Article:

Hietarinta, J., Martin, P. orcid.org/0000-0002-8141-9465 and Rowell, E.C. (2024) Solutions to the constant Yang–Baxter equation: additive charge conservation in three dimensions. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 480 (2294). 20230810. ISSN 1364-5021

https://doi.org/10.1098/rspa.2023.0810

Solutions to the constant Yang-Baxter equation: additive charge conservation in three dimensions

J.Hietarinta[*], P.Martin[†], E.C.Rowell[‡]

January 6, 2025

Abstract

We find all solutions to the constant Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in three dimensions, subject to an additive charge-conservation ansatz. This ansatz is a generalisation of (strict) charge-conservation, for which a complete classification in all dimensions was recently obtained. Additive charge-conservation introduces additional sector-coupling parameters – in 3 dimensions there are 4 such parameters. In the generic dimension 3 case, in which all of the 4 parameters are nonzero, we find there is a single 3 parameter family of solutions. We give a complete analysis of this solution, giving the structure of the centraliser (symmetry) algebra in all orders. We also solve the remaining cases with three, two, or one nonzero sector-coupling parameter(s).

1 Introduction

The Yang-Baxter equation (YBE) reads (in shorthand form)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1}$$

It is a fundamental equation for many applications — see for example [2, 3, 20, 4], [8, 11, 12, 23] and references therein.

To make (1) explicit, one first fixes a dimension N for a vector space $V = \mathbf{C}^N$. We can also pick bases for V and $V \otimes V$. Then we have an underlying matrix R acting on $V \otimes V$. Each matrix R_{ij} acts on $V \otimes V \otimes V$, acting on the i-th and j-th factors as R, and on the other factor as the identity. Thus in explicit form the Yang-Baxter equation reads

$$\sum_{\alpha_1,\alpha_2,\alpha_3} \mathcal{R}^{i_1 i_2}_{\alpha_1\alpha_2}\mathcal{R}^{\alpha_1 i_3}_{j_1\alpha_3}\mathcal{R}^{\alpha_2\alpha_3}_{j_2 j_3} = \sum_{\beta_1,\beta_2,\beta_3} \mathcal{R}^{i_2 i_3}_{\beta_2\beta_3}\mathcal{R}^{i_1\beta_3}_{\beta_1 j_3}\mathcal{R}^{\beta_1\beta_2}_{j_1 j_2}, \tag{2}$$

where the indices range over $0, 1, \ldots, N-1$ and $\mathcal{R}^{i_1 i_2}_{\alpha_1\alpha_2}$ is the appropriate matrix entry of R. (See also §2.1.)

[*]hietarin@utu.fi, University of Turku, Finland
[†]p.p.martin@leeds.ac.uk, University of Leeds, UK
[‡]rowell@tamu.edu, Texas A& M University, USA

With various applications in mind, we impose

$$\det(R) \neq 0. \tag{3}$$

For some applications the R matrices depend on spectral parameters that can be different for each R_{ij} [20, 4], but in this paper we will consider the constant YBE. By construction, any $\check{R}$ gives a representation of the braid group B_n for each n.

Observe that R will have $N^2 \times N^2$ entries and there will be, in principle, $N^3 \times N^3$ equations. It is clear that such an overdetermined set of nonlinear equations is difficult to solve, even in this constant form. Indeed, while many individual solutions are known, a complete solution is known only for dimension two [9] and for higher dimensions knowledge is far from complete. The three dimensional upper triangular case was solved in [10], but for further progress it is important to make a meaningful ansatz.

Recently Martin and Rowell proposed [15] charge-conservation of the form

$$\mathcal{R}_{ij}^{kl} = 0, \text{ if } \{i, j\} \neq \{k, l\} \text{ as a set}, \tag{4}$$

as an effective constraint and with it they were able to find all solutions for all dimensions. The above constraint may be called "strict charge conservation" (SCC). In this paper we will explore the results obtained by relaxing the SCC rule to "additive charge conservation" (ACC) defined by

$$\mathcal{R}_{ij}^{kl} = 0, \text{ if } i + j \neq k + l. \tag{5}$$

Observe that ACC differs from SCC first in dimension 3. In practice this change increases the complexity of the underlying computational problem by introducing four further 'mixing' parameters (SCC itself having fifteen parameters in dimension 3).

The paper is organized as follows. In Section 2 we discuss notational matters and symmetries of the problem. In Section 3 we present the solutions. The set of solutions is organized according to the non-vanishing conditions on the four mixing parameters (together with their symmetries). Thus the first family of solutions is the generic case, with all parameters non-zero – it is solved in detail in §3.2. The various possibilities are then addressed in turn, the last case being the set of solutions where all but one mixing parameter vanishes - §3.6.

It turns out that in several solutions have the 'Hecke' property (i.e., having precisely two distinct eigenvalues). In §4.1 we use this to analyse the representations, giving a complete analysis for the generic case.

One natural realisation of the constant Yang–Baxter problem is as a problem in categorical representation theory, and this is the perspective largely taken in [15] (see also [16], for example). However here we will keep to a simple analytical setting. Direct transliteration of results between the settings is a routine exercise.

Acknowledgements. We thank Frank Nijhoff for various important contributions, including initiating our collaboration. PM thanks EPSRC for funding under grant EP/W007509/1; and Paula Martin for useful conversations. ECR was partially funded by US NSF grants DMS-2000331 and DMS-2205962.

2 The setup

For the braid group point of view we first define

$$\check{R} = P\,R, \text{ where } \mathcal{P}_{ij}^{kl} = \delta_i^l \delta_j^k, \quad \text{i.e. } \check{\mathcal{R}}_{ij}^{kl} = \mathcal{R}_{ji}^{kl}. \tag{6}$$

and furthermore

$$(PR)_{12} \;=\; \check{R}_1 \;:=\; \check{R}\otimes 1 \quad \text{and} \quad (PR)_{23} = \check{R}_2 \;:=\; 1\otimes \check{R}$$

acting on $V\otimes V\otimes V$. Then the YBE in (1) becomes

$$\check{R}_1\check{R}_2\check{R}_1 = \check{R}_2\check{R}_1\check{R}_2, \tag{7}$$

i.e., the braid group version of the YBE.

2.1 Presenting matrices

Set $V = \mathbf{C}^3$ with basis labeled by $\{0,1,2\}$. We will order this basis as the symbols suggest. Using the standard ket notation, i.e. $i\otimes j =: |ij\rangle$, we may order the basis of $V\otimes V$ for example using lexicographical order

$$|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle$$

or reverse lexicographical order (rlex)

$$|00\rangle, |10\rangle, |20\rangle, |01\rangle, |11\rangle, |21\rangle, |02\rangle, |12\rangle, |22\rangle$$

Still another possibility is to use a 'graded' reverse lexicographical ordering (grlex)

$$|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle, |21\rangle, |12\rangle, |22\rangle$$

The name is borrowed from monomial orderings, in which setting the symbols *are* numbers, rather than being arbitrarily associated to numbers as in our case.

The matrix entries are defined as:

$$\mathcal{R}_{ij}^{kl} := \langle ij|R|kl\rangle$$

In the present case with ACC (5) and the rlex ordering we get the matrix

$$R_{rlex} = \begin{pmatrix}
\mathcal{R}^{0,0}_{0,0} & . & . & . & . & . & . & . & . \\
. & \mathcal{R}^{1,0}_{1,0} & . & \mathcal{R}^{0,1}_{1,0} & . & . & . & . & . \\
. & . & \mathcal{R}^{2,0}_{2,0} & . & \mathcal{R}^{1,1}_{2,0} & . & \mathcal{R}^{0,2}_{2,0} & . & . \\
. & \mathcal{R}^{1,0}_{0,1} & . & \mathcal{R}^{0,1}_{0,1} & . & . & . & . & . \\
. & . & \mathcal{R}^{2,0}_{1,1} & . & \mathcal{R}^{1,1}_{1,1} & . & \mathcal{R}^{0,2}_{1,1} & . & . \\
. & . & . & . & . & \mathcal{R}^{2,1}_{2,1} & . & \mathcal{R}^{1,2}_{2,1} & . \\
. & . & \mathcal{R}^{2,0}_{0,2} & . & \mathcal{R}^{1,1}_{0,2} & . & \mathcal{R}^{0,2}_{0,2} & . & . \\
. & . & . & . & . & \mathcal{R}^{2,1}_{1,2} & . & \mathcal{R}^{1,2}_{1,2} & . \\
. & . & . & . & . & . & . & . & \mathcal{R}^{2,2}_{2,2}
\end{pmatrix} \tag{8}$$

Indeed the 'shape' - the non-vanishing pattern - is the same for R, R_{rlex} and $\check{R}$. The grlex matrix is obtained from this with

$$R_{grlex} = P_G R_{rlex} P_G,$$

where P_G implements the transpositions $|01\rangle \leftrightarrow |20\rangle$ and $|21\rangle \leftrightarrow |02\rangle$. Then an ACC matrix takes the block form exemplified by

$$R_{grlex} = \begin{pmatrix} \mathcal{R}^{0,0}_{0,0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{R}^{1,0}_{1,0} & \mathcal{R}^{0,1}_{1,0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{R}^{1,0}_{0,1} & \mathcal{R}^{0,1}_{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}^{2,0}_{2,0} & \mathcal{R}^{1,1}_{2,0} & \mathcal{R}^{0,2}_{2,0} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}^{2,0}_{1,1} & \mathcal{R}^{1,1}_{1,1} & \mathcal{R}^{0,2}_{1,1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}^{2,0}_{0,2} & \mathcal{R}^{1,1}_{0,2} & \mathcal{R}^{0,2}_{0,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}^{2,1}_{2,1} & \mathcal{R}^{1,2}_{2,1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}^{2,1}_{1,2} & \mathcal{R}^{1,2}_{1,2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}^{2,2}_{2,2} \end{pmatrix} \tag{9}$$

In order to save space we will in the following just give the blocks as

$$R_{grlex} = \begin{bmatrix} \mathcal{R}^{0,0}_{0,0} \end{bmatrix} \begin{bmatrix} \mathcal{R}^{1,0}_{1,0} & \mathcal{R}^{0,1}_{1,0} \\ \mathcal{R}^{1,0}_{0,1} & \mathcal{R}^{0,1}_{0,1} \end{bmatrix} \begin{bmatrix} \mathcal{R}^{2,0}_{2,0} & \mathcal{R}^{1,1}_{2,0} & \mathcal{R}^{0,2}_{2,0} \\ \mathcal{R}^{2,0}_{1,1} & \mathcal{R}^{1,1}_{1,1} & \mathcal{R}^{0,2}_{1,1} \\ \mathcal{R}^{2,0}_{0,2} & \mathcal{R}^{1,1}_{0,2} & \mathcal{R}^{0,2}_{0,2} \end{bmatrix} \begin{bmatrix} \mathcal{R}^{2,1}_{2,1} & \mathcal{R}^{1,2}_{2,1} \\ \mathcal{R}^{2,1}_{1,2} & \mathcal{R}^{1,2}_{1,2} \end{bmatrix} \begin{bmatrix} \mathcal{R}^{2,2}_{2,2} \end{bmatrix}. \tag{10}$$

Recall that $\check{R}$ is obtained from R by exchanging lower indices, which corresponds to up-down reflection within the block. In order to match with [15] (using shifted basis labels $\{0,1,2\} \rightsquigarrow \{1,2,3\}$), highlight the new parameters, and save from writing many double indices we introduce shorthand notation for $\check{R}$:

$$\check{R} = PR = \begin{pmatrix} a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{12} & \cdot & b_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{13} & \cdot & x_1 & \cdot & b_{13} & \cdot & \cdot \\ \cdot & c_{12} & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_2 & \cdot & a_2 & \cdot & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{23} & \cdot & b_{23} & \cdot \\ \cdot & \cdot & c_{13} & \cdot & x_4 & \cdot & d_{13} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_{23} & \cdot & d_{23} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_3 \end{pmatrix} \tag{11}$$

Then the block form is

$$\begin{bmatrix} a_1 \end{bmatrix} \begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix} \begin{bmatrix} a_{13} & x_1 & b_{13} \\ x_2 & a_2 & x_3 \\ c_{13} & x_4 & d_{13} \end{bmatrix} \begin{bmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{bmatrix} \begin{bmatrix} a_3 \end{bmatrix} \tag{12}$$

2.2 Symmetries

Naturally it is useful to consider additive charge-conserving solutions to (7) up to transformations that preserve (7) and the additive charge-conserving condition.

1. **Scaling** symmetry: Equation (7) and the additive charge-conserving condition is invariant under rescaling $\check{R}$ by a non-zero complex number.

2. **Transpose** symmetry: The additive charge-conservation is preserved under transpose: $\check{R} \mapsto \check{R}^T$; and of course (7) is satisfied by $\check{R}^T$ if it is satisfied by $\check{R}$ quite generally. Indeed, notice that from the form of (2) it is easy to see that if $\check{\mathcal{R}}_{i,j}^{k,l}$ solves the equation, then so does $\check{\mathcal{R}}_{k,l}^{i,j}$. The effect on the variable choices in (11) are $b_{ij} \leftrightarrow c_{ij}$ and $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$.

3. **Left-Right** (LR) symmetry: Changing the ordering of the basis from lex to rlex the resulting matrix will also be a solution. This can be seen in matrix entries because if $\check{\mathcal{R}}_{i,j}^{k,l}$ solves equation (7), then so does $\check{\mathcal{R}}_{j,i}^{l,k}$. This corresponds to reflecting each of the blocks in (12) across both the diagonal and the skew-diagonal, i.e. $a_{ij} \leftrightarrow d_{ij}$, $b_{ij} \leftrightarrow c_{ij}$ as well as $x_1 \leftrightarrow x_4$ and $x_2 \leftrightarrow x_3$.

4. **02-** or $|0\rangle \leftrightarrow |2\rangle$ symmetry: while (7) is clearly invariant under local basis changes, the additive charge-conserving condition is not. However, the local basis change (permutation) $|j\rangle \leftrightarrow |2-j\rangle$ with indices $\{0, 1, 2\}$ taken modulo 3 does preserve the form of an additive charge-conserving matrix: the span of the $|ij\rangle$ with $i + j = 2$ is preserved, while the $|ij\rangle$ with $i + j = 1$ and $i + j = 3$ are interchanged as are the vectors $|00\rangle$ and $|22\rangle$. The effect on the block form (12) is to interchange the pairs of 1×1 and 2×2 blocks followed by a reflection across both the diagonal and skew-diagonal of each block.

Of course these symmetries can be composed with one another and, discounting the rescaling, one finds that the group of such symmetries is the dihedral group of order 8. This can be seen by tracking the orbit of the 2×2 matrix $\begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix}$, since there are no symmetries that fix it. Indeed, we see that there are 4 forms it can take, generated by the reflections across the diagonal and, independently across the skew-diagonal, and two positions in (12) it can occupy.

3 The solutions

For constant Yang–Baxter solutions, a necessary and sufficient set of constraint equations on the indeterminate matrix entries arise as follows. Firstly compute, say,

$$A_R := \check{R}_1 \check{R}_2 \check{R}_1 - \check{R}_2 \check{R}_1 \check{R}_2 \tag{13}$$

which we call the *braid anomaly* so that the constraints are obtained from $A_R = 0$.

The SCC case in which all x_i vanish was solved in [15]. Note that in ACC some x_i can be nonzero there will be mixing between more states $|ij\rangle$, but always with $i + j$ constant, so this is a computationally relatively modest generalisation. However the full symmetry of indices that exists for SCC is now broken. This ansatz-relaxing obviously increases the complexity of the system of cubic equations, but they can still be solved, as given below.

We organise the solutions according to which x_is are vanishing. In principle there are $2^4-1=15$ cases (excluding the SCC case), but we can use the above symmetries in order to omit some x configurations. This leads to the following classification into 6 cases:

1. All x_i are nonzero. See §3.1 and §3.2.
2. Precisely one x vanishes, by symmetry it can be assumed to be x_4. See §3.3.
3. $x_3x_4 \neq 0$ and $x_1 = x_2 = 0$, related by the LR symmetry to $x_1x_2 \neq 0$ and $x_3 = x_4 = 0$. See §3.4.
4. $x_1x_3 \neq 0$ and $x_2 = x_4 = 0$, related to $x_2x_4 \neq 0$ and $x_1 = x_3 = 0$ by transposition. See §3.3.
5. $x_1x_4 \neq 0$, $x_2 = x_3 = 0$, related to $x_2x_3 \neq 0$ and $x_1 = x_4 = 0$ by transposition, §3.5
6. Only one x is nonzero, by symmetry it can be assumed to be x_4, §3.6.

As noted, solution of constant Yang–Baxter is equivalent to solving $A_R = 0$. We write out A_R explicitly in Appendix A.1. We solve for the various cases as above in the following sections 3.1–3.6. In the first of these we treat Case 1 relatively gently. After that we will proceed more rapidly though all cases.

3.1 The $x_1x_2x_3x_4 \neq 0$ solutions

Recall the ACC ansatz for $\check{R}$, which is as in (11). Consider now the refinement of this ansatz indicated by the block structure

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} a & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1x_3+b}{b} & x_3 \\ \frac{{x_1}^2{x_3}^2}{b^3} & -\frac{x_1(ab+x_1x_3)}{ab^2} & -\frac{x_1x_3}{ab} \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

that is

$$\check{R}_j = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & x_1 & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{x_3(a-1)}{b} & \cdot & \frac{x_1x_3+b}{b} & \cdot & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{{x_3}^2{x_1}^2}{b^3} & \cdot & -\frac{x_1(ab+x_1x_3)}{a\,b^2} & \cdot & -\frac{x_3x_1}{ab} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{14}$$

Here the parameters $a_{13},\ b_{13},\ x_1,\ x_3$ are indeterminate (we write $a = a_{13},\ b = b_{13}$) but the remaining parameters are replaced with functions of these four as shown.

Proposition 3.1. *(I) Consider the ansatz for $\check{R}$ in (11). If we leave parameters $a_{13},\ b_{13},\ x_1,\ x_3$ indeterminate (here we write $a = a_{13},\ b = b_{13}$) but replace the remaining parameters with functions of these four as shown in (14) then the braid anomaly A_R has an overall factor*

$$x_3^2 x_1^2 a + a^2 b^2 + x_1 b x_3 a - a\, b^2 - x_1 b x_3 \;=\; b^2 a(a-1) + b x_3 x_1 (a-1) + a x_3^2 x_1^2$$

That is, we have a family of solutions obeying $\check{R}_1\check{R}_2\check{R}_1 = \check{R}_2\check{R}_1\check{R}_2$ with free non-zero parameters (say) a, x_1, x_3, and parameter b determined by

$$\frac{b}{x_1 x_3} = \frac{-\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - \frac{4}{a-1}}}{2} \qquad \textit{or} \qquad \frac{x_1 x_3}{b} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4a^2(a-1)}}{2a} \tag{15}$$

and the remaining entries determined as in (14) above.
(II) If $x_1x_2x_3x_4 \neq 0$ then the above (with $a \neq 1$) gives the complete set of solutions up to overall rescaling.

Proof. (I) is simply a brutal but straightforward calculation, plugging in to A_R as given for example in Appendix A.1. For (II) we proceed as follows. The matrix A_R is rather large to write out (again see Appendix A.1), but a subset of its entries is

$$\begin{aligned}
\mathsf{S}_R \;=\; \{ & -a_{12}b_{12}c_{12} - a_1 a_{12}(a_{12} - a_1), \qquad -b_{23}c_{23}d_{23} - a_3 d_{23}(d_{23} - a_3), && (16)\\
& -b_{23}x_1x_3, \qquad -b_{12}x_1x_3, \qquad (a_{12} - d_{12})x_1x_3, \qquad ((-d_{12} + a_1)b_{13} - b_{12}^2)x_1, && (17)\\
& a_{13}(b_{12}c_{12} - b_{23}c_{23}) + a_{12}a_{23}(a_{12} - a_{23}), \qquad (d_{12} - d_{23})x_1x_2, \qquad (d_{23} - a_{23})x_1x_3, \\
& -a_{12}x_1x_3 - a_{13}b_{13}d_{13}, \qquad -a_{12}x_2x_4 - a_{13}c_{13}d_{13}, \qquad a_{12}c_{12}d_{12}, \qquad a_{23}c_{23}d_{23}, && (18)\\
& -a_{13}b_{13}x_4 + (a_1a_{12} - a_{12}a_2 - a_1a_{13})x_1, && (19)\\
& (a_1d_{13} + a_2d_{12} - a_1d_{12})x_3 + b_{13}d_{13}x_2, \qquad (a_{13}d_{13} + a_2a_{23} - a_{13}a_{23})x_3 + (a_3b_{13} - b_{23}^2)x_2, && (20)\\
& a_{13}c_{13}x_3 + (a_{13}a_3 - a_{23}a_3 + a_2a_{23})x_2, \qquad a_{12}c_{13}x_3 + (-a_2d_{23} + a_{13}d_{23} - b_{23}c_{12} + a_2^2)x_2, && (21)\\
& (a_1c_{13} - c_{12}^2)x_3 + (a_{13}d_{13} - d_{12}d_{13} + a_2d_{12})x_2, && (22)\\
& -a_2x_1x_2 + a_{13}d_{12}^2 - a_{13}^2d_{12} - a_{23}b_{13}c_{13} + a_{23}b_{12}c_{12}, && (23)\\
& a_1x_3x_4 + d_{13}x_1x_2 - a_2d_{12}^2 - b_{12}c_{12}d_{12} + a_2^2d_{12}, && (24)\\
& a_{13}x_3x_4 + a_3x_1x_2 - a_{23}b_{23}c_{23} - a_2a_{23}^2 + a_2^2a_{23}, \quad \ldots\} && (25)
\end{aligned}$$

Imposing $A_R = 0$ and $x_1x_3 \neq 0$ we thus get $b_{12} = b_{23} = 0$ and $a_{12} = d_{12}$ from (17). Note that $\check{R}$ is invertible, so $a_1, a_3 \neq 0$ and

$$a_{12}d_{12} - b_{12}c_{12} \neq 0, \qquad a_{23}d_{23} - b_{23}c_{23} \neq 0. \tag{26}$$

Thus $a_{12}, d_{12}, a_{23}, d_{23} \neq 0$. Thus $d_{12} = a_{23} = a_{12} = a_1 = d_{23} = a_3$ and $c_{12} = c_{23} = 0$ from (18). Note that if $\check{R}$ is a solution then so is any non-zero scalar multiple, so we first scale $\check{R}$ by an overall factor, so that $a_1 = 1$. This confirms the form for $\check{R}$ above outside the 3x3 block.

Observe now from (18) that $a_{13}b_{13}d_{13} = -x_1x_3 \neq 0$, and that we may either replace $a_{13} = -\frac{x_1x_3}{b_{13}d_{13}}$, or $d_{13} = -\frac{x_1x_3}{a_{13}b_{13}}$. The latter gives the form of d_{13} in the Proposition.

Before proceeding we will need to show that $d = 1$ cannot occur here. (Recall $a = a_{13}$, and write also $d = d_{13}$.) Comparing (24) and (25) we find

$$(a-1)x_3x_4 = (d-1)x_1x_2.$$

So $a = 1$ if and only if $d = 1$. So if $a = 1$ then $b_{13} = -x_1x_3$ and $c_{13} = -x_2x_4$ (consider (18i/ii)).

Evaluating (21ii)-(22) here we find:

$$a_2^2 - a_2 + a - a_2 + d - ad \;=\; (a_2-1)^2 - (a-1)(d-1) \;= 0$$

So if $a = 1$ then $a_2 = 1$. Further, if $a = 1$ then (23) becomes $-x_1x_2 - b_{13}c_{13} = -x_1x_2 - x_1x_2x_3x_4 = 0$ so $x_3x_4 = -1$. But if $a = 1$ then (18) gives $x_4 = (-x_1)/(-x_1x_3) = 1/x_3$ – a contradiction. We conclude here that $(a-1) \neq 0$; and hence $(d-1) \neq 0$.

From (20) we have two formulae for $b_{13}x_2$. Equating we have

$$\frac{(a_1d_{13} + a_2d_{12} - a_1d_{12})}{d_{13}} \;= (a_{13}d_{13} + a_2a_{23} - a_{13}a_{23})$$

that is

$$\frac{(d_{13} + a_2 - 1)}{d_{13}} \;= (a_{13}d_{13} + a_2 - a_{13}), \quad \text{thus} \quad \frac{(a_2-1)(1-d_{13})}{d_{13}} \;= -a_{13}(1-d_{13}).$$

Since $d_{13} - 1 \neq 0$ we have $a_2 - 1 = -a_{13}d_{13} = \frac{x_1x_3}{b_{13}}$ giving a_2 as in the Proposition. Plugging back in we find $(d_{13} + a_2 - 1)x_3 = -b_{13}d_{13}x_2$, $\frac{(a_{13}-1)x_3}{b_{13}} = x_2$ as in the Proposition.

From (19) we have

$$x_4 \;=\; x_1\frac{1-a_2-a_{13}}{a_{13}b_{13}} \;=\; x_1\frac{-x_1x_3 - a_{13}b_{13}}{a_{13}b_{13}^2}$$

as in the Proposition. Finally from (21) we now have

$$c_{13} = -\frac{(a_{13}+a_2-1)x_2}{a_{13}x_3} = -\frac{(a_{13}b_{13}+x_1x_3)(a_{13}-1)}{a_{13}b_{13}^2}$$

$$c_{13}x_3 = -(a_{13} + a_2(a_2-1))x_2 = -(a + \frac{(x_1x_3+b)}{b}\frac{x_1x_3}{b})\frac{(a_{13}-1)x_3}{b}$$
$$= \frac{-x_3((ab^2 + x_1x_3(x_1x_3+b_{13}))(a_{13}-1))}{b_{13}^3}$$

Equating the two formulae for c_{13}, and noting that $a_{13} - 1 \neq 0$, we have

$$a_{13}(x_1x_3)^2 + (a_{13}-1)b_{13}x_1x_3 + (a_{13}-1)a_{13}b_{13}^2 = 0$$

Plugging back in to (21) we obtain c_{13} as in the Proposition, so we are done. □

3.2 Case 1: $x_1x_2x_3x_4 \neq 0$ revisited

In this section we solve the case in which $x_1x_2x_3x_4 \neq 0$ again, but leaning directly on the Appendix (as we shall below for the remaining cases). Since all x_i are nonzero, we conclude from equations (A.5)-(A.8) that $b_{12} = c_{12} = b_{23} = c_{23} = 0$, and from (A.11)-(A.13) $a_{12} = d_{12} = a_{23} = d_{23}$. Then since $a_{12} \neq 0$ from (A.29) we get $a_{12} = 1$ and since $a_3 \neq 0$ from (A.30) $a_3 = 1$.

Now from (A.18) we find $a_{13}b_{13}d_{13} \neq 0$ and we can solve $d_{13} = -\frac{x_1x_3}{a_{13}b_{13}}$, and from (A.23) $c_{13} = \frac{b_{13}x_2x_4}{x_1x_3}$. Then from (A.70) we find $x_4 = \frac{x_1(1-a_2-a_{13})}{a_{13}b_{13}}$. Now it turns out that some equations factorize, for example (A.94) can be written as $x_1(a_{13}-1)[(a_2-1)b_{13} - x_1x_3] = 0$. If we were to choose $a_{13} = 1$ we reach a contradiction: from (A.58) we get $a_2 = 1$ and then (A.50) and (A.66) are contradictory since $x_i \neq 0$. Thus we can solve $a_2 = \frac{b_{13}+x_1x_3}{b_{13}}$, and then from (A.50) $x_2 = \frac{x_3(a_{13}-1)}{b_{13}}$.

After this all remaining nonzero equations simplify to

$$b_{13}^2a_{13}(a_{13}-1) + b_{13}x_1x_3(a_{13}-1) + a_{13}x_1^2x_3^2 = 0.$$

This biquadratic equation can be resolved using Weierstrass elliptic function $\wp$:

$$a = -\wp + \frac{5}{12}, \quad \beta = 6\frac{12\wp + 7 + 12\wp'}{(12\wp - 5)(12\wp + 7)}, \quad (\wp')^2 = 4\wp^3 - \frac{1}{12}\wp + \frac{7 \cdot 23}{2^3\,3^3}$$

where $a_{13} = a$ and $b_{13} = x_1x_3\beta$. The solution in block form now reads

$$[1]\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}\begin{bmatrix} a & x_1 & \beta x_1x_3 \\ \frac{a-1}{\beta x_1} & \frac{\beta+1}{\beta} & x_3 \\ \frac{1}{\beta^3x_1x_3} & \frac{-(a\beta+1)}{a\beta^2x_3} & \frac{-1}{a\beta} \end{bmatrix}\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}[1] \tag{27a}$$

with constraint

$$\beta^2a(a-1) + \beta(a-1) + a = 0. \tag{27b}$$

3.3 Cases 2 and 4: $x_1x_3 \neq 0$ and $x_4x_2 = 0$

From (A.7),(A.8) we get $b_{12} = b_{23} = 0$ and then since the matrix is non-singular we must have $a_{12}d_{12}a_{23}d_{23} \neq 0$. Then from (A.1),(A.2) we get $c_{12} = c_{23} = 0$, from (A.29),(A.31) $a_{12} = d_{12} = 1$ (recall that we have scaled $a_1 = 1$) and from (A.84),(A.85) $a_{23} = d_{23} = 1$. Next from (A.30) $a_3 = 1$. Since $x_1x_3 \neq 0$ we have from (A.18) that $a_{13}b_{13}d_{13} \neq 0$ and then from (A.24) we find $c_{13} = 0$.

To continue we consider first the case $x_2 = 0$, x_4 free. Then from (A.52) we get $a_{13} = 1$ and from (A.76) $d_{13} = 1 - a_2$. Then since $d_{13} \neq 0$ we cannot have $a_2 = 1$ but (A.106) is $x_3(a_2-1)^2 = 0$, a contradiction.

Next assume $x_2 \neq 0$, $x_4 = 0$. Then from (A.66) we get $d_{13} = 1$ and from (A.70) $a_{13} = 1 - a_2$ but (A.98) yields $a_2 = 1$ which is in contradiction with $a_{13} \neq 0$.

Thus there are no solutions in this case.

3.4 Case 3: $x_1 = x_2 = 0$, $x_3 x_4 \neq 0$

From (A.16) we get $a_{23} = a_{12}$ and from (A.94) and (A.104) $b_{13} = b_{12}^2$ and $c_{13} = c_{12}^2$. On the basis of (A.42) and (A.46) we can divide the problem into 2 branches: Case 3.1: $a_{12} = 0$, $b_{12}c_{12} \neq 0$, and Case 3.2: $a_{12}d_{12} \neq 0$, $b_{12} = c_{12} = 0$.

Case 3.1: $a_{12} = 0$**,** $b_{12}c_{12} \neq 0$**.** From (A.46) we get $a_{13} = 0$ and then from (A.41) $d_{12} = 1 - b_{12}c_{12}$ and from (A.76) $d_{13} = (1 - a_2)(1 - b_{12}c_{12})$. Then from (A.100) and (A.101), $b_{23} = a_2^2/c_{12}$, $c_{23} = a_2^2/b_{12}$ and from (A.95) $a_3 = a_2^4/(b_{12}c_{12})^2$. Since $a_3 \neq 0$ we have $a_2 \neq 0$ and can solve d_{23} from (A.43): $d_{23} = (a_2^4 - (b_{12}c_{12})^3)/(b_{12}c_{12})^2$.

Now (A.82) factorizes as $(a_2^2 - b_{12}c_{12})(a_2^2 + a_2 b_{12}c_{12} + (b_{12}c_{12})^2) = 0$.

Case 3.1.1: If we choose the first factor and set $b_{12} = a_2^2/c_{12}$ the remaining equations simplify to $x_3 = a_2(a_2^2 - 1)^2/x_4$, yielding the first solution ($a_2 \to a,\ c_{12} \to c$);

$$[1] \begin{bmatrix} \cdot & \frac{a^2}{c} \\ c & 1 - a^2 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{a^4}{c^2} \\ \cdot & a & \frac{a(a^2-1)^2}{x_4} \\ c^2 & x_4 & (a+1)(a-1)^2 \end{bmatrix} \begin{bmatrix} \cdot & \frac{a^2}{c} \\ c & 1 - a^2 \end{bmatrix} [1] \tag{28}$$

The eigenvalues of this solution are $1, -a^2$ and a^3 with multiplicities $5, 3$ and 1, respectively.

Case 3.1.2: For the second solution we solve (A.82) by $b_{12} = a\omega/c_{12}$, where ω is a cubic root of unity $\omega \neq 1$. Then the remaining equation is solved by $x_3 = a_2(a_2 - 1)(1 - \omega a_2)/x_4$ and we have

$$[1] \begin{bmatrix} \cdot & \frac{\omega a}{c} \\ c & 1 - \omega a \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{\omega^2 a^2}{c^2} \\ \cdot & a & \frac{(\omega^2 - a)(a-1)a}{x_4} \\ c^2 & x_4 & (1 - \omega a)(1 - a) \end{bmatrix} \begin{bmatrix} \cdot & \frac{a^2}{c} \\ \omega^2 ac & \omega a(a - 1) \end{bmatrix} [\omega a^2] \tag{29}$$

The eigenvalues are $\{1, -\omega a, \omega a^2\}$ each with multiplicity 3.

For both solutions $a \neq 0, 1$ and for the first $a \neq -1$. Note that for the second case if $a = -1$ there are only two eigenvalues: 1 and ω.

Case 3.2: $a_{12}d_{12} \neq 0$, $b_{12} = c_{12} = 0$. From equations (A.9), (A.29) we get $a_{12} = d_{12} = 1$ and from (A.44), (A.47) $b_{23} = c_{23} = 0$. Due to non-singularity we may now assume $a_{23}d_{23} \neq 0$ and then from (A.10), (A.16) we get $a_{23} = d_{23} = 1$. Since $a_3 \neq 0$ (A.30) yields $a_3 = 1$. Now from (A.69) and (A.100) we get $a_2 = 1, d_{13} = 0$, after which we get a contradiction in (A.50).

3.5 Case 5: $x_2 x_3 \neq 0$, $x_1 = x_4 = 0$

This case contains many solutions and therefore it is necessary to do some basic classification first. We do this on the basis of the 2×2 blocks.

For the first 2×2 block (the "12" block), consider equations (A.1), (A.2), (A.9), (A.29) and (A.31). The solutions to these equations can be divided into the following:

α: $a_{12}d_{12} \neq 0$. Then one finds $b_{12} = c_{12} = 0$ and $a_{12} = d_{12} = a_1$.

β: $a_{12} \neq 0$, $d_{12} = 0$ and $b_{12}c_{12} \neq 0$, then $a_{12} = a_1 - b_{12}c_{12}/a_1$

γ: $d_{12} \neq 0$, $a_{12} = 0$ and $b_{12}c_{12} \neq 0$, then $d_{12} = a_1 - b_{12}c_{12}/a_1$

δ: $a_{12} = d_{12} = 0$.

The results for the other 2×2 block (the "23" block) are obtained by index changes, including $a_1 \to a_3$, we denote them as α' etc.

In principle there would be $4 \times 4 = 16$ cases, but we can omit several using the known symmetries. First of all for the "12" block we can omit γ because it is related to β by LR symmetry. The list of cases is as follows:

1. (α, α'): $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 & \cdot \\ \cdot & a_3 \end{bmatrix} [a_3]$.

2. (α, β'): $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 - b_{23}c_{23}/a_3 & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

3. (α, δ'): $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

4. (β, β'): $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 - b_{23}c_{23}/a_3 & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

5. (β, γ'): $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & a_3 - b_{23}c_{23}/a_3 \end{bmatrix} [a_3]$.

6. (β, δ'): $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

7. (δ, δ'): $[a_1] \begin{bmatrix} \cdot & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

Here we have omitted (α, γ'), (β, α'), (δ, α'), (δ, β') and (δ, γ'), because they are related to entries in the above list of seven by some symmetry. Specifically, notice that the vanishing of x_1 and x_4 and the non-vanishing of x_2x_3 is preserved under LR-symmetry and the $|0\rangle \leftrightarrow |2\rangle$ symmetry, but *not* the transpose symmetry. Moreover the composition of the LR and 02-symmetries has the effect of simply interchanging the pairs of 2×2 and 1×1 blocks.

Case 5.1 (α, α')

We scale to $a_1 = 1$ and from (A.30) get $a_3 = 1$. According to (A.86) $a_2^2 - a_2 = 0$ and then from (A.106) and (A.108) we get $d_{13} = -b_{13}x_2/x_3$ and $c_{13} = -c_{13}x_2/x_3$ but then the 3×3 block matrix becomes singular. Therefore no solutions for this subcase.

Case 5.2 (α, β')

From (A.39) and (A.43) we get $c_{13} = c_{23}^2/a_3$ and $b_{13} = b_{23}a_3/c_{23}$. Next from (A.58) $d_{13} = 0$ and from (A.76) $a_2 = 1$ and from (A.54) $a_{23} = a_{13}$. After setting $a_3 = -x_3c_{23}^2/x_2$ from (A.104), the GCD of the remaining equations is $(x_3c_{23} - x_2b_{23})(x_2 + x_3c_{23})^2$ and we get two solutions: ($c_{23} \to c$, $b_{23} \to b$)

5.2.1: $x_2 = -x_3c_{23}^2$

$$[1]\begin{bmatrix}1 & . \\ . & 1\end{bmatrix}\begin{bmatrix}1-bc & . & \frac{b}{c} \\ -x_3c^2 & 1 & x_3 \\ c^2 & . & .\end{bmatrix}\begin{bmatrix}1-bc & b \\ c & .\end{bmatrix}[1] \tag{30}$$

noindentThe eigenvalues are $-bc$ with multiplicity 2 and 1 with multiplicity 7.

5.2.2: $x_2 = x_3c_{23}/b_{23}$

$$[1]\begin{bmatrix}1 & . \\ . & 1\end{bmatrix}\begin{bmatrix}1-bc & . & -b^2 \\ \frac{x_3c}{b} & 1 & x_3 \\ \frac{-c}{b} & . & .\end{bmatrix}\begin{bmatrix}1-bc & b \\ c & .\end{bmatrix}[-bc] \tag{31}$$

The eigenvalues are $-bc$ with multiplicity 3 and 1 with multiplicity 6.

Case 5.3 (α, δ')

From (A.54) and (A.62) we get $a_{13} = d_{13} = 0$ and from (A.72) $a_2 = 1$. Next (A.40) and (A.43) yield $c_{13} = b_{13} = a_3$ and (A.102) $x_3 = -x_2a_3$ The remaining equations are satisfied with $a_3 = \epsilon_1$ and $c_{23} = \epsilon_2$, where $\epsilon_j^2 = 1$. The result is

$$[1]\begin{bmatrix}1 & . \\ . & 1\end{bmatrix}\begin{bmatrix}. & . & \epsilon_1 \\ x_2 & 1 & -x_2\epsilon_1 \\ \epsilon_1 & . & .\end{bmatrix}\begin{bmatrix}. & \epsilon_2 \\ \epsilon_2 & .\end{bmatrix}[\epsilon_1]$$

The eigenvalues are 1 and -1 with multiplicity 7 and 2 if $\epsilon_1 = 1$ and 6 and 3 otherwise.

However, when $\epsilon_1 = -1$ this is a special case of (31) by setting $b = c = -1$. For $\epsilon_1 = 1$ we may take $b = c = 1$ in (30). Thus this case may be discarded *a posteriori* as a subcase.

Case 5.4 (β, β')

From (A.37) and (A.39) we get $c_{13} = c_{12}^2$ and $a_3 = c_{23}^2/c_{12}^2$. Next since $a_{12} = 1 - b_{12}c_{12} \neq 0$ we get $d_{13} = 0$ from (A.61). From (A.41) $b_{13} = b_{12}/c_{12}$ and from (A.78) $a_{13} = 1 - b_{12}c_{12}$. For nonsingularity we must have $a_2 \neq 0$ and then from (A.82) we get $b_{23} = b_{12}$ and from (A.43) $c_{23} = c_{12}$. Now from (A.74) we find $a_2 = -x_3c_{12}^2/x_2$ and after that the remaining equations factorize and we have two solutions:

5.4.a $x_2 = -c_{12}^2x_3$

$$[1]\begin{bmatrix}1-bc & b \\ c & .\end{bmatrix}\begin{bmatrix}1-bc & . & \frac{b}{c} \\ -x_3c^2 & 1 & x_3 \\ c^2 & . & .\end{bmatrix}\begin{bmatrix}1-bc & b \\ c & .\end{bmatrix}[1] \tag{32}$$

noindent Eigenvalues are 1 with multiplicity 6 and $-bc$ with multiplicity 3.

5.4.b $x_2 = c_{12}x_3/b_{12}$

$$[1]\begin{bmatrix}1-bc & b\\ c & .\end{bmatrix}\begin{bmatrix}1-bc & . & \frac{b}{c}\\ \frac{x_3 c}{b} & -bc & x_3\\ c^2 & . & .\end{bmatrix}\begin{bmatrix}1-bc & b\\ c & .\end{bmatrix}[1] \tag{33}$$

Eigenvalues are 1 with multiplicity 5 and $-bc$ with multiplicity 4.

Case 5.5 (β, γ')

Since the matrix is non-singular we must have $a_2 \neq 0$. From (A.38) and (A.41) we get $c_{13} = c_{12}^2$ and $b_{13} = b_{12}/c_{12}$, and from (A.39) and (A.43) $b_{12} = b_{23}^2 c_{12}/a_3$, $c_{23} = b_{23}c_{12}^2/a_3$. Then we get from several equations the condition $a_{13}d_{13} = 0$. If both $a_{13} = d_{13} = 0$, we would get from (A.78) $a_{12} = 0$, which would lead to case δ'. Therefore we have two branches:

5.5.1 Assume $a_{13} = 0$, $d_{13} \neq 0$. From (A.76) we get $x_3 = -x_2 b_{23}^2/a_3$ and then since $a_{12} \neq 0$ equation (A.102) yields $a_2 = 1$. From (A.66) we get $d_{13} = 1 - b_{23}^2 c_{12}^2/a_3$. If we use (A.81) to eliminate second and higher powers of a_3 of equation (A.82), it factorizes as $(a_3 - 1)(1 + b_{23}c_{12}) = 0$, and we get two branches:

5.5.1.1 If we choose $a_3 = 1$ all other equations are satisfied with $b_{23} = \omega^2/c_{12}$, where $\omega^3 = 1$ but we must have $\omega \neq 1$ to stay in the (β, γ') case.

$$[1]\begin{bmatrix}1-\omega & \frac{\omega}{c}\\ c & .\end{bmatrix}\begin{bmatrix}. & . & \frac{\omega}{c^2}\\ x_2 & 1 & \frac{-x_2\omega}{c^2}\\ c^2 & . & 1-\omega\end{bmatrix}\begin{bmatrix}. & \frac{\omega^2}{c}\\ c\omega^2 & 1-\omega\end{bmatrix}[1] \tag{34}$$

The eigenvalues are 1 with multiplicity 6 and ω with multiplicity 3.

5.5.1.2 Now we choose $b_{23} = -1/c_{12}$ and then the remaining equations are satisfied with $a_3 = \varsigma = \pm i$.

$$[1]\begin{bmatrix}\varsigma+1 & \frac{-\varsigma}{c}\\ c & .\end{bmatrix}\begin{bmatrix}. & . & \frac{-\varsigma}{c^2}\\ x_2 & 1 & \frac{x_2\varsigma}{c^2}\\ c^2 & . & \varsigma+1\end{bmatrix}\begin{bmatrix}. & \frac{-1}{c}\\ \varsigma c & \varsigma+1\end{bmatrix}[\varsigma] \tag{35}$$

The eigenvalues are 1 with multiplicity 5 and ς with multiplicity 4.

5.5.2 The case $d_{13} = 0$, $a_{13} \neq 0$ is obtained by 02-symmetry from 5.5.1. Indeed, we see that the form (β, γ') is invariant under the $|0\rangle \leftrightarrow |2\rangle$ symmetry, with the 3×3 block having the following pairs interchanged (a_{13}, d_{13}), (b_{13}, c_{13}), (x_2, x_3) and $(x_1, x_4) = (0, 0)$. Thus any solution obtained for $d_{13} = 0$ and $a_{13} \neq 0$ may be transformed into a solution with $d_{13} \neq 0$ and $a_{13} = 0$.

Case 5.6 (β, δ')

Since in this case $a_{12} \neq 0$ we have from (A.54) and (A.63) that $a_{13} = d_{13} = 0$ but then (A.78) implies $a_{12} = 0$, a contradiction.

Case 5.7 (δ, δ').

From $\det \neq 0$ we get $a_2 \neq 0$ and then (A.81) and (A.82) imply $c_{23} = c_{12}$ and $b_{23} = b_{12}$. Next from (A.38) and (A.42) we get $c_{13} = c_{12}^2$ and $b_{13} = b_{12}^2$. Equation (A.39) then gives $a_3 = 1$ and (A.40) implies $c_{12} = 1/b_{12}$. After this (A.52) and (A.66) yield $a_{13} = d_{13} = 0$. The remaining equations are satisfied with $a_2 = \epsilon$, $\epsilon = \pm 1$.

$$[1] \begin{bmatrix} \cdot & b \\ \frac{1}{b} & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & b^2 \\ x_2 & \epsilon & x_3 \\ \frac{1}{b^2} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & b \\ \frac{1}{b} & \cdot \end{bmatrix} [1] \tag{36}$$

The eigenvalues are 1 and -1 with multiplicities 5 and 4 if $\epsilon = -1$ and multiplicities 6 and 3 otherwise.

3.6 Case 6: $x_1 = x_2 = x_3 = 0$, $x_4 \neq 0$

From the outset it is best to divide this into two cases depending on whether or not b_{12} vanishes.

Case 6.1: $b_{12} = 0$ therefore $a_{12}d_{12} \neq 0$. Then from (A.1) $c_{12} = 0$ and from (A.29) and (A.31) $a_{12} = d_{12} = 1$. From (A.94) we get $b_{13} = 0$, and hence $a_{13}a_2d_{13} \neq 0$ and then from (A.90), (A.86) and (A.66) $a_{13} = a_2 = d_{13} = 1$, which leads to a contradiction with (A.68).

Case 6.2: Now that $b_{12} \neq 0$ we get from (A.68) and (A.72) $a_{13} = d_{13} = 0$. From (A.72) $a_{13} = a_{12}$ and from (A.94) $b_{13} = b_{12}^2$ and then from (A.98) and (A.99) $a_{12} = a_{23} = 0$ and therefore $c_{12}b_{23}c_{23} \neq 0$. Next from (A.46) $c_{13} = c_{12}^2$ and from (A.100) and (A.101) $b_{23} = a_2^2/c_{12}$, $b_{12} = a_2^2/c_{23}$ and from (A.95) $a_3 = c_{23}^2/c_{12}^2$.

At this point we divide the problem into two branches on whether or not d_{12} vanishes.

Case 6.2.1: $d_{12} = 0$. Then from (A.68) $d_{13} = 0$ and from (A.95) $c_{23} = a_2^2 c_{12}$ and from (A.83) $d_{23} = a_2(1 - a_2^2)$. After this the remaining equations imply that we must have either $a_2 = \epsilon$, $\epsilon = \pm 1$ or $a_2 = \omega$ with $\omega^3 = 1$, $\omega \neq 1$. After changing $c_{12} \to c$ one solution is

$$[1] \begin{bmatrix} \cdot & \frac{1}{c} \\ c & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2} \\ \cdot & \epsilon & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{1}{c} \\ c & \cdot \end{bmatrix} [1] \tag{37}$$

The eigenvalues are 1 and -1, with multiplicities 5 and 4 if $\epsilon = -1$, otherwise multiplicities 6 and 3. Note that this solution may be obtained from (36) by setting $x_2 = 0$ and taking the transpose, but this violates the Case 5 assumption that $x_2 \neq 0$.

The second solution is

$$[1] \begin{bmatrix} \cdot & \frac{1}{c} \\ c & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2} \\ \cdot & \omega & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{\omega^2}{c} \\ \omega^2 c & \omega - 1 \end{bmatrix} [\omega] \tag{38}$$

The eigenvalues are $1, \omega$ and -1, each with multiplicity 3.

Case 6.2.2: $d_{12} \neq 0$. From (A.64) and (A.68) we get $d_{13} = d_{23} = d_{12}(1 - a_2)$ and then from (A.63) $a_2 = 1$. The remaining equations are solved by $d_{12} = (c_{23} - c_{12})/c_{23}$ and $c_{23} = c_{12}\omega$ with $\omega^3 = 1$, $\omega \neq 1$, yielding:

$$[1] \begin{bmatrix} \cdot & \frac{1}{c\omega} \\ c & \omega + 2 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2\omega^2} \\ \cdot & 1 & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{1}{c} \\ \omega c & \cdot \end{bmatrix} [\omega^2] \tag{39}$$

The eigenvalues are $1, \omega^2$ and $-\omega^2$, each with multiplicity 3.

Altogether we have established the following:

Theorem 3.2. *For the Yang-Baxter equation (7) in three dimensions, the complete list of solutions satisfying ACC but not SCC (see [15] for SCC) is given, up to noted symmetries (see Sec.2.2), in the formulae* (27)-(39), *and collected in the Table 1.* □

4 Analysis of the generic representations (14/15).

Constant Yang–Baxter solutions can be of considerable intrinsic interest. But they are also often interesting because of their symmetry algebras. In the XXZ case (one of the strict charge-conserving cases) for example, the symmetry algebra is the quantum group $U_q sl_2$. This holds true in all ranks (i.e. all system sizes n) - as we go up in ranks we simply see more of the symmetry algebra -i.e. the action of the symmetry algebra on n-fold tensor space has a smaller kernel as n increases. It is not immediate that such a strong outcome would hold in general. But it is interesting to investigate.

In §4.1 we analyse our new solutions. (In §B we recall some classical facts about the classical cases for comparison.)

4.1 Analysis of the generic solution: spectrum of $\check{R}$

Now we consider the solution in (14/15). Observe that the trace of the 3×3 block is

$$a + \frac{x_1 x_3 + b}{b} - \frac{x_1 x_3}{ab} = \frac{a^2 b + abx_1 x_3 + ab^2 - x_1 x_3}{ab}$$

Consider $\check{R}_j - 1$, so that all but the 3x3 block is zero. Restricting to the 3x3 block of $\check{R}$, call it $\check{r}$, we have

$$\check{r} - 1_3 = \begin{bmatrix} a & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1 x_3 + b}{b} & x_3 \\ \frac{x_3{}^2 x_1{}^2}{b^3} & -\frac{x_1(ab + x_1 x_3)}{ab^2} & -\frac{x_3 x_1}{ab} \end{bmatrix} - 1_3 = \begin{bmatrix} a-1 & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1 x_3}{b} & x_3 \\ \frac{x_3{}^2 x_1{}^2}{b^3} & -\frac{x_1(ab + x_1 x_3)}{ab^2} & -\frac{b(x_3 x_1 + ab)}{ab^2} \end{bmatrix} \tag{40}$$

Note that $\frac{x_3^2 x_1^2}{b^3} = \frac{-(a-1)(ab + x_1 x_3)}{ab^2}$ so this is clearly rank 1. Thus only one eigenvalue of $\check{R} - 1$ is not 0, and so only one eigenvalue of $\check{R}$ is not 1. We have

$$Trace(\check{R} - 1) = a - 1 + \frac{x_1 x_3}{b} - \frac{b(x_1 x_3 + ab)}{ab^2} = \frac{a^2 b^2 + abx_1 x_3 - b(x_1 x_3 + ab)}{ab^2} - 1$$

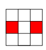
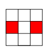
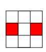
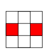
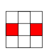
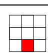
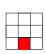
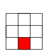
soln. name	non-zero x_is	block form	parameters cont./discrete	eigenvalues, degeneracies
1		(27)	3 /0	$\begin{pmatrix} 1 & , & x \\ \times 8 & & \times 1 \end{pmatrix}$
2		-		-
3.1.1		(28)	3 /0	$\begin{pmatrix} 1 & , & -x^2 & , & x^3 \\ \times 5 & & \times 3 & & \times 1 \end{pmatrix}$
3.1.2		(29)	3 /1	$\begin{pmatrix} 1 & , & -\omega x & , & \omega x^2 \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$
4		-		-
5.2.1		(30)	3 /0	$\begin{pmatrix} 1 & , & x \\ \times 7 & & \times 2 \end{pmatrix}$
5.2.2		(31)	3 /0	$\begin{pmatrix} 1 & , & x \\ \times 6 & & \times 3 \end{pmatrix}$
5.4.a		(32)	3 /0	$\begin{pmatrix} 1 & , & x \\ \times 6 & & \times 3 \end{pmatrix}$
5.4.b		(33)	3 /0	$\begin{pmatrix} 1 & , & x \\ \times 5 & & \times 4 \end{pmatrix}$
5.5.1.1		(34)	2 /1	$\begin{pmatrix} 1 & , & \omega \\ \times 6 & & \times 3 \end{pmatrix}$
5.5.1.2		(35)	2 /1	$\begin{pmatrix} 1 & , & \varsigma \\ \times 5 & & \times 4 \end{pmatrix}$
5.7		(36)	3 /1	$\begin{pmatrix} 1 & , & -1 \\ \times 5 & & \times 4 \end{pmatrix} / \begin{pmatrix} 1 & , & -1 \\ \times 6 & & \times 3 \end{pmatrix}$
6.2.1		(37)	2 /1	$\begin{pmatrix} 1 & , & -1 \\ \times 5 & & \times 4 \end{pmatrix} / \begin{pmatrix} 1 & , & -1 \\ \times 6 & & \times 3 \end{pmatrix}$
6.2.1'		(38)	2 /1	$\begin{pmatrix} 1 & , & \omega & , & -1 \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$
6.2.2		(39)	2 /1	$\begin{pmatrix} 1 & , & \omega & , & -\omega \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$

Table 1: Table of all ACC solutions to the Yang-Baxter equation (7) in rank-3. Here x denotes a non-zero variable possibly with further constraints described in the text, ω is a primitive 3rd root of unity; and ς is a primitive 4th root of unity. In continuous/discrete parameter column entry 3/1 means a 3-free-parameter family, not counting overall scaling, with 1 discrete parameter (which always take on exactly 2 values). (Hyphens and omitted 'names' correspond to choices leading to no solution.)

The other eigenvalue of $\check{R}$ is

$$\lambda_2 = \frac{a(a-1)b^2 + (a-1)bx_1x_3}{ab^2} = -(x_1x_3/b)^2 \ = -\left(\frac{-(a-1) \pm \sqrt{(a-1)^2 - 4a^2(a-1)}}{2a}\right)^2$$

— note from (15) that this depends only on a.

In particular each of our braid representations (varying the parameters appropriately) is a Hecke representation.

We see that the eigenvalue λ_2 can be varied over an open interval (in each branch it is a continuous function of a, small for a close to 1; and large for large negative a). So (by Hecke representation theory, specifically that the Hecke algebras are generically semisimple, and abstract considerations [7, 13]) the representation is generically semisimple.

Returning to (40) we have

$$\check{r} - 1_3 \ = \ \left[1, \frac{x_3}{b}, \frac{-ab - x_1x_3}{ab^2}\right]^t \ [a-1, x_1, b] = \ \frac{1}{ab^2} \begin{bmatrix} ab^2 \\ abx_3 \\ -ab - x_1x_3 \end{bmatrix} [a-1, x_1, b]$$

and

$$\check{R} - 1_9 \ = \ \left[0, 0, 1, 0, \frac{x_3}{b}, 0, \frac{-ab - x_1x_3}{ab^2}, 0, 0\right]^t \ [0, 0, a-1, 0, x_1, 0, b, 0, 0]$$

Armed with this, we have a Temperley–Lieb category representation (i.e. an embedded TQFT - we assume familiarity with the standard $U_q sl_2$ version which can be used for comparison - see e.g. [6, Sec.6.2] and references therein). In this form the duality is going to be skewed (not a simple conjugation) but should be workable. In particular the loop parameter is

$$[0, 0, a-1, 0, x_1, 0, b, 0, 0] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ x_3/b \\ 0 \\ (-ab - x_1x_3)/ab^2 \\ 0 \\ 0 \end{bmatrix} = \ (a-1) + \frac{x_1x_3}{b} - \frac{b(ab + x_1x_3)}{ab^2}$$

$$= \frac{a^2b^2 - 2ab^2 + (a-1)bx_1x_3}{ab^2} = \frac{a-1}{a}(a + \frac{x_1x_3}{b}) - 1 = \lambda_2 - 1$$

which, note, depends only on a.

4.2 Irreducible representation content of the generic solution ρ_n

The following analysis gives us an invariant, and thus a way to classify solutions $\check{R}$ (or equivalently R).

Thus in principle we can classify R-matrices according to the B_n-representation structure (the irrep content and so on) for each (and all) n. In general such an approach is very hard (due to the limits on knowledge of the braid groups B_n and their representation theory). Certain properties can, however, make the problem more tractable.

In our case call the representation ρ_n (or just ρ if no ambiguity arises, or to denote the monoidal functor from the braid category, as in [15]). Depending on the field we are working over, this might mean the rep with indeterminate parameters, or a generic point in parameter space (i.e. the rep variety or a point on that variety).

Since this $\check{R}$ has two eigenvalues (see §4.1) we have a Hecke representation — a representation of the algebra $H_n = H_n(q)$, a quotient of the group algebra of B_n for each n, for some q. (With the same understanding about parameters.)

Since eigenvalue $\lambda_1 = 1$ this $H_n(q)$ is essentially in the 'Lusztig' convention — we can write t_i for the braid generators in H_n, so

$$R_i = \rho(t_i) \;=\; \rho_n(t_i)\;;$$

then the quotient relation is

$$(t_i - 1)(t_i + q) = 0 \tag{41}$$

for some $q = -\lambda_2$, as in [17]. Here it is convenient to define

$$U_i = \frac{t_i - 1}{\alpha}$$

so $\alpha U_i(\alpha U_i + 1 + q) = 0$, i.e. $\alpha U_i^2 = -(1+q)U_i$.

In a convention/parameterisation as in (41) the operator

$$e' = 1 - t_1 - t_2 + t_1 t_2 + t_2 t_1 - t_1 t_2 t_1$$

is an unnormalised idempotent, and hence

$$\rho_n(e') = 1 - R_1 - R_2 + R_1 R_2 + R_2 R_1 - R_1 R_2 R_1$$

is an unnormalised (possibly zero) idempotent, whenever $\check{R}$ gives such a Hecke representation.

In our case in fact e' is zero (by direct computation):

$$\rho_n(e') = 0 \tag{42}$$

Note that

$$\alpha^3 U_1 U_2 U_1 = (t_1 - 1)(t_2 - 1)(t_1 - 1) = t_1 t_2 t_1 - t_1 t_2 - t_2 t_1 - t_1 t_1 + 2t_1 + t_2 - 1$$

so in our case

$$\rho_n(\alpha^3 U_1 U_2 U_1) = (R_1 - 1)(R_2 - 1)(R_1 - 1) = R_1 R_2 R_1 - R_1 R_2 - R_2 R_1 - R_1 R_1 + 2R_1 + R_2 - 1$$

so

$$\rho_n(\alpha^3 U_1 U_2 U_1) = -R_1^2 + R_1 = -R_1(R_1 - 1) = \rho_n(q\alpha U_1) \qquad \text{so} \qquad \rho_n(U_1 U_2 U_1) = \rho_n(\frac{q}{\alpha^2} U_1)$$

so if we put $\alpha = \pm\sqrt{q}$ then we have the relations of the usual generators for Temperley–Lieb [22].

We assume familiarity with the generic irreducible representations of H_n, which we write, up to isomorphism, as L_λ with $\lambda \vdash n$ an integer partition of n. The idempotent e' induces the irrep L_{1^3}. The unnormalised idempotent inducing the irrep L_3 is

$$e'_3 \;=\; 1 + \frac{1}{q}(R_1 + R_2) + \frac{1}{q^2}(R_1R_2 + R_2R_1) + \frac{1}{q^3}R_1R_2R_1 \tag{43}$$

This gives

$$L_3(e'_3) = \frac{1}{q^3}(1+q)(1+q+q^2)$$

which gives the normalisation factor, so

$$e_3 = \frac{q^3}{(1+q)(1+q+q^2)}e'_3$$

The generalisation to irrep L_n in rank n will hopefully be clear (in fact we won't really need it except for checking).

We can write χ_λ for the irreducible character associated to irrep L_λ. That is,

$$\chi_\lambda(t_i) = Trace(L_\lambda(t_i)).$$

We can evaluate these characters in various ways, but a simple device is the restriction rule for the inclusion $H_{n-1} \otimes 1_1 \hookrightarrow H_n$; together with the easy cases:

$$\chi_n(t_i) = 1, \qquad\qquad \chi_{1^n}(t_i) = \lambda_2 \tag{44}$$

For example

$$\chi_{2,1}(t_i) = \chi_2(t_i) + \chi_{1^2}(t_i) = 1 + \lambda_2$$

and so on.

Observe that the eigenvalues of R_i, specifically $R_1 = \check{R} \otimes 1_3$, are three copies each of the eigenvalues of $\check{R}$. Hence there are 24 eigenvalues $\lambda_1 = 1$ and 3 copies of the other eigenvalue, call it λ_2:

$$\chi_\rho(t_i) = 3(8 + \lambda_2) = 24 + 3\lambda_2$$

The 1d irrep L_3, when present, contributes 1 eigenvalue $\lambda_1 = 1$. The 2d irrep $L_{2,1}$ contributes 1 eigenvalue $\lambda_1 = 1$ and 1 of the other eigenvalue λ_2. The 1d irrep L_{1^3} contributes just 1 of the other eigenvalue λ_2. Since $e' = 0$ the multiplicity of this irrep in ρ is 0. Therefore all the 3 eigenvalues λ_2 come from $L_{2,1}$ summands. The identity (42) therefore tells us that the irreducible content of our representation of H_3 (the Hecke quotient of B_3) is

$$\rho \;=\; 21\;L_3 \;+\; 3\;L_{2,1} \tag{45}$$

(the sum is generically but not necessarily always direct). In particular we have re-verified:

Proposition 4.1. *Representation ρ is a representation of Temperley–Lieb.*

Note that it follows from the tensor construction that this TL property holds (i.e. the image of e' continues to vanish) for all n.
Next we address the question of faithfulness of ρ_n as a TL representation, and determine the centraliser, for all n.

Write m_λ for the multiplicity of the generic irrep L_λ in our rep ρ (the generic character is well-defined in all specialisations, but the corresponding rep is not irreducible in all specialisations):

$$\chi_{\rho_n} = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{46}$$

Note that integer partitions can be considered as vectors ('weights' in Lie theory) and hence added. For example if $\mu = (\mu_1, \mu_2, \mu_3, ..., \mu_l)$ then

$$\mu + 11 = \mu + (1,1) = \mu + (1,1,0,...,0) = (\mu_1 + 1, \mu_2 + 1, \mu_3, ..., \mu_l).$$

Stability Lemma. The multiplicity m_μ at level $n-2$ is the same as $m_{\mu+11}$ at level n.

Outline Proof. The method of 'virtual Lie theory' works here (see e.g. [13, 18]). Let us define

$$U_i = \check{R}_i - 1$$

our rank-1 operator. Thus U_i is itself an unnormalised idempotent - indeed it is, up to scalar, the image of the cup-cap operator in the TL diagram algebra.

Write T_n for TL on n strands. Recall that $U_1 T_n$ is a left T_{n-2} right T_n bimodule. Recall the algebra isomorphism $U_1 T_n U_1 \cong T_{n-2}$; and recall that $T_n / T_n U_1 T_n \cong k$ where k is the ground field (for us it is **C**). It follows that the category $T_{n-2} - mod$ embeds in $T_n - mod$, with embedding functor given by:

$$M \mapsto T_n U_1 \otimes_{T_{n-2}} M \tag{47}$$

The irrep $L_\mu = L_{\mu_1,\mu_2}$ is taken to $L_{\mu+11} = L_{\mu_1+1,\mu_2+1}$. Here L_n is the module not hit by the embedding — this is the module corresponding to $T_n / T_n U_1 T_n \cong k$, so the one that is annihilated by the localisation $M \mapsto U_1 M$. □

The Theorem below is a corollary of this Lemma.

It might also be of interest to show how to compute the further multiplicities m_μ by direct calculation. For $n = 4$ we have $\chi_{\rho_4}(t_i) = 3\chi_{\rho_3}(t_i) = 72 + 9\lambda_2$. A direct calculation gives $\chi_{\rho_4}(e_4) = 55$ so $m_4 = 55$, and we have $\chi_{\rho_4}(t_1) = 55 + m_{3,1}(2+\lambda_2) + m_{2,2}(1+\lambda_2)$. We have $2m_{31} + m_{22} = 72 - 55 = 17$ and $m_{31} + m_{22} = 9$, giving $m_{3,1} = 8$ and $m_{2,2} = 1$.

Observe that this is in agreement with the Stability Lemma.

For $n = 5$ we have $\chi_{\rho_5}(t_i) = 3\chi_{\rho_4}(t_i) = 216 + 27\lambda_2$ A direct calculation gives $\chi_{\rho_5}(e_5) = 144$ and so we have $\chi_{\rho_5}(t_1) = 144 + m_{4,1}(3+\lambda_2) + m_{3,2}(3+2\lambda_2)$ We have $3m_{41} + 3m_{32} = 216 - 144 = 72$ and $m_{41} + 2m_{32} = 27$, giving $m_{4,1} = 21$ and $m_{3,2} = 3$.

We observe a pattern of repeated multiplicities, in agreement with the Stability Lemma:

m_λ	1	3	8	21	55	144
λ	11		2			
		21		3		
	22		31		4	
		32		41		5
	33					

Besides the Stability Lemma or a direct calculation, the last entry above may be guessed based on Perron–Frobenius applied to the Hamiltonian $H = \sum_i \check{R}_i$ — if some power of H is positive then there is a unique largest magnitude eigenvalue, and hence the corresponding multiplicity is 1. We know from the XXZ chain, which has the same eigenvalues but different multiplicities, that $\lambda = mm$ gives the largest eigenvalue when $n = 2m$.

Theorem 4.2. *The multiplicity m_n in (46) is given by A001906 from Sloane/OEIS [21], with all other multiplicities m_μ determined by the Stability Lemma.* □

The Temperley–Lieb algebras are generically semisimple; and a representation of a semisimple algebra is faithful if and only if every irrep appears as a summand. The latter is immediate from the Theorem, so generical faithfulness of our representations ρ_n is similarly immediate.

This brings us back to the original question about the stability of the centraliser as n varies - the possibility of an overarching symmetry algebra analogous to $U_q sl_2$ in the XXZ case. Of course by Schur's Lemma the Stability Lemma exactly says that there is a limit symmetry algebra, with all finite cases simply quotients of this limit. But the combinatorial fact does not of itself imply that the symmetry algebra is something as beautiful as a quantum group (cf. Appendix B).

A Appendix: The equations

A.1 The cubic constraints

Here we write out the system of cubics corresponding to entries in A_R as in (13), hence the cubics that must vanish, in the ACC ansatz.

In fact the first few cubics in A_R are unchanged (ordering 000 001 002 010 011 012 020 021 022 100 101 102 ... 222) from the strict CC ansatz. Row 000 has vanishing anomaly. Row 001 gives:

$$\langle 001|A_R|001\rangle = -a_{12}b_{12}c_{12} - a_1 a_{12}^2 + a_1^2 a_{12}, \qquad \langle 001|A_R|010\rangle = -a_{12}b_{12}d_{12}$$

with all other entries vanishing. The first departure from SCC is in the 002 row, which is:

$$\begin{aligned}\langle 002|A_R \;=\; & [0,0,\; -a_{12}x_1x_2 - a_{13}b_{13}c_{13} - a_1 a_{13}^2 + a_1^2 a_{13},\\ & 0,\; -a_{13}b_{13}x_4 + (a_1a_{12} - a_{12}a_2 - a_1a_{13})x_1, 0,\; -a_{12}x_1x_3 - a_{13}b_{13}d_{13},\; 0,0,0,\\ & -b_{13}c_{12}x_1 - a_{12}b_{12}x_1 + a_1b_{12}x_1,\; 0, -b_{13}d_{12}x_1 + a_1b_{13}x_1 - b_{12}^2x_1,\; 0,0,0,0,0,0,0,0,0,0,0,0,0,0]\end{aligned}$$

A.2 List of equations

We give the complete list equations that are distinct up to an overall sign, organised by the number of terms (in computations we use the scale freedom to assume $a_1 = 1$).

$$a_{12}\,c_{12}\,d_{12} = 0, \tag{A.1}$$

$$a_{12}\, b_{12}\, d_{12} = 0, \tag{A.2}$$
$$a_{23}\, c_{23}\, d_{23} = 0, \tag{A.3}$$
$$a_{23}\, b_{23}\, d_{23} = 0, \tag{A.4}$$
$$x_2\, x_4\, c_{12} = 0, \tag{A.5}$$
$$x_2\, x_4\, c_{23} = 0, \tag{A.6}$$
$$x_1\, x_3\, b_{12} = 0, \tag{A.7}$$
$$x_1\, x_3\, b_{23} = 0 \tag{A.8}$$
$$a_{12}\, d_{12}\, (a_{12} - d_{12}) = 0, \tag{A.9}$$
$$a_{23}\, d_{23}\, (a_{23} - d_{23}) = 0, \tag{A.10}$$
$$x_1\, x_2\, (d_{12} - d_{23}) = 0, \tag{A.11}$$
$$x_1\, x_3\, (a_{12} - d_{12}) = 0, \tag{A.12}$$
$$x_1\, x_3\, (a_{23} - d_{23}) = 0, \tag{A.13}$$
$$x_2\, x_4\, (a_{12} - d_{12}) = 0, \tag{A.14}$$
$$x_2\, x_4\, (a_{23} - d_{23}) = 0, \tag{A.15}$$
$$x_3\, x_4\, (a_{12} - a_{23}) = 0, \tag{A.16}$$
$$x_1\, x_3\, c_{12} - a_{12}\, b_{12}\, d_{12} = 0, \tag{A.17}$$
$$x_1\, x_3\, d_{23} + a_{13}\, b_{13}\, d_{13} = 0, \tag{A.18}$$
$$x_1\, x_3\, a_{12} + a_{13}\, b_{13}\, d_{13} = 0, \tag{A.19}$$
$$x_1\, x_3\, c_{23} - a_{23}\, b_{23}\, d_{23} = 0, \tag{A.20}$$
$$x_1\, x_3\, d_{12} + a_{13}\, b_{13}\, d_{13} = 0, \tag{A.21}$$
$$x_1\, x_3\, a_{23} + a_{13}\, b_{13}\, d_{13} = 0, \tag{A.22}$$
$$x_2\, x_4\, d_{12} + a_{13}\, c_{13}\, d_{13} = 0, \tag{A.23}$$
$$x_2\, x_4\, a_{12} + a_{13}\, c_{13}\, d_{13} = 0, \tag{A.24}$$
$$x_2\, x_4\, b_{12} - a_{12}\, c_{12}\, d_{12} = 0, \tag{A.25}$$
$$x_2\, x_4\, d_{23} + a_{13}\, c_{13}\, d_{13} = 0, \tag{A.26}$$
$$x_2\, x_4\, a_{23} + a_{13}\, c_{13}\, d_{13} = 0, \tag{A.27}$$
$$x_2\, x_4\, b_{23} - a_{23}\, c_{23}\, d_{23} = 0, \tag{A.28}$$
$$a_{12}\, (a_1^2 - a_1\, a_{12} - c_{12}\, b_{12}) = 0, \tag{A.29}$$
$$a_{23}\, (c_{23}\, b_{23} - a_3^2 + a_3\, a_{23}) = 0, \tag{A.30}$$
$$d_{12}\, (a_1^2 - a_1\, d_{12} - c_{12}\, b_{12}) = 0, \tag{A.31}$$
$$d_{23}\, (c_{23}\, b_{23} - a_3^2 + a_3\, d_{23}) = 0, \tag{A.32}$$
$$x_1\, (a_1\, b_{12} - c_{12}\, b_{13} - a_{12}\, b_{12}) = 0, \tag{A.33}$$
$$x_1\, (a_1\, b_{13} - d_{12}\, b_{13} - b_{12}^2) = 0, \tag{A.34}$$
$$x_1\, (c_{23}\, b_{13} - a_3\, b_{23} + a_{23}\, b_{23}) = 0, \tag{A.35}$$
$$x_1\, (a_3\, b_{13} - d_{23}\, b_{13} - b_{23}^2) = 0, \tag{A.36}$$
$$x_2\, (a_1\, c_{12} - c_{12}\, a_{12} - c_{13}\, b_{12}) = 0, \tag{A.37}$$
$$x_2\, (a_1\, c_{13} - c_{12}^2 - c_{13}\, d_{12}) = 0, \tag{A.38}$$

$$x_2\,(c_{13}\,a_3 - c_{13}\,d_{23} - c_{23}^2) = 0, \tag{A.39}$$
$$x_2\,(c_{13}\,b_{23} - c_{23}\,a_3 + c_{23}\,a_{23}) = 0, \tag{A.40}$$
$$x_3\,(a_1\,b_{12} - c_{12}\,b_{13} - d_{12}\,b_{12}) = 0, \tag{A.41}$$
$$x_3\,(a_1\,b_{13} - a_{12}\,b_{13} - b_{12}^2) = 0, \tag{A.42}$$
$$x_3\,(c_{23}\,b_{13} - a_3\,b_{23} + d_{23}\,b_{23}) = 0, \tag{A.43}$$
$$x_3\,(a_3\,b_{13} - a_{23}\,b_{13} - b_{23}^2) = 0, \tag{A.44}$$
$$x_4\,(a_1\,c_{12} - c_{12}\,d_{12} - c_{13}\,b_{12}) = 0, \tag{A.45}$$
$$x_4\,(a_1\,c_{13} - c_{12}^2 - c_{13}\,a_{12}) = 0, \tag{A.46}$$
$$x_4\,(c_{13}\,a_3 - c_{13}\,a_{23} - c_{23}^2) = 0, \tag{A.47}$$
$$x_4\,(c_{13}\,b_{23} - c_{23}\,a_3 + c_{23}\,d_{23}) = 0, \tag{A.48}$$
$$x_3\,x_4\,(a_{12} - a_{23}) + x_1\,x_2\,(-d_{12} + d_{23}) = 0, \tag{A.49}$$
$$x_3\,x_4\,a_{23} - x_2\,x_1\,d_{23} + d_{13}\,a_{13}\,(d_{13} - a_{13}) = 0, \tag{A.50}$$
$$x_3\,x_4\,a_{12} - x_2\,x_1\,d_{12} + d_{13}\,a_{13}\,(d_{13} - a_{13}) = 0, \tag{A.51}$$
$$x_1\,x_2\,a_{12} + a_{13}\,(-a_1^2 + a_1\,a_{13} + c_{13}\,b_{13}) = 0, \tag{A.52}$$
$$x_1\,x_2\,a_{23} + a_{13}\,(c_{13}\,b_{13} - a_3^2 + a_3\,a_{13}) = 0, \tag{A.53}$$
$$x_1\,x_2\,b_{12} + b_{23}\,(-d_{23}\,a_{13} + a_{12}\,a_{13} - a_{12}\,a_{23}) = 0, \tag{A.54}$$
$$x_1\,x_2\,b_{23} + b_{12}\,(-d_{12}\,a_{13} - a_{12}\,a_{23} + a_{13}\,a_{23}) = 0, \tag{A.55}$$
$$x_1\,x_2\,c_{12} + c_{23}\,(-d_{23}\,a_{13} + a_{12}\,a_{13} - a_{12}\,a_{23}) = 0, \tag{A.56}$$
$$x_1\,x_2\,c_{23} + c_{12}\,(-d_{12}\,a_{13} - a_{12}\,a_{23} + a_{13}\,a_{23}) = 0, \tag{A.57}$$
$$x_1\,x_3\,a_2 + b_{13}\,(d_{13}\,a_{12} - d_{23}\,a_{12} + d_{23}\,a_{13}) = 0, \tag{A.58}$$
$$x_1\,x_3\,a_2 + b_{13}\,(d_{12}\,a_{13} - d_{12}\,a_{23} + d_{13}\,a_{23}) = 0, \tag{A.59}$$
$$x_2\,x_4\,a_2 + c_{13}\,(d_{12}\,a_{13} - d_{12}\,a_{23} + d_{13}\,a_{23}) = 0, \tag{A.60}$$
$$x_2\,x_4\,a_2 + c_{13}\,(d_{13}\,a_{12} - d_{23}\,a_{12} + d_{23}\,a_{13}) = 0, \tag{A.61}$$
$$x_3\,x_4\,b_{12} + b_{23}\,(d_{12}\,d_{13} - d_{12}\,d_{23} - d_{13}\,a_{23}) = 0, \tag{A.62}$$
$$x_3\,x_4\,b_{23} + b_{12}\,(-d_{12}\,d_{23} + d_{13}\,d_{23} - d_{13}\,a_{12}) = 0, \tag{A.63}$$
$$x_3\,x_4\,c_{12} + c_{23}\,(d_{12}\,d_{13} - d_{12}\,d_{23} - d_{13}\,a_{23}) = 0, \tag{A.64}$$
$$x_3\,x_4\,c_{23} + c_{12}\,(-d_{12}\,d_{23} + d_{13}\,d_{23} - d_{13}\,a_{12}) = 0, \tag{A.65}$$
$$x_3\,x_4\,d_{12} + d_{13}\,(-a_1^2 + a_1\,d_{13} + c_{13}\,b_{13}) = 0, \tag{A.66}$$
$$x_3\,x_4\,d_{23} + d_{13}\,(c_{13}\,b_{13} - a_3^2 + a_3\,d_{13}) = 0, \tag{A.67}$$
$$x_4\,(a_1\,d_{12} - a_1\,d_{13} - a_2\,d_{12}) - x_1\,c_{13}\,d_{13} = 0, \tag{A.68}$$
$$x_4\,(a_2\,d_{23} + a_3\,d_{13} - a_3\,d_{23}) + x_1\,c_{13}\,d_{13} = 0, \tag{A.69}$$
$$x_4\,a_{13}\,b_{13} + x_1\,(a_2\,a_{23} + a_3\,a_{13} - a_3\,a_{23}) = 0, \tag{A.70}$$
$$x_4\,a_{13}\,b_{13} + x_1\,(-a_1\,a_{12} + a_1\,a_{13} + a_2\,a_{12}) = 0, \tag{A.71}$$
$$x_4\,b_{12}\,(a_{12} - a_{13}) + x_1\,c_{12}\,(-d_{12} + d_{13}) = 0, \tag{A.72}$$
$$x_4\,b_{23}\,(a_{13} - a_{23}) + x_1\,c_{23}\,(-d_{13} + d_{23}) = 0, \tag{A.73}$$
$$x_2\,(a_1\,a_{12} - a_1\,a_{13} - a_2\,a_{12}) - x_3\,c_{13}\,a_{13} = 0, \tag{A.74}$$
$$x_2\,(a_2\,a_{23} + a_3\,a_{13} - a_3\,a_{23}) + x_3\,c_{13}\,a_{13} = 0, \tag{A.75}$$

$$x_2\, d_{13}\, b_{13} + x_3\, (-a_1\, d_{12} + a_1\, d_{13} + a_2\, d_{12}) = 0, \tag{A.76}$$
$$x_2\, d_{13}\, b_{13} + x_3\, (a_2\, d_{23} + a_3\, d_{13} - a_3\, d_{23}) = 0, \tag{A.77}$$
$$x_2\, b_{12}\, (d_{12} - d_{13}) + x_3\, c_{12}\, (-a_{12} + a_{13}) = 0, \tag{A.78}$$
$$x_2\, b_{23}\, (d_{13} - d_{23}) + x_3\, c_{23}\, (-a_{13} + a_{23}) = 0, \tag{A.79}$$
$$x_1\, (a_2\, b_{12} - a_2\, b_{23} + a_{12}\, b_{23} - a_{23}\, b_{12}) = 0, \tag{A.80}$$
$$x_2\, (c_{12}\, a_2 - c_{12}\, a_{23} - a_2\, c_{23} + c_{23}\, a_{12}) = 0, \tag{A.81}$$
$$x_3\, (a_2\, b_{12} - a_2\, b_{23} + d_{12}\, b_{23} - d_{23}\, b_{12}) = 0, \tag{A.82}$$
$$x_4\, (c_{12}\, a_2 - c_{12}\, d_{23} - a_2\, c_{23} + c_{23}\, d_{12}) = 0, \tag{A.83}$$
$$c_{12}\, d_{13}\, b_{12} - c_{23}\, d_{13}\, b_{23} + d_{12}^2\, d_{23} - d_{12}\, d_{23}^2 = 0, \tag{A.84}$$
$$c_{12}\, a_{13}\, b_{12} - c_{23}\, a_{13}\, b_{23} + a_{12}^2\, a_{23} - a_{12}\, a_{23}^2 = 0, \tag{A.85}$$
$$x_1\, x_2\, a_1 + x_3\, x_4\, a_{13} + a_{12}\, (-a_{12}\, a_2 - b_{12}\, c_{12} + a_2^2) = 0, \tag{A.86}$$
$$x_1\, x_2\, a_3 + x_3\, x_4\, a_{13} + a_{23}\, (-a_{23}\, a_2 - b_{23}\, c_{23} + a_2^2) = 0, \tag{A.87}$$
$$x_1\, x_2\, d_{13} + x_3\, x_4\, a_3 + d_{23}\, (-b_{23}\, c_{23} + a_2^2 - a_2\, d_{23}) = 0, \tag{A.88}$$
$$x_1\, x_2\, d_{13} + x_3\, x_4\, a_1 + d_{12}\, (-b_{12}\, c_{12} + a_2^2 - a_2\, d_{12}) = 0, \tag{A.89}$$
$$x_1\, x_2\, a_2 - c_{12}\, a_{23}\, b_{12} + c_{13}\, a_{23}\, b_{13} - d_{12}^2\, a_{13} + d_{12}\, a_{13}^2 = 0, \tag{A.90}$$
$$x_1\, x_2\, a_2 + c_{13}\, a_{12}\, b_{13} - c_{23}\, a_{12}\, b_{23} - d_{23}^2\, a_{13} + d_{23}\, a_{13}^2 = 0, \tag{A.91}$$
$$x_3\, x_4\, a_2 - c_{12}\, d_{23}\, b_{12} + c_{13}\, d_{23}\, b_{13} + d_{13}^2\, a_{12} - d_{13}\, a_{12}^2 = 0, \tag{A.92}$$
$$x_3\, x_4\, a_2 + c_{13}\, d_{12}\, b_{13} - c_{23}\, d_{12}\, b_{23} + d_{13}^2\, a_{23} - d_{13}\, a_{23}^2 = 0, \tag{A.93}$$
$$x_1\, (a_{13}\, d_{13} + a_2\, d_{12} - d_{12}\, d_{13}) + x_4\, (-b_{12}^2 + b_{13}\, a_1) = 0, \tag{A.94}$$
$$x_1\, (a_{13}\, d_{13} + a_2\, d_{23} - d_{13}\, d_{23}) + x_4\, (b_{13}\, a_3 - b_{23}^2) = 0, \tag{A.95}$$
$$x_1\, (a_1\, c_{13} - c_{12}^2) + x_4\, (-a_{12}\, a_{13} + a_{12}\, a_2 + a_{13}\, d_{13}) = 0, \tag{A.96}$$
$$x_1\, (c_{13}\, a_3 - c_{23}^2) + x_4\, (-a_{13}\, a_{23} + a_{13}\, d_{13} + a_{23}\, a_2) = 0, \tag{A.97}$$
$$x_1\, (a_{13}\, d_{12} - b_{23}\, c_{12} + a_2^2 - a_2\, d_{12}) + x_4\, a_{23}\, b_{13} = 0, \tag{A.98}$$
$$x_1\, (a_{13}\, d_{23} - b_{12}\, c_{23} + a_2^2 - a_2\, d_{23}) + x_4\, a_{12}\, b_{13} = 0, \tag{A.99}$$
$$x_1\, c_{13}\, d_{12} + x_4\, (-a_{23}\, a_2 + a_{23}\, d_{13} - b_{23}\, c_{12} + a_2^2) = 0, \tag{A.100}$$
$$x_1\, c_{13}\, d_{23} + x_4\, (-a_{12}\, a_2 + a_{12}\, d_{13} - b_{12}\, c_{23} + a_2^2) = 0, \tag{A.101}$$
$$x_2\, (a_1\, b_{13} - b_{12}^2) + x_3\, (a_2\, a_{12} + d_{13}\, a_{13} - a_{12}\, a_{13}) = 0, \tag{A.102}$$
$$x_2\, (a_3\, b_{13} - b_{23}^2) + x_3\, (a_2\, a_{23} + d_{13}\, a_{13} - a_{13}\, a_{23}) = 0, \tag{A.103}$$
$$x_2\, (a_2\, d_{12} - d_{12}\, d_{13} + d_{13}\, a_{13}) + x_3\, (a_1\, c_{13} - c_{12}^2) = 0, \tag{A.104}$$
$$x_2\, (a_2\, d_{23} - d_{13}\, d_{23} + d_{13}\, a_{13}) + x_3\, (c_{13}\, a_3 - c_{23}^2) = 0, \tag{A.105}$$
$$x_2\, d_{12}\, b_{13} + x_3\, (a_2^2 - a_2\, a_{23} - c_{23}\, b_{12} + d_{13}\, a_{23}) = 0, \tag{A.106}$$
$$x_2\, d_{23}\, b_{13} + x_3\, (-c_{12}\, b_{23} + a_2^2 - a_2\, a_{12} + d_{13}\, a_{12}) = 0, \tag{A.107}$$
$$x_2\, (c_{12}\, b_{23} - a_2^2 + a_2\, d_{23} - d_{23}\, a_{13}) - x_3\, c_{13}\, a_{12} = 0, \tag{A.108}$$
$$x_2\, (a_2^2 - a_2\, d_{12} - c_{23}\, b_{12} + d_{12}\, a_{13}) + x_3\, c_{13}\, a_{23} = 0, \tag{A.109}$$

B Appendix: Aside on further analysing solutions

A step even further than the all-ranks representation theory analysis in Sec.4 above would be to give an *intrinsic* characterisation of the centraliser algebra. We do not do this, but we can briefly set the scene.

For an example $\check{R} = P$ as in (6) is itself a solution — this specific case, and also the corresponding P for each N. This solution is relatively simple, and completely understood in all cases, but still highly non-trivial. Of course it factors through the symmetric group. (It is the Schur–Weyl dual to the natural general linear group action on tensor space.) Its kernel as a symmetric group representation depends on N as well as n. Assuming we work over the complex field, then the kernel is generated exactly by the rank $N+1$ antisymmetriser. Thus in particular for $N = 2$ we have a faithful representation of 'classical' Temperley–Lieb. While for $N = 3$ the rank-3 antisymmetriser does not vanish (so faithful on the corresponding algebras — see e.g. [5]).

More explicitly we have the charge-conserving decomposition

$$\rho = (\rho_{111} \oplus \rho_{222} \oplus \rho_{333}) \oplus (\rho_{112} \oplus \rho_{122} \oplus \rho_{113} \oplus \rho_{133} \oplus \rho_{223} \oplus \rho_{233}) \oplus (\rho_{123})$$

$$\cong 3\rho_{111} \oplus 6\rho_{112} \oplus \rho_{123} \cong 10L_3 \oplus 8L_{21} \oplus L_{1^3} \tag{B.1}$$

where the bracketed sums are of isomorphic reps, and ρ_{111} is trivial; $\rho_{112} = L_3 \oplus L_{21}$; $\rho_{123} = L_3 \oplus 2L_{21} \oplus L_{1^3}$ (i.e. the regular rep). Observe that the multiplicities 10, 8, 1 are the dimensions of the corresponding GL_3 irreps (recall these may be indexed by integer partitions of at most 2 rows, or equivalently of at most 3 rows where we delete all length-3 columns) as dictated by the duality. Note that this structure will be preserved by any generic deformation.

We can characterise this in the classical way, starting with the spectrum of $\check{R}$ itself:

$$\square \otimes \square \quad = \quad \ydiagram{2} \oplus \ydiagram{1,1} \tag{B.2}$$

$$3 \times 3 \quad = \quad 6 \;+\; \overline{3} \tag{B.3}$$

$$\square \otimes \square \otimes \square \quad = \quad \left(\ydiagram{2} \oplus \ydiagram{1,1} \right) \otimes \square \quad = \quad \ydiagram{3} \oplus 2.\ydiagram{2,1} \oplus \emptyset \tag{B.4}$$

$$3 \times 3 \times 3 \quad = \quad (6 \;+\; \overline{3}) \times 3 \quad = \quad 10 \;+\; 2.8 \;+\; 1 \tag{B.5}$$

cf. (B.1). Recall that this continues

$$\square \otimes \square \otimes \square \otimes \square \;= \ydiagram{4} \oplus 3.\ydiagram{3,1} \oplus 2.\ydiagram{2,2} \oplus 3.\square$$

$$3 \times 3 \times 3 \times 3 \;=\; 15 \;+\; 3.15 \;+\; 2.\overline{6} \;+\; 3.3$$

(Side note for future reference: Here in each third rank up the reps from three ranks down reappear (along with some more). This 'three' is one sign that we are with gl_3 or sl_3 in this case.)

Observe that the solution for $\check{R}$ in (14) (in §3.2) certainly does not have the multiplicities in (B.3). Indeed it agrees formally initially with

$$\begin{aligned} \ydiagram{1} \otimes \ydiagram{1,1} &= \ydiagram{2,1} \oplus \emptyset \\ 3 \times \overline{3} &= 8 + 1 \end{aligned}$$

(see e.g. [14]) - formally, in the sense that the symmetry needed for the symmetric group(/Hecke/braid) action is broken here. In this formal picture it is not clear how the labels would correspond with the Hecke algebra/symmetric group labels — we are in rank-2 (but at least there are two summands). And it is not clear how to continue. We have

$$\ydiagram{1} \otimes \ydiagram{2,1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{1} \tag{B.6}$$

$$3 \times 8 = 15 + \overline{6} + 3 \tag{B.7}$$

for example (so at least the centralised algebra of $\ydiagram{1} \otimes \ydiagram{1,1} \otimes \ydiagram{1}$ is -miraculously - isomorphic to the Hecke quotient of B_3). But this is nowhere close to what we have. This suggests that it is at least time to pass to the Lie supergroups again, such as $GL(2|1)$ (cf. e.g. [1, 19, 14]). (Alternatively it could be that the construction is not dual to a quantum group action.)

References

[1] D. Arnal, H. B. Amor, and G. Pinczon. The structure of sl(2,1) supersymmetry: irreps and primitive ideals. *Pacific J Math*, 1994.

[2] R. J. Baxter. Partition function of the eight-vertex lattice model. *Ann. Phys.*, 70:193–228, 1972.

[3] R. J. Baxter. Asymptotically degenerate maximum eigenvalues of the eight-vertex model transfer matrix and interfacial tension. *J. Stat. Phys.*, 8:25–55, 1973.

[4] R. J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1982.

[5] T. Brzezinski and J. Katriel. Representation-theoretic derivation of the Temperley-Lieb-Martin algebras. *Journal of Physics A: Mathematical and General*, 28(18):5305, sep 1995.

[6] A. Bullivant, A. Kimball, P. Martin, and E. C. Rowell. Representations of the necklace braid group: topological and combinatorial approaches. *Comm. Math. Phys.*, 375(2):1223–1247, 2020.

[7] E. Cline, B. Parshall, and L. Scott. Generic and q–rational representation theory. *Publ RIMS*, 35:31–90, 1999.

[8] V. G. Drinfeld. Hopf algebras and the quantum Yang-Baxter equation. *Dokl. Akad. Nauk SSSR*, 283(5):1060–1064, 1985.

[9] J. Hietarinta. All solutions to the constant quantum Yang-Baxter equation in two dimensions. *Phys. Lett. A*, 165(3):245–251, 1992.

[10] J. Hietarinta. The upper triangular solutions to the three-state constant quantum yang-baxter equation. *Journal of Physics A: Mathematical and General*, 26(23):7077, dec 1993.

[11] M. Jimbo. A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Lett. Math. Phys.*, 10(1):63–69, 1985.

[12] V. F. R. Jones. Baxterization. In *Proceedings of the Conference on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory*, volume 4, pages 701–713, 1990.

[13] P. Martin. *Potts models and related problems in statistical mechanics*. World Scientific Publishing Co., 1991.

[14] P. Martin and V. Rittenberg. A template for quantum spin chain spectra. *Int Rev Mod Phys B*, 1992.

[15] P. Martin and E. C. Rowell. Classification of spin-chain braid representations. *arXiv:2112.04533*, 2021.

[16] P. Martin, E. C. Rowell, and F. Torzewska. Classification of charge-conserving loop braid representations. *arXiv:2301.13831*, 2023.

[17] P. P. Martin. *Lecture Notes in Representation Theory*. unpublished lecture notes (online), 2009.

[18] P. P. Martin and S. Ryom-Hansen. Virtual algebraic Lie theory: Tilting modules and Ringel duals for blob algebras. *Proc LMS*, 89:655–675, 2004. (math.RT/0210063).

[19] M. Scheunert, W. Nahm, and V. Rittenberg. Irreducible representations of osp(2,1) and spl(2,1) graded Lie algebras. *J Math Phys*, 18, 1977.

[20] E. K. Skljanin, L. A. Tahtadžjan, and L. D. Faddeev. Quantum inverse problem method. I. *Teoret. Mat. Fiz.*, 40(2):194–220, 1979.

[21] N. J. A. Sloane (editor). The on-line encyclopedia of integer sequences, 2003. published electronically at http://oeis.org.

[22] H. N. V. Temperley and E. H. Lieb. Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: some exact results for the percolation problem. *Proceedings of the Royal Society A*, 322:251–280, 1971.

[23] V. G. Turaev. The Yang-Baxter equation and invariants of links. *Invent. Math.*, 92(3):527–553, 1988.