{st}} \tag{30}$$

(recall that $\langle ji|S|ij\rangle$, for example, is the bottom left entry in the 2×2 matrix corresponding to the 12 edge). If instead $j < i$ then

$$c_{ij} = \frac{1}{\langle ii|S|ii\rangle\langle ji|S|ij\rangle}, \;\; C_{st} = \frac{c_{ij}\mu_{st}}{\langle ii|S|ii\rangle\langle ji|R|ij\rangle}. \tag{31}$$

Observe that the operator spectrum depends on C but is invariant under $C \leftrightarrow 1/C$.

Write $\underline{\chi}(R,S)$ for this collection of parameters. Note that there is potential ambiguity in the organisation of the collection, resolved here by the total order on nations, as in (6.1).

(**7.11**) We claim that $\mathsf{R}^{\circ}((\pi_{\mathsf{r}}(R,S), \underline{\chi}(R,S)))$ is gauge-equivalent to F. (N.B. This implies (AII).)

To see this, first observe that in the case all restrictions of F to a pair of indices i, j are 0-type solutions, then $\mathsf{R}^{\circ}((\pi_{\mathsf{r}}(R,S), \underline{\chi}(R,S)))$ is precisely F. In general, it is enough to show that, for all pairs $i < j$ in $\underline{N}$, there exists a 2×2 invertible diagonal matrix $X(i,j)$ with $X(i,j)A(i,j)X(i,j)^{-1} \in \underline{\alpha}(R)$ equal to $A'(i,j) \in \underline{\alpha}(F(\sigma))$ and $X(i,j)B(i,j)X(i,j)^{-1} \in \underline{\alpha}(S)$ equal to $B'(i,j) \in \underline{\alpha}(F(s))$. Let X be the matrix such that $\underline{\alpha}(X) = (1,\ldots,1,X(i,j),\ldots)$, and observe that this is an invertible diagonal matrix such that $XRX^{-1} = \mathsf{F}(\sigma)$ and $XSX^{-1} = \mathsf{F}(s)$. Thus, this implies $\mathsf{R}^{\circ}((\pi_{\mathsf{r}}(R,S), \underline{\chi}(R,S)))$ is gauge-equivalent to F.

The different forms of $X(i,j)$ that can appear depend only on where i, j appear in $\pi_{\mathsf{r}}(R,S)$ relative to one another - for example, i, j in different rows of different nations with the same sign, i, j in the different rows of the same nation which has sign minus etc. In fact, all forms of $X(i,j)$ appear even in the case $N = 2$. We check these forms in some rank 3 contexts to illustrate that no new issues arise from the relative positions of i and j that appear in higher rank.

1. Suppose $\pi_{\mathsf{r}}(R,S) = \left(\begin{array}{|c|}\hline 1\\ \hline \end{array}, \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline \end{array} \right)$. Necessarily then $\underline{\chi}(R,S)$ gives values for two nations, thus

$\alpha_1, \alpha_2, \beta_2, \mu_{12}, C_{12}, c_{12}, c_{13}, c_{23}$. For all pairs i, j we are in case (30), thus we know from this and Lem.5.2 that S takes the form

$$\underline{\alpha}(S) = (1, -1, 1, \begin{pmatrix} 0 & \frac{1}{c_{12}} \\ c_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{c_{13}} \\ c_{13} & 0 \end{pmatrix}, -\begin{pmatrix} 0 & \frac{1}{c_{23}} \\ c_{23} & 0 \end{pmatrix})$$

for some c_{12}, c_{13}, c_{23}, with $\langle 33|S|33\rangle = 1$ forced. Note that Lem.5.2 puts $-c_{23}$ here, since $\langle 22|S|22\rangle = -1$. And given this S then R must be

$$\underline{\alpha}(R) = \left(\alpha_1, \alpha_2, \beta_2, \begin{pmatrix} 0 & \frac{\mu_{12}}{c_{12}C_{12}} \\ \mu_{12}c_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{12}}{c_{13}C_{12}} \\ \mu_{12}c_{13}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2+\beta_2 & \alpha_2/c_{23} \\ -c_{23}\beta_2 & 0 \end{pmatrix}\right)$$

It follows from (6.6) that the R° image takes the form

$$\underline{\alpha}(F(s)) = (1, -1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), \qquad \underline{\alpha}(F(\sigma)) = \left(\alpha_1, \alpha_2, \beta_2, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2+\beta_2 & -\alpha_2 \\ \beta_2 & 0 \end{pmatrix}\right)$$

Observe that this is indeed in the X-orbit of (R, S), by putting $c_{12} = c_{13} = 1$, $c_{23} = -1$. In terms of choosing appropriate local matrices as described above, this corresponds to

$$X(1,2) = \begin{pmatrix} c_{12} & 0 \\ 0 & 1 \end{pmatrix}, \qquad X(1,3) = \begin{pmatrix} c_{13} & 0 \\ 0 & 1 \end{pmatrix}, \qquad X(2,3) = \begin{pmatrix} c_{23} & 0 \\ 0 & -1 \end{pmatrix}$$

2. Suppose $\pi_{\mathsf{r}}(R, S) = (\boxed{1}\boxed{3}, \boxed{2})$. Necessarily then $\underline{\chi}(R, S)$ gives values for two nations, thus $\alpha_1, \alpha_2, \mu_{12}, C_{12}, c_{12}, c_{23}$. Here we are in case (31) when $i = 3, j = 2$, and case (30) otherwise, using this together with Lem.5.2 we know that S takes the form

$$\underline{\alpha}(S) = (1, -1, 1, \begin{pmatrix} 0 & \frac{1}{c_{12}} \\ c_{12} & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, -\begin{pmatrix} 0 & c_{23} \\ \frac{1}{c_{23}} & 0 \end{pmatrix})$$

for some c_{12}, c_{23}, with $\langle 33|S|33\rangle = 1$ forced. Note that Lem.5.2 puts $-c_{23}$ here, since $\langle 22|S|22\rangle = -1$. And given this S then R must be

$$\underline{\alpha}(R) = \left(\alpha_1, \alpha_2, \alpha_1, \begin{pmatrix} 0 & \frac{\mu_{12}}{c_{12}C_{12}} \\ \mu_{12}c_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \\ & \alpha_1 \end{pmatrix}, -\begin{pmatrix} 0 & \mu_{12}c_{23}C_{12} \\ \frac{\mu_{12}}{c_{23}C_{12}} & 0 \end{pmatrix}\right)$$

It follows from (6.6) that the R° image takes the form

$$\underline{\alpha}(F(s)) = (1, -1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

$$\underline{\alpha}(F(\sigma)) = \left(\alpha_1, \alpha_2, \alpha_1, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \\ & \alpha_1 \end{pmatrix}, \begin{pmatrix} 0 & C_{12}\mu_{12} \\ \frac{\mu_{12}}{C_{12}} & 0 \end{pmatrix}\right)$$

Observe that this is indeed in the X-orbit of (R, S), by putting $c_{12} = 1$, $c_{23} = -1$.

3. Suppose $\pi_{\mathsf{r}}(R, S) = (\square^3, \emptyset)$. We know from Lem.5.2 that S takes the form

$$\underline{\alpha}(S) = (1, 1, 1, \begin{pmatrix} 0 & \frac{1}{c_{12}} \\ c_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{c_{13}} \\ c_{13} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{c_{23}} \\ c_{23} & 0 \end{pmatrix})$$

for some c_{12}, c_{13}, c_{23}. And given this then R must in particular be

$$\underline{\alpha}(R) = \left(A_1, A_2, A_3, \begin{pmatrix} 0 & \frac{\mu_{12}}{c_{12}C_{12}} \\ \mu_{12}c_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{c_{13}C_{13}} \\ \mu_{13}c_{13}C_{13} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{23}}{c_{23}C_{23}} \\ \mu_{23}c_{23}C_{23} & 0 \end{pmatrix}\right)$$

for some values of the parameters. In this case it follows from (6.6) that the R° image takes the form

$$\underline{\alpha}(F(s)) = (1, 1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

$$\underline{\alpha}(F(\sigma)) = \left(A_1, A_2, A_3, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12}C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{C_{13}} \\ \mu_{13}C_{13} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{23}}{C_{23}} \\ \mu_{23}C_{23} & 0 \end{pmatrix}\right)$$

This is in the X-orbit taking $c_{12} = c_{13} = c_{23} = 1$.

4. Suppose $\pi_{\mathsf{r}}(R, S) = (\boxed{1}\boxed{2}, \boxed{3})$. The main new observation here is that $\langle 22|S|22\rangle = 1$ by the nature of 0-edges. So again the recipe agrees up to gauge. $\square$

7.4 (BII) Proof

Let us write the action of Σ_N on $\lambda \in \mathbb{S}_N$ simply permuting the entries as $w.\lambda$. For (BII) it is enough to show that the f^w (with notation as in 3.7) and naive actions agree on $\mathbb{S}_N$, since $\mathbb{J}^{\pm}_N$ is clearly a transversal of $\mathbb{S}_N$ under the dot action.

(**7.12**) We claim that $w.\pi_{\mathsf{r}}(\mathsf{F}) = \pi_{\mathsf{r}}(f^w\mathsf{F})$.
(Note that the identity is one of varieties, not of individual solutions. We can see from the construction that the size of the variety is not affected by w.)

Proof. Let us work through some specific kinds of cases. We will be done when we have verified the action of (generating) elementary transpositions across the various contexts that arise. We must consider $f^{(ij)}\mathsf{F}$ under the following scenarios:
1. i, j in the same county;
2. i, j in different counties in the same nation;
3. i, j in different nations with the same sign;
4. i, j in different nations with different sign.
For case-1 both actions are trivial (see also 6.15).
For case-2, consider applying $(3, X) \in \Sigma_N$ to our F in (29). This flips the 3X edge, taking it from type $\underline{\mathsf{a}}$ to type a (cf. Example 6.9). It also swaps the 1X and 13 edges (a 0 edge and an a edge), and leaves all other edges and S unchanged. Looking at the construction of π_{r} we have that $\pi_{\mathsf{r}}(f^{(3X)}\mathsf{F})$ is

$$\left(\begin{array}{c} 5 \\ 7 \end{array} \;\; \begin{array}{cc} 9 & 3 \\ X & \end{array} \;\; , \;\; \begin{array}{ccc} 6 & 8 & \\ 1 & 2 & 4 \end{array} \right)$$

which is immediately seen to be $(3X).\pi_{\mathsf{r}}(\mathsf{F})$.

For case-4, suppose we apply $(1, 9) \in \Sigma_N$ to our F in (29), i.e. $\mathsf{F} \mapsto f^{(1,9)}\mathsf{F}$ as in (3.7). Then we claim $\pi_{\mathsf{r}}(\mathsf{F}) \rightsquigarrow \pi_{\mathsf{r}}(f^{(1,9)}\mathsf{F})$ is given by

$$\left(\begin{array}{c} 5 \\ 7 \end{array} \;\; \begin{array}{cc} 9 & X \\ 3 & \end{array} \;\; , \;\; \begin{array}{ccc} 6 & 8 & \\ 1 & 2 & 4 \end{array} \right) \rightsquigarrow \left(\begin{array}{c} 5 \\ 7 \end{array} \;\; \begin{array}{cc} 1 & X \\ 3 & \end{array} \;\; , \;\; \begin{array}{ccc} 6 & 8 & \\ 9 & 2 & 4 \end{array} \right)$$

(and cf. (6.13)). This can be seen by noting the following. In this new equivalent solution $f^{(1,9)}\mathsf{F}$ vertex 1 will have the the α parameter value from the old 9X3 nation, and vertex 9 the β parameter from the 68124 nation (so it is apt to think of the parameter staying with the nation under exchange, even though two nations could entirely swap populations — we can call this Mach's other principle [18]). Vertex 1 will also have 'exchanged' its S eigenvalue with vertex 9 (although this is +1 to +1 so no change, here). The 0-edge at 9X is now a 0-edge at 1X. The /-edge at 69 is now a /-edge at 61 (or 16, note that the off diagonal entries will also flip), and so on. Similarly, the /-edge at 1X is now a /-edge at 9X. The $\underline{\mathsf{a}}$ edge 93 is now an a edge 13, and similarly with 61 to 69.
Case-3 is similar.

□

This concludes the proof of the Theorem.

8 Discussion and future directions

The input $J^{\pm}_N$ to our recipe yields some interesting combinatorial questions.

(**8.1**) Note from (6.17) that the integer sequence for $|J_N|$ is the Euler transform of 1,2,3,4,..., which begins 1,3,6,13,... This is also MacMahon's sequence for plane partitions (since this is the same Euler transform, as noted in [40, OEIS A000219]). An explicit bijection may be obtained from [37, Thm 2.1].

(**8.2**) Let us determine the size of J_N by another method. First note that there are exactly k distinct compositions of k into at most 2 parts. To build the elements of J_N we form a multiset of such compositions so that the sum of their sizes is N. The generating function for the number of ordinary partitions is $\sum_{i=0}^{\infty} p(n)x^n = \prod_{k\geq 1} \frac{1}{1-x^k}$: indeed, partitions can be regarded as a multiset of natural numbers $k \geq 1$. The modification for our set up is that we are selecting multisets of compositions into two parts, and there are k distinct compositions of each size k. Thus each factor should be taken k times, giving

$$\sum_{n=0}^{\infty} |J_n| x^n = \prod_{k\geq 1} \frac{1}{(1-x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + \cdots$$

This indeed coincides with MacMahon's GF for the sequence of sizes of sets of plane partitions. It is intriguing that the proof here is obtained very neatly from the very classical (due to Euler) ordinary-partitions case. The proof for plane partitions appears more involved — see e.g. [41].

(**8.3**) The sequence $|J_N^{\pm}|$ for $N = 0, 1, \ldots$ is (1), 2, 7, 18, 47, 110, ... This is also the count for the set of ordered pairs of plane partitions of total degree N (see e.g. [40, A161870]). Of course if a sequence has generating function $A(z)$ (in our case MacMahon's function) then the sequence for ordered pairs has generating function $A^2(z)$.

It is an important question as to the measure of the set of charge-conserving representations in the set of all representations up to a suitable notion of isomorphism (i.e. how restrictive were the constraints we put on in order to get a complete solution?). One way to think about this is by analogy with the corresponding problem for Hecke representations. To this end it is instructive to compare with the machinery of classical Hecke representation theory as in [26]. There we see, by a method involving Bruhat orders, that cc representations are, in a suitable sense, eventually faithful (note in particular [26, §2.3(7)]). This raises several further representation-theoretic questions, such as the following.
What is the smallest (monoidal) subcategory of Mat that contains all perm matrices? What is the smallest that contains all perm matrices as "additive" components (subblocks)? Is there a fusion trick? What about monomial matrices?

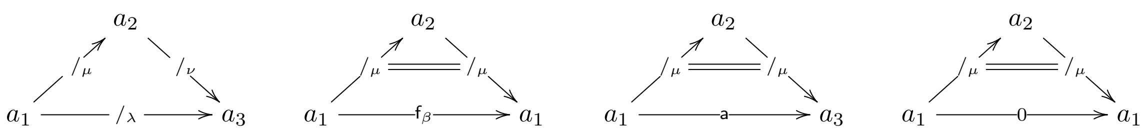
This work can be seen as part of a more general programme on Representation theory of natural categories. Here one starts by observing the naturalness/singular-ness of natural categories in the wider realm of monoidal categories (cf., for example, [19]), as way of unifying groups and algebras *a la* statistical mechanics (cf. for example [29]); and observes key rigidification aspects. Many questions arise here, for example around generalisations and around applications (in particular to topological quantum computation - cf. for example [3, 35, 14, 44]).

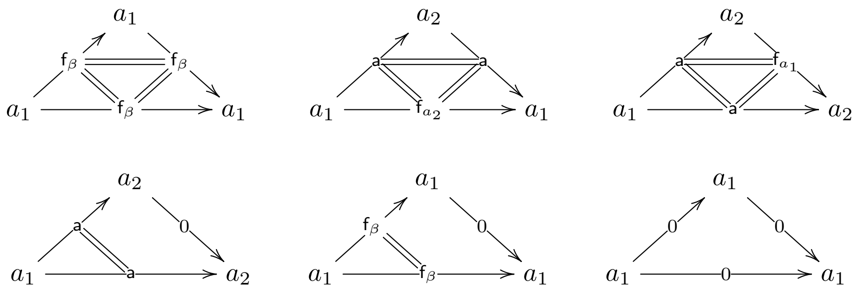
Appendices

A Background for the Proof of Lemma 7.1

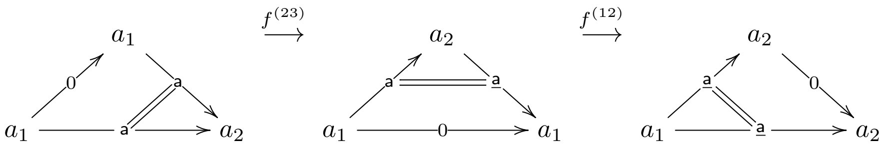
The simplicially-directed K_3 graph has edge orientations $1 \to 2$, $2 \to 3$, $1 \to 3$. The following decorated simplicially-directed K_3 diagrams should be understood as in [27], giving $\underline{\alpha}(R) = (a_1, a_2, a_3, A(1,2), A(1,3), A(2,3))$ with vertex parameters a_1, a_2, a_3 given by vertex labels, and for example, $A(1,2)$ given by the label on the edge from vertex 1 to vertex 2.

(**A.1**) **Proposition**. [27, Prop.5.1] *For $N = 3$ the following types of configurations yield a charge-conserving functor* $\mathsf{F} : \mathsf{B} \to \mathsf{Match}^3$ *(showing one variety per $\mathbb{Z}_2 \times \Sigma_3$ orbit):*



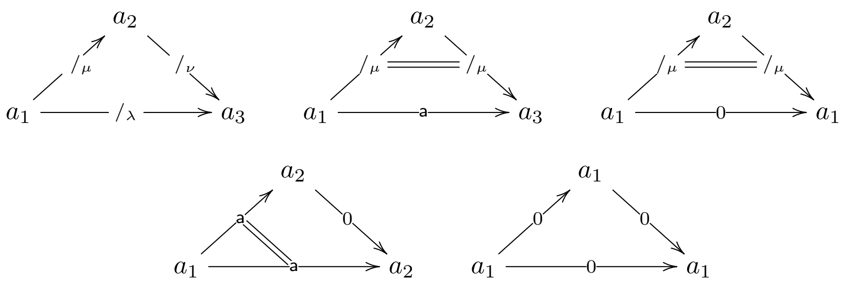


(**A.2**) For example, the aa0 case shown is $\begin{smallmatrix}1\\2&3\end{smallmatrix}$ (second row coloured). In this orbit we have also $\begin{smallmatrix}2\\1&3\end{smallmatrix}$ and $\begin{smallmatrix}3\\1&2\end{smallmatrix}$, collectively labelled . On the other hand, in the  orbit (not explicitly represented above) we have $\begin{smallmatrix}1&2\\3\end{smallmatrix} \xrightarrow{f^{(23)}} \begin{smallmatrix}1&3\\2\end{smallmatrix} \xrightarrow{f^{(12)}} \begin{smallmatrix}2&3\\1\end{smallmatrix}$ which unpacks to



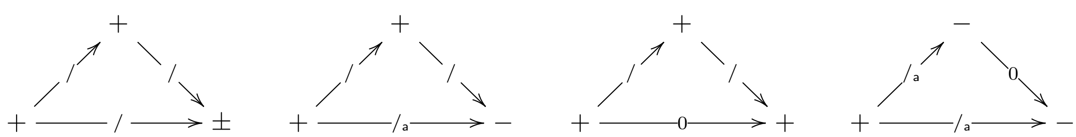
- note that after the first move the 23 edge decoration is reversed, but the parameters on this edge are now tied to those on edge-12.

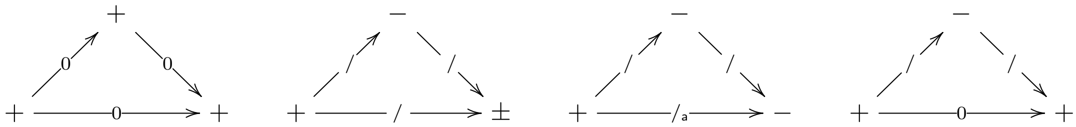
(**A.3**) Eliminating from (A.1) the cases with f, which by Lem.5.2 do not extend, we have:



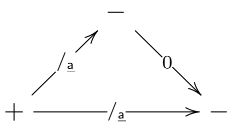
with corresponding indices:

Note that we do not know *ab initio* that any of these extend. The formally possible extensions in each case (according to the application of Lem.5.2 to each K_2 subgraph) are represented by giving the corresponding decorations for $F(s)$. We write $+$ and $-$ for the diagonal eigenvalues $+1, -1$. We write $/$ for the edge decoration $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (taking advantage of the gauge freedom to fix this), but $/_{\mathsf{a}}$ to indicate that the vertex eigenvalues are forced opposite. For the 0 edges the vertex eigenvalues are forced the same. Note in particular that the diagonal eigenvalues within a nation are all determined by a_1. Indeed they would be over-determined if there are more than two a edges — more than two counties, but this is already impossible. On the other hand the relative eigenvalues between nations are not formally determined. Thus the possibilities overall correspond to a choice of sign for (the lead a_i in) each nation. We have:





together with the overall $s \rightsquigarrow -s$ flips. For completeness (since we have not introduced the flip move here) we should include here, say,



We find by direct calculation that all satisfy the $N = 3$ constraints.

(**A.4**) Overall from (A.3), we have the following signed-multiset indices up to flip and symmetry:

$(\square^3,)$ $(\square$ ▯■$,)$ $(\square\ \square\square,)$ (▯▯/■$,)$ $(\square\square\square,)$

$(\square^2, \square)$ (▯■$, \square)$ $(\square\square, \square)$

B Ket calculus: the Texas braid relation

Here we give a proof of Lemma 4.5 by direct calculation. In fact, we prove a slightly more general statement:

(**B.1**) **Lemma**. Suppose $z, Z, \zeta \in \mathsf{Match}^3(2,2)$ satisfy

$$z_1 Z_2 \zeta_1 = \zeta_2 Z_1 z_2 \tag{32}$$

in $\mathsf{Match}^3(3,3)$ (recall $z_1 = z \otimes \mathbf{1}$ and so on). For invertible $X \in \mathsf{Match}^3(2,2)$ let $z^X = XzX^{-1}$ and so on. If X is diagonal then

$$z_1^X Z_2^X \zeta_1^X = \zeta_2^X Z_1^X z_2^X$$

.

Proof. Consider the equation

$$z_1 Z_2 \zeta_1 = \zeta_2 Z_1 z_2 \tag{33}$$

where $z, Z, \zeta \in \mathsf{Match}^3(2,2)$ with nonzero entries lower case Roman $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$, upper case Roman $(A_{ij}, B_{ij}, C_{ij}, D_{ij})$ and lower case Greek $(\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij})$, respectively, where $i \leq j$. We have (for z, with analogous conventions for Z, ζ):

$$z|ij\rangle = \begin{cases} d_{ij}|ij\rangle + b_{ij}|ji\rangle & i < j \\ a_{ji}|ij\rangle + c_{ji}|ji\rangle & j < i \\ a_{ii}|ii\rangle & i = j \end{cases}$$

We will show that if z, Z and ζ satisfy equation (33) then so do z^X, Z^X and ζ^X.

First note that conjugating each of z, Z and ζ by a diagonal matrix leaves the diagonal entries $(a_{ij}, A_{ij}, \alpha_{ij})$ and $(d_{ij}, D_{ij}, \delta_{ij})$ invariant, while the effect on the other entries is: $\sqsupset_{ij} \mapsto m_{ij}\sqsupset_{ij}$ for $\sqsupset \in \{b, B, \beta\}$ and $\urcorner_{ij} \mapsto \urcorner_{ij}/m_{ij}$ for $\urcorner \in \{c, C, \gamma\}$ for some m_{ij}. To see that this leaves the polynomial equations unchanged we compute a few entries. First it should be clear that the $|iii\rangle$ entries give trivial conditions. For the $|ijk\rangle$ entries with i, j, k distinct it is enough to consider $|123\rangle$ by a local permutation argument: one simply applies the permutation to all indices and then defines, for $i < j$, $a_{ji} := d_{ij}$, $d_{ji} := a_{ij}$, $b_{ji} := c_{ij}$ and $c_{ji} := b_{ij}$ and similarly for the A, B, C, D and $\alpha, \beta, \gamma, \delta$.

We have:

$$\begin{aligned}
&z_1 Z_2 \zeta_1|123\rangle = \\
&z_1 Z_2(\beta_{12}|213\rangle + \delta_{12}|123\rangle) = \\
&z_1(\beta_{12}(B_{13}|231\rangle + D_{13}|213\rangle) + \delta_{12}(B_{23}|132\rangle + D_{23}|123\rangle)) = \\
&\beta_{12}(B_{13}(b_{23}|321\rangle + d_{23}|231\rangle) + D_{13}(a_{12}|213\rangle + c_{12}|123\rangle)) + \\
&\delta_{12}(B_{23}(b_{13}|312\rangle + d_{13}|132\rangle) + D_{23}(b_{12}|213\rangle + d_{12}|123\rangle))
\end{aligned}$$

whereas:

$$\begin{aligned}
&\zeta_2 Z_1 z_2|123\rangle = \\
&\zeta_2 Z_1(b_{23}|132\rangle + d_{23}|123\rangle) = \\
&\zeta_2(b_{23}(B_{13}|312\rangle + D_{13}|132\rangle) + d_{23}(B_{12}|213\rangle + D_{12}|123\rangle)) = \\
&b_{23}(B_{13}(\beta_{12}|321\rangle + \delta_{12}|312\rangle) + D_{13}(\alpha_{23}|132\rangle + \gamma_{23}|123\rangle)) + \\
&d_{23}(B_{12}(\beta_{13}|231\rangle + \delta_{13}|213\rangle) + D_{12}(\beta_{23}|132\rangle + \delta_{23}|123\rangle))
\end{aligned}$$

This yields 6 equations the last of which is trivial (denoted by (*)):

$$\beta_{12}D_{13}c_{12} + \delta_{12}D_{23}d_{12} = b_{23}D_{13}\gamma_{23} + d_{23}D_{12}\delta_{23} \tag{34}$$
$$\beta_{12}D_{13}a_{12} + \delta_{12}D_{23}b_{12} = d_{23}B_{12}\delta_{13} \tag{35}$$
$$\delta_{12}B_{23}d_{13} = b_{23}D_{13}\alpha_{23} + d_{23}D_{12}\beta_{23} \tag{36}$$
$$\beta_{12}B_{13}d_{23} = d_{23}B_{12}\beta_{13} \tag{37}$$
$$\delta_{12}B_{23}b_{13} = b_{23}B_{13}\delta_{12} \tag{38}$$
$$\beta_{12}B_{13}b_{23} = b_{23}B_{13}\beta_{12} \quad (*)$$

Next we compute the polynomial consequences of the generic braid relation on $|112\rangle$. Here we have:

$$\begin{aligned}
&z_1 Z_2 \zeta_1|112\rangle = \\
&\alpha_{11}(D_{12}a_{11}|112\rangle + B_{12}(d_{12}|121\rangle + b_{12}|211\rangle))
\end{aligned}$$

whereas

$$\begin{aligned}
&\zeta_2 Z_1 z_2|112\rangle = \\
&d_{12}A_{11}(\delta_{12}|112\rangle + \beta_{12}|121\rangle) + \\
&b_{12}(D_{12}(\alpha_{12}|121\rangle + \gamma_{12}|112\rangle) + B_{12}\alpha_{11}|211\rangle)
\end{aligned}$$

These yield 3 equations, that last of which is trivial:

$$\alpha_{11}D_{12}a_{11} = d_{12}A_{11}\delta_{12} + b_{12}D_{12}\gamma_{12} \tag{39}$$
$$\alpha_{11}B_{12}d_{12} = d_{12}A_{11}\beta_{12} + b_{12}D_{12}\alpha_{12} \tag{40}$$
$$\delta_{11}B_{12}b_{12} = b_{12}B_{12}\delta_{11} \quad (*)$$

Similar computations equating $z_1 Z_2 \zeta_1$ and $\zeta_2 Z_1 z_2$ on $|121\rangle$ and $|211\rangle$ yield:

$$\delta_{12}A_{12}a_{12} + \beta_{12}D_{12}c_{12} = a_{12}D_{12}\alpha_{12} + c_{12}A_{11}\beta_{12} \tag{41}$$
$$\alpha_{12}C_{12}a_{11} = a_{12}D_{12}\gamma_{12} + c_{12}A_{11}\delta_{12} \tag{42}$$
$$\delta_{12}A_{12}b_{12} + \beta_{12}D_{12}a_{12} = a_{12}B_{12}\alpha_{11} \tag{43}$$
$$\alpha_{12}A_{11}a_{12} + \gamma_{12}A_{12}b_{12} = a_{11}\alpha_{11}A_{12} \tag{44}$$
$$\alpha_{12}A_{11}c_{12} + \gamma_{12}A_{12}a_{12} = a_{11}C_{12}\delta_{12} \tag{45}$$
$$\gamma_{12}C_{12}a_{11} = a_{11}C_{12}\gamma_{12} \quad (*)$$

Again, we omit the calculations for $|122\rangle, |212\rangle$ and $|221\rangle$ as the corresponding polynomial equations can be obtained from the above by applying the transposition (1 2) and defining $b_{21} := c_{12}$, $a_{21} = d_{12}$ etc.

Now we observe that in each equation either

1. Each term has the same number of b_{ij}, B_{ij} or β_{ij} (resp. c_{ij}, C_{ij} or γ_{ij}) terms so that the equation obtained by simultaneous X-conjugation has a negligible overall non-zero factor, or

2. Each term with a $\{b_{ij}, B_{ij}, \beta_{ij}\}$ term also has a corresponding $\{c_{ij}, C_{ij}, \gamma_{ij}\}$ term (not necessarily matching) so that the effect of X-conjugation is nugatory.

This completes the proof. □

Lemma 4.5 now follows by substituting R, S for z, Z, ζ as appropriate.

(**B.2**) *Remark.* The superficially *braid*-like equation (32) seems too general to have such a geometric-topological meaning. Some of its specialisations can be endowed with such meaning, as we can see from our context. But not all. Nonetheless, in Match^3 it is useful.

C The loop braid category L: technical asides

Here we aim to provide, in support of Sec.2, more of a rough substitute for the braid laboratory of [27] - by working with an appropriate motion groupoid; and hence to give a bit more of a *feel* for the relations in the presentation (3.1). And to prove C.14, on non-monoidal presentation.

C.1 Topological Background

In what follows we denote the unit interval $[0, 1] \subset \mathbb{R}$ by $\mathbb{I}$. We denote the unit ball $D^3 = \{x \in \mathbb{R}^3 \mid |x| \leq 1\}$ and the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$.

(**C.1**) Let M be a manifold, with boundary ∂M, and $\underline{M} = (M, \partial M)$. Let $\mathrm{TOP}^h(M, M)$ denote the set of self-homeomorphisms of M, made a space with the compact-open topology.

A *flow* of $\underline{M}$ is a path $f\colon \mathbb{I} \to \mathrm{TOP}^h(M, M)$ with $f_0 = \mathrm{id}_M$ and all f_t pointwise fixing ∂M; or equivalently an automorphism $f'\colon M \times \mathbb{I} \to M \times \mathbb{I}$ such that for each $t \in \mathbb{I}$, f' restricts to a homeomorphism $f'_t\colon M \times \{t\} \to M \times \{t\}$ fixing $\partial M \times \{t\}$, and such that f'_0 is the identity.

A *motion* of $\underline{M}$ taking N to N' is a triple (f, N, N') consisting of a flow f of $\underline{M}$, a subset $N \subset M$, and the image $N' = f_1(N)$.

Two motions in $\underline{M}$, $(f, N, N'), (g, N, N')$ are *motion equivalent* if f can be continuously deformed into g through motions taking N to N'.

For each $\underline{M} = (M, \partial M)$, there is a groupoid with objects the power set of M and morphisms from $N \subset M$ to $N' \subset M$, motions up to motion equivalence. Following [42] (where all details can be found), we denote this groupoid $\mathrm{Mot}_{\underline{M}}$.

(**C.2**) For some manifold M and set Q of subsets of M, we use $\mathrm{Mot}_{\underline{M}}|_Q$ to denote the full subgroupoid of $\mathrm{Mot}_{\underline{M}}$ with objects Q.

For example, if $p_n \subset (0, 1)^2$ is a collection of n points, for each $n \in \mathbb{N}$; and p_* is the set of all the p_ns, then $\mathrm{Mot}_{\underline{\mathbb{I}^2}}(p_n, p_n)$ is a realisation of a braid group, and $\mathrm{Mot}_{\underline{\mathbb{I}^2}}|_{p_*}$ is a realisation of the braid category. This realisation bridges between the purely presentational one useful in representation theory, and the braid laboratory. Of course the connection to the braid laboratory makes several distinct uses of the continuum hypothesis. What we do next will make several more!

Fix a small number $\mathfrak{r} > 0$. For each $n \in \mathbb{N}$, we fix an embedding $e_n\colon (S^1 \sqcup \ldots \sqcup S^1) \to \mathrm{int}(D^3)$ of n copies of S^1 into the interior of D^3 such that $l_n = e_n(S^1 \sqcup \ldots \sqcup S^1)$ is a configuration of unlinked circles of radius $\mathfrak{r}/n$ in the xy-plane with the i-th loop centred at $(\frac{i}{n} - \frac{1}{2n}, 0, 0)$. (Up to isomorphism it will not matter precisely which configuration we take. The present choice is different from the one in the body of the paper, simply because we prefer not to have to unpick here the technical details of the limit required to have both the correct motion group on the nose, and the natural monoidal structure.)

Let l_* be the set of all such l_n. Note that $\mathrm{Mot}_{\underline{D^3}}|_{l_*} \cong \sqcup_{l_n}\mathrm{Mot}_{\underline{D^3}}(l_n, l_n)$, since morphisms in $\mathrm{Mot}_{\underline{D^3}}$ exist only between homeomorphic subsets of D^3, and, by construction, l_* contains precisely one element in each isomorphism class.

The reader will readily confirm that for each pair $n, m \in \mathbb{N}$ there is a function taking $D^3 \sqcup D^3$ to D^3 that takes the corresponding $l_n \sqcup l_m$ to l_{n+m} in the spirit of Fig.7 (up to some rescalings), and hence in the spirit of the braid laboratory of [27]. It follows from (C.1) that this collection of functions induces a (natural) monoidal structure on $\sqcup_{n\in\mathbb{N}}\mathrm{Mot}_{\underline{D^3}}(l_n, l_n)$, and hence on $\mathrm{Mot}_{\underline{D^3}}|_{l_*}$ (rigorous details will appear in [30]). As a brief abuse of notation we will use the same symbols for the monoidal categories.

Let

$$\mathsf{L}^{\mathrm{ext}} \;=\; \mathrm{Mot}_{\underline{D^3}}|_{l_*} \;\cong\; \sqcup_{n\in\mathbb{N}}\mathrm{Mot}_{\underline{D^3}}(l_n, l_n).$$

(**C.3**) Next we consider some morphisms in $\mathsf{L}^{\mathrm{ext}}$. We have in mind motions inducing the circle trajectories of type ς and ϱ as in §2. That such trajectories extend to motions as in (C.1) could be shown by giving an explicit construction, or proving a suitable isotopy extension theorem, but here we simply lean on the (correct) intuition that there is one. (See [42, §5.1] for brief technicalities. Complete details will appear in [30].)

Let ς_i denote a morphism represented by a motion which swaps the positions of the ith and $i+1$th circle, such that the circles remain parallel to the xy-plane, and such that the ith circle 'passes through' the disk parallel to the xy-plane bounded by the $i+1$th circle. The circle trajectory for such a motion is represented in Fig. 6. (The *support* of a motion is the set of points that are not static throughout. A key property of motions such as ς_i is that they include representatives where the support is localised in the region of i and $i+1$. Thus the local-view Figure conveys the motion - it is recovered by suitably appending and prepending static loops.)

Let ϱ_i denote a morphism represented by a motion which swaps the positions of the ith and $i+1$th circle, such that the circles remain parallel to the xy-plane, and neither circle ever intersects the disk parallel to the xy-plane bounded by the other. Such a motion is represented in Fig. 5.

Let τ_i denote a morphism represented by a motion which rotates the ith circle by π about an axis passing through the circle twice and its centre (thus the restriction of the endpoint of the motion to the ith circle is an orientation reversing homeomorphism).

The proof of (C.4)-(C.6) below is essentially contained in [9], and reproduced in [17], although we note that, as pointed out by [46], the proof requires the additional assumption that each motion, and each homotopy, has compact support, rather than just each self homeomorphism. With this additional assumption it can then be shown that the motion group of loops in $\mathbb{R}^3$ constructed in [9, 17] coincides with $\mathrm{Mot}_{\underline{D^3}}|_{l_*}$.

(**C.4**) Consider the bouquet of n loops homotopic to the complement of l_n in D^3 (see e.g. [11] and references therein). Each motion induces an automorphism on the fundamental group of this bouquet. Clearly this fundamental group is generated by the n simple loop paths. In fact it is free on these generators (for example by a universal cover argument).

Let $\mathrm{Aut}(\mathrm{F}(x_1, \ldots, x_n))$ be the group of automorphisms of the free group generated by $\{x_1, \ldots, x_n\}$. The given construction yields a homomorphism

$$\mathfrak{D}\colon \mathsf{L}^{\mathrm{ext}}(l_n, l_n) \to \mathrm{Aut}(\mathrm{F}(x_1, \ldots, x_n)).$$

(**C.5**) Let us consider the following automorphisms in $\mathrm{Aut}(\mathrm{F}(x_1, \ldots, x_n))$.

$$\begin{aligned}
&\mathfrak{t}_i\colon x_i \mapsto x_i^{-1},\;\; x_k \mapsto x_k,\;\; k \neq i\\
&\mathfrak{s}_i\colon x_i \mapsto x_{i+1},\;\; x_{i+1} \mapsto x_i,\;\; x_k \mapsto x_k,\;\; k \neq i, i+1\\
&\mathfrak{r}_i\colon x_i \mapsto x_{i+1}x_i x_{i+1}^{-1},\;\; x_{i+1} \mapsto x_i,\;\; x_k \mapsto x_k,\;\; k \neq i
\end{aligned}$$

While it is not easy to see that the construction in C.4 yields a well-defined group homomorphism from $\mathsf{L}^{\mathrm{ext}}(l_n, l_n)$ it is straightforward to see that the images of our motions in (C.3) above are these automorphisms, in particular $\mathfrak{D}(\tau_i) = \mathfrak{t}_i$, $\mathfrak{D}(\varrho_i) = \mathfrak{s}_i$ and $\mathfrak{D}(\varsigma_i) = \mathfrak{r}_i$. Moreover $\mathfrak{D}$ is an isomorphism onto the subgroup generated by $\mathfrak{t}_i, \mathfrak{s}_i, \mathfrak{r}_i$. We thus have the following.

(**C.6**) The elements $\tau_i, \varrho_i, \varsigma_i$ generate the group $\mathsf{L}^{\text{ext}}(l_n, l_n)$.

(**C.7**) Observe that $\mathsf{L}(n,n)$ as we have introduced it in the paper obeys $\mathsf{L}(n,n) \subset \mathsf{L}^{\text{ext}}(n,n)$. It is the subgroup generated by ϱ_i and ς_i, at each l_n.

Intuitively, we are restricting to classes which contain a motion which keeps all loops parallel to the xz-plane throughout the motion.

It follows from from (C.1) and (C.6) that there is an isomorphism (see also [9]) from $\mathsf{L}(l_n, l_n)$ onto the subgroup of $\mathrm{Aut}(F(x_1, \ldots, x_n))$ generated by the $\mathfrak{s}_i, \mathfrak{r}_i$.

The following identities are readily verified in $\mathrm{Aut}(F(x_1, \ldots, x_n))$:

$$\begin{cases} \mathfrak{r}_i\mathfrak{r}_j = \mathfrak{r}_j\mathfrak{r}_i & |i-j| > 1 \\ \mathfrak{r}_i\mathfrak{r}_{i+1}\mathfrak{r}_i = \mathfrak{r}_{i+1}\mathfrak{r}_i\mathfrak{r}_{i+1} & i = 1, \ldots, n-2 \\ \mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_i\mathfrak{s}_j & |i-j| > 1 \\ \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1} & i = 1, \ldots n-2 \\ \mathfrak{s}_i^2 = 1 & i = 1, \ldots, n-1 \\ \mathfrak{s}_i\mathfrak{r}_j = \mathfrak{r}_j\mathfrak{s}_i & |i-j| > 1 \\ \mathfrak{r}_i\mathfrak{r}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{r}_i\mathfrak{r}_{i+1} & i = 1, \ldots n-2 \\ \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{r}_1 = \mathfrak{r}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1} & i = 1, \ldots n-2. \end{cases} \tag{46}$$

(**C.8**) As proved by Savushkina [39], the subgroup of $\mathrm{Aut}(F(x_1, \ldots, x_n))$ generated by the $\mathfrak{s}_i, \mathfrak{r}_i$ has *presentation* as in (46) above.

C.2 Generators and relations

Here we give a little background on two further technical aspects (again not strictly needed for this paper, but useful contextually). One is the nature of functors from finitely presented natural categories; and the other is on monoidal versus non-monoidal presentation.

(**C.9**) Let C, D be monoidal categories, with monoidal identity objects denoted by $\emptyset_{\mathrm{C}}$ and $\emptyset_{\mathrm{D}}$, and monoidal composition maps denoted $\otimes_{\mathrm{C}}, \otimes_{\mathrm{D}}$ respectively.
A *strict monoidal functor* $\mathrm{F}\colon \mathrm{C} \to \mathrm{D}$ is a functor such that $\mathrm{F}(\emptyset_{\mathrm{C}}) = \emptyset_{\mathrm{D}}$, $F(X) \otimes_{\mathrm{D}} F(Y) = F(X \otimes_{\mathrm{C}} Y)$ for all pairs X, Y of objects in C, and $F(f) \otimes_{\mathrm{D}} F(g) = F(f \otimes_{\mathrm{C}} g)$ for all pairs f, g of morphisms in C.

(**C.10**) *Strong* and *lax* monoidal functors weaken the equalities in Def. C.9 to isomorphisms and morphisms respectively. Specifically a strong (resp. lax) monoidal functor consists of a functor $F\colon \mathrm{C} \to \mathrm{D}$, together with an isomorphism (resp. morphism) $F_\emptyset\colon \mathrm{F}(\emptyset_{\mathrm{C}}) \to \emptyset_{\mathrm{D}}$ and a natural isomorphism (resp. transformation) $\{F_2\colon F(X) \otimes_{\mathrm{D}} F(Y) \to F(X \otimes_{\mathrm{C}} Y)\}_{X,Y}$, with various coherence conditions.

In the diagonal categories considered here, there are no morphisms between distinct objects so the conditions of strong and lax monoidal functors imply the equalities on objects as in Def. C.9. Although this a priori leaves open the possibility of non-trivial automorphisms $F_\emptyset$ and F_2.

We can codify the strict monoidal functors from C to D as follows.

Lemma C.11. *Let* C, D *be natural monoidal categories such that* C *is finitely presented with non-empty set of generating morphisms* $G(\mathrm{C})$ *and a corresponding set of relations* $R(\mathrm{C})$*. Of course any functor* F *restricts to a pair consisting of the value of* $F_0(1)$*; and a function* $F_1 : G(\mathrm{C}) \to \mathrm{D}$*. In particular if* $F_0(1) = M$ *then* $F_1(f : n \to m) \in \mathrm{D}(Mn, Mm)$*. Let* $M \in \mathbb{N}$*, and let* $\mathfrak{F}\colon G(\mathrm{C}) \to \mathrm{D}$ *be a map sending each* $f\colon n \to m$ *in* $G(\mathrm{C})$ *to a morphism in* $\mathrm{D}(nM, mM)$*. Then* $\mathfrak{F}\colon G(\mathrm{C}) \to \mathrm{D}$ *extends to a functor* $\mathsf{F}\colon \mathrm{C} \to \mathrm{D}$*, which is unique, precisely if the images under* $\mathfrak{F}$ *of the relations* $R(\mathrm{C})$ *are satisfied in* D.

Proof. Since $G(\mathrm{C})$ generates, every morphism can be built (albeit non-uniquely) as a category-and-monoidal product of elements and identity morphisms. There is an extension of $\mathfrak{F}$ to a formal map $\mathfrak{F}' \colon \mathrm{C} \to \mathrm{D}$, which on objects is the map $F_0 \colon \mathbb{N} \to \mathbb{N}$ with $F_0(1) = M$, and on morphisms is defined by the relations $\mathfrak{F}'(1_X) = 1_{\mathfrak{F}'(X)}$, $\mathfrak{F}'(f \circ g) = \mathfrak{F}'(f) \circ F(g)$, and $\mathfrak{F}'(f \otimes g) = \mathfrak{F}'(f) \otimes \mathfrak{F}'(g)$. This $\mathfrak{F}'$ is a well defined map $\mathsf{F} \colon \mathrm{C} \to \mathrm{D}$ precisely when the relations $R(C)$ are satisfied, in which case it is a strict monoidal functor. By construction it is unique. $\square$

(**C.12**) Recall that the object monoid of the monoidal groupoid L' is the natural numbers, and that both generating morphisms are of the form $f \colon n \to n$, where n=2, and therefore all $\mathsf{L}'(n, n') = \emptyset$ unless $n = n'$.

(**C.13**) After fixing n, we denote the element $\sigma \otimes 1 \otimes \ldots \otimes 1 \in \mathsf{L}'(n, n)$ of length $n-1$ by σ_1. We similarly label $\sigma_2, \ldots, \sigma_{n-1}$, and $s_1, \ldots, s_{n-1}$. Notice, using that $\otimes$ is a functor and thus preserves composition, that these elements generate the group $\mathsf{L}'(n, n)$.

Lemma C.14. *There is an isomorphism of categories (not using the monoidal structure)*

$$\Theta \colon \mathsf{L}' \to \mathsf{L}$$

which, at each n, sends $\sigma_i \mapsto \varsigma_i$ and $s_i \mapsto \varrho_i$.

Proof. First observe that since the σ_i, and s_i generate L', there is a unique map Θ satisfying the conditions of the Lemma and preserving composition. For this Θ to a be a well defined functor we must check that the relations in (46) are satisfied in L'. Surjectivity is satisfied since the ς_i and ϱ_i generate L.

We can prove both well-definedness and injectivity by noting that each relation in (46) corresponds directly to a relation in L' and vice versa. The relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ and $\rho_i \rho_j = \rho_j \rho_i$ when $|i - j| > 1$ correspond to relations coming from the fact that $\otimes$ is a functor. All other relations in (46) correspond directly to the relations (2), (3) and (4). $\square$

D Applying the recipe, and variations, to $J_3^{\pm}$

(**D.1**) Consider the set of solutions for $N = 3$, thus associated to $J_3^{\pm}$ as in (6.18) by the recipe as in (6.22). Here firstly we have four solution sets associated to $\Box^3$ (or eight ignoring Σ_3 symmetry):

$$\underline{\alpha}(F(s)) = (1, 1, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)), \qquad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_3, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12} C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{C_{13}} \\ \mu_{13} C_{13} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{23}}{C_{23}} \\ \mu_{23} C_{23} & 0 \end{pmatrix})$$

(we use up the gauge symmetry on s and absorb the effect of the gauge choice in free variables C_{ij});

$$\underline{\alpha}(F(s)) = (1, 1, -1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)), \qquad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_3, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12} C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{C_{13}} \\ \mu_{13} C_{13} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{23}}{C_{23}} \\ \mu_{23} C_{23} & 0 \end{pmatrix})$$

(here the -1 can be in each position); and similarly to those above with $s \rightsquigarrow -s$. The other two solutions from $J_3^{\pm}$ are given by two and three -1's in $\underline{\alpha}(F(s))$. Note that overall this is every possible way of extending $\Box^3$ in each $N = 2$ subspace, given that these solutions interlock at the vertices.

Next we have four solution sets associated to $\Box^1\ \Box\!\Box^1$:

$$\underline{\alpha}(F(s)) = (\pm 1, 1, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)), \qquad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_2, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12} C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{C_{13}} \\ \mu_{13} C_{13} & 0 \end{pmatrix}, \left(\begin{smallmatrix} A_2 & 0 \\ 0 & A_2 \end{smallmatrix}\right))$$

and $s \rightsquigarrow -s$ (with corresponding re-gauging if required).

Four sets associated to $\Box^1\ \boxminus^1$: for example ($\Box^1\ \boxminus^1$,) gives

$$\underline{\alpha}(F(s)) = (1, 1, -1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)), \qquad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_3, \begin{pmatrix} 0 & \frac{\mu_{12}}{C_{12}} \\ \mu_{12} C_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu_{13}}{C_{13}} \\ \mu_{13} C_{13} & 0 \end{pmatrix}, \left(\begin{smallmatrix} A_2 + A_3 & A_2 \\ -A_3 & 0 \end{smallmatrix}\right))$$

while $(\square^1,\ \ydiagram{1,1}^{\,1})$ gives

$$\underline{\alpha}(F(s)) = (1,-1,1,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)), \quad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_3, \left(\begin{smallmatrix}0&\frac{\mu_{12}}{C_{12}}\\ \mu_{12}C_{12}&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&\frac{\mu_{13}}{C_{13}}\\ \mu_{13}C_{13}&0\end{smallmatrix}\right), \left(\begin{smallmatrix}A_2+A_3&-A_2\\A_3&0\end{smallmatrix}\right))$$

Two sets associated to $\ydiagram{3}^1$:

$$F(s) = \pm 1_9, \qquad F(\sigma) = A_1 1_9$$

Two sets associated to $\ydiagram{2,1}^{\,1}$:

$$\underline{\alpha}(F(s)) = (1,1,-1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)), \quad \underline{\alpha}(F(\sigma)) = (A_1, A_1, A_3, \left(\begin{smallmatrix}A_1&0\\0&A_1\end{smallmatrix}\right), \left(\begin{smallmatrix}A_1+A_3&A_1\\-A_3&0\end{smallmatrix}\right), \left(\begin{smallmatrix}A_1+A_3&A_1\\-A_3&0\end{smallmatrix}\right))$$

and $s \rightsquigarrow -s$.
Two sets associated to $\ydiagram{1,2}^{\,1}$:

$$\underline{\alpha}(F(s)) = (1,-1,-1,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)), \quad \underline{\alpha}(F(\sigma)) = (A_1, A_2, A_2, \left(\begin{smallmatrix}A_1+A_2&A_1\\-A_2&0\end{smallmatrix}\right), \left(\begin{smallmatrix}A_1+A_2&A_1\\-A_2&0\end{smallmatrix}\right), \left(\begin{smallmatrix}A_2&0\\0&A_2\end{smallmatrix}\right))$$

and $s \rightsquigarrow -s$.

Note from [27, Prop.5.1] that these exhaust the possibilities for $F(\sigma)$ and visit every possibility for $F(s)$ consistent with $N = 2$ and the interlocking conditions.

References

[1] N. Andruskiewitsch and H.-J. Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.

[2] E. Artin. The theory of braids. *American Scientist*, 38(1):112–119, 1950.

[3] J. C. Baez and U. Schreiber. Higher gauge theory. In *Categories in algebra, geometry and mathematical physics. Conference and workshop in honor of Ross Street's 60th birthday, Sydney and Canberra, Australia, July 11–16/July 18–21, 2005*, pages 7–30. Providence, RI: American Mathematical Society (AMS), 2007.

[4] J. C. Baez, D. K. Wise, and A. S. Crans. Exotic statistics for strings in 4d BF theory. *Advances in Theoretical and Mathematical Physics*, 11(5):707–749, 2007.

[5] V. G. Bardakov. Extending representations of braid groups to the automorphism groups of free groups. *J. Knot Theory Ramifications*, 14(8):1087–1098, 2005.

[6] R. J. Baxter. *Exactly solved models in statistical mechanics.* 1982.

[7] T. E. Brendle and A. Hatcher. Configuration spaces of rings and wickets. *Commentarii Mathematici Helvetici*, 88(1):131–162, 2013.

[8] A. Bullivant, J. Faria Martins, and P. Martin. Representations of the loop braid group and Aharonov-Bohm like effects in discrete $(3+1)$-dimensional higher gauge theory. *Adv. Theor. Math. Phys.*, 23(7):1685–1769, 2019.

[9] D. M. Dahm. *A generalization of braid theory.* PhD thesis, Princeton University, Mathematics, 1962.

[10] C. Damiani. A journey through loop braid groups. *Expositiones Mathematicae*, 35(3):252–285, 2017.

[11] C. Damiani, J. Faria Martins, and P. P. Martin. On a canonical lift of Artin's representation to loop braid groups. *Journal of Pure and Applied Algebra*, 225(12):106760, 2021.

[12] C. Damiani, P. P. Martin, and E. C. Rowell. Generalisations of Hecke algebras from loop braid groups. *Pacific J. Math.*, to appear.

[13] J. Elgueta. Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces. *Advances in Mathematics*, 213(1):53 – 92, 2007.

[14] J. Faria Martins and R. Picken. Surface holonomy for non-Abelian 2-bundles via double groupoids. *Adv. Math.*, 226(4):3309–3366, 2011.

[15] R. Fenn, R. Rimányi, and C. Rourke. The braid-permutation group. *Topology*, 36(1):123–135, 1997.

[16] M. H. Freedman, A. Kitaev, and Z. Wang. Simulation of topological field theories by quantum computers. *Communications in Mathematical Physics*, 227:587–603, 2002.

[17] D. L. Goldsmith. The theory of motion groups. *The Michigan Mathematical Journal*, 28(1):3–17, 1981.

[18] S. Hawking and G. Ellis. *The large scale structure of space-time.* 1973.

[19] A. Joyal and R. Street. The geometry of tensor calculus, i. *Advances in Mathematics*, 88(1):55–112, 1991.

[20] Z. Kádár, P. Martin, E. Rowell, and Z. Wang. Local representations of the loop braid group. *Glasg. Math. J.*, 59(2):359–378, 2017.

[21] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[22] C. Kassel. *Quantum Groups.* Springer–Verlag, 1995.

[23] T. Lan, L. Kong, and X.-G. Wen. Classification of $(3+1)$D bosonic topological orders: The case when pointlike excitations are all bosons. *Phys. Rev. X*, 8:021074, Jun 2018.

[24] X.-S. Lin. *Nankai Tracts in Mathematics: Volume 12*, chapter Xiao-Song Lin's Unpublished Papers, pages 411–417. World Scientific, 2008.

[25] S. MacLane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer New York, 1998.

[26] P. Martin. On Schur-Weyl duality, A_n Hecke algebras and quantum $SL(N)$ on $\mathbb{C}_N^n$. *Int J Mod Phys A7 Suppl.1B*, page 645, 1992.

[27] P. Martin and E. C. Rowell. Classification of spin-chain braid representations. *arXiv:2112.04533*, 2022.

[28] P. Martin and E. C. Rowell. Maple worksheet n3aa0checknew.mw available., 2022.

[29] P. P. Martin. On diagram categories, representation theory and statistical mechanics. *AMS Contemp Math*, 456:99–136, 2008.

[30] J. F. Martins, P. P. Martin, and F. Torzewska. Monoidal motion groupoids: Examples, points and loops. *in preparation*, 2023. preprint in preparation.

[31] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80:1083–1159, Sep 2008.

[32] S.-Q. Ning, Z.-X. Liu, and P. Ye. Fractionalizing global symmetry on looplike topological excitations. *Phys. Rev. B*, 105:205137, May 2022.

[33] J. K. Pachos. *Introduction to topological quantum computation.* Cambridge University Press, 2012.

[34] M. Palmer and A. Soulié. The Burau representations of loop braid groups. *C. R. Math. Acad. Sci. Paris*, 360:781–797, 2022.

[35] T. Porter and V. Turaev. Formal homotopy quantum field theories. I: Formal maps and crossed $\mathcal{C}$-algebras. *J. Homotopy Relat. Struct.*, 3(1):113–159, 2008.

[36] E. C. Rowell and Z. Wang. Mathematics of Topological Quantum Computing. *ArXiv e-prints*, May 2017.

[37] B. E. Sagan. Combinatorial proofs of hook generating functions for skew plane partitions. *Theoretical Computer Science*, 117(1):273–287, 1993.

[38] S. Sahai. Composition tableaux basis for Schur functors and the Plücker algebra, 1811.10687, 2018.

[39] A. G. Savushkina. On a group of conjugating automorphisms of a free group. *Mat. Zametki*, 60(1):92–108, 159, 1996.

[40] N. J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2020.

[41] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

[42] F. Torzewska, J. Faria Martins, and P. P. Martin. Motion groupoids and mapping class groupoids. *Communications in Mathematical Physics*, 402(2):1621–1705, 2023.

[43] V. V. Vershinin. On homology of virtual braids and Burau representation. volume 10, pages 795–812. 2001. Knots in Hellas '98, Vol. 3 (Delphi).

[44] J. C. Wang and X.-G. Wen. Non-abelian string and particle braiding in topological order: Modular sl (3, z) representation and (3+ 1)-dimensional twisted gauge theory. *Physical Review B*, 91(3):035134, 2015.

[45] Z. Wang. *Topological quantum computation.* Number 112. American Mathematical Soc., 2010.

[46] F. Wattenberg. Differentiable motions of unknotted, unlinked circles in 3-space. *Mathematica Scandinavica*, 30(1):107–135, 1972.