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Hameomorphism groups of positive genus surfaces

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Abstract. In their 2021 and 2022 papers, Cristofaro-Gardiner, Humilière, Mak, Seyfaddini, and Smith defined links spectral invariants on connected compact surfaces and used them to show various results on the algebraic structure of the group of area-preserving homeomorphisms of surfaces, particularly in cases where the surfaces have genus zero. We show that on surfaces with higher genus, for a certain class of links, the invariants will satisfy a local quasimorphism property. Subsequently, we generalize their results to surfaces of any genus. This extension includes the non-simplicity of (i) the group of hameomorphisms of a closed surface, and (ii) the kernel of the Calabi homomorphism inside the group of hameomorphisms of a surface with non-empty boundary. Moreover, we prove that the Calabi homomorphism extends (non-canonically) to the C^0-closure of the set of Hamiltonian diffeomorphisms of any surface. The local quasimorphism property is a consequence of a quantitative Künneth formula for a connected sum in Heegaard–Floer homology, inspired by the results of Ozsváth and Szabó.

1. Introduction

Let Σ be a compact connected orientable surface (possibly with boundary) equipped with an area from ω. In the 1980s, Fathi defined the mass-flow homomorphism [13],

$$\mathrm{Homeo}_{0,c}(\Sigma, \omega) \to \mathbb{R},$$

from the identity component of the group of area-preserving homeomorphisms supported in the interior of Σ to $\mathbb{R}$. Whether its kernel is a simple group was an open question for a long time and has recently been resolved negatively using techniques from symplectic geometry. The case of the sphere was answered by [10] using periodic Floer homology, building on the work of [11, 16]. The case of positive genus surfaces was answered by [8] using Lagrangian Floer theory, borrowing ideas from [19, 21, 25].

Symplectic geometry enters the picture because ω is a symplectic form and the kernel of the mass-flow homomorphism can be identified with the C^0 closure of the group $\mathrm{Ham}(\Sigma)$ of Hamiltonian diffeomorphisms supported in the interior of Σ. The

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Hofer metric on $\mathrm{Ham}(\Sigma)$, a bi-invariant and non-degenerate metric, enables us to define two natural normal subgroups of $\overline{\mathrm{Ham}}(\Sigma)$, namely, the group of hameomorphisms $\mathrm{Hameo}(\Sigma)$ and the group of finite energy homeomorphisms $\mathrm{FHomeo}(\Sigma)$ (see Section 2.1 for the precise definitions, and also [10, 20] for more discussions). Indeed, the authors of [8] show that the subgroup $\mathrm{Hameo}(\Sigma)$ is always a proper normal subgroup.

Since then, the method has been pushed further to answer more refined questions about the algebraic structure of $\overline{\mathrm{Ham}}(\Sigma)$, especially when Σ has genus 0, using a property called the *quasimorphism* property. The goal of this paper is to generalize the results of [9] to all surfaces even though we no longer have the quasimorphism property for positive genus surfaces.

1.1. Link spectral invariants and known results for genus zero surfaces

Link spectral invariants are introduced in [8] as the main tool to study $\overline{\mathrm{Ham}}(\Sigma)$. Given a Lagrangian link (i.e., a union of disjoint circles) $\underline{L} = L_1 \cup \cdots \cup L_k$ satisfying certain monotonicity conditions on a closed surface (Σ, ω), we can associate a spectral invariant $c_{\underline{L}} : C^\infty(S^1 \times \Sigma, \mathbb{R}) \to \mathbb{R}$ which satisfies several useful properties (Proposition 13).

In particular, the homotopy invariance permits defining $c_{\underline{L}}(\varphi)$ for $\varphi \in \widetilde{\mathrm{Ham}}(\Sigma)$ by the formula

$$c_{\underline{L}}(\{\varphi_H^t\}_{t\in[0,1]}) = c_{\underline{L}}(H)$$

for a mean normalized Hamiltonian function H. The homogenization $\mu_{\underline{L}}$ of $c_{\underline{L}}$ is defined by

$$\mu_{\underline{L}}(\varphi) = \lim_{n\to\infty} \frac{c_{\underline{L}}(\varphi^n)}{n}.$$

In the case of $\Sigma = S^2$, we have the following theorem.

Theorem 1 ([8, Theorem 7.7]). *$c_{\underline{L}} : \widetilde{\mathrm{Ham}}(S^2) \to \mathbb{R}$ is a quasimorphism with defect $D \leqslant \frac{k+1}{k}\lambda$, where λ is the monotonicity constant of $\underline{L}$. Moreover, $\mu_{\underline{L}}$ descends to a homogeneous quasimorphism on* $\mathrm{Ham}(S^2)$*, with defect bounded by* $2D$.

The fact that $\mu_{\underline{L}}$ is a quasimorphism and that we can quantify its defect is the key ingredient to prove the following results (the definition of Cal will be recalled in Section 2.1).

(1) The Calabi homomorphism $\mathrm{Cal} : \mathrm{Hameo}(D^2) \to \mathbb{R}$ can be extended to $\overline{\mathrm{Ham}}(D^2) = \mathrm{Homeo}(D^2, \omega)$ ([9, Theorem 1.9]).

(2) $\mathrm{Ker}(\mathrm{Cal}) \cap \mathrm{Hameo}(D^2)$ is not simple ([9, Theorem 1.3]).

(3) $\mathrm{Hameo}(S^2)$ is not simple ([9, Theorem 1.3]).

1.2. Main results for positive genus surfaces

The purpose of this paper is to generalize (1) and (2) to any compact oriented surface Σ (of any genus) with non-empty boundary.

Theorem 2. *The Calabi homomorphism* $\mathrm{Hameo}(\Sigma) \to \mathbb{R}$ *can be extended to* $\overline{\mathrm{Ham}}(\Sigma)$.

Theorem 3. $\mathrm{Ker(Cal)} \cap \mathrm{Hameo}(\Sigma)$ *is not simple.*

We also generalize (3) to any connected closed oriented surface (Σ, ω).

Theorem 4. $\mathrm{Hameo}(\Sigma, \omega)$ *is not simple.*

Theorems 3 and 4 together answer a question in [20, Problem 4] for all surfaces.

There is a fundamental difference between the genus 0 and positive genus case: $c_{\underline{L}}$ and $\mu_{\underline{L}}$ are never quasimorphisms for positive genus surfaces for any $\underline{L}$ (cf. Proposition 17). To remedy this, we need to prove a local version of the quasimorphism property when Σ has positive genus and combine it with the fragmentation technique. This requires a slightly different class of Lagrangian links (see Definition 14) than those in [8]. We define the spectral invariants $c_{\underline{L}}$ for this new class of links show that they satisfy all the usual spectral invariant properties listed in Proposition 13, as well as the following *local quasimorphism* property.

Theorem 5. *Let $\underline{L}$ be an admissible link with k contractible components, with monotonicity constant λ (see Definition 14). Let $D \subset \Sigma$ be a disk that does not intersect the non-contractible components of $\underline{L}$, and denote by* $\mathrm{Ham}_D(\Sigma)$ *the Hamiltonian diffeomorphisms supported in D. Then, the restriction of $c_{\underline{L}}$ to* $\mathrm{Ham}_D(\Sigma)$ *is a quasimorphism with defect bounded by $\frac{k+1}{k+g}\lambda$.*

The construction of $c_{\underline{L}}$ and the proof of its local quasimorphism property rely on the following Künneth formula for connected sums in Heegaard–Floer homology, similar to the stabilization result of [21], which is proved by identifying moduli spaces of holomorphic maps under degeneration.

Theorem 6. *Consider two transverse η-monotone admissible Lagrangian links $\underline{L}$ and $\underline{K}$ with k components on a closed surface (Σ, ω). Let (E, ω_E) denote the two-dimensional torus, and α be a non-contractible circle on E. Let α' be a small Hamiltonian deformation of α such that α and α' are transverse. Then, for an appropriate choice of almost complex structure, there is an isomorphism of filtered chain complexes*

$$CF^*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K}) \otimes CF^*(\alpha, \alpha') \xrightarrow{\sim} CF^*(\mathrm{Sym}(\underline{L} \cup \alpha), \mathrm{Sym}(\underline{K} \cup \alpha')),$$

where the left-hand side is computed considering the links $\underline{L}$ and $\underline{K}$ in (Σ, ω), α and α' in (E, ω_E), while on the right-hand side, $\underline{L} \cup \alpha$ and $\underline{K} \cup \alpha'$ are links in the *connected sum* $(\Sigma \# E, \omega')$ *(where we perform the connected sum between a point* $\sigma_1 \in \Sigma$ *away from the links* $\underline{L}$ *and* $\underline{K}$*, and a point* $\sigma_2 \in E$ *away from the isotopy between* α *and* α'*).*

If we forget the filtration, Theorem 6 is an identification of generators and differentials so it does not depend on the symplectic form. To guarantee that the filtration also agrees, the symplectic form ω' on $\Sigma \# E$ is chosen such that it is equal to ω away from a neighbourhood $B(\sigma_1)$ of σ_1 which does not intersect $\underline{L} \cup \underline{K}$, equal to ω_E over the support K_α of the Hamiltonian isotopy from α to α', and satisfies $\omega'(\Sigma \# E) = \omega(\Sigma)$ (so we need to assume that $\omega_E(K_\alpha) < \omega(B(\sigma_1))$ for ω' to exist).

Structure of the paper. We collect some preliminaries in Section 2. The new class of Lagrangian links and the proof of their local quasimorphism property (Theorem 5) are given in Section 3. Section 4 is devoted to the proof of the main results, Theorems 2, 3, and 4. Theorem 6 is proved in Section 5.

2. Preliminaries

2.1. Subgroups of $\overline{\mathrm{Ham}}(\Sigma)$

Let Σ be a compact connected surface equipped with an area form ω. We start by introducing some conventions and notations, closely following [8].

- Given a Hamiltonian $H : S^1 \times \Sigma \to \mathbb{R}$, the Hamiltonian diffeomorphism ϕ_H^1 is the time 1 flow of the Hamiltonian vector field X_{H_t} defined by $\iota_{X_{H_t}} \omega = dH_t$.
- Given two Hamiltonians H and K, we define the composition by $(H \# K)_t(x) := H_t(x) + K_t((\phi_H^t)^{-1}(x))$.
- We denote by $\mathrm{Ham}(\Sigma)$ the group of Hamiltonian diffeomorphisms of Σ supported in the interior of Σ (it is often denoted $\mathrm{Ham}_c(\Sigma)$ in the literature).
- $\overline{\mathrm{Ham}}(\Sigma)$ denotes its closure for the C^0 distance inside $\mathrm{Homeo}_c(\Sigma)$.
- The Hofer norm of a Hamiltonian is $\|H\|_{\mathrm{Hof}} := \int_0^1 \mathrm{osc}\, H_t dt = \int_0^1 (\max H_t - \min H_t) dt$.
- The Hofer norm of a Hamiltonian diffeomorphism is $\|\varphi\|_{\mathrm{Hof}} := \inf_{H, \varphi = \phi_H^1} \|H\|_{\mathrm{Hof}}$.
- The Hofer distance on $\mathrm{Ham}(\Sigma)$ is $d_H(\varphi, \psi) := \|\varphi \psi^{-1}\|_{\mathrm{Hof}}$.

  We define some subgroups of $\overline{\mathrm{Ham}}(\Sigma)$ (cf. [10, 20]).

Definition 7. We call $\varphi \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ a *finite energy homeomorphism* if there exists a sequence of smooth Hamiltonians H_i such that

- $\phi_{H_i}^1 \xrightarrow{C^0} \varphi$,
- there exists $C \geqslant 0$ such that for every i, $\|H_i\|_{\mathrm{Hof}} \leqslant C$.

Definition 8. We call $\varphi \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ a *hameomorphism* if there exists an isotopy $(\psi^t)_{t\in[0,1]}$ in $\overline{\mathrm{Ham}}(\Sigma)$ from Id to φ and a sequence of smooth Hamiltonians H_i supported in a compact subset K of the interior of Σ such that

- $\phi^t_{H_i} \xrightarrow{C^0} \psi^t$ uniformly in $t \in [0, 1]$,
- (H_i) is a Cauchy sequence for the Hofer norm.

We denote the group of finite energy homeomorphisms by $\mathrm{FHomeo}(\Sigma, \omega)$, and the group of hameomorphisms by $\mathrm{Hameo}(\Sigma, \omega)$. When Σ has non-empty boundary, one can define the Calabi invariant $\mathrm{Cal} : \mathrm{Ham}(\Sigma) \to \mathbb{R}$ as follows: let $\varphi \in \mathrm{Ham}(\Sigma)$, and H_t be a Hamiltonian supported in the interior of Σ such that $\varphi = \phi^1_{H_t}$. Then,

$$\mathrm{Cal}(\varphi) = \int_0^1 \int_\Sigma H_t \omega dt.$$

This definition does not depend on the choice of the Hamiltonian H_t, and Cal is a group homomorphism.

As shown in [8], Cal can be extended canonically to a group homomorphism $\mathrm{Hameo}(\Sigma) \to \mathbb{R}$ by the formula $\mathrm{Cal}(\varphi) = \lim_{i\to\infty} \mathrm{Cal}(\varphi^1_{H_i})$, where we consider any sequence (H_i) as in the definition of a hameomorphism.

The purpose of this paper is to study the algebraic structure of $\overline{\mathrm{Ham}}(\Sigma)$ and its subgroups for a general surface Σ.

Here is what was known before this paper.

(1) $\overline{\mathrm{Ham}}(\Sigma)$ is not simple since $\mathrm{FHomeo}(\Sigma)$ is a proper normal subgroup ([8]).

(2) $\mathrm{Hameo}(S^2)$ is not simple ([9]).

(3) $\mathrm{FHomeo}(S^2)$ is not simple since $\mathrm{Hameo}(S^2)$ is a proper normal subgroup ([4]).

(4) When Σ has non-empty boundary:

  (a) $\mathrm{Hameo}(\Sigma)$ is not simple since it contains the kernel of the (extended) Calabi homomorphism ([8]);

  (b) $\mathrm{FHomeo}(\Sigma)$ is not simple, since either $\mathrm{Hameo}(\Sigma)$ is a proper normal subgroup, or they coincide and by the previous point they are not simple ([8]);

  (c) $\mathrm{Hameo}(D^2) \cap \mathrm{Ker}(\mathrm{Cal})$ is not simple ([9]).

(5) All normal subgroups of $\overline{\mathrm{Ham}}(\Sigma)$ contain the commutator subgroup, which is perfect and simple ([9, Section 6]).

We will extend this picture with a generalization of (2), (3), and (4)(c), respectively.

- When Σ is closed, $\mathrm{Hameo}(\Sigma)$ is not simple (Theorem 4).

- When Σ is closed, FHomeo(Σ) is not simple, since either Hameo(Σ) is a proper normal subgroup, or they coincide and by the previous point they are not simple.
- When Σ has non-empty boundary, Hameo(Σ) $\cap$ Ker(Cal) is not simple (Theorem 3).

2.2. Spectral invariants and quasimorphisms

Let (M, ω) be a closed symplectic manifold and $L \subset M$ a monotone Lagrangian, i.e.,

$$\omega|_{\pi_2(M,L)} = \tau\mu|_{\pi_2(M,L)}$$

for some constant $\tau > 0$, where μ is the Maslov homomorphism. Then, by [17], for a Lagrangian L' Hamiltonian isotopic to L, and a Hamiltonian H such that $\varphi_H^1(L) \pitchfork L'$, the Floer cohomology $HF^*(L, L', H)$ is well defined.

We follow the convention in [8, Section 6] and define the Floer cohomology $HF^*(L, L', H)$ as a vector space over $\Lambda := \mathbb{C}[[T]][T^{-1}]$. In particular, there is an action filtration on the Floer complex. We define $CF_\lambda(L, L', H)$ to be the subcomplex of $CF(L, L', H)$ generated by capped Hamiltonian chords of action less than or equal to λ. The inclusion of this subcomplex gives rise to a map

$$i_\lambda : HF_\lambda(L, L', H) \to HF(L, L', H).$$

We assume that either $L' = L$ or $L' \pitchfork L$. In the former case, there is the PSS isomorphism $QH(L) \to HF(L, L, H)$. In the latter case, there is the continuation isomorphism $HF(L, L', 0) \to HF(L, L', H)$. By an abuse of notation, we denote $QH(L)$ by $HF(L, L, 0)$ and the isomorphism (in either case) by κ. Given a homology class $a \in HF^*(L, L', 0) \setminus \{0\}$, one can define a spectral invariant

$$c_{L,L'}(a, H) := \inf\{\lambda \mid \kappa(a) \in \operatorname{Im} i_\lambda\}.$$

This spectral invariant satisfies a homotopy invariance property, which enables us to define $c_{L.L'}$ on $\widetilde{\operatorname{Ham}}(M, \omega)$, the universal cover of $\operatorname{Ham}(M, \omega)$. When $L = L'$ and $a = e_L$ is the unit of $QH^*(L)$, we will simply denote

$$c_L(H) := c_{L,L}(e_L, H).$$

We recall the definition of a quasimorphism.

Definition 9. Let G be a group. A *quasimorphism* on G is a map $\mu : G \to \mathbb{R}$ that satisfies

$$\exists D \geqslant 0, \forall g, h \in G, \quad |\mu(gh) - \mu(g) - \mu(h)| \leqslant D.$$

The infimal value of D such that this property holds is called the defect of μ.

Moreover, μ is a *homogeneous quasimorphism* if it also satisfies

$$\forall n \in \mathbb{Z}, \forall g \in G, \quad \mu(g^n) = n\mu(g).$$

When $(M, \omega) = (\mathbb{CP}^n, \omega_{FS})$ and L is a monotone Lagrangian submanifold with $HF(L) \neq 0$, c_L is a quasimorphism on $\widetilde{\mathrm{Ham}}(M, \omega)$. This is a consequence of the same result for the Hamiltonian spectral invariant c (cf. [12]), and the inequality $c_L \leqslant c$ (cf. [17, Proposition 4]).

Proposition 10 (Homogenization). *Let $\mu : G \to \mathbb{R}$ be a quasimorphism. Then,*

$$\widetilde{\mu}(g) := \lim_{n\to\infty} \frac{\mu(g^n)}{n}$$

is well defined, and it is a homogeneous quasimorphism, called the homogenization of μ.

Now, we explain the construction of spectral invariants for Lagrangian links as defined in [8].

Consider a closed symplectic surface (Σ, ω), with a compatible complex structure j. A Lagrangian link on Σ is a disjoint union $\underline{L} = L_1 \cup \cdots \cup L_k$ of smooth simple curves in Σ.

Definition 11. Denote by B_j, $1 \leqslant j \leqslant s$, the connected components of $\Sigma \setminus \underline{L}$. Let k_j be the number of boundary components of B_j, and A_j the ω-area of B_j. Let $\eta \geqslant 0$. We say that $\underline{L}$ is *η-monotone* if

$$\lambda := 2\eta(k_j - 1) + A_j$$

does not depend on j. λ is called the *monotonicity constant* of $\underline{L}$.

A Lagrangian link $\underline{L}$ on a compact surface Σ_0 with non-empty boundary is called η-monotone if there exists a symplectic embedding of Σ_0 into a closed surface Σ such that $\underline{L}$ is η-monotone inside Σ.

Remark 12. The constant λ is equal to the area of the disks bounded by contractible components of the link. Therefore, if $\underline{L}$ has m components bounding pairwise disjoint disks, then $\lambda \leqslant \frac{1}{m} \times \mathrm{Area}(\Sigma)$.

Let $\underline{L} = L_1 \cup \cdots \cup L_k$ be a Lagrangian link on the closed surface Σ. Denote by $\mathrm{Sym}\,\underline{L}$ the image of $L_1 \times \cdots \times L_k$ in the symmetric product $\mathrm{Sym}^k(\Sigma) := \Sigma^k/\mathfrak{S}_k$, where $\mathfrak{S}_k$ is the permutation group permuting the factors. Suppose $\underline{L}$ is η-monotone and $\underline{L}'$ is Hamiltonian isotopic to $\underline{L}$. Let $H : S^1 \times \Sigma \to \mathbb{R}$ be a Hamiltonian and $\mathrm{Sym}^k(H) : S^1 \times \mathrm{Sym}^k(\Sigma) \to \mathbb{R}$ be given by $\mathrm{Sym}^k(H)_t(x_1, \ldots, x_k) := \sum_{i=1}^k H_t(x_i)$.

We recall in Section 5.1 how from such a link one can define a Floer cohomology[1],

$$HF(\underline{L}, \underline{L}', H) := HF^*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}', \mathrm{Sym}^k(H)). \tag{2.1}$$

It was shown in [8] that $HF^*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}', \mathrm{Sym}^k(H))$ is isomorphic to $H^*(\mathrm{Sym}\,\underline{L})$ as a vector space (without filtration) so it is non-zero. Moreover, they show that Lagrangian spectral invariants $c_{\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}'}(a, \mathrm{Sym}^k(H))$ are well defined. Therefore, one can define link spectral invariants

$$c_{\underline{L}} := \frac{1}{k} c_{\mathrm{Sym}\,\underline{L}} = \frac{1}{k} c_{\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}}(e_{\mathrm{Sym}\,\underline{L}}, \cdot).$$

Proposition 13. *This invariant inherits all the properties of Lagrangian spectral invariants.*

- *(Spectrality)* $c_{\underline{L}}(H)$ *lies in the action spectrum* $\mathrm{Spec}(H, \underline{L})$.
- *(Hofer–Lipschitz)* $|c_{\underline{L}}(H) - c_{\underline{L}}(K)| \leqslant \|H - K\|_{\mathrm{Hof}}$.
- *(Monotonicity) If* $H \leqslant K$, *then* $c_{\underline{L}}(H) \leqslant c_{\underline{L}}(K)$.
- *(Lagrangian control) If* $H_t|_{L_i} = s_i(t)$ *for each* i, *then*

  $$c_{\underline{L}}(H) = \frac{1}{k} \sum_{i=1}^{k} \int s_i(t)dt.$$

  Moreover,

  $$\frac{1}{k} \sum_{i=1}^{k} \int_{S^1} \min_{L_i} H_t dt \leqslant c_{\underline{L}}(H) \leqslant \frac{1}{k} \sum_{i=1}^{k} \int_{S^1} \max_{L_i} H_t dt.$$

- *(Triangle inequality)* $c_{\underline{L}}(H \# K) \leqslant c_{\underline{L}}(H) + c_{\underline{L}}(K)$.
- *(Homotopy invariance) If* H, K *are mean normalized,* $\phi_H^1 = \phi_K^1$ *and* $(\phi_H^t)_{t\in[0,1]}$ *is homotopic to* $(\phi_K^t)_{t\in[0,1]}$ *relative to endpoints, then* $c_{\underline{L}}(H) = c_{\underline{L}}(K)$.
- *(Shift)* $c_{\underline{L}}(H + s(t)) = c_{\underline{L}}(H) + \int s(t)dt$.

The homotopy invariance permits to define $c_{\underline{L}}(\{\varphi^t\}_{t\in[0,1]})$ for $\{\varphi^t\}_{t\in[0,1]} \in \widetilde{\mathrm{Ham}}(\Sigma)$ by the formula $c_{\underline{L}}(\{\phi_H^t\}_{t\in[0,1]}) = c_{\underline{L}}(H)$ for a mean normalized H.

Moreover, $c_{\underline{L}}$ is a quasimorphism when $\Sigma = S^2$ (i.e., Theorem 1). It is proved using the fact that $\mathrm{Sym}^k(S^2) \cong \mathbb{CP}^k$ (cf. [12]).

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[1] The function $\mathrm{Sym}^k(H)$ is not smooth along the diagonal of $\mathrm{Sym}^k(\Sigma)$ but it turns out that any smooth Hamiltonian that agrees with $\mathrm{Sym}^k(H)$ outside a sufficiently small neighbourhood of the diagonal will give the same Floer cohomology up to canonical isomorphisms as a filtered vector space. Therefore, $HF^*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}', \mathrm{Sym}^k(H))$ is defined to be the filtered vector space.

3. Construction of the new invariants

Let (Σ, ω) be a compact surface of genus g. We suppose that Σ has area 1. We introduce the following class of links, which is slightly different from the ones in [8] (cf. [5,6] for the study of this class of links in the cylindrical setting).

Definition 14. A Lagrangian link $\underline{L} = L_1 \cup \cdots \cup L_k \cup \alpha_1 \cup \cdots \cup \alpha_g$ is called *admissible* if the following statements hold:

- the circles $L_1, \ldots, L_k, \alpha_1, \ldots, \alpha_g$ are all disjoint,
- $\alpha_1, \ldots, \alpha_g$ are non-contractible,
- there exists a decomposition of Σ as a connected sum of a genus zero surface Σ_0 and g tori such that each α_i lives in a different torus and L_i lives in Σ_0,
- $\underline{L}_0 := L_1 \cup \cdots \cup L_k \subset \Sigma_0$ is η-monotone for some $\eta \geqslant 0$, with respect to a symplectic form ω_0 on Σ_0 which coincides with ω outside a small neighbourhood of the connected sum region away from the link such that $\omega_0(\Sigma_0) = 1$.

We define the monotonicity constant of $\underline{L}$ as the monotonicity constant of $\underline{L}_0$ (see Definition 11). See Figure 1.

Remark 15 (A remark about the links considered in [8]). The links considered in [8] satisfy a planarity assumption, which, in particular, implies that they have at least $2g$ non-contractible components. This hypothesis was needed in their paper to show that the Heegaard–Floer homology of the link was non-vanishing. However, in our case, we only allow links to have the same number of non-contractible components as the genus of the surface; otherwise, the statement and proof of Theorem 6 would be much more complicated. The Heegaard–Floer chain complex for our admissible links is still defined in the same way as in [8], and applying Theorem 6 proves that its homology does not vanish.

Remark 16 (A remark on the third bullet of Definition 14). Suppose that $\underline{L} = L_1 \cup \cdots \cup L_k \cup \alpha_1 \cup \cdots \cup \alpha_g$ satisfies the first two bullets of Definition 14. Let B be the image of $H_1(\partial\Sigma) \to H_1(\Sigma)$, V be the image of $H_1(\alpha_1 \cup \cdots \cup \alpha_g) \to H_1(\Sigma)$ and l_i be the image of $H_1(L_i) \to H_1(\Sigma)$. Topologically, if V is a g dimensional subspace which intersects B only at 0 and $l_i \subset B$ for all i, then there is a decomposition of Σ as a connected sum of a genus zero surface Σ_0 and g tori such that the third bullet of Definition 14 is satisfied.

To see this, for simplicity, we first assume that there is no L_i and Σ is closed (so $B = 0$). Then, V is a Lagrangian subspace with respect to the intersection form Ω on $H_1(\Sigma)$. Let $a_i := [\alpha_i] \in H_1(\Sigma)$. We can complete $\{a_i\}$ to a basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_1(\Sigma; \mathbb{Z})$ such that $\Omega(a_i, b_i) = 1$ and $\Omega(a_i, b_j) = 0$ if $i \neq j$, and $\Omega(b_i, b_j) = 0$ for all i, j. We can find circles $\beta_i \subset \Sigma$, $i = 1, \ldots, g$ such that the geometric intersection

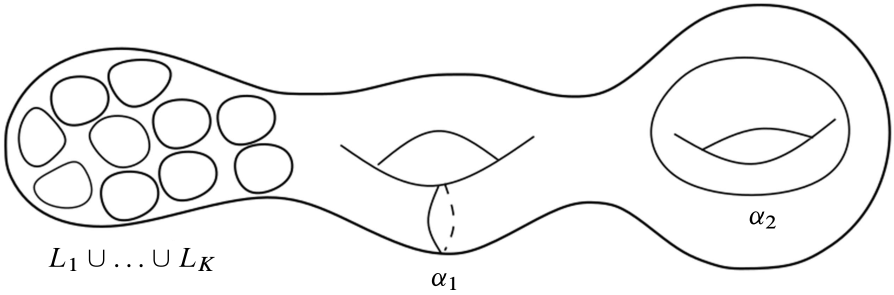

Figure 1. An admissible link.

number between any two circles in $\{\alpha_i, \beta_j\}$ agrees with the homological intersection number. The regular neighbourhood of $\alpha_i \cup \beta_i$ gives the splitting of the i^{th} torus in the connected sum decomposition. The case when Σ has boundary components can be proved by first embedding it to a closed surface by capping off the boundary components by disks (and choosing β_i to avoid the capping disks). The case when there is L_i can be reduced to the case with no L_i by running the argument above, for the positive genus components, in the complement of $\cup_i L_i$ (in particular, L_i are allowed to be non-contractible separating circles).

We assume that Σ is closed. Then, given an admissible link $\underline{L}$, there exists a decomposition of Σ as a connected sum $\Sigma = S^2 \# E_1 \# \cdots \# E_g$, where the E_i are copies of the 2-torus such that $\underline{L}_0 := L_1 \cup \cdots \cup L_k \subset S^2$ is η-monotone, and for all $1 \leqslant i \leqslant g$, $\alpha_i \subset E_g$. (Here, we inflate the symplectic form near the connected sum point in S^2 so that S^2 has area 1. This choice of symplectic form makes $\underline{L}_0$ η-monotone by the fourth bullet of Definition 14, and it is compatible with the one in Theorem 6.)

The authors of [8] show that $HF^*(\underline{L}_0, \underline{L}_0)$ is well defined and non-zero.

By applying Theorem 6 g times, we get that $HF^*(\underline{L}, \underline{L})$ is non-zero, and therefore, for a non-degenerate Hamiltonian H, $HF^*(\underline{L}, \underline{L}, H)$ is also non-zero. As a result, one can define spectral invariants

$$c_{\underline{L}}(H) := \frac{1}{k+g} c_{\mathrm{Sym}\,\underline{L}}(\mathrm{Sym}^{k+g}(H))$$

for non-degenerate H and then extend them to all Hamiltonians by continuity (i.e., the Hofer–Lipschitz property in Proposition 13).

If Σ_0 has non-empty boundary, then one can embed Σ_0 into a closed surface Σ such that $\underline{L}$ remains admissible in Σ. Indeed, by the definition of η-monotonicity for surfaces with boundary, there exists an embedding into a closed surface Σ such that

$\underline{L}$ is still η-monotone inside Σ. Then, one can define the link spectral invariant for Σ_0 by restricting $c_{\underline{L}}$ to $\mathrm{Ham}(\Sigma_0) \subset \mathrm{Ham}(\Sigma)$.

The fact that this invariant satisfies all the properties listed in Proposition 13, as in [8], is a straightforward consequence of the properties of Lagrangian spectral invariants (see [14], for instance).

Before proving the local quasimorphism property 5, we will show the following proposition.

Proposition 17. *Let Σ be a surface of genus $g > 0$. Let $\underline{L} = L_1 \cup \cdots \cup L_k \cup \alpha_1 \cup \cdots \cup \alpha_g$ be a monotone admissible Lagrangian link on Σ, where $\alpha_1, \ldots, \alpha_g$ are the non-contractible components of $\underline{L}$. Then, $c_{\underline{L}}$ is not a quasimorphism.*

Proof. It is enough to show that there exists a sequence of Hamiltonians $(H_n)_n$ such that $\gamma_{\underline{L}}(H_n) := c_{\underline{L}}(H_n) + c_{\underline{L}}(\overline{H}_n)$ is not bounded. We pick a non-contractible circle in Σ that intersects $\underline{L}$ at a single point in α_1. Such a circle always exists, take, for instance, β_1 as in Remark 16. Then, we pick a small neighbourhood U of this circle, diffeomorphic to the annulus $A = S^1 \times (-1, 1)$ (we denote by $\psi : U \to A$ such a diffeomorphism) such that $U \cap \underline{L} = U \cap \alpha_1$ is connected and sent to a vertical $\{\theta_0\} \times (-1, 1)$ by ψ. Let $H : (-1, 1) \to \mathbb{R}$ be a smooth function such that

- H is compactly supported,
- H admits a single local maximum at 0 and no other critical point in the interior of its support,
- $H(0) = 1$.

We define K_n on A by $K_n(\theta, t) = nH(t)$, and H_n on Σ by

- $H_n(x) = K_n(\psi(x))$ if $x \in U$,
- $H_n(x) = 0$ if $x \notin U$.

Now, we compute the sequence $(\gamma_{\underline{L}}(H_n))_n$ for this choice of Hamiltonians.

We know that $c_{\underline{L}}(H_n)$ lies in $\frac{1}{k+g}\,\mathrm{Spec}(\mathrm{Sym}(H_n))$. In order to compute this spectrum, we consider the critical points of the action that are in the same connected component as a chosen reference path in $\mathcal{P}(\mathrm{Sym}(\underline{L}), \mathrm{Sym}(\underline{L}))$ (see Section 5.1 for a definition of the Heegaard–Floer complex and the action functional). We pick $x_1 \in L_1, \ldots, x_k \in L_k, y_1 \in \alpha_1, \ldots, y_g \in \alpha_g$ fixed by the flow of H_n, and take the constant path $\zeta := \{x_1, \ldots, x_k, y_1, \ldots, y_g\}$ in $\mathrm{Sym}(\underline{L})$ as the reference path.

Then, the only critical points of the action that are in the same connected component as ζ in $\mathcal{P}(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L})$ are symmetric products of points fixed by H_n. They all have zero action except when we choose in α_1 the point y_1' for which H_n is maximal. For any choice of $x_i' \in L_i, 1 \leqslant i \leqslant k$, and $y_i' \in \alpha_i, 2 \leqslant i \leqslant g$, the critical point $\{x_1', \ldots, x_k', y_1', \ldots, y_g'\}$ has action n.

Hence, $\mathrm{Spec}(\mathrm{Sym}(H_n)) = \{0, n\}$. Similarly, $\mathrm{Spec}(\mathrm{Sym}(\bar{H}_n)) = \{-n, 0\}$ because $\bar{H}_n = -H_n$. Therefore, $\gamma_{\underline{L}}(H_n) \in \{-\frac{n}{k+g}, 0, \frac{n}{k+g}\}$.

Since $\gamma_{\underline{L}}$ is non-negative, we can rule out $-\frac{n}{k+g}$. Moreover, by the energy-capacity inequality [15, Lemma 6], which also holds for monotone Lagrangians in a compact manifold by [17], we have that $\gamma_{\underline{L}}$ is non-zero on Hamiltonians that do not fix $\mathrm{Sym}(\underline{L})$. Therefore, $\gamma_{\underline{L}}(H_n)$ is non-zero.

Finally, we get that $\gamma_{\underline{L}}(H_n) = \frac{n}{k+g}$, which is unbounded as n goes to infinity. ∎

Remark 18. We expect that $c_{\underline{L}}$ is not a quasimorphism for the links defined in [8]. Indeed, their links can be related to ours by a series of handleslide moves to reduce the number of non-contractible components, and we expect that the proof of the handleslide invariance of Heegaard–Floer homology in [21, Section 9] could also be adapted to show that handleslide moves preserve the spectral invariants up to homogenization (cf. (3.1)).

We now prove that this invariant satisfies Theorem 5.

Proof of Theorem 5. We consider an admissible link $\underline{L} = L_1 \cup \cdots \cup L_k \cup \alpha_1 \cup \cdots \cup \alpha_g$, and a disk D that does not intersect $\underline{\alpha} := \alpha_1 \cup \cdots \cup \alpha_g$. Then, one can find a decomposition of Σ as a connected sum $\Sigma = S^2 \# E_1 \# \cdots \# E_g$ such that $\underline{L}_0 := L_1 \cup \cdots \cup L_k \subset S^2$ is η-monotone for all $1 \leqslant i \leqslant g$, $\alpha_i \subset E_g$, and $D \subset S^2$.

Let H be a Hamiltonian supported in D, and let H_ε be an ε-perturbation of H in small neighbourhoods of the link's components so that $HF(\underline{L}, \underline{L}, H_\varepsilon)$ is well defined (cf. (2.1)). We can assume that H_ε is chosen such that it is supported away from the connected sum neighbourhoods of the decomposition $\Sigma = S^2 \# E_1 \# \cdots \# E_g$.

Then, by applying Theorem 6 g times, we have that

$$CF^*(\underline{L}, \underline{L}, H_\varepsilon) \simeq CF^*(\underline{L}_0, \underline{L}_0, H_\varepsilon|_{S^2}) \otimes \bigotimes_{i=1}^{g} CF^*(\alpha_i, \alpha_i, H_\varepsilon|_{E_i}).$$

Then, we claim that this isomorphism maps representatives of $\kappa(e_{\mathrm{Sym}(\underline{L})})$ in $CF^*(\underline{L}, \underline{L}, H_\varepsilon)$ to tensor products of representatives of unit classes in $CF^*(\underline{L}_0, \underline{L}_0, H_\varepsilon|_{S^2})$ and $CF^*(\alpha_i, \alpha_i, H_\varepsilon|_{E_i})$. This is the content of Corollary 57, which we prove in Section 5.5. It follows from the proof of Theorem 6 that this one-to-one correspondence preserves the action (which is defined as the sum of the actions on the tensor product).

Since H_ε is ε-small on E_i, we get

$$c_{\mathrm{Sym}\,\underline{L}}(\mathrm{Sym}^{k+g}(H)) = c_{\mathrm{Sym}\,\underline{L}_0}(\mathrm{Sym}^k(H)),$$

where $c_{\mathrm{Sym}\,\underline{L}_0}$ is computed inside $\mathrm{Sym}^k(S^2)$. Since L_0 is η-monotone inside S^2, with monotonicity constant λ, applying Theorem 1 gives that the restriction of $c_{\mathrm{Sym}\,\underline{L}}$ to $\mathrm{Ham}_D(\Sigma)$ is a quasimorphism with defect bounded by $\frac{k+1}{k+g}\lambda$. ∎

We define the homogenized spectral invariant $\mu_{\underline{L}}$ by the formula

$$\mu_{\underline{L}}(H) := \lim_{n\to\infty} \frac{c_{\underline{L}}(H^{\#n})}{n}. \tag{3.1}$$

This is well defined by the triangle inequality and Fekete's lemma.

Proposition 19. *The invariant $\mu_{\underline{L}}$ satisfies the following properties.*

- *(Hofer–Lipschitz)* $|\mu_{\underline{L}}(H) - \mu_{\underline{L}}(K)| \leqslant \|H - K\|_{\mathrm{Hof}}$.
- *(Lagrangian control) Suppose H is mean-normalized, $H_t|_{L_i} = s_i(t)$, $H_t|_{\alpha_j} = s'_j(t)$. Then,*

$$\mu_{\underline{L}}(H) = \frac{1}{k+g}\left(\sum_{i=1}^{k}\int_0^1 s_i(t)dt + \sum_{j=1}^{g}\int_0^1 s'_j(t)dt\right).$$

  Moreover,

$$\frac{1}{k+g}\left(\sum_{i=1}^{k}\int_0^1 \min_{L_i} H_t dt + \sum_{j=1}^{g}\int_0^1 \min_{\alpha_j} H_t dt\right) \leqslant \mu_{\underline{L}}(H) \leqslant \frac{1}{k+g}\left(\sum_{i=1}^{k}\int_0^1 \max_{L_i} H_t dt + \sum_{j=1}^{g}\int_0^1 \max_{\alpha_j} H_t dt\right).$$

- *(Homotopy invariance) $\mu_{\underline{L}}$ descends to a map* $\mathrm{Ham}(\Sigma) \to \mathbb{R}$.
- *(Support control) If* $\mathrm{supp}(\varphi) \subset \Sigma \setminus \underline{L}$*, then* $\mu_{\underline{L}}(\varphi) = -\mathrm{Cal}(\varphi)$.
- *(Conjugacy invariance)* $\mu_{\underline{L}}(\psi\varphi\psi^{-1}) = \mu_{\underline{L}}(\varphi)$.

Proof. These are all straightforward consequences of the properties of $c_{\underline{L}}$ (Proposition 13) and the definition of $\mu_{\underline{L}}$. ∎

The following might be of independent interest but we will not use it in the rest of the paper.

Theorem 20. *Suppose $\underline{L}$ and $\underline{L}'$ are two admissible η-monotone links with the same number of components $k+g$, that share the same non-contractible components $\underline{\alpha}$. Suppose k is even. Then, the homogenized spectral invariants $\mu_{\underline{L}}$ and $\mu_{\underline{L}'}$ coincide.*

Proof. Let $*$ denote the pants product

$$HF^*(\underline{L}, \underline{L}') \otimes HF^*(\underline{L}', \underline{L}) \to HF^*(\underline{L}, \underline{L}).$$

Using Theorem 6, we can view it as a map

$$HF^*(\underline{L}_0, \underline{L}'_0) \otimes HF^*(\underline{\alpha}, \underline{\alpha}) \otimes HF^*(\underline{L}'_0, \underline{L}_0) \otimes HF^*(\underline{\alpha}, \underline{\alpha}) \to HF^*(\underline{L}_0, \underline{L}_0) \otimes HF^*(\underline{\alpha}, \underline{\alpha}).$$

Recall that $\underline{L}_0$ and $\underline{L}_0'$ are two η-monotone links with k components in S^2. We want to first show that $\underline{L}_0$ is Floer theoretically isomorphic to $\underline{L}_0'$ (i.e., there exist classes $a_0 \in HF^*(\underline{L}_0, \underline{L}_0')$ and $b_0 \in HF^*(\underline{L}_0', \underline{L}_0)$ such that $a_0 * b_0 = e_{\mathrm{Sym}\,\underline{L}_0} \in HF^*(\underline{L}_0, \underline{L}_0)$ and $b_0 * a_0 = e_{\mathrm{Sym}\,\underline{L}_0'} \in HF^*(\underline{L}_0', \underline{L}_0')$). As explained in [8, Remark 6.10] (see also [8, Section 7.2]), $CF(\underline{L}_0, \underline{L}_0)$ is the same as the Lagrangian Floer cochain complex for the monotone Lagrangian $\mathrm{Sym}(\underline{L}_0)$ in $\mathbb{CP}^k$ with respect to a constant multiple of the Fubini study form. Also, as explained in [8, Lemma 5.14] (see the second last paragraph of the proof), the potential functions of $\mathrm{Sym}(\underline{L}_0)$ and $\mathrm{Sym}(\underline{L}_0')$ are both equal to $\frac{1}{x_1,\ldots,x_k} + \sum_{i=1}^k x_i$ with respect to appropriate coordinates. Recall that the quantum multiplication $c_1 * : QH(\mathbb{CP}^k) \to QH(\mathbb{CP}^k)$ has $k+1$ simple eigenvalues given by $(k+1)e^{\frac{2\pi j}{k+1}}$ for $j = 0, \ldots, k$, which are the critial values of $\frac{1}{x_1,\ldots,x_k} + \sum_{i=1}^k x_i$ (cf. [27, Proposition 1.2]). By [27, Corollary 1.12], both $\mathrm{Sym}(\underline{L}_0)$ and $\mathrm{Sym}(\underline{L}_0')$ split generate the monotone Fukaya category $\mathcal{F}(\mathbb{CP}^k)_{k+1}$ at the critical value $k+1$. To show that $\mathrm{Sym}(\underline{L}_0)$ is Floer theoretically isomorphic to $\mathrm{Sym}(\underline{L}_0')$, it suffices to show that they define isomorphic objects in $\mathcal{F}(\mathbb{CP}^k)_{k+1}$. By [7, Theorem 5.6], we know that the Fukaya algebra of $\mathrm{Sym}(\underline{L}_0)$ and $\mathrm{Sym}(\underline{L}_0')$ are both the Clifford algebra Cl_k, which is intrinsically formal [27, Corollary 6.4]. Since we assumed that k is even, $\mathcal{F}(\mathbb{CP}^k)_{k+1}$ is quasi-equivalent to the dg category of vector spaces $D^b(\Lambda)$ [27, Corollary 6.5]. An object in $D^b(\Lambda)$ is the same as a chain complex over Λ. Since Λ is a field, every chain complex is quasi-isomorphic to its homology. Therefore, every object is isomorphic to a direct sum of copies of graded shifts of Λ. Any object whose endomorphism algebra equals Λ is a graded shift of Λ. Since Cl_k is the endomorphism algebra of k copies of Λ [27, Lemma 6.1][2], $\mathrm{Sym}(\underline{L}_0)$ is a direct sum of k copies of Λ as objects in $D^b(\Lambda)$, and so is $\mathrm{Sym}(\underline{L}_0')$. It implies that $\mathrm{Sym}(\underline{L}_0)$ is isomorphic to $\mathrm{Sym}(\underline{L}_0')$. As a result, there exist classes $a_0 \in HF^*(\underline{L}_0, \underline{L}_0')$ and $b_0 \in HF^*(\underline{L}_0', \underline{L}_0)$ such that $a_0 * b_0 = e_{\mathrm{Sym}\,\underline{L}_0} \in HF^*(\underline{L}_0, \underline{L}_0)$.

Let a be the image of $a_0 \otimes e_{\underline{\alpha}}$ in $HF^*(\underline{L}, \underline{L}') \cong HF^*(\underline{L}_0, \underline{L}_0') \otimes HF^*(\underline{\alpha}, \underline{\alpha})$, and b the image of $b_0 \otimes e_{\underline{\alpha}}$ in $HF^*(\underline{L}', \underline{L}) \cong HF^*(\underline{L}_0', \underline{L}_0) \otimes HF^*(\underline{\alpha}, \underline{\alpha})$.

Then, $a * b$ is the image of $(a_0 * b_0) \otimes (e_{\underline{\alpha}} * e_{\underline{\alpha}}) = e_{\mathrm{Sym}\,\underline{L}_0} \otimes e_{\underline{\alpha}}$, i.e., $a * b = e_{\mathrm{Sym}\,\underline{L}}$ is the unit of $HF^*(\underline{L}, \underline{L})$.

Then, by the subadditivity property of Lagrangian spectral invariants, we have for any Hamiltonian H:

$$\begin{aligned} &c(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}, e_{\mathrm{Sym}\,\underline{L}}, H) \\ &\quad \leqslant c(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}', a, H) + c(\mathrm{Sym}\,\underline{L}', \mathrm{Sym}\,\underline{L}, b, 0) \\ &\quad \leqslant c(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{L}', a, 0) + c(\mathrm{Sym}\,\underline{L}', \mathrm{Sym}\,\underline{L}', e_{\mathrm{Sym}\,\underline{L}'}, H) \\ &\qquad + c(\mathrm{Sym}\,\underline{L}', \mathrm{Sym}\,\underline{L}, b, 0), \end{aligned}$$

[2]The paper [27] works over $\mathbb{C}$ but we work over the Novikov field.

i.e.,

$$c_{\underline{L}}(H) \leqslant c_{\underline{L}'}(H) + \frac{1}{k+g}\big(c(\operatorname{Sym}\underline{L}, \operatorname{Sym}\underline{L}', a, 0) + c(\operatorname{Sym}\underline{L}', \operatorname{Sym}\underline{L}, b, 0)\big).$$

We get for all $n > 0$,

$$\frac{c_{\underline{L}}(H^{\#n})}{n} \leqslant \frac{c_{\underline{L}'}(H^{\#n})}{n} + \frac{c(\operatorname{Sym}\underline{L}, \operatorname{Sym}\underline{L}', a, 0) + c(\operatorname{Sym}\underline{L}', \operatorname{Sym}\underline{L}, b, 0)}{(k+g)n},$$

and therefore, $\mu_{\underline{L}}(H) \leqslant \mu_{\underline{L}'}(H)$. Swapping the roles of $\underline{L}$ and $\underline{L}'$, we get the other inequality, and finally, $\mu_{\underline{L}} = \mu_{\underline{L}'}$. ■

Remark 21. The assumption that k is even in Theorem 20 is unnecessary. The assumption is only used to show that $\underline{L}_0$ is Floer theoretically isomorphic to $\underline{L}'_0$. Indeed, by generalizing the argument in [21, Section 9], one can give a geometric proof that $\underline{L}_0$ is Floer theoretically isomorphic to $\underline{L}'_0$ but this is beyond the scope of the paper so we leave interested readers to investigate it.

Since the homogenized spectral invariants are conjugacy invariant, $\mu_{\underline{L}\cup\underline{\alpha}} = \mu_{\underline{L}'\cup\underline{\alpha}'}$ when $\underline{L}\cup\underline{\alpha}$, $\underline{L}'\cup\underline{\alpha}'$ are Hamiltonian isotopic, and therefore, we can write $\mu_{[\underline{L}],[\underline{\alpha}]} := \mu_{\underline{L}\cup\underline{\alpha}}$, where $[\underline{L}]$ (resp., $[\underline{\alpha}]$) is the class of $\underline{L}$ (resp., $\underline{\alpha}$).

We fix a decomposition of Σ as a connected sum $\Sigma = \Sigma_0 \# E_1 \# \cdots \# E_g$, where Σ_0 is a genus zero surface, and the E_i are copies of the 2-torus. Recall that we modify the symplectic form in a neighbourhood of the connected sum points so that Σ_0 has area 1.

Let $\beta^1_{j,\theta}$ be the circle $\{\theta\} \times S^1 \subset S^1 \times S^1 \cong E_j$, and $\beta^2_{j,\theta}$ the circle $S^1 \times \{\theta\} \subset E_j$.

For $\underline{\theta} = (\theta_1, \ldots, \theta_g) \in (S^1)^g$, and $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_g) \in \{1, 2\}^g$, let $\underline{\alpha}^{\underline{\varepsilon}}_{\underline{\theta}} := \beta^{\varepsilon_1}_{1,\theta_1} \cup \cdots \cup \beta^{\varepsilon_g}_{g,\theta_g}$. When the components of $\underline{\alpha}^{\underline{\varepsilon}}_{\underline{\theta}}$ do not intersect the connected sum regions, this defines a Lagrangian link on Σ.

Proposition 22. (1) *Let $\{\mu_i\}, i \in I$ be a family of homogenized spectral invariants of the form $\{\mu_{\underline{L}_i}\}$ associated to links whose respective monotonicity constant λ_i are pairwise distinct. Then, the family $\{\mu_i\}$ is linearly independent.*

(2) *For fixed $[\underline{L}]$, the $\{\mu_{[\underline{L}],[\underline{\alpha}^{\underline{\varepsilon}}_{\underline{\theta}}]}\}$ are linearly independent. See Figure 2.*

Proof. The proof of the first bullet is identical to that of [8, Theorem 7.7 (ii)].

As for the second bullet, let $E_{[\underline{L}]}$ be the vector space generated by the $\{\mu_{[\underline{L}],[\underline{\alpha}^{\underline{\varepsilon}}_{\underline{\theta}}]}\}$. For $1 \leqslant j \leqslant g$, let $E_{[\underline{L}],\beta^{\varepsilon}_{j,\theta}}$ be the subspace generated by the $\mu_{[\underline{L}],[\underline{\alpha}^{\underline{\varepsilon}}_{\underline{\theta}}]}$ that satisfy $\varepsilon_j = \varepsilon$ and $\theta_j = \theta$. We are going to show that for every $j = 1, \ldots, g$, we have

$$E_{[\underline{L}]} = \bigoplus_{\varepsilon,\theta} E_{[\underline{L}],\beta^{\varepsilon}_{j,\theta}}. \tag{3.2}$$

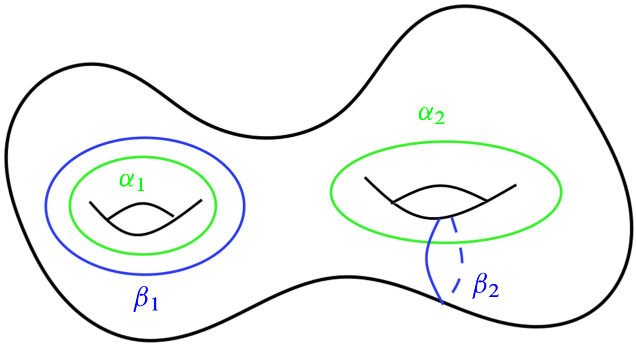

Figure 2. Two links $\underline{\alpha}$ and $\underline{\beta}$ inducing independent invariants $\mu_{\underline{\alpha}}$ and $\mu_{\underline{\beta}}$.

Let l and m be non-negative integers. Pick l different elements $\theta_1, \dots, \theta_l$ in S^1, and μ_i be an element of $E_{[\underline{L}],\beta^1_{j,\theta_i}}$. We can pick m different elements $\theta_{l+1}, \dots, \theta_{l+m}$ in S^1, and let μ_{l+i} be an element of $E_{[\underline{L}],\beta^2_{j,\theta_{l+i}}}$. Let a_i be real numbers such that

$$\sum_{i=1}^{l+m} a_i \mu_i = 0.$$

We want to show that for all i, $a_i = 0$.

Let V be a small neighbourhood of β^1_{j,θ_1} that does not intersect the connected sum points and the β^1_{j,θ_i} for $2 \leqslant i \leqslant l$. Let H be a Hamiltonian supported in V such that $H|_{\beta^1_{j,\theta_1}} \equiv 1$.

Let $\theta_0 \in S^1 - \{\theta_1, \dots, \theta_l\}$ be such that β^1_{j,θ_0} is away from the connected sum points. For $0 \leqslant i \leqslant l$, let ρ_i be the rotation of the torus defined by

$$\rho_i(\theta, \varphi) = (\theta + \theta_i - \theta_1, \varphi).$$

We can assume that V is small enough such that for any $i = 0, \dots, l$, $\rho_i(V)$ is a neighbourhood of β^1_{j,θ_i} that does not intersect the connected sum points and the β^1_{j,θ_s} for $s \neq i$. Let $H_i := H \circ \rho_i^{-1}$, which is supported in $\rho_i(V)$.

Then, by the Lagrangian control property, for $0 \leqslant i \leqslant l$ and $1 \leqslant p \leqslant l$, $\mu_p(H_i) = \frac{1}{k+g}\delta_{i,p}$. Moreover, for $l+1 \leqslant p \leqslant l+m$, since the ρ_i stabilize the β^2_{j,θ_p}, we get that $\mu_p(H_i)$ does not depend on i.

Therefore, applying the equality $\sum_{i=1}^{l+m} a_i \mu_i = 0$ at H_i for $1 \leqslant i \leqslant l$ gives

$$\frac{a_i}{k+g} = -\sum_{p=l+1}^{l+m} a_p \mu_p(H_1)$$

and at H_0

$$\sum_{p=l+1}^{l+m} a_p \mu_p(H_1) = 0.$$

Thus, $a_i = 0$ for $1 \leqslant i \leqslant l$. Since the β^1 and β^2 play symmetric roles, we can show in the same way that $a_i = 0$ for $l+1 \leqslant i \leqslant l+m$.

Therefore, we have (3.2) for all j, and hence, $\{\mu_{[\underline{L}],[\underline{\alpha}_\theta^\varepsilon]}\}$ are linearly independent. ∎

We conclude this section with the following lemma. It will not be used anywhere in the paper but might be of interest on its own. Let $\mu_{\mathrm{PFH},k}$ be the homogenized periodic Floer spectral invariant of period k (see [10, 11, 16]).

Lemma 23. *For any k, there is no constant C such that $\mu_{\mathrm{PFH},k}(H) \leqslant \mu_{\underline{L}^k}(H) + C$ for all H.*

Proof. By contradiction, suppose such C exists. By Chen [6], $\mu_{\mathrm{PFH},k}(H) \geqslant \mu_{\underline{L}^k}(H)$. We consider H supported in a small neighbourhood of a component of $\underline{L}^k$ such that $\mu_{\underline{L}^k}(H) > C$. Then, $\mu_{\mathrm{PFH},k}(H) > C$ by Chen. However, $\mu_{\mathrm{PFH},k}$ is conjugation invariant by any symplectomorphisms. By conjugating φ_H by a small flux translate such that its support is disjoint from $\underline{L}^k$, we get a contradiction. ∎

Remark 24. Lemma 23 uses the standing assumption that $g \geq 1$ essentially. If Σ were a sphere, then [6] proved that there is a $C > 0$ such that $\mu_{\mathrm{PFH},k}(H) \leqslant \mu_{\underline{L}^k}(H) + C$ for all H.

4. Extending the Calabi homomorphism and simplicity

The proofs of Theorems 2, 3, and 4 will be given in the following three subsections, respectively. The main idea is to replace the quasimorphism property (Theorem 1), which no longer exists for positive genus surfaces, by Theorem 5 and the fragmentation technique. Some of the estimates are a bit more delicate than the ones in [9].

4.1. Proof of Theorem 2

We can now give a proof of Theorem 2, inspired by the proof of (1) found in [9].

Definition 25. A sequence of admissible Lagrangian links $(\underline{L}^k)$ is called *equidistributed* if the following statements hold:

- all the $\underline{L}^k$ share the same non-contractible components $\alpha_1, \dots, \alpha_g$,
- $\underline{L}^k$ has k contractible components $L_1^k, \dots, L_k^k$,
- the L_i^k bound disjoint disks D_i^k, and $\mathrm{diam}(\underline{L}^k) := \max(\mathrm{diam}\, D_i^k) \xrightarrow[k\to\infty]{} 0$.

Given such a sequence, we get a sequence of link spectral invariants $c_{\underline{L}^k}$ which satisfies the Calabi property.

Proposition 26. *For any smooth Hamiltonian $H : S^1 \times \Sigma \to \mathbb{R}$,*

$$c_{\underline{L}^k}(H) = \int_{S^1} \int_{\Sigma} H_t \omega dt + \mathrm{O}_{k\to\infty}(\mathrm{diam}(\underline{L}^k)).$$

In particular, for any smooth Hamiltonian diffeomorphism φ,

$$c_{\underline{L}^k}(\varphi) = \mathrm{O}_{k\to\infty}(\mathrm{diam}(\underline{L}^k)).$$

Proof. We fix a point x_i^k in each of the disks D_i^k. Then, one can find smooth Hamiltonians G^k such that

- $G_t^k \equiv H_t(x_i^k)$ on D_i^k,
- $G_t^k = H_t$ on α_i,
- $\|G_t^k - H_t\|_\infty \leqslant \mathrm{diam}(\underline{L}^k) \sup_{S^1\times\Sigma} \|dH_t\|$.

Then, using the Hofer–Lipschitz property and Lagrangian control (Proposition 13), we get

$$\begin{aligned}
&\left| c_{\underline{L}^k}(H) - \int_{S^1}\int_{\Sigma} H_t \omega dt \right| \\
&\leqslant \left| c_{\underline{L}^k}(H) - c_{\underline{L}^k}(G^k) \right| + \left| c_{\underline{L}^k}(G^k) - \int_{S^1}\int_{\Sigma} G_t^k \omega dt \right| + \left| \int_{S^1}\int_{\Sigma} (G_t^k - H_t)\omega dt \right| \\
&\leqslant \|H - G^k\|_{\mathrm{Hof}} + \left| \frac{1}{k+g} \sum_{i=1}^{k} \int_{S^1} G_t^k(x_i^k) dt - \int_{S^1}\int_{\Sigma} G_t^k \omega dt \right| \\
&\quad + \frac{1}{k+g} \sum_{i=1}^{g} \int_{S^1} \max_{\alpha_i} |G_t^k| dt + \|G^k - H\|_{\mathrm{Hof}} \\
&\leqslant 2\,\mathrm{diam}(\underline{L}^k) \sup_{S^1\times\Sigma} \|dH_t\| + \left| \frac{1}{k+g} \sum_{i=1}^{k} \int_{S^1} G_t^k(x_i^k) dt - \int_{S^1}\int_{\Sigma} G_t^k \omega dt \right| + \mathrm{O}\left(\frac{1}{k}\right).
\end{aligned}$$

Let $A := \mathrm{Area}(D_i^k)$. We have $\frac{1}{k+1} \leqslant A \leqslant \frac{1}{k}$, so $\mathrm{diam}(\underline{L}^k) \geqslant \frac{C}{\sqrt{k}}$ for some positive constant C, and therefore, $\frac{1}{k} = \mathrm{O}_{k\to\infty}(\mathrm{diam}(\underline{L}^k))$. Moreover,

$$\begin{aligned}
&\left| \frac{1}{k+g} \sum_{i=1}^{k} \int_{S^1} G_t^k(x_i^k) dt - \int_{S^1}\int_{\Sigma} G_t^k \omega dt \right| \\
&= \left| \frac{1}{(k+g)A} \sum_{i=1}^{k} \int_{S^1}\int_{D_i^k} G_t^k \omega dt - \int_{S^1}\int_{\Sigma} G_t^k \omega dt \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \left(\frac{1}{(k+g)A} - 1 \right) \int_{S^1} \int_{\bigcup D_i^k} G_t^k \omega dt - \int_{S^1} \int_{\Sigma \setminus (\bigcup D_i^k)} G_t^k \omega dt \right| \\
&\leqslant \left| \frac{1}{(k+g)A} - 1 \right| \|G^k\|_\infty + \|G^k\|_\infty \int_{\Sigma \setminus (\bigcup D_i^k)} \omega \\
&\leqslant \left(\left| \frac{1}{(k+g)A} - 1 \right| + 1 - kA \right) (\|H\|_\infty + \|H - G^k\|_\infty) \\
&= \mathrm{O}\left(\frac{1}{k}\right).
\end{aligned}
$$

■

Since the invariants $c_{\underline{L}^k}$ satisfy all the properties listed in Proposition 13, the same proof as in [8, Proposition 3.4] shows the following proposition.

Proposition 27. *The map $f_{\underline{L}^k} := c_{\underline{L}^k} + \mathrm{Cal} : \mathrm{Ham}(\Sigma) \to \mathbb{R}$ is uniformly continuous with respect to the C^0 topology of* $\mathrm{Ham}(\Sigma)$*, and therefore, it extends continuously to* $\overline{\mathrm{Ham}}(\Sigma)$.

Now, we define the following relation on the space of real valued sequences $\mathbb{R}^{\mathbb{N}}$: we say that $x \sim y$ if $\lim x - y = 0$. This is an equivalence relation, and the quotient $\mathbb{R}^{\mathbb{N}} / \sim$ is a real vector space. Then, we can define a map

$$
\begin{aligned}
f : \overline{\mathrm{Ham}}(\Sigma) &\to \mathbb{R}^{\mathbb{N}} / \sim, \\
\varphi &\mapsto (f_{\underline{L}^1}(\varphi), f_{\underline{L}^2}(\varphi), \ldots).
\end{aligned}
$$

We claim that f is a group homomorphism, that is, for every φ, ψ in $\overline{\mathrm{Ham}}(\Sigma)$,

$$
f_{\underline{L}^k}(\varphi\psi) - f_{\underline{L}^k}(\varphi) - f_{\underline{L}^k}(\psi) \to 0.
$$

Since the spectral invariants satisfy the triangle inequality, and since Cal is a group homomorphism, we have the following inequality for every φ, ψ in $\mathrm{Ham}(\Sigma)$ and k in $\mathbb{N}$,

$$
f_{\underline{L}^k}(\varphi\psi) - f_{\underline{L}^k}(\varphi) - f_{\underline{L}^k}(\psi) \leqslant 0.
$$

This inequality still holds for the extension of $f_{\underline{L}^k}$ to $\overline{\mathrm{Ham}}(\Sigma)$. We also have, using the triangle inequality,

$$
\begin{aligned}
f_{\underline{L}^k}(\varphi\psi) - f_{\underline{L}^k}(\varphi) - f_{\underline{L}^k}(\psi) &\geqslant f_{\underline{L}^k}(\varphi\psi) - f_{\underline{L}^k}(\varphi) - f_{\underline{L}^k}(\varphi\psi) - f_{\underline{L}^k}(\varphi^{-1}) \\
&\geqslant -(f_{\underline{L}^k}(\varphi) + f_{\underline{L}^k}(\varphi^{-1})).
\end{aligned}
$$

Hence, the following property is enough to show that f is a group homomorphism.

Proposition 28. *For all $\varphi \in \overline{\mathrm{Ham}}(\Sigma)$, $\gamma_{\underline{L}^k}(\varphi) := f_{\underline{L}^k}(\varphi) + f_{\underline{L}^k}(\varphi^{-1})$ goes to zero when k goes to infinity.*

Proof. Fix $\varphi \in \overline{\mathrm{Ham}}(\Sigma)$. Using a standard fragmentation result (see, for instance, [1] or [26] for a more quantitative version), one can find $\varphi_1, \ldots, \varphi_n$ supported in disks $D_1, \ldots, D_n$ such that $\varphi = \varphi_1 \circ \cdots \circ \varphi_n$. For each $1 \leqslant i \leqslant n$, we pick a smooth ψ_i sending D_i to a disk that does not intersect $\underline{\alpha} := \alpha_1 \cup \cdots \cup, \alpha_g$.

Then, we have, using the triangle inequality,

$$\begin{aligned} 0 \leqslant \gamma_{\underline{L}^k}(\varphi) &\leqslant \sum \gamma_{\underline{L}^k}(\varphi_i) \\ &\leqslant \sum (\gamma_{\underline{L}^k}(\psi_i \varphi_i \psi_i^{-1}) + \gamma_{\underline{L}^k}(\psi_i) + \gamma_{\underline{L}^k}(\psi_i^{-1})) \\ &\leqslant \sum (\gamma_{\underline{L}^k}(\psi_i \varphi_i \psi_i^{-1}) + 2\gamma_{\underline{L}^k}(\psi_i)). \end{aligned}$$

Since for all i, $\psi_i \varphi_i \psi_i^{-1}$ is supported in a disk away from $\underline{\alpha}$, we can apply Theorem 5 and Remark 12 to get that $\gamma_{\underline{L}^k}(\psi_i \varphi_i \psi_i^{-1}) \leqslant \frac{k+1}{k+g}\lambda \leqslant \frac{k+1}{k(k+g)}$, and hence, this term goes to zero.

As for the other terms, since the ψ_i are smooth, the Calabi property (Proposition 26) implies that $\gamma_{\underline{L}^k}(\psi_i)$ goes to $\mathrm{Cal}(\psi_i) + \mathrm{Cal}(\psi_i^{-1}) = 0$. Thus, $\gamma_k(\varphi)$ goes to zero for any φ, and hence, f is a group homomorphism. ■

For φ smooth, we have $f(\varphi) = (\mathrm{Cal}(\varphi), \mathrm{Cal}(\varphi), \ldots)$ since $f_{\underline{L}^k}(\varphi)$ converges to $\mathrm{Cal}(\varphi)$. Let Δ denote the vector $(1, 1, 1, \ldots)$ in $\mathbb{R}^{\mathbb{N}}/\sim$. Using Zorn's lemma, we complete this vector into a base $(a_1 = \Delta, a_2, a_3, \ldots)$ of $\mathbb{R}^{\mathbb{N}}/\sim$. Now, let s be the following map:

$$\begin{aligned} \mathbb{R}^{\mathbb{N}}/\sim &\to \mathbb{R}, \\ \sum \lambda_i a_i &\mapsto \lambda_1. \end{aligned}$$

Then, $s \circ f$ is a group homomorphism from $\overline{\mathrm{Ham}}(\Sigma)$ to $\mathbb{R}$ that extends Cal. This completes the proof of Theorem 2.

4.2. Proof of Theorem 3

We now give a proof of Theorem 3. Once again, it is inspired by the proof of (2) found in [9].

We start by fixing an equidistributed sequence of links $\underline{L}^k = L_1^k \cup \cdots \cup L_k^k \cup \alpha_1 \cup \cdots \cup \alpha_g$ such that $\mathrm{diam}(\underline{L}^k) = O(\frac{1}{\sqrt{k}})$. We define

$$N((\underline{L}^k)_{k\in\mathbb{N}}) := \left\{\varphi \in \mathrm{Hameo}(\Sigma, \omega) \mid \sqrt{k}(f_{\underline{L}^k}(\varphi) - \mathrm{Cal}(\varphi)) \text{ is bounded}\right\}.$$

We claim the following proposition.

Proposition 29. *The set $N((\underline{L}^k)_{k\in\mathbb{N}})$ is a normal subgroup of* $\mathrm{Hameo}(\Sigma, \omega)$.

Proof. Let $\varphi, \psi \in N((\underline{L}^k)_{k\in\mathbb{N}})$, and $\theta \in \mathrm{Hameo}(\Sigma)$. Then, by the triangle inequality and the fact that Cal is a homomorphism,

$$-\sqrt{k}\gamma_{\underline{L}^k}(\varphi) \leqslant \sqrt{k}(f_{\underline{L}^k}(\varphi\psi) - \mathrm{Cal}(\varphi\psi)) - \sqrt{k}(f_{\underline{L}^k}(\varphi) - \mathrm{Cal}(\varphi)) \\ - \sqrt{k}(f_{\underline{L}^k}(\psi) - \mathrm{Cal}(\psi)) \leqslant 0.$$

Moreover, we have that

$$\sqrt{k}(f_{\underline{L}^k}(\varphi^{-1}) - \mathrm{Cal}(\varphi^{-1})) = \sqrt{k}\gamma_{\underline{L}^k}(\varphi) - \sqrt{k}(f_{\underline{L}^k}(\varphi) - \mathrm{Cal}(\varphi)),$$

and the triangle inequality also implies that

$$-\sqrt{k}\gamma_{\underline{L}^k}(\theta) \leqslant \sqrt{k}(f_{\underline{L}^k}(\varphi) - f_{\underline{L}^k}(\theta\varphi\theta^{-1})) \leqslant \sqrt{k}\gamma_{\underline{L}^k}(\theta).$$

Therefore, the following lemma proves $\varphi\psi$, φ^{-1}, and $\theta\varphi\theta^{-1}$ are in $N((\underline{L}^k)_{k\in\mathbb{N}})$, which concludes the proof of the proposition. ■

Lemma 30. *For all $\varphi \in \overline{\mathrm{Ham}}(\Sigma)$, $(\sqrt{k}\gamma_{\underline{L}^k}(\varphi))$ is bounded.*

Proof. Writing $\varphi = \varphi_1, \dots, \varphi_N$, where each φ_i is supported in a disk D_i, and choosing some ψ_i displacing D_i away from $\underline{\alpha}$, we get as in the proof of Proposition 28 that

$$\begin{aligned}\sqrt{k}\gamma_{\underline{L}^k}(\varphi) &\leqslant \sqrt{k}\sum_{i=1}^{N}(\gamma_{\underline{L}^k}(\psi_i\varphi_i\psi_i^{1}) + 2\gamma_{\underline{L}^k}(\psi_i)) \\ &\leqslant \sqrt{k}\sum_{i=1}^{N}\left(\frac{k+1}{k(k+g)} + \mathrm{O}\left(\frac{1}{\sqrt{k}}\right)\right) \\ &\leqslant \frac{k+1}{\sqrt{k}(k+g)}N + \mathrm{O}(1),\end{aligned}$$

where we used the Calabi property (Proposition 26), with $\mathrm{diam}(\underline{L}^k) = \mathrm{O}(\frac{1}{\sqrt{k}})$. This shows that $\sqrt{k}\gamma_{\underline{L}^k}(\varphi)$ is bounded for every φ. ■

Remark 31. We see in this proof that the terms $\gamma_{\underline{L}^k}(\psi_i)$ are of higher order than the other ones. If we manage to show that for smooth elements, $k\gamma_{\underline{L}^k}$ is bounded, then it would be the case for all $\varphi \in \overline{\mathrm{Ham}}(\Sigma)$, and we could define an even smaller normal subgroup by considering $k\gamma_{\underline{L}^k}$ instead of $\sqrt{k}\gamma_{\underline{L}^k}$.

It remains to show that for a certain choice of $(\underline{L}^k)$, this subgroup is proper. The proof is similar to the one in [9] in the case of the disk. There are three steps.

- We show that $N((\underline{L}^k)_{k\in\mathbb{N}}) \cap \mathrm{Ker}(\mathrm{Cal})$ contains all the smooth elements.
- We construct a hameomorphism T, and choose an equidistributed sequence $(\underline{L}^k)$ such that T does not belong to $N((\underline{L}^k)_{k\in\mathbb{N}})$.

- From this T we can construct another hameomorphism with the same property that lies in Ker(Cal).

Lemma 32. *The subgroup* $N((\underline{L}^k)_{k\in\mathbb{N}})\cap \mathrm{Ker(Cal)}$ *contains all the smooth elements.*

Proof. It is a corollary of the Calabi property (Proposition 26). ∎

Now, we construct the hameomorphism and the sequence of links. Fix g disjoint and homologically independent circles $\alpha_1, \dots, \alpha_g$. Let D be a disk of area $1/2$ in Σ away from $\underline{\alpha}$, and pick a point z_0 in its interior. We fix a symplectomorphism $\Phi : (D \setminus \{z_0\}, \omega) \xrightarrow{\sim} (S^1 \times (0, \frac{1}{\sqrt{2\pi}}], r dr \wedge d\theta)$.

We define an autonomous Hamiltonian H on $\Sigma \setminus \{z_0\}$ (not on Σ!) as follows:

- $H = 0$ outside $\Phi^{-1}(S^1 \times (0, \frac{1}{2\sqrt{\pi}}]) \subset \Sigma \setminus \{z_0\}$;
- $H(\theta, r) = h(\pi r^2)$ is radial;
- $h : (0, \frac{1}{4}] \to [0, +\infty)$ is decreasing, $h(r) \leqslant r^{-a}$ with equality on $(0, \frac{1}{8}]$ for some $\frac{1}{2} + \frac{1}{2\sqrt{2}} < a < 1$.

Then, φ_H^1 defines a Hamiltonian diffeomorphism on $\Sigma \setminus \{z_0\}$ which acts as a rotation around the origin inside $D \setminus \{z_0\}$. Therefore, it extends continuously to a homeomorphism T that fixes z_0. We claim the following proposition.

Proposition 33. *We have* $T \in \mathrm{Hameo}(\Sigma)$.

Proof. We have to find a sequence of Hamiltonians (K_n), supported in a compact subset of the interior of Σ such that

- $\phi_{K_n}^1 \xrightarrow{C^0} T$;
- $(\phi_{K_n}^t)$ is Cauchy for the C^0 distance, uniformly in $t \in [0, 1]$;
- the sequence (K_n) is Cauchy for the Hofer norm.

  Let $D_n := \{z_0\} \cup \Phi^{-1}(S^1 \times (0, \frac{1}{\sqrt{\pi 2^{n/a}}})) \subset \Sigma$. It has area $\frac{1}{2^{n/a}}$.

  We start with a sequence of smooth Hamiltonians (H_n) such that

- H_n coincides with H outside of D_n;
- $H_n \approx 2^n$ in D_n;
- $\|H_{n+1} - H_n\|_{\mathrm{Hof}} \leqslant 2^n$.

To construct such a sequence, we flatten H inside D_n.

Since H_n coincides with H outside of D_n, we have that $\phi_{H_n}^1 \circ T^{-1} = \mathrm{Id}$ outside of D_n, and therefore, $\phi_{H_n}^1 \xrightarrow{C^0} T$.

We will now construct a sequence (K_n) such that $\phi_{K_n}^1 = \phi_{H_n}^1$, (K_n) is Cauchy for the Hofer norm, and $(\phi_{K_n}^t)$ is Cauchy for the C^0 distance uniformly in t.

We will use a lemma from [9, Lemma 4.5].

Lemma 34. *Let Δ be a Euclidean 2-disk equipped with an area form ω of total area A. Suppose $D \subset \Delta$ is diffeomorphic to D^2 and that* $\mathrm{Area}(D) < \frac{A}{N}$ *some integer $N > 0$. Let F be a smooth Hamiltonian supported in the interior of D. Then, we have*

$$d_{\mathrm{H}}(\phi_F^1, \mathrm{Id}) \leqslant \frac{\|F\|_{\mathrm{Hof}}}{N} + 2A,$$

where d_H denotes the Hofer distance on $\mathrm{Ham}_c(\Delta, \omega)$.

Let b be a real number such that $1 < b < \frac{1}{a}$. Let $N = 2^{\lfloor bn \rfloor}$, and $A_n = (N + 1)2^{-\frac{n}{a}}$. If $n \geqslant n_0$, where n_0 is large enough, $A < \frac{1}{2}$, and we can define $\Delta_n := \{z_0\} \cup \Phi^{-1}(S^1 \times (0, \frac{1}{\sqrt{\pi A_n}}))$. It is a disk of area A_n. $H_{n+1} - H_n$ is supported inside D_n, which has area $2^{-\frac{n}{a}} < \frac{A_n}{N}$, so we can apply the lemma and get that

$$d_{\mathrm{H}}(\phi^1_{H_{n+1}-H_n}, \mathrm{Id}) \leqslant \frac{\|H_{n+1} - H_n\|_{\mathrm{Hof}}}{N} + 2A_n.$$

Therefore, there exists G_n supported in Δ_n such that $\phi^1_{G_n} = \phi^1_{H_{n+1}-H_n} = \phi^{-1}_{H_n} \circ \phi^1_{H_{n+1}}$ and $\|G_n\|_{\mathrm{Hof}} \leqslant 2^{n-\lceil(1-b)n\rceil} + (2^{\lfloor bn \rfloor} + 1)2^{-\frac{a}{n}}$.

By definition of b, the series $\sum_{n=n_0}^{\infty} \|G_n\|_{\mathrm{Hof}}$ is summable. Since G_n is supported inside Δ_n, $d_{C^0}(\phi^t_{G_n}, \mathrm{Id}) \leqslant \mathrm{diam}\, \Delta_n = \mathrm{O}(2^{(b-\frac{1}{a})n})$, so $\sum_{n=n_0}^{\infty} d_{C^0}(\phi^t_{G_n}, \mathrm{Id})$ is also summable, uniformly in t.

Then, we define (K_n) recursively by

- $K_n = H_n$ for $n \leqslant n_0$;
- $K_{n+1} = K_n \# G_n$ for $n \geqslant n_0$.

We get that $\phi^1_{K_n} = \phi^1_{H_n}$ for $n \leqslant n_0$, and for $n > n_0$,

$$\begin{aligned}\phi^1_{K_n} &= \phi^1_{H_{n_0}} \phi^1_{G_{n_0}} \cdots \phi^1_{G_{n-1}} \\ &= \phi^1_{H_{n_0}} \phi^{-1}_{H_{n_0}} \phi^1_{H_{n_0+1}} \cdots \phi^{-1}_{H_{n-1}} \phi^1_{H_n} \\ &= \phi^1_{H_n}.\end{aligned}$$

Moreover, the summability of $\sum_{n=n_0}^{\infty} \|G_n\|_{\mathrm{Hof}}$ and $\sum_{n=n_0}^{\infty} d_{C^0}(\phi^t_{G_n}, \mathrm{Id})$ implies that (K_n) is Cauchy for the Hofer norm, and $(\phi^t_{K_n})$ is Cauchy for the C^0 distance uniformly in t.

This concludes the proof that $T \in \mathrm{Hameo}(\Sigma)$. ■

We now construct an equidistributed sequence of admissible links as follows.

Fix an integer $k \geqslant 1$. For $0 \leqslant i \leqslant \lfloor \frac{\sqrt{k}}{2} \rfloor$, denote by A_i the annulus $S^1 \times (\frac{i}{\sqrt{\pi k}}, \frac{i+1}{\sqrt{\pi k}}) \subset D \setminus \{z_0\}$.

Let us assume that L_1^k is the circle $S^1 \times \left\{\sqrt{\frac{1}{\pi(k+1)}}\right\}$. It bounds a disk of area $\frac{1}{k+1}$. For $i \geqslant 1$, each annulus A_i has area $\frac{1}{k}((i+1)^2 - i^2) = \frac{2i+1}{k}$, hence, we can fit inside

A_i $2i+1$ disjoint circles $L^k_{i^2+1}, \dots, L^k_{(i+1)^2}$ that bound disjoint disks of area $\frac{1}{k+1}$ and of diameter bounded by $\frac{C}{\sqrt{k}}$, where C is a constant that does not depend on k.

The union of all the annuli covers a disk of area $\frac{(\lfloor\frac{\sqrt{k}}{2}\rfloor+1)^2}{k}$. The remaining area in Σ is $\frac{k-(\lfloor\frac{\sqrt{k}}{2}\rfloor+1)^2}{k}$ which is enough to fit $k-(\lfloor\frac{\sqrt{k}}{2}\rfloor+1)^2$ disjoint circles $L^k_{(\lfloor\frac{\sqrt{k}}{2}\rfloor+1)^2+1}$, $\dots, L^k_k$ that bound disjoint disks of area $\frac{1}{k+1}$ and of diameter bounded by $\frac{C'}{\sqrt{k}}$, where C' is a constant that does not depend on k.

Let $\underline{L}^k := L^k_1 \cup \dots \cup L^k_k \cup \alpha_1 \cup \dots \cup \alpha_g$ (see Figure 3). Then, $(\underline{L}^k)$ is an equidistributed sequence of monotone Lagrangian links, and moreover, we have the following statement.

Proposition 35. *For this choice of equidistributed sequence, $T \notin N((\underline{L}^k)_{k\in\mathbb{N}})$.*

Proof. We want to show that $\sqrt{k}(f_{\underline{L}^k}(T) - \mathrm{Cal}(T))$ is unbounded.

First, we observe that

$$\mathrm{Cal}(T) = \lim_{n\to\infty} \mathrm{Cal}(\phi^1_{K_n}) = \lim_{n\to\infty} \mathrm{Cal}(\phi^1_{H_n}) = \lim_{n\to\infty} \int_\Sigma H_n\omega = \int_\Sigma H\omega$$

and

$$f_{\underline{L}^k}(T) = \lim_{n\to\infty} f_{\underline{L}^k}(\phi^1_{K_n}) = \lim_{n\to\infty} f_{\underline{L}^k}(\phi^1_{H_n}) = \lim_{n\to\infty} c_{\underline{L}^k}(H_n).$$

Since H_n coincides with H outside of D_n, for n sufficiently large, H_n coincides with H on $\underline{L}^k$, and therefore, $c_{\underline{L}^k}(H_n) = c_k(H)$. Thus, $f_{\underline{L}^k}(T) = c_{\underline{L}^k}(H)$.

We start by estimating $c_{\underline{L}^k}(H)$. Since H is supported inside $S^1 \times (0, \frac{1}{2\sqrt{\pi}}]$, and by Lagrangian control, we have

$$\begin{aligned}
c_{\underline{L}^k}(H) &\leqslant \frac{1}{k+g} \sum_{i=1}^{(\lfloor\frac{\sqrt{k}}{2}\rfloor+1)^2} \max_{L^k_i} H \\
&\leqslant \frac{1}{k}\left(\max_{L^k_1} H + \sum_{i=1}^{\lfloor\frac{\sqrt{k}}{2}\rfloor} (2i+1)\max_{A_i} H\right) \\
&\leqslant \frac{1}{k}\left(h\left(\frac{1}{k+1}\right) + \sum_{i=1}^{\lfloor\frac{\sqrt{k}}{2}\rfloor} (2i+1)h\left(\frac{i^2}{k}\right)\right) \\
&\leqslant \frac{1}{k}\left(h\left(\frac{1}{k+1}\right) + 3h\left(\frac{1}{k}\right) + 2\sum_{i=2}^{\lfloor\frac{\sqrt{k}}{2}\rfloor} h\left(\frac{i^2}{k}\right) + \sum_{i=2}^{\lfloor\frac{\sqrt{k}}{2}\rfloor} (2i-1)h\left(\frac{i^2}{k}\right)\right).
\end{aligned}$$

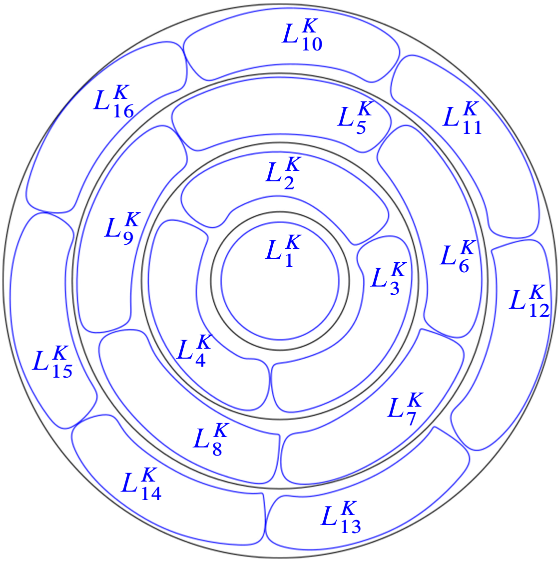

Figure 3. The sequence $\underline{L}^k$.

Using that h is decreasing, and comparing the sums with integrals, we get

$$\begin{aligned}
c_{\underline{L}^k}(H) &\leqslant \frac{(k+1)^a}{k} + 3k^{a-1} + \frac{2}{\sqrt{k}} \int_{\frac{1}{\sqrt{k}}}^{\frac{1}{2}} h(r^2)dr + \sum_{i=1}^{\lfloor \frac{\sqrt{k}}{2} \rfloor - 1} \frac{2i+1}{k} \min_{A_i} H \\
&\leqslant \left(\frac{k+1}{k}\right)^a k^{a-1} + 3k^{a-1} + \frac{2}{\sqrt{k}} \left[\frac{1}{1-2a} r^{1-2a}\right]_{\frac{1}{\sqrt{k}}}^{\frac{1}{2}} + \int_{\Sigma \setminus A_0} H\omega \\
&\leqslant \left(1 + \frac{a}{k}\right) k^{a-1} + 3k^{a-1} + \frac{2}{2a-1} k^{a-1} + \int_{\Sigma \setminus A_0} H\omega.
\end{aligned}$$

Therefore, for $k \geqslant 8$,

$$\begin{aligned}
\sqrt{k}(f_{\underline{L}^k}(T) - \mathrm{Cal}(T)) &= \sqrt{k}(c_{\underline{L}^k}(H) - \sqrt{k} \int_\Sigma H\omega) \\
&\leqslant \left(4 + \frac{a}{k} + \frac{2}{2a-1}\right) k^{a-\frac{1}{2}} - \sqrt{k} \int_{A_0} H\omega \\
&\leqslant \left(4 + \frac{a}{k} + \frac{2}{2a-1}\right) k^{a-\frac{1}{2}} - \sqrt{k} \int_0^{\frac{1}{k}} h(r)dr \\
&\leqslant \left(4 + \frac{a}{k} + \frac{2}{2a-1}\right) k^{a-\frac{1}{2}} - \sqrt{k} \frac{1}{1-a} k^{a-1} \\
&\leqslant \left(4 + \frac{a}{k} + \frac{2}{2a-1} - \frac{1}{1-a}\right) k^{a-\frac{1}{2}} \leqslant \left(\frac{-8a^2 + 8a - 1}{(2a-1)(1-a)} + \frac{a}{k}\right) k^{a-\frac{1}{2}}.
\end{aligned}$$

Since $\frac{1}{2}+\frac{1}{2\sqrt{2}} < a < 1$, $\frac{-8a^2+8a-1}{(2a-1)(1-a)} + \frac{a}{k} < 0$ for k large enough, and $\sqrt{k}(f_k(T) - \mathrm{Cal}(T))$ goes to $-\infty$. ■

In a similar fashion, we also compute a lower bound for $c_{\underline{L}^k}(H) - \int_\Sigma H\omega$ (when k is large enough) that we will need in the following section:

$$\begin{aligned}
&c_{\underline{L}^k}(H) - \int_\Sigma H\omega \\
&\quad \geqslant \frac{1}{k+g} \sum_{i=1}^{(\lfloor \frac{\sqrt{k}}{2} \rfloor + 1)^2} \min_{L_i^k} H - \int_\Sigma H\omega \\
&\quad \geqslant \frac{1}{k+g} \Bigg(\min_{L_1^k} H + \sum_{i=1}^{\lfloor \frac{\sqrt{k}}{2} \rfloor} (2i+1) \min_{A_i} H \Bigg) - \int_\Sigma H\omega \\
&\quad \geqslant \frac{1}{k+g} \Bigg(h\Big(\frac{1}{k+1}\Big) + \sum_{i=1}^{\lfloor \frac{\sqrt{k}}{2} \rfloor} (2i+1) h\Big(\frac{(i+1)^2}{k}\Big)\Bigg) - \int_\Sigma H\omega \\
&\quad \geqslant \frac{1}{k+g} \Bigg(h\Big(\frac{1}{k+1}\Big) - 3h\Big(\frac{1}{k}\Big) - 2\sum_{i=2}^{\lfloor \frac{\sqrt{k}}{2} \rfloor} h\Big(\frac{i^2}{k}\Big) + \sum_{i=1}^{\lfloor \frac{\sqrt{k}}{2} \rfloor} (2i+1) h\Big(\frac{i^2}{k}\Big)\Bigg) \\
&\qquad - \int_\Sigma H\omega \frac{k}{k+g} \Bigg(\frac{(k+1)^a}{k} - 3k^{a-1} - \frac{2}{\sqrt{k}} \int_{\frac{1}{\sqrt{k}}}^{\frac{1}{2}} h(r^2)dr \\
&\qquad\qquad + \sum_{i=1}^{\lfloor \frac{\sqrt{k}}{2} \rfloor - 1} \frac{2i+1}{k} \max_{A_i} H - \int_\Sigma H\omega \Bigg) - \frac{g}{k+g} \int_\Sigma H\omega.
\end{aligned}$$

Once again, we used that h is decreasing and compared the first sum with an integral. Doing the same for the second sum, we obtain

$$\begin{aligned}
&c_{\underline{L}^k}(H) - \int_\Sigma H\omega \\
&\geqslant \frac{1}{1+\frac{g}{k}} \Bigg(\Big(\frac{k+1}{k}\Big)^a k^{a-1} - 3k^{a-1} - \frac{2}{\sqrt{k}} \Big[\frac{1}{1-2a} r^{1-2a}\Big]_{\frac{1}{\sqrt{k}}}^{\frac{1}{2}} + \int_{\Sigma \setminus A_0} H\omega - \int_\Sigma H\omega \Bigg) \\
&\quad - \frac{g}{k} \int_0^{\frac{1}{4}} h(r)dr \\
&\geqslant \Big(1 - \frac{g}{k}\Big) \Bigg(\Big(1+\frac{1}{k}\Big)^a k^{a-1} - 3k^{a-1} - \frac{2}{2a-1} k^{a-1} - \int_{A_0} H\omega \Bigg) - \frac{g}{k} \int_0^{\frac{1}{4}} r^{-\alpha} dr
\end{aligned}$$

$$\geqslant \left(1-\frac{g}{k}\right)\left(\left(1-3-\frac{2}{2a-1}\right)k^{a-1}-\int_0^{\frac{1}{k}} h(r)dr\right)-\frac{g}{k}\left[\frac{r^{1-\alpha}}{1-\alpha}\right]_0^{\frac{1}{4}}$$
$$\geqslant \left(1-\frac{g}{k}\right)\left(\left(-2-\frac{2}{2a-1}\right)k^{a-1}-\frac{1}{1-a}k^{a-1}\right)-\frac{4^{\alpha-1}g}{(1-\alpha)k}$$
$$\geqslant \left(1-\frac{g}{k}\right)\left(-2-\frac{2}{2a-1}-\frac{1}{1-a}\right)k^{a-1}-\frac{4^{\alpha-1}g}{(1-\alpha)k}.$$

Now, it remains to construct a hameomorphism in Ker(Cal). Choose a smooth Hamiltonian diffeomorphism θ such that $\mathrm{Cal}(\theta)=\mathrm{Cal}(T)$. Then, $T' := T\theta^{-1} \in \mathrm{Hameo}(\Sigma)\cap\mathrm{Ker}(\mathrm{Cal})$. Since θ is smooth, by Lemma 32, $\theta \in N(\Sigma)$, and therefore, $T' \notin N((\underline{L}^k)_{k\in\mathbb{N}})$.

Hence, $N((\underline{L}^k)_{k\in\mathbb{N}}) \cap \mathrm{Ker}(\mathrm{Cal})$ is a proper normal subgroup of $\mathrm{Hameo}(\Sigma)\cap \mathrm{Ker}(\mathrm{Cal})$, which concludes the proof of Theorem 3.

We give more precision on the subgroups we defined.

Note that since the Hamiltonian H we constructed in this section is radial, we have $H^{\#n}=nH$, and therefore,

$$\mu_{\underline{L}^k}(H)=\lim_{n\to\infty}\frac{c_{\underline{L}^k}(H^{\#n})}{n}=c_{\underline{L}^k}(H).$$

Then, by Theorem 20, $c_{\underline{L}^k}(H)$ only depends on k, $\underline{\alpha}$ and the monotonicity constant of the link. Therefore, for any other equidistributed sequence of links $(\underline{L}'^k)$ having the same non-contractible components $\underline{\alpha}$, and the same monotonicity constant, we have that $T \notin N_{(\underline{L}'^k)}$.

Since we fixed an arbitrary $\underline{\alpha}$ at the start of the proof, and since we could modify the definition of $(\underline{L}^k)$ to change its monotonicity constant while keeping similar inequalities for $c_{\underline{L}^k}(H)$, we get the following proposition.

Proposition 36. *Let $(\underline{L}'^k)$ be an equidistributed sequence of admissible links, satisfying* $\mathrm{diam}(\underline{L}'^k)=\mathrm{O}(\frac{1}{\sqrt{k}})$. *Then, $N((\underline{L}'^k)_{k\in\mathbb{N}})$ is a proper normal subgroup of* $\mathrm{Hameo}(\Sigma)\cap\mathrm{Ker}(\mathrm{Cal})$.

Moreover, taking the intersection of those subgroups over all such sequences of links, we get an even smaller proper normal subgroup N, which contains all smooth elements.

Remark 37. We need the assumption on the diameter to ensure that $N((\underline{L}'^k)_{k\in\mathbb{N}})$ is a normal subgroup (Proposition 29).

4.3. Proof of Theorem 4

This time we consider a connected, closed, oriented surface (Σ,ω).

Fix an equidistributed sequence of admissible links $(\underline{L}^k)$ with

$$\mathrm{diam}(\underline{L}^k) = \mathrm{O}\Big(\frac{1}{\sqrt{k}}\Big).$$

Fix a such that $\frac{1}{2} + \frac{1}{2\sqrt{2}} < a < 1$. Then, for k_0 large enough,

$$\frac{1}{1-a} - 4 - \frac{a}{2^{2k_0}} - \frac{2}{2a-1} > 0.$$

Fix such a k_0, then for N large enough,

$$-2 - \frac{2}{2a-1} - \frac{1}{1-a} + 2^{N(1-a)}\Big(\frac{1}{1-a} - 4 - \frac{a}{2^{2k_0}} - \frac{2}{2a-1}\Big) > 0.$$

Fix such an integer, and define $g_k := c_{\underline{L}^{2^{2k}}} - c_{\underline{L}^{2^{2k-N}}}$.

Assume that

$$N((\underline{L}^k)_{k\in\mathbb{N}}) := \big\{\varphi \in \mathrm{Hameo}(\Sigma), (2^k g_k(\varphi)) \text{ is bounded}\big\}.$$

Remark 38. As in [9], the definition of the g_k's is more involved for closed surfaces than when there is boundary. This is because we want them to be C^0-continuous so that they extend to Homeo(Σ). When there is boundary, we can subtract the Calabi homomorphism to the spectral invariant, however, this homomorphism is not defined on closed surfaces, therefore, we need to take differences of spectral invariants, whose indices are chosen to make the computations work.

Proposition 39. *The set* $N((\underline{L}^k)_{k\in\mathbb{N}})$ *is a normal subgroup of* Hameo(Σ).

Proof. Let $\varphi, \psi \in N((\underline{L}^k)_{k\in\mathbb{N}})$, and $\theta \in \mathrm{Hameo}(\Sigma)$. Using the triangle inequality, we have

$$\gamma_{\underline{L}^{2^{2k}}}(\varphi) \leqslant g_k(\varphi\psi) - g_k(\varphi) - g_k(\psi) \leqslant \gamma_{\underline{L}^{2^{2k-N}}}(\varphi),$$
$$g_k(\varphi^{-1}) = \gamma_{\underline{L}^{2^{2k}}}(\varphi) - \gamma_{\underline{L}^{2^{2k-N}}}(\varphi) - g_k(\varphi)$$

and

$$-(\gamma_{\underline{L}^{2^{2k}}}(\theta) + \gamma_{\underline{L}^{2^{2k-N}}}(\theta)) \leqslant g_k(\theta\varphi\theta^{-1}) - g_k(\varphi) \leqslant \gamma_{\underline{L}^{2^{2k}}}(\theta) + \gamma_{\underline{L}^{2^{2k-N}}}(\theta).$$

Therefore, Lemma 30 implies that $\varphi\psi$, φ^{-1} and $\theta\varphi\theta^{-1}$ are in $N(\Sigma)$. ■

We claim that for a certain choice of link, $N((\underline{L}^k)_{k\in\mathbb{N}})$ is a proper subgroup of Hameo(Σ).

In fact, once again Proposition 26 shows that it contains all the smooth elements.

We define a hameomorphism T and a sequence of links $(\underline{L}^k)$ as in the previous section, with the parameter $\frac{1}{2} + \frac{1}{2\sqrt{2}} < a < 1$ we fixed earlier.

We claim that $T \notin N((\underline{L}^k)_{k\in\mathbb{N}})$.

Let k be a sufficiently large integer. Then, by the estimates of the previous section, we have

$$\begin{aligned}
&2^k g_k(T)\\
&= 2^k\big(c_{\underline{L}^{2^{2k}}}(H) - c_{\underline{L}^{2^{2k-N}}}(H)\big)\\
&= 2^k\bigg(\Big(c_{\underline{L}^{2^{2k}}}(H) - \int_\Sigma H\omega\Big) + \Big(\int_\Sigma H\omega - c_{\underline{L}^{2^{2k-N}}}(H)\Big)\bigg)\\
&\geqslant 2^k\bigg(\Big(1-\frac{g}{2^{2k}}\Big)\Big(-2-\frac{2}{2a-1}-\frac{1}{1-a}\Big)(2^{2k})^{a-1} - \frac{4^{\alpha-1}g}{(1-\alpha)2^{2k}}\bigg)\\
&\quad + 2^k\Big(\frac{1}{1-a}-4-\frac{a}{2^{2k-N}}-\frac{2}{2a-1}\Big)(2^{2k-N})^{a-1}\\
&\geqslant 2^{k(2a-1)}\bigg(-2-\frac{2}{2a-1}-\frac{1}{1-a}+2^{N(1-a)}\Big(\frac{1}{1-a}-4-\frac{a}{2^{2k-N}}-\frac{2}{2a-1}\Big)\bigg)\\
&\quad - 2^{k(2a-3)}g\Big(-2-\frac{2}{2a-1}-\frac{1}{1-a}\Big) - 2^{-k}\frac{4^{\alpha-1}g}{1-\alpha}.
\end{aligned}$$

By definition of N, for $k \geqslant k_0 + \frac{N}{2}$,

$$-2-\frac{2}{2a-1}-\frac{1}{1-a}+2^{N(1-a)}\Big(\frac{1}{1-a}-4-\frac{a}{2^{2k-N}}-\frac{2}{2a-1}\Big) > 0,$$

and therefore, $2^k g_k(T)$ goes to infinity, which concludes the proof of Theorem 4.

Using the same argument as in the previous section, we also get the following proposition.

Proposition 40. *Let $(\underline{L}'^k)$ be an equidistributed sequence of admissible links satisfying* $\operatorname{diam}(\underline{L}'^k) = O(\frac{1}{\sqrt{k}})$. *Then, $N((\underline{L}'^k)_{k\in\mathbb{N}})$ is a proper normal subgroup of* $\operatorname{Hameo}(\Sigma)$. *Moreover, taking the intersection of those subgroups over all such sequences of links, we get an even smaller proper normal subgroup N, which contains all smooth elements.*

5. The Künneth formula for connected sums

This section is devoted to the proof of Theorem 6.

5.1. Heegaard–Floer homology

Let us start by discussing how we define Heegaard Floer homology. Indeed, we decided to use the original construction, which computes Lagrangian Floer homology in the symmetric product. Alternatively, one could work in a cylindrical setting and define

Heegaard–Floer homology by counting pseudo-holomorphic curves in the 4-manifold $\Sigma \times [0, 1] \times \mathbb{R}$. We believe that this cylindrical reformulation (formulated by Lipshitz in [18]) could be used to prove the statements of this paper since similar results are proved in [18] and [22] in the cylindrical setting. Indeed, a cylindrical reformulation of the Lagrangian link spectral invariants is considered in [5, 6] so it is likely that the cylindrical approach together with our arguments in the previous sections can be combined to obtain Theorems 2, 3, and 4. However, since our main results are inspired by [8, 9, 21], which are all using the symmetric product setting, we will do the same. This setting is the following.

Consider a closed symplectic surface (Σ, ω), with a compatible complex structure j. Let $\underline{L} = L_1 \cup \cdots \cup L_k$ be a Lagrangian link in Σ. Denote by $\operatorname{Sym} \underline{L}$ the image of $L_1 \times \cdots \times L_k$ in the symmetric product $\operatorname{Sym}^k(\Sigma) := \Sigma^k/\mathfrak{S}_k$.

Denote by π the projection $\Sigma^k \to \operatorname{Sym}^k(\Sigma)$. The symmetric product is naturally endowed with a singular symplectic form $\operatorname{Sym}(\omega) := \pi_*(\omega^{\oplus k})$, which is smooth away from the diagonal $\Delta := \pi(\{(x_1, \ldots, x_k), \exists i \neq j, x_i = x_j\})$. It is also endowed with a complex structure $\operatorname{Sym}^k(j) := \pi_* j^{\oplus k}$ making it a smooth complex manifold.

Since the circles composing $\underline{L}$ are disjoint, $\operatorname{Sym} \underline{L}$ does not intersect Δ. Let V be a neighbourhood of Δ that does not intersect $\operatorname{Sym} \underline{L}$. By the result in [24, Section 7], one can find a smooth symplectic form ω_V on $\operatorname{Sym}^k(\Sigma)$ which agrees with $\operatorname{Sym}(\omega)$ away from V, and is compatible with $\operatorname{Sym}^k(j)$. Then, $\operatorname{Sym} \underline{L}$ is a Lagrangian submanifold inside $(\operatorname{Sym}^k(\Sigma), \omega_V)$.

Let $(\underline{L}, \underline{K})$ be a pair of Lagrangian links with k components. Let $H : S^1 \times \Sigma \to \mathbb{R}$ be a smooth Hamiltonian. Then, one can define a Hamiltonian $\operatorname{Sym}^k(H)$ on $\operatorname{Sym}^k(\Sigma)$ by the formula $\operatorname{Sym}^k(H)_t(\{x_1, \ldots, x_k\}) := H_t(x_1) + \cdots + H_t(x_k)$.

Let V be a neighbourhood of Δ that does not intersect $\operatorname{Sym}(\varphi_H^t(\underline{L}))$, $\operatorname{Sym}(\varphi_H^t(\underline{K}))$ for $t \in [-1, 1]$.

Definition 41. An almost complex structures J over $\operatorname{Sym}^k(\Sigma)$ is V-nearly symmetric if it agrees with $\operatorname{Sym}^k(j)$ over V, and tames $\operatorname{Sym}(\omega)$ outside of V.

If $\operatorname{Sym} \underline{L}$ and $\operatorname{Sym} \underline{K}$ are both monotone Lagrangian submanifolds that are Hamiltonian isotopic and $\operatorname{Sym}(\varphi_H^1(\underline{L})) \pitchfork \operatorname{Sym} \underline{K}$, then given a path of V-nearly symmetric almost complex structures $(J_t)_{t\in[0,1]}$, one can define the Lagrangian Floer cohomology

$$HF^*(\operatorname{Sym} \underline{L}, \operatorname{Sym} \underline{K}, \operatorname{Sym}^k(H))$$

in the standard way (cf. [8, Section 6]). One can show that this construction will not depend on the choice of V and J_t.

In order to clarify some notations, we will recall briefly how $HF^*(L, K, H)$ is constructed for two Hamiltonian isotopic monotone Lagrangians L and K inside a closed monotone symplectic manifold (M, ω), and a smooth Hamiltonian H.

We define the space $\mathcal{P}(L, K)$ of smooth paths $\gamma : [0, 1] \to M$ with $\gamma(0) \in L$ and $\gamma(1) \in K$.

Fix ζ in $\mathcal{P}(L, K)$, and let $\widetilde{P}_\zeta(L, K)$ be the universal cover of the connected component of ζ (with base point ζ).

Given a Hamiltonian H, we can define an action functional on $\widetilde{P}_\zeta(L, K)$, by

$$\mathcal{A}_H([\gamma, w]) := -\int w^*\omega + \int H(t, \gamma(t))dt.$$

Here, w is a homotopy from γ to ζ in $\mathcal{P}(L, K)$.

By the definition of $\widetilde{P}_\zeta(L, K)$, two cappings $[\gamma, w]$ and $[\gamma', w']$ agree in $\widetilde{P}_\zeta(L, K)$ if $\gamma = \gamma'$ and w and w' coincide in the set $\pi_2(\zeta, \gamma)$ of homotopy classes of cappings from ζ to γ with boundary in L and K.

Definition 42. Two cappings $[\gamma, w]$ and $[\gamma', w']$ are defined to be equivalent if $\gamma = \gamma'$ and w and w' have the same image in $\frac{\pi_2(\zeta,\gamma)}{\operatorname{Ker}\omega}$.

Let $CF^*_\circ(L, K, H; \zeta)$ be the $\mathbb{C}$-vector space generated by the equivalent classes of critical points of the action functional $\mathcal{A}_H$. It is naturally a $\mathbb{C}[T^{\pm 1}]$-module where T acts by adjoining the smallest positive area disk class in $\pi_2(\zeta, \zeta)$ to the capping. The Lagrangian Floer complex is

$$CF^*(L, K, H) := \bigoplus_{\zeta \in \pi_0(\mathcal{P}(L,K))} CF^*(L, K, H; \zeta),$$
$$CF^*(L, K, H; \zeta) := CF^*_\circ(L, K, H; \zeta) \otimes_{\mathbb{C}[T^{\pm 1}]} \mathbb{C}[[T]][T^{-1}].$$

One can also think of $CF^*(L, K, H)$ as a $\mathbb{C}[[T]][T^{-1}]$-vector space generated by the critical points of the circle-valued action functional on $\mathcal{P}(L, K)$ induced by $\mathcal{A}_H$. These critical points are trajectories of φ^t_H from L to K, and are in one-to-one correspondence with $\varphi^1_H(L) \cap K$. This complex is graded by the Maslov index, and the Novikov parameter carries a grading given by the minimal Maslov number of L.

To define the differential, we fix a path of ω-compatible almost complex structures J_t on M. Then, given two Hamiltonian trajectories γ and γ' from L to K, and a homotopy class β of Maslov index 1, we define the space $\widetilde{\mathcal{M}}_{J_t,\beta}(\gamma, \gamma')$ of smooth maps $u : \mathbb{R} \times [0, 1] \to M$ satisfying the following:

- $\frac{\partial u}{\partial s} + J_t(\frac{\partial u}{\partial t} - X_H(u)) = 0$ (where $(s, t) \in \mathbb{R} \times [0, 1]$);
- u has finite energy;
- $u(\mathbb{R}, 0) \subset L$ and $u(\mathbb{R}, 1) \subset K$;
- $[u] = \beta$;
- u is asymptotic to γ at $s = -\infty$, and to γ' at $s = +\infty$.

Denote by $\mathcal{M}_{J_t,\beta}(\gamma,\gamma')$ the quotient of this moduli space by the action of $\mathbb{R}$ by translation, and by $\pi_2(\gamma,\gamma')$ the set of homotopy classes of Floer trajectories between γ and γ'. The differential is then given by

$$\partial[\gamma', w] := \sum_{[\gamma, w\#\beta]\in \mathrm{Crit}(\mathcal{A}_H)} \#\mathcal{M}_{J_t,\beta}(\gamma,\gamma')[\gamma, w\#\beta].$$

5.2. Identification of the vector spaces

Let $\underline{L}$ and $\underline{K}$ be two transverse Lagrangian links with k components on a closed Riemann surface (Σ,ω,j). Let (E,ω_E,j_E) denote the two-dimensional torus with complex structure j_E and Kähler form ω_E. Let α be a non-contractible circle on E and α' be a small Hamiltonian deformation of α such that α and α' are transverse. Let $\sigma_1 \in \Sigma\setminus(\underline{L}\cup\underline{K})$, and σ_2 be a point in E away from the isotopy between α and α'.

We denote by $\Sigma\#_T E$ the connected sum of Σ and E along the points σ_1 and σ_2, which we construct in the following way.

Pick small real numbers r_1 and r_2, and fix conformal identifications $\Phi_1 : B_{r_1}(\sigma_1)\setminus\{\sigma_1\} \xrightarrow{\simeq} [0,\infty)\times S^1$ and $\Phi_2 : B_{r_2}(\sigma_2)\setminus\{\sigma_2\} \xrightarrow{\simeq} [0,\infty)\times S^1$ (where $B_r(z)$ denotes the closed ball of radius r centred at z). Let $\Sigma(2T) := \Sigma\setminus(\Phi_1^{-1}((2T,\infty)\times S^1)\cup\{\sigma_1\})$ and $E(2T) := E\setminus(\Phi_2^{-1}((2T,\infty)\times S^1)\cup\{\sigma_2\})$. Then, $\Sigma\#_T E$ is the union of $\Sigma(2T)$ and $E(2T)$ modulo the identification of the cylinders $[0,2T]\times S^1\subset\Sigma(2T)$ and $[0,2T]\times S^1\subset E(2T)$ via the involution $(t,\theta)\sim(2T-t,\theta)$. We denote the resulting complex structure on $\Sigma\#_T E$ by $j'(T)$, which agrees with j over $\Sigma\setminus B_{r_1}(\sigma_1)$, agrees with j_E over $E\setminus B_{r_2}(\sigma_2)$ and agrees with the standard complex structure over the tube $[0,2T]\times S^1$.

We assume the Hamiltonian isotopy from α to α' is small enough such that the area of its support is less than $\omega(B_{r_1}(\sigma_1))$. In this case, we can equip $\Sigma\#_T E$ with a symplectic form $\omega'(T)$ which agrees with ω over $\Sigma\setminus B_{r_1}(\sigma_1)$, agrees with ω_E over the support of the Hamiltonian isotopy from α to α', is compatible with $j'(T)$, and

$$\omega'(T)(\Sigma\#_T E) = \omega(\Sigma).$$

Let $W := \{\sigma_1\}\times\mathrm{Sym}^{k-1}(\Sigma)\subset\mathrm{Sym}^k(\Sigma)$. Let σ be a point that lies in the same connected component of $\Sigma\setminus(\underline{L}\cup\underline{K})$ as σ_1, but away from $B_{r_1}(\sigma_1)$.

For any $z\in\Sigma\setminus(\underline{L}\cup\underline{K})$ and $\varphi\in H_2(\mathrm{Sym}^k(\Sigma),\mathrm{Sym}(\underline{L})\cup\mathrm{Sym}(\underline{K}))$, we denote by $n_z(\varphi)$ the intersection number of φ with $\{z\}\times\mathrm{Sym}^{k-1}(\Sigma)\subset\mathrm{Sym}^k(\Sigma)$. Similarly, for $z'\in\Sigma\#_T E\setminus(\underline{L}\cup\underline{K}\cup\alpha\cup\alpha')$ and $\varphi'\in H_2(\mathrm{Sym}^{k+1}(\Sigma\#_T E),\mathrm{Sym}(\underline{L}\cup\alpha)\cup\mathrm{Sym}(\underline{K}\cup\alpha'))$, we denote by $n'_{z'}(\varphi')$ the intersection number of φ' with $\{z'\}\times\mathrm{Sym}^k(\Sigma\#_T E)\subset\mathrm{Sym}^{k+1}(\Sigma\#_T E)$. For $z_E\in E\setminus(\alpha\cup\alpha')$, and φ_E in $H_2(E,\alpha\cup\alpha')$, we denote by $n^E_{z_E}(\varphi_E)$ the intersection number of φ_E with z_E.

In order to prove Theorem 6, we start by establishing an isomorphism of vector spaces between the Floer complexes. We will show that there is a one-to-one correspondence between generators.

Given an intersection point $x = \{x_1, \dots, x_k\} \in \mathrm{Sym}^k(\Sigma)$ between $\mathrm{Sym}\,\underline{L}$ and $\mathrm{Sym}\,\underline{K}$, and an intersection point $c \in E$ between α and α', we get an intersection point $x \times \{c\} \in \mathrm{Sym}^k(\Sigma \setminus B_{r_1}(\sigma_1)) \times \mathrm{Sym}^1(E \setminus B_{r_2}(\sigma_2)) \subset \mathrm{Sym}^{k+1}(\Sigma \#_T E)$ between $\mathrm{Sym}(\underline{L} \cup \alpha)$ and $\mathrm{Sym}(\underline{K} \cup \alpha')$.

Since α and α' do not intersect $\underline{L}$ and $\underline{K}$, any intersection point between $\mathrm{Sym}(\underline{L} \cup \alpha)$ and $\mathrm{Sym}(\underline{K} \cup \alpha')$ can be decomposed in a single way as $x \times \{c\}$ where $x \in \mathrm{Sym}(\underline{L}) \cap \mathrm{Sym}(\underline{K})$ and $c \in \alpha \cap \alpha'$, and therefore, there is a one-to-one correspondence between $(\mathrm{Sym}(\underline{L}) \cap \mathrm{Sym}(\underline{K})) \times (\alpha \cap \alpha')$ and $\mathrm{Sym}(\underline{L} \cup \alpha) \cap \mathrm{Sym}(\underline{K} \cup \alpha')$.

Now, we need to consider the cappings. In fact, recall that a generator of the Floer complex is an equivalence class of an intersection point x together with a capping w (cf. Definition 42).

For each connected component of $\mathcal{P}(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K})$ that contains an intersection point of $\mathrm{Sym}\,\underline{L}$ and $\mathrm{Sym}\,\underline{K}$, we choose the reference path ζ for that connected component to be a constant path at one of the intersection points that are contained in that component. On E, since we assumed that α' is a Hamiltonian perturbation of α, we can assume that all intersection points between them are in the same connected component of $\mathcal{P}(\alpha, \alpha')$. We choose as a reference path ζ_E to be a constant path equal to an intersection point between α and α'. For every ζ chosen above, we define $\zeta' := \zeta \times \zeta_E$, and choose it as a reference path in $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$. We will compare the cappings with respect to these reference paths in Proposition 47. The following definition and lemmas are preparation for the proof of Proposition 47.

Definition 43. Let $\underline{L}$ and $\underline{K}$ be two Lagrangian links on Σ away from a point z. Let x be a path between $\mathrm{Sym}\,\underline{L}$ and $\mathrm{Sym}\,\underline{K}$. A class φ in $\pi_2(x, x)$ is said to be periodic if $n_z(\varphi) = 0$. The set of periodic classes will be denoted by $\Pi_x(z)$.

Then, we show the following lemma (which is a generalization of [21, Proposition 2.15], which corresponds to the case $k = g$).

Lemma 44. *Let (Σ, z) be a pointed surface. Let $\underline{L}$ and $\underline{K}$ be two Lagrangian links on Σ, away from z, with k components. Then, for any path x from* $\mathrm{Sym}\,\underline{L}$ *to* $\mathrm{Sym}\,\underline{K}$,

$$\pi_2(x, x) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus \Pi_x(z).$$

Proof. The set $\pi_2(x, x)$ is the fundamental group of $\mathcal{P}(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K})$ based at the point x. The evaluation at both ends of the path gives rise to a fibration

$$\Omega\,\mathrm{Sym}^k(\Sigma) \to \mathcal{P}(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K}) \to \mathrm{Sym}\,\underline{L} \times \mathrm{Sym}\,\underline{K}.$$

The corresponding long exact sequence gives

$$0 \to \pi_2(\mathrm{Sym}^k(\Sigma)) \to \pi_1(\mathcal{P}(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K})) \to \pi_1(\mathrm{Sym}\,\underline{L} \times \mathrm{Sym}\,\underline{K}) \xrightarrow{f} \pi_1(\mathrm{Sym}^k(\Sigma)).$$

One can rewrite it as a short exact sequence

$$0 \to \pi_2(\mathrm{Sym}^k(\Sigma)) \to \pi_2(x, x) \to \mathrm{Ker}(f) \to 0.$$

Since $k \geq g$, we have either $\pi_2(\mathrm{Sym}^k(\Sigma)) \cong \mathbb{Z}$ or $\pi_2(\mathrm{Sym}^k(\Sigma)) = 0$, and hence, $\pi_2(\mathrm{Sym}^k(\Sigma)) = 0$ happens only when $k = g = 1$ (cf. [3, Theorem 5.4]). When $\pi_2(\mathrm{Sym}^k(\Sigma)) \cong \mathbb{Z}$, the map $\pi_2(x, x) \xrightarrow{n_z} \mathbb{Z}$ gives a splitting of the short exact sequence, and $\pi_2(x, x) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus \mathrm{Ker}(f)$. When $\pi_2(\mathrm{Sym}^k(\Sigma)) = 0$, we also have $\pi_2(x, x) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus \mathrm{Ker}(f)$. Since n_z gives a splitting of the sequence, $\mathrm{Ker}(f)$ can be identified with $\mathrm{Ker}(n_z) = \Pi_x(z)$, which shows that

$$\pi_2(x, x) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus \Pi_x(z). \qquad \blacksquare$$

Lemma 45. *Given an intersection point $x = \{x_1, \ldots, x_k\} \in \mathrm{Sym}^k(\Sigma)$ between $\mathrm{Sym}\,\underline{L}$ and $\mathrm{Sym}\,\underline{K}$, all cappings are of the form $[x, nS\#w]$, where S is a generator of $\pi_2(\mathrm{Sym}^k(\Sigma))$ whose intersection number with W is $n_{\sigma_1}(S) = 1$, w is a capping from x to ζ that does not intersect W, and n is an integer.*

Proof. We can construct a representative of the generator S as follows. Let $p_1, \ldots, p_{k-2} \in \Sigma \setminus \{\sigma_1\}$. Let $\pi_\Sigma : \Sigma \to \mathbb{CP}^1$ be a topological 2-to-1 branched covering such that σ_1 is not a critical point. We can define a continuous map $u : \mathbb{CP}^1 \to \mathrm{Sym}^k(\Sigma)$ by $u(z) = (\pi_\Sigma^{-1}(z), p_1, \ldots, p_{k-2})$, where $\pi_\Sigma^{-1}(z)$ is understood to be two copies of the critical points when z is a critical value of π_Σ. Note that u intersects W exactly at the point $u(\pi_\Sigma(\sigma_1))$ and it can be checked that the intersection is transversal. Let $\bar{u}$ be the same as u but equip the domain with the opposite orientation. We have $n_{\sigma_1}(u) = 1$ and $n_{\sigma_1}(\bar{u}) = -1$.

Let w be a capping from ζ to x which intersects W transversely. For each intersection point q between w and W, we can choose $p_1, \ldots, p_{k-2}$ and π_Σ such that the corresponding u intersects W at the same point q. By applying a connected sum with u or $\bar{u}$ depending on the sign of the intersection point, we obtain another capping from ζ to x with one less intersection point with W. Continuing this procedure, we will obtain a capping w_x from ζ to x that does not intersect W.

Then, $\pi_2(x, \zeta) = w_x \# \pi_2(\zeta, \zeta)$. By the previous lemma, $\pi_2(\zeta, \zeta) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus \Pi_\zeta(\sigma_1)$. Therefore,

$$\pi_2(x, \zeta) \cong \pi_2(\mathrm{Sym}^k(\Sigma)) \oplus (w_x \# \Pi_\zeta(\sigma_1)),$$

and elements of $\Pi_\zeta(\sigma_1) \# w_x$ do not intersect W. $\blacksquare$

Given an intersection point $c \in E$ between α and α', since E is aspherical, all cappings from c to ζ_E do not intersect σ_2. Moreover, since α and α' are Hamiltonian isotopic to each other, any two cappings would have the same areas, and hence, descend to a unique equivalence class.

Lemma 46. *Given an intersection point $x \times \{c\}$ between* $\mathrm{Sym}(\underline{L} \cup \alpha)$ *and* $\mathrm{Sym}(\underline{K} \cup \alpha')$*, a capping w from x to ζ that does not intersect W, and w_E from c to ζ_E that does not intersect σ_2, $w \times w_E$ is a capping from $x \times \{c\}$ to ζ' such that $n'_\sigma(w \times w_E) = 0$. Moreover, all cappings from $x \times \{c\}$ to ζ' are of the form $[x \times \{c\}, nS'\#(w \times w_E)]$ for some integer n and cappings w and w_E as above, and where S' is a generator of* $\pi_2(\mathrm{Sym}^{k+1}(\Sigma\#_T E))$ *with $n'_\sigma(S') = 1$.*

Proof. The first part of the lemma is straightforward. The proof of the second part is identical to that of the previous lemma. ■

Proposition 47. *The linear map defined by*

$$\Phi : CF^*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}\,\underline{K}) \otimes CF^*(\alpha, \alpha') \to CF^*(\mathrm{Sym}(\underline{L} \cup \alpha), \mathrm{Sym}(\underline{K} \cup \alpha')),$$
$$[x, nS\#w] \otimes [c, w_E] \mapsto [x \times \{c\}, nS'\#(w \times w_E)]$$

is an isomorphism of vector spaces.

Proof. We already know there is a one-to-one correspondence between $(\mathrm{Sym}(\underline{L}) \cap \mathrm{Sym}(\underline{K})) \times (\alpha \cap \alpha')$ and $\mathrm{Sym}(\underline{L} \cup \alpha) \cap \mathrm{Sym}(\underline{K} \cup \alpha')$, and according to the previous lemma, the mapping $\Psi : \pi_2(x, \zeta) \times \pi_2(c, \zeta_E) \to \pi_2(x \times \{c\}, \zeta')$ is also a one-to-one correspondence. It remains to show that this mapping descends to a one-to-one correspondence between equivalence classes of cappings

$$\pi_2(x,\zeta)/\operatorname{Ker}(\mathrm{Sym}(\omega)) \times \pi_2(c,\zeta_E)/\operatorname{Ker}(\omega_E) \to \pi_2(x \times \{c\},\zeta')/\operatorname{Ker}(\mathrm{Sym}(\omega'(T))).$$

First, note that in the torus, $\pi_2(c,\zeta_E)/\operatorname{Ker}(\omega_E)$ is trivial (see the paragraph before Lemma 46).

Then, recall that $\omega'(T)$ is chosen such that it agrees with ω over $\Sigma \setminus B_{r_1}(\sigma_1)$, agrees with ω_E over the support of the Hamiltonian isotopy from α to α', is compatible with $j'(T)$ and $\omega'(T)(\Sigma\#_T E) = \omega(\Sigma)$. These conditions guarantee that for all w and w_E, we have $\mathrm{Sym}(\omega'(T))(\Psi(w, w_E)) = \mathrm{Sym}(\omega)(w) + \omega_E(w_E)$. It implies the result. ■

To show that it is an isomorphism of chain complexes, we need to show that it preserves the differential, i.e., that for all such x and c, $\partial(x \times \{c\}) = \Phi((\partial x) \otimes c + x \otimes (\partial c))$.[3]

[3]More precisely, we need to identify the differentials of the *capped* intersection points, but the identification of the cappings is straightforward so we focus on the intersection points.

In order to do this, we compare the moduli space $\mathcal{M}(x, y)$ of Maslov index 1 Floer trajectories from x to some y in $\mathrm{Sym}^k(\Sigma)$ to $\mathcal{M}(x \times \{c\}, y \times \{c\})$, and $\mathcal{M}(c, c')$ to $\mathcal{M}(x \times \{c\}, x \times \{c'\})$.

In fact, we will show that these moduli spaces can be identified when considering a complex structure on $\Sigma \#_T E = \Sigma \# E$ that stretches sufficiently the connected sum tube.

This is a generalization of a statement in [21] which only considers the case of links with g components (where g is the genus of Σ), and circles α and α' with a single intersection point. The proof of this statement still works in our setting. We will recall the main steps of this proof and emphasize where one has to be careful when generalizing.

Before discussing the moduli spaces of Floer trajectories, which are pseudo-holomorphic disks, we have to fix paths of almost complex structures on each manifold.

We fix a path of V-nearly symmetric almost complex structures $(J_t)_{t\in[0,1]}$ on $\mathrm{Sym}^k(\Sigma)$, for some neighbourhood V of $\Delta \cup \mathrm{Sym}^{k-1}(\Sigma) \times \{\sigma_1\} \subset \mathrm{Sym}^k(\Sigma)$.

Recall that $j'(T)$ is the complex structure on $\Sigma \#_T E$ that coincides with j on $\Sigma \setminus B_{r_1}(\sigma_1)$, with j_E on $E \setminus B_{r_2}(\sigma_2)$, and is the standard cylindrical complex structure on the connected sum tube $[0, 2T] \times S^1$ between Σ and E.

Then, the symmetric product $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$ endowed with the complex structure $\mathrm{Sym}^{k+1}(j'(T))$ admits an open subset holomorphically identified with

$$\mathrm{Sym}^k(\Sigma - B_{r_1}(\sigma_1)) \times \mathrm{Sym}^1(E - B_{r_2}(\sigma_2)).$$

Fix $R_1 > r_1$ and $R_2 > r_2$. We choose a path of almost complex structures $J'_t(T)$ on $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$ satisfying the following conditions:

- $J'_t(T) \equiv J_t \times j_E$ on $\mathrm{Sym}^k(\Sigma - B_{R_1}(\sigma_1)) \times \mathrm{Sym}^1(E - B_{R_2}(\sigma_2))$;
- $J'_t(T) = J_{t,r} \times j_E$ on $\mathrm{Sym}^k(\Sigma - B_{r_1}(\sigma_1)) \times \mathrm{Sym}^1(B_{R_2}(\sigma_2) - B_{r_2}(\sigma_2))$, where $J_{t,r}$ is V-nearly symmetric for all r and connects J_t to $\mathrm{Sym}^k(j)$ as r, the normal parameter to σ_2, goes from R_2 to r_2;
- $J'_t(T) \equiv \mathrm{Sym}^{k+1}(j'(T))$ on the rest of $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$.

In particular, $J'_t(T)$ is V'-nearly symmetric for some neighbourhood V' of the diagonal $\Delta' \subset \mathrm{Sym}^{k+1}(\Sigma \#_T E)$.

Let $x, y \in \mathrm{Sym}(\underline{L}) \cap \mathrm{Sym}(\underline{K})$, and $c, c' \in \alpha \cap \alpha'$.

Given $\varphi \in \pi_2(x, y)$, there is a single class $\varphi'_c \in \pi_2(x \times \{c\}, y \times \{c\})$ such that for any $z \in \Sigma \setminus (\underline{L} \cup \underline{K})$, $n_z(\varphi) = n'_z(\varphi'_c)$. Similarly, for any $\varphi_E \in \pi_2(c, c')$, there is a single class $\varphi'_{E,x} \in \pi_2(x \times \{c\}, x \times \{c'\})$ such that for any $z_E \in E \setminus (\alpha \cup \alpha')$,

$$n^E_{z_E}(\varphi_E) = n'_{z_E}(\varphi'_{E,x}).$$

Then, Theorem 6 is a consequence of the following statement.

Theorem 48. *Let $\varphi \in \pi_2(x, y)$ and $\varphi_E \in \pi_2(c, c')$ be two classes of Maslov index* 1. *Then, for sufficiently large T, $\mathcal{M}_{J_t,\varphi}(x, y) \simeq \mathcal{M}_{J'_t(T),\varphi'_c}(x \times \{c\}, y \times \{c\})$ and $\mathcal{M}_{j_E,\varphi_E}(c, c') \simeq \mathcal{M}_{J'_t(T),\varphi'_{E,x}}(x \times \{c\}, x \times \{c'\})$.*

Remark 49. The isomorphisms above are identifications between deformation theories, and therefore, $\mu(\varphi'_c) = \mu(\varphi) = 1$, and $\mu(\varphi'_{E,x}) = \mu(\varphi_E) = 1$.

The proof of this theorem consists of two steps.

- Given a pseudo-holomorphic disk in $\mathrm{Sym}^k(\Sigma)$, we construct a corresponding disk in $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$ by gluing spheres.
- By a Gromov compactness argument, we show that all Maslov index 1 pseudo-holomorphic disks in $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$ can be constructed in this way.

These two steps will be addressed in the next two subsections, respectively.

5.3. Gluing

Let u be a pseudo-holomorphic disk in $\mathcal{M}_{j_E,\varphi_E}(c, c')$. Then, u does not intersect σ_2, and therefore, $x \times u$ defines a trajectory in $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$ between $x \times \{c\}$ and $x \times \{c'\}$, which is $J'_t(T)$-holomorphic. Moreover, for any $z_E \in E \setminus (\alpha \cup \alpha')$, $n'_{z_E}(u) = n^E_{z_E}(\varphi_E)$, so $x \times u \in \mathcal{M}_{J'_t(T),\varphi'_{E,x}}(x \times \{c\}, x \times \{c'\})$.

Let u be a pseudo-holomorphic disk in $\mathcal{M}_{J_t,\varphi}(x, y)$. When $n_{\sigma_1}(\varphi) = 0$, we can construct $u' := u \times \{c\}$, and as before it lives in $\mathcal{M}_{J'_t(T),\varphi'_c}(x \times \{c\}, y \times \{c\})$.

However, when $n := n_{\sigma_1}(\varphi) \neq 0$, we need to glue n spheres to the disk u to construct a disk in $\mathrm{Sym}^{k+1}(\Sigma \#_T E)$. We follow the construction of [21], which was done in the case $k = g$, but still works in this more general case. We will only give an outline of the proof without going into the more technical details, which are exactly the same as in [21, Sections 10.2 and 10.3].

Suppose that u meets $W = \{\sigma_1\} \times \mathrm{Sym}^{k-1}(\Sigma)$ transversally in n distinct points $q_1, \ldots, q_n$. We identify $\mathbb{R} \times [0, 1]$ with $D \setminus \{\pm i\}$, where D denotes the unit disk in $\mathbb{C}$. Then, u extends continuously to D by setting $u(-i) = x, u(i) = y$.

We fix constants $0 < r_1 < R_1$ such that

$$u(D) \cap \left(B_{r_1}(\sigma_1) \times \mathrm{Sym}^{k-1}(\Sigma - B_{r_1}(\sigma_1))\right) \subset B_{r_1}(\sigma_1) \times \mathrm{Sym}^{k-1}(\Sigma - B_{R_1}(\sigma_1)).$$

There exists $\varepsilon > 0$ such that for every $1 \leqslant i \leqslant n$, $B_\varepsilon(q_i)$ is mapped by u into this subset.

We fix conformal identifications $B_{r_1}(\sigma_1) - \sigma_1 \cong [0, \infty) \times S^1$, and $B_\varepsilon(q_i) \cong [0, \infty) \times S^1$.

We will use Sobolev spaces with weight function $e^{\delta \tau_1}$, where

- δ is a positive constant;

- $\tau_1 : D - \{q_1, \dots, q_n\} \to [0, \infty)$ is supported inside the $B_\varepsilon(q_i)$;
- $\tau_1(s, \varphi) = s$ for $s \geqslant 1$ in each $B_\varepsilon(q_i) \cong [0, \infty) \times S^1$.

Then, for each i there exists $w_i \in \mathrm{Sym}^{k-1}(\Sigma)$, and $(t_i, \theta_i) \in \mathbb{R} \times S^1$ such that the restriction of u to $B_\varepsilon(q_i) - \{q_i\} \cong [0, \infty) \times S^1$ differs by a $L^p_{1,\delta}$ map from the smooth map

$$a_{t_i,\theta_i,w_i} : [0, \infty) \times S^1 \to \mathrm{Sym}^{k-1}(\Sigma) \times [0, \infty) \times S^1 \subset \mathrm{Sym}^k(\Sigma)$$

defined by

$$a_{t_i,\theta_i,w_i}(s, \varphi) = (w_i, s + t_i, \varphi + \theta_i),$$

where we cut off $s + t_i$ if it is negative ([21, Section 10.2]).

Given $T > 0$, we define $X_1(T) := \tau_1^{-1}([0, T])$ and $X_1(\infty) = D - \{q_1, \dots, q_n\}$.

Let $h : \mathbb{R} \to [0, 1]$ be a smooth, increasing function such that $h(t) = 0$ for $t < 0$ and $h(t) = 1$ for $t > 1$.

We can define a map $\tilde{u}_T : X_1(\infty) \to \mathrm{Sym}^k(\Sigma)$ which agrees with u away from the $B_\varepsilon(q_i)$, by

$$\tilde{u}_T(s, \varphi) = h(s - T)a_{t_i,\theta_i,w_i}(s, \varphi) + (1 - h(s - T))u(s, \varphi)$$

over $B_\varepsilon(q_i) \setminus \{q_i\} \cong [0, \infty) \times S^1$, and extends smoothly over q_i (where the convex combination is to be interpreted using the exponential map).

We also fix a constant $\delta_0 > 0$, and define $\tau_0 : \mathbb{R} \times [0, 1] \cong D \to \mathbb{R}$ supported away from the $B_\varepsilon(q_i)$, and such that $\tau_0(s, t) = |s|$ for sufficiently large s.[4]

Then, according to [21, Lemma 10.6], for the Sobolev norm with weight $e^{\delta_0\tau_0+\delta\tau_1}$, there are constants $\kappa > 0$, $S_0 > 0$ and $C > 0$ such that for all $S > S_0$

$$\|\bar{\partial}_{J_t}\tilde{u}_S\|_{L^p_{\delta,\delta_0}(\Lambda^{0,1}\tilde{u}_S)} \leqslant Ce^{-\kappa S}.$$

Now, we consider spheres in $\mathrm{Sym}^2(E)$. Let v be a holomorphic map from S^2 to $\mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E)$, constant on the first factor, and such that $n_{\sigma_2}([v]) = 1$. Denote by $\mathcal{M}(S^2 \to \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E))$ the moduli space of such maps, modulo holomorphic reparametrization. According to [21, Lemma 10.7], we have the following lemma.

Lemma 50. *For such a map v, there exists a unique pair (w, c) in $\mathrm{Sym}^{k-1}(\Sigma) \times E$ such that $(w, \{c, \sigma_2\}) \in \mathrm{Im}(v)$.*[5]

[4]In [21], they use $(s, t) \in [0, 1] \times \mathbb{R}$ and require that τ_0 equals $|t|$ for sufficiently large t, but we follow the more standard convention that $(s, t) \in \mathbb{R} \times [0, 1]$.

[5]In [21], they use the notation (w, y) instead of (w, c). We use y to denote an intersection point between $\mathrm{Sym}\,\underline{L}$ and $\mathrm{Sym}\,\underline{K}$ so we use c here.

The map $[v] \mapsto (w, c)$ *is then a one-to-one correspondence between* $\mathcal{M}(S^2 \to \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E))$ *and* $\mathrm{Sym}^{k-1}(\Sigma) \times E$.

We fix v as above, and normalize it so that $v(0) = w \times \{c, \sigma_2\}$ (where we view S^2 as $\mathbb{C} \cup \{\infty\}$).

We will only be interested in the case that $c \in \alpha \cap \alpha'$. The intuitive reason is that we are going to glue $u \times \{c\} : D \to \mathrm{Sym}^k(\Sigma) \times E$ and n many $v : S^2 \to \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E)$ together (one v for each q_i), so c has to be an intersection point between α and α' for $u \times \{c\}$ to satisfy the Lagrangian boundary conditions (cf. the degeneration (5.1), where the Gromov limit lives). Therefore, we assume from now on that $c \neq \sigma_2$.

We identify a neighbourhood of $v(0)$ with

$$\mathrm{Sym}^{k-1}(\Sigma) \times B_{r_2}(\sigma_2) \times (E - B_{R_2}(\sigma_2)) \subset \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E)$$

for some $0 < r_2 < R_2$.

Fix $\varepsilon > 0$ such that $B_\varepsilon(0)$ is sent by v to this neighbourhood. Fix conformal identifications $[0, \infty) \times S^1 \cong B_\varepsilon(0) - \{0\}$ and $[0, \infty) \times S^1 \cong B_{r_2}(\sigma_2) - \{\sigma_2\}$.

Then, there are unique $w \in \mathrm{Sym}^{k-1}(\Sigma)$, $c \in E$, $(t, \theta) \in [0, \infty) \times S^1$ such that v restricted to $[0, \infty) \times S^1 \cong B_\varepsilon(0) - \{0\}$ differs by a $L^p_{1,\delta}$ map from the map

$$b_{(t,\theta,w,y)}(s, \varphi) = (w, s + t, \theta + \varphi, c),$$

where $L^p_{1,\delta}$ is defined with a weight function $e^\delta \tau_2$ with $\tau_2 : S^2 - \{0\} \to \mathbb{R}^+$ defined in a similar fashion as τ_1.

For $S > 0$, let $S^2(S) := \tau_2^{-1}([0, S])$. We define a map[6]

$$v_S : S^2 - \{0\} \to \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E - \{\sigma_2\}),$$

which agrees with v over $S^2(S)$, and such that over $[0, \infty) \times S^1 \cong B_\varepsilon(0) - \{0\}$,

$$v_S(s, \varphi) = h(s - S) b_{(t,\theta,w,c)}(s, \varphi) + (1 - h(s - S)) v(s, \varphi).$$

In [21, Definition 10.8], the authors define a normalization condition on holomorphic spheres called being "*centred*". They show that the moduli space of centred maps $\mathcal{M}^{\mathrm{cent}}(S^2 \to \mathrm{Sym}^{k-1}(\Sigma) \times \mathrm{Sym}^2(E))$ is diffeomorphic to $\mathrm{Sym}^{k-1}(\Sigma) \times \mathbb{R} \times S^1 \times E$ through the assignment $v \mapsto (w, t, \theta, c)$.

Denote by $v_{(w,t,\theta,c)}$ the pre-image of (w, t, θ, c) by this diffeomorphism.

Using the conformal identifications $B_{r_1}(\sigma_1) - \sigma_1 \cong [0, \infty) \times S^1$ and $B_{r_2}(\sigma_2) - \sigma_2 \cong [0, \infty) \times S^1$, one can think of $\Sigma \#_T E$ as the union of $\Sigma(2T)$ and $E(2T)$ modulo

[6] In [21], the domain of v_S is $S^2 - \{\infty\}$ which we believe is a typo.

the identification of the cylinders $[0, 2T] \times S^1 \subset \Sigma(2T)$ and $[0, 2T] \times S^1 \subset E(2T)$ via the involution $(t, \theta) \sim (2T - t, \theta)$.

Let $X_2(T) := \bigsqcup_{i=1}^{n} S^2(T)_i$ and $X_1 \cup_T X_2$ be the union of $X_1(T)$ and $X_2(T)$ glued at their common boundary. We have that

$$X_1 \cup_T X_2 \cong D^2.$$

There exists some constant $t > 0$ such that for any real numbers S and T such that $0 < S < T - t$, given the pseudo-holomorphic disk $u \in \mathcal{M}_{J_s,\varphi}(x, y)$ we fixed earlier, and the intersection point $c \in \alpha \cap \alpha' \subset E$, one can define a map

$$\widehat{\gamma}_c(u, S, T) : D \cong X_1 \cup_T X_2 \to \mathrm{Sym}^{k+1}(\Sigma \#_T E),$$

which agrees with $\tilde{u}_S \times \{c\}$ over $X_1(T)$ and with $v_{(w_i, -t_i, \theta_i, c), S}$ on $S^2(T)_i \subset X_2(T)$.

Following [21, Lemma 10.9], if S is sufficiently large, then for large T this map is smooth, and for an appropriate Sobolev norm there are some positive constants C and a such that

$$\|\overline{\partial}_{J'_t(T)} \widehat{\gamma}_c(u, S, T)\| \leqslant C e^{-aS}.$$

One can show ([21, Proposition 10.12]) that when T is sufficiently large, the linearization of $\overline{\partial}_{J'_t(T)}$ for the spliced map from $X_1 \cup_T X_2$ admits a right inverse whose norm is bounded independent of T.

Then, applying the inverse theorem function ([21, Proposition 10.14]), there is an $\varepsilon > 0$ such that for sufficiently large T, there is a unique holomorphic curve $\gamma_c(u)$ which lies in an ε-neighbourhood of $\widehat{\gamma}_c(u, S, T)$ (measured in the appropriate Sobolev norm).

This $\gamma_c(u)$ lives in $\mathcal{M}_{J'_t(T),\varphi'_c}(x \times \{c\}, y \times \{c\})$.

5.4. Gromov compactness

In this section, we need to show that every map in $\mathcal{M}_{J'_t(T),\varphi'_c}(x \times \{c\}, y \times \{c\})$ and $\mathcal{M}_{J'_t(T),\varphi'_{E,x}}(x \times \{c\}, x \times \{c'\})$ can be attained by the construction of the previous section. Once again, the argument is similar to the one in [21].

Let $x', y' \in \mathrm{Sym}^{k+1}(\Sigma \#_T E)$ be two critical points of the action functional (i.e., intersection points of the Lagrangians $\mathrm{Sym}(\underline{L} \cup \alpha) \cap \mathrm{Sym}(\underline{K} \cup \alpha')$). Let φ' be a Maslov index 1 class in $\pi_2(x', y')$.

Then, according to [21, Proposition 10.15], any sequence $u_T \in \mathcal{M}_{J'_t(T),\varphi'}(x', y')$ with T going to infinity has, up to passing to a subsequence, a Gromov limit u_∞ mapping to

$$\mathrm{Sym}^{k+1}(\Sigma \vee E) = \bigcup_{i=0}^{k+1} \mathrm{Sym}^{k+1-i}(\Sigma) \times \mathrm{Sym}^i(E). \tag{5.1}$$

We can think of the wedge sum $\Sigma \vee E$ as the degeneration of the connected sum $\Sigma \#_T E$ when the neck length T goes to infinity. The limit u_∞ is analyzed in Lemmas 54 and 52.

Remark 51. An alternative way to think of this Gromov compactness is to consider the relative Hilbert scheme $\mathrm{Hilb}^{k+1}(\pi)$ of a Lefschetz fibration $\pi : E \to D$ over a disk D, where generic fibres are smooth and the singular fibre is $\Sigma \vee E$. The relative Hilbert scheme is smooth [23, Proposition 3.7] and one can equip $\mathrm{Hilb}^{k+1}(\pi)$ with a one-parameter family of almost complex structures such that the projection to D are pseudo-holomorphic, they are fibrewise V-nearly symmetric, and they agree with $J'_t(T)$ over some fibres such that $T \to \infty$ corresponds to degenerating to the central fibre. The central fibre of $\mathrm{Hilb}^{k+1}(\pi)$ has a canonical 'cycle map' to $\mathrm{Sym}^{k+1}(\Sigma \vee E)$ (cf. [23, Section 1.5.1]) and u_∞ is the same as applying Gromov compactness inside $\mathrm{Hilb}^{k+1}(\pi)$ and then applying the cycle map to $\mathrm{Sym}^{k+1}(\Sigma \vee E)$.

Lemma 52 (cf. [21, Proposition 10.16]). *If (x', y') is of the form $(x \times \{c\}, y \times \{c\})$, then u_∞ consists of a main component of the form $u \times \{c\}$, where u is a Maslov index 1 trajectory from x to y in $\mathrm{Sym}^k(\Sigma)$, together with possibly some sphere components of the form $\{w\} \times v$, where $w \in \mathrm{Sym}^{k-1}(\Sigma)$ and v is a holomorphic sphere in $\mathrm{Sym}^2(E)$ with Chern number 2 (i.e., a sphere in the ruling).*

Proof. Since the u_T are Floer trajectories between x' and y', u_∞ consists of a (possibly) broken Floer trajectory between x' and y', as well as sphere bubbles and disk bubbles. Let u^i_∞ be the components of u_∞ in $\mathrm{Sym}^{k+1-i}(\Sigma) \times \mathrm{Sym}^i(E)$. By projection to factors, we can write it as $u^i_\infty = (u^{\Sigma,i}_\infty, u^{E,i}_\infty)$.

Let $D_{\Sigma,i} = \mathrm{Sym}^{k-i}(\Sigma) \times \{\sigma_1\} \subset \mathrm{Sym}^{k+1-i}(\Sigma)$ and $D_{E,i} = \mathrm{Sym}^{i-1}(E) \times \{\sigma_2\} \subset \mathrm{Sym}^i(E)$. We define the adjusted Maslov index $\widetilde{\mu}(u^i_\infty)$ of u^i_∞ relative to $D_{\Sigma,i} \times \mathrm{Sym}^i(E) + \mathrm{Sym}^{k+1-i}(\Sigma) \times D_{E,i}$ as the Maslov index of u with respect to the log canonical line bundle with a simple pole along $D_{\Sigma,i} \times \mathrm{Sym}^i(E) + \mathrm{Sym}^{k+1-i}(\Sigma) \times D_{E,i}$. In other words, the adjusted Maslov index of u^i_∞ is its usual Maslov viewed as a map to $\mathrm{Sym}^{k+1-i}(\Sigma) \times \mathrm{Sym}^i(E)$ subtracted by $2[u^i_\infty] \cdot (D_{\Sigma,i} \times \mathrm{Sym}^i(E) + \mathrm{Sym}^{k+1-i}(\Sigma) \times D_{E,i})$. The additivity of Maslov index under passing to a Gromov limit implies that the sum of the adjusted Maslov indices of the components of u_∞ is equal to the Maslov index of φ', which is 1. This is because $D_{\Sigma,i} \times \mathrm{Sym}^i(E) + \mathrm{Sym}^{k+1-i}(\Sigma) \times D_{E,i}$ is precisely the locus, where $\mathrm{Sym}^{k+1-i}(\Sigma) \times \mathrm{Sym}^i(E)$ intersects other irreducible components of $\mathrm{Sym}^{k+1}(\Sigma \vee E)$.

Since $\mathrm{Sym}(\underline{L} \cup \alpha)$, $\mathrm{Sym}(\underline{K} \cup \alpha')$ are contained in $\mathrm{Sym}^k(\Sigma) \times E \subset \mathrm{Sym}^{k+1}(\Sigma \vee E)$, the broken Floer trajectory and disk bubbles are contained in $\mathrm{Sym}^k(\Sigma) \times E$. We denote the spherical components of u^1_∞ by $u^1_{\infty,S}$ and the other components of u^1_∞ by $u^1_{\infty,D}$.

Now, we analyze the adjusted Maslov indices of the spherical components of u_∞. Recall from [2, Theorem 9.2] that the rank of $\pi_2(\mathrm{Sym}^j(\Sigma)) \to H_2(\mathrm{Sym}^j(\Sigma))$ is 1 when $j \geq 2$ or Σ has genus 0, and 0 otherwise. Moreover, the Chern number of a spherical class u is given by $(j+1-g)[u] \cdot [\mathrm{Sym}^{j-1}(\Sigma) \times \{\sigma_1\}]$ (cf. [8, Remark 4.18]). Therefore, its adjusted Maslov index relative to $\mathrm{Sym}^{j-1}(\Sigma) \times \{\sigma_1\}$ is given by

$$2c_1 \cdot [u] - 2[u] \cdot [\mathrm{Sym}^{j-1}(\Sigma) \times \{\sigma_1\}] = 2(j-g)[u] \cdot [\mathrm{Sym}^{j-1}(\Sigma) \times \{\sigma_1\}]. \quad (5.2)$$

The adjusted Maslov index $\widetilde{\mu}(u^i_\infty)$ is the sum of the adjusted Maslov indices $\widetilde{\mu}(u^{\Sigma,i}_\infty) + \widetilde{\mu}(u^{E,i}_\infty)$, where the two terms in the sum are, respectively, relative to $D_{\Sigma,i} \times \mathrm{Sym}^i(E)$ and $\mathrm{Sym}^{k+1-i}(\Sigma) \times D_{E,i}$. For spherical components, they can be computed by the formula (5.2).

For $i \neq 1$, we define

$$N_{\Sigma,i} := [u^{\Sigma,i}_\infty] \cdot D_{\Sigma,i}, \quad N_{E,i} := [u^{E,i}_\infty] \cdot D_{E,i}.$$

For $i = 1$, we define

$$N_{\Sigma,1} := [u^{\Sigma,1}_{\infty,S}] \cdot D_{\Sigma,1}, \quad N_{E,1} := [u^{E,1}_{\infty,S}] \cdot D_{E,1}$$

and

$$P_{\Sigma,1} := [u^{\Sigma,1}_{\infty,D}] \cdot D_{\Sigma,1}, \quad P_{E,1} := [u^{E,1}_{\infty,D}] \cdot D_{E,1}.$$

The terms $P_{\Sigma,1}$ and $P_{E,1}$ make sense because the Lagrangian boundary condition splits as a product, and they are disjoint from the divisor $D_{\Sigma,1}$, $D_{E,1}$. Clearly, $N_{E,0} = N_{E,1} = 0$ and $N_{\Sigma,k+1} = 0$ because the spherical class is trivial. Note that $D_{\Sigma,i} \times \mathrm{Sym}^i(E) \subset \mathrm{Sym}^{k+1-i}(\Sigma) \times \mathrm{Sym}^i(E)$ and $\mathrm{Sym}^{k-i}(\Sigma) \times D_{E,i+1} \subset \mathrm{Sym}^{k-i}(\Sigma) \times \mathrm{Sym}^{i+1}(E)$ are, precisely, the locus where these two components of $\mathrm{Sym}^{k+1}(\Sigma \vee E)$ meet each other. Therefore, we have

$$N_{\Sigma,i} = N_{E,i+1}$$

for $i \geq 2$, and

$$N_{\Sigma,1} + P_{\Sigma,1} = N_{E,2}, \quad N_{\Sigma,0} = N_{E,1} + P_{E,1} = P_{E,1}.$$

Recall that α' is a Hamiltonian push-off of α such that the Hamiltonian isotopy from α to α' does not pass through σ_2. Therefore, any Floer trajectory between two intersection points of α and α' does not pass through σ_2 in E. Also, there is no non-constant disk bubble in E so $P_{E,1} = 0$. Therefore, $N_{\Sigma,0} = 0$ and u_∞ does not intersect the component $\mathrm{Sym}^{k+1}(\Sigma) \subset \mathrm{Sym}^{k+1}(\Sigma \vee E)$ at all.

The total sum of the adjusted Maslov indices of the components of u_∞ is given by

$$
\begin{aligned}
&\widetilde{\mu}(u^1_\infty) + \sum_{i=2}^{k+1} \widetilde{\mu}(u^i_\infty)\\
&\quad = \widetilde{\mu}(u^1_{\infty,D}) + \widetilde{\mu}(u^{\Sigma,1}_{\infty,S}) + \widetilde{\mu}(u^{E,1}_{\infty,S}) + \sum_{i=2}^{k+1} \widetilde{\mu}(u^{\Sigma,i}_\infty) + \sum_{i=2}^{k+1} \widetilde{\mu}(u^{E,i}_\infty)\\
&\quad = (\mu(u^1_{\infty,D}) - 2P_{\Sigma,1}) + 2(k-g)N_{\Sigma,1} + 0 + \sum_{i=2}^{k+1} 2(k+1-i-g)N_{\Sigma,i}\\
&\qquad + \sum_{i=2}^{k+1} 2(i-1)N_{E,i}\\
&\quad = \mu(u^1_{\infty,D}) - 2P_{\Sigma,1} + 2(k-g)N_{\Sigma,1} + \sum_{i=2}^{k} 2(k+1-i-g)N_{\Sigma,i} + 2N_{E,2}\\
&\qquad + \sum_{i=2}^{k} 2iN_{E,i+1}\\
&\quad = \mu(u^1_{\infty,D}) - 2P_{\Sigma,1} + 2(k-g)N_{\Sigma,1} + 2(N_{\Sigma,1} + P_{\Sigma,1}) + \sum_{i=2}^{k} 2(k+1-g)N_{\Sigma,i}\\
&\quad = \mu(u^1_{\infty,D}) + 2(k-g+1)N_{\Sigma,1} + \sum_{i=2}^{k} 2(k+1-g)N_{\Sigma,i}.
\end{aligned}
$$

As we said earlier, this total sum has to be 1. By regularity, each Floer trajectory component of $u^1_{\infty,D}$ contributes at least 1 to the Maslov index. Any non-constant disk bubble in $u^1_{\infty,D}$ contributes at least 2 to the Maslov index. Since $k \geq g$, we have $(k-g+1) > 0$. Therefore, the sum is 1 only if $N_{\Sigma,i} = 0$ for all i, and $u^1_{\infty,D}$ consists of a single component. It implies that $P_{\Sigma,1} = N_{E,2}$.

By genericity, we can assume that the component $u^1_{\infty,D}$ intersects $D_{\Sigma,1} \times E$ transversely. Therefore, u^2_∞ intersects $\mathrm{Sym}^{k-1}(\Sigma) \times D_{E,1}$ transversely. Note that every component of u^2_∞ projects to a constant in $\mathrm{Sym}^{k-1}(\Sigma)$ because $N_{\Sigma,2} = 0$. Since the domain has genus 0, the bubbling is modelled on a tree, and hence, no component of u^2_∞ can be a multiple cover of an underlying holomorphic sphere. It implies that the sphere components of u_∞ are of the form $\{w\} \times v$, where $w \in \mathrm{Sym}^{k-1}(\Sigma)$ and v is a holomorphic sphere in $\mathrm{Sym}^2(E)$ with Chern number 2. The Floer trajectory component of u_∞ is u^1_∞, which goes between $x \times \{c\}$ and $y \times \{c\}$. Therefore, it is of the form $u \times \{c\}$, where u is a Maslov index 1 trajectory from x to y in $\mathrm{Sym}^k(\Sigma)$. ∎

Remark 53. The bubbling analysis in the proof of Lemma 52 provides the details of the phrase 'by a dimension count' in the proof of [21, Proposition 10.16] and at the same time confirms that we can generalize it to all $k \geq g$.

Lemma 54. *If (x', y') is of the form $(x \times \{c\}, x \times \{c'\})$, then u_∞ consists of a single component and it is of a product form $\{x\} \times u$, where u is a Maslov index 1 trajectory from c to c' in E.*

Proof. The proof is the same as Lemma 52. The only difference is that when (x', y') is of the form $(x \times \{c\}, x \times \{c'\})$, the Floer trajectory $u^1_{\infty,D}$ is of the form $\{x\} \times u$, where u is a Maslov index 1 trajectory from c to c'. Therefore, we have $P_{\Sigma,1} = 0$, and hence, there are no sphere bubbles. ■

Lemma 55. *If (x', y') is of the form $(x \times \{c\}, y \times \{c'\})$, where $x \neq y$ and $c \neq c'$, then $M_{J'_t(T),\varphi'}(x', y')$ is empty for large T.*

Proof. We need to show that there is no possible u_∞. The proof is the same as Lemma 52 again. More precisely, the reasoning for $N_{\Sigma,i} = 0$ for all i, and $u^1_{\infty,D}$ consists of a single component of Maslov index 1 can be applied without any change. Since $x \neq y$ and $c \neq c'$, there is no regular Maslov 1 Floer trajectory connecting x' and y'. Therefore, there is no possible u_∞. ■

Now, we show that for sufficiently large T, any map in $\mathcal{M}_{J'_t(T),\varphi'}(x \times \{c\}, y \times \{c\})$ is attained by the construction of the previous section.

We proceed by contradiction: suppose that there is a sequence (T_m) going to infinity, and a sequence of disks $u_{T_m} \in \mathcal{M}_{J'_s(T_m),\varphi'}(x \times \{c\}, y \times \{c\})$ that are not attained by the gluing construction.

By what precedes, we can extract a subsequence converging to a bubbletree u_∞, consisting of a disk $u \times \{c\}$ and n spheres $\{w_i\} \times v_i$.

Then, the authors of [21] show that for sufficiently large m, u_{T_m} is in an ε-neighbourhood of the nearly holomorphic map $\widehat{\gamma}_c(u, S, T_m)$ for some $0 < S < T_m - t$ (for the suitable Sobolev distance).

But we showed in the previous section that there was a single $J'_t(T)$ holomorphic curve in such a neighbourhood, namely, the curve $\gamma_c(u, T_m)$. Therefore, for large m,

$$u_{T_m} = \gamma_c(u, T_m),$$

which contradicts our assumption.

Hence, by contradiction for large T, we have

$$\mathcal{M}_{J_s,\varphi}(x, y) \simeq \mathcal{M}_{J'_t(T),\varphi'_c}(x \times \{c\}, y \times \{c\}).$$

This concludes the proof of Theorem 48.

5.5. Continuation maps

In this section, we study how the isomorphism of Theorem 6 behaves with respect to continuation maps.

We consider an admissible link $\underline{L}' = \underline{L} \cup \alpha$ with $k+1$ components on a surface Σ', where α is non-contractible, and a disk D that does not intersect α. Then, one can find a decomposition of Σ' as a connected sum $\Sigma' = \Sigma \# E$ such that $\underline{L} \subset \Sigma$ is η-monotone, $\alpha \subset E$, and $D \subset \Sigma$.

Let $H^i, i = 0, 1$ be two Hamiltonians supported in D, and let H^i_ε be an ε-perturbation of H^i in small neighbourhoods of the link's components so that $HF^*(\underline{L}', \underline{L}', H^i_\varepsilon)$ is well defined (cf. (2.1)). We can assume that H^i_ε is chosen such that it is supported away from the connected sum neighbourhood of the decomposition $\Sigma' = \Sigma \# E$, and that H^0_ε and H^1_ε coincide away from D. Let J_t (resp., J_E, resp., $J'_t(T)$) be a family of complex structures on Σ (resp., E, resp., Σ') as in the proof of Theorem 6. Let $H^s_\varepsilon, s \in \mathbb{R}$ be a smooth family of Hamiltonians satisfying $H^s_\varepsilon = H^0_\varepsilon$ for $s \leq 0$ and $H^s_\varepsilon = H^1_\varepsilon$ for $s \geq 1$, supported away from the connected sum neighbourhood. We also assume that H^s_ε is equal to $H^0_\varepsilon = H^1_\varepsilon$ outside D, and $(H^s_\varepsilon, J_t(T))$ is regular, by perturbing H^s_ε away from the connected sum region, for $0 < s < 1$.

Lemma 56. *The following diagram commutes:*

$$\begin{array}{ccc} CF^*(\underline{L}', \underline{L}', H^0_\varepsilon) & \longrightarrow & CF^*(\underline{L}, \underline{L}, H^0_\varepsilon|_\Sigma) \otimes CF^*(\alpha, \alpha, H^0_\varepsilon|_E) \\ \downarrow & & \downarrow \\ CF^*(\underline{L}', \underline{L}', H^1_\varepsilon) & \longrightarrow & CF^*(\underline{L}, \underline{L}, H^1_\varepsilon|_\Sigma) \otimes CF^*(\alpha, \alpha, H^1_\varepsilon|_E) \end{array}$$

where the horizontal arrows come from the isomorphism of Theorem 6, and the vertical arrows are the continuation maps induced by H^s_ε, and the tensor product of the continuation maps induced by the restriction of H^s_ε to the different components of the connected sum.

Proof. The proof is analogous to that of Theorem 6, we only need to identify the moduli spaces of curves involved in the definition of the continuation map. Denote by $\underline{L}_i$ (resp., α_i) the image of $\underline{L}$ (resp., α) by $\Phi_{H^i_\varepsilon}$. Note that by definition of H^i_ε, $\alpha_0 = \alpha_1$. Let x_i be an intersection point between $\operatorname{Sym} \underline{L}$ and $\operatorname{Sym} \underline{L}_i$, and c_i be an intersection point between α and α_i. We need to show that $\mathcal{M}_{J'_t(T)}(x_0 \times \{c_0\}, x_1 \times \{c_1\})$ is in one-to-one correspondence with $\mathcal{M}_{J_t}(x_0, x_1) \times \mathcal{M}_{J_E}(c_0, c_1)$.

We recall the definition of those moduli spaces: $\mathcal{M}_{J_E}(c_0, c_1)$ consists of the strips $u : \mathbb{R} \times [0, 1] \to E$ that are J_E-holomorphic, have finite energy, have Maslov index 0, satisfy the boundary condition $u(\mathbb{R}, 0) \subset \alpha$, $u(s, 1) \subset \Phi_{H^s_\varepsilon}(\alpha)$, and are asymptotic to c_0 at $s = -\infty$ and c_1 at $s = +\infty$. $\mathcal{M}_{J'_t(T)}(x_0 \times \{c_0\}, x_1 \times \{c_1\})$ (resp., $\mathcal{M}_{J_t}(x_0, x_1)$) is defined in a similar fashion on $\operatorname{Sym}^{k+1}(\Sigma')$ (resp., $\operatorname{Sym}^k(\Sigma)$).

Since we constructed H_ε^s to be a constant homotopy on E, the moduli spaces $\mathcal{M}_{J_E}(c_0, c_1)$ are empty when $c_0 \neq c_1$, and they consist of a single point (the constant strip) when $c_0 = c_1$. Therefore, we only have to show that $\mathcal{M}_{J_t'(T)}(x_0 \times \{c_0\}, x_1 \times \{c_1\})$ is empty whenever $c_0 \neq c_1$, and can be identified with $\mathcal{M}_{J_t}(x_0, x_1)$ when $c_0 = c_1$. The former is once again proved in the same way as Lemma 52. The latter is proved exactly as in Theorem 48. ∎

Corollary 57. *The following diagram commutes:*

$$\begin{array}{ccc} QH^*(\underline{L}', \underline{L}') & \xrightarrow{f_0} & QH^*(\underline{L}, \underline{L}) \otimes QH^*(\alpha, \alpha) \\ {\scriptstyle \mathrm{PSS}_{\Sigma'}}\downarrow & & \downarrow{\scriptstyle \mathrm{PSS}_\Sigma \otimes \mathrm{PSS}_E} \\ HF^*(\underline{L}', \underline{L}', H_\varepsilon^1) & \xrightarrow{f_1} & HF^*(\underline{L}, \underline{L}, H_\varepsilon^1|_\Sigma) \otimes HF^*(\alpha, \alpha, H_\varepsilon^1|_E) \end{array}$$

where the horizontal arrows come from the isomorphism of Theorem 6, and the vertical arrows are PSS isomorphisms. In particular, we have

$$f_1(\mathrm{PSS}_{\Sigma'}(e_{\underline{L}'})) = \mathrm{PSS}_\Sigma \otimes \mathrm{PSS}_E(f_0(e_{\underline{L}'})) = \mathrm{PSS}_\Sigma \otimes \mathrm{PSS}_E(e_{\underline{L}} \otimes e_\alpha).$$

Proof. There is a canonical isomorphism between $QH^*(\underline{L}', \underline{L}')$ and $HF^*(\underline{L}', \underline{L}', H_\varepsilon^0)$ when H_ε^0 is C^2 small. Therefore, the commutativity follows from Lemma 56. Finally, $f_0(e_{\underline{L}'}) = e_{\underline{L}} \otimes e_\alpha$ follows from the fact that the unit with respect to quantum multiplication is the same as the Morse theoretical unit, so $e_{\underline{L}'}$ is sent to $e_{\underline{L}} \otimes e_\alpha$ by Künneth theorem. ∎

Remark 58. We could also get Corollary 57 by showing that the isomorphism of Theorem 6 preserves the product structure, that is that the following diagram commutes:

$$\begin{array}{ccc} CF^*(\underline{L}_0', \underline{L}_1') \otimes CF^*(\underline{L}_!', \underline{L}_2') & \longrightarrow & CF^*(\underline{L}_0, \underline{L}_1) \otimes CF^*(\underline{\alpha}_0, \alpha_1) \otimes CF^*(\underline{L}_1, \underline{L}_2) \otimes CF^*(\alpha_1, \alpha_2) \\ \Big\downarrow & & \downarrow \\ & & CF^*(\underline{L}_0, \underline{L}_1) \otimes CF^*(\underline{L}_1, \underline{L}_2) \otimes CF^*(\alpha_0, \alpha_1) \otimes CF^*(\alpha_1, \alpha_2) \\ & & \downarrow \\ CF^*(\underline{L}_0', \underline{L}_2') & \longrightarrow & CF^*(\underline{L}_0, \underline{L}_2) \otimes CF^*(\alpha_0, \alpha_2) \end{array}$$

We believe that this can be proved once again by identifying moduli spaces of triangles involved in the definition of the product structure, as in Theorem 48.

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References

[1] A. Banyaga, *The structure of classical diffeomorphism groups*. Math. Appl. 400, Kluwer, Dordrecht, 1997 Zbl 0874.58005 MR 1445290

[2] A. Bertram and M. Thaddeus, On the quantum cohomology of a symmetric product of an algebraic curve. *Duke Math. J.* **108** (2001), no. 2, 329–362 Zbl 1050.14052 MR 1833394

[3] M. Bökstedt and N. M. Romão, On the curvature of vortex moduli spaces. *Math. Z.* **277** (2014), no. 1-2, 549–573 Zbl 1294.32005 MR 3205783

[4] L. Buhovsky, On two remarkable groups of area-preserving homeomorphisms. *J. Math. Phys. Anal. Geom.* **19** (2023), no. 2, 339–373 Zbl 1539.53091 MR 4633987

[5] G. Chen, Closed-open morphisms on periodic Floer homology. [v1] 2021, [v5] 2025, arXiv:2111.11891v5

[6] G. Chen, On PFH and HF spectral invariants. [v1] 2022, [v5] 2025, arXiv:2209.11071v5

[7] C.-H. Cho, Products of Floer cohomology of torus fibers in toric Fano manifolds. *Comm. Math. Phys.* **260** (2005), no. 3, 613–640 Zbl 1109.53079 MR 2183959

[8] D. Cristofaro-Gardiner, V. Humilière, C. Y. Mak, S. Seyfaddini, and I. Smith, Quantitative Heegaard Floer cohomology and the Calabi invariant. *Forum Math. Pi* **10** (2022), article no. e27 Zbl 1508.53090 MR 4524364

[9] D. Cristofaro-Gardiner, V. Humilière, C. Y. Mak, S. Seyfaddini, and I. Smith, Subleading asymptotics of link spectral invariants and homeomorphism groups of surfaces. [v1] 2022, [v2] 2025, arXiv:2206.10749v2

[10] D. Cristofaro-Gardiner, V. Humilière, and S. Seyfaddini, Proof of the simplicity conjecture. *Ann. of Math. (2)* **199** (2024), no. 1, 181–257 Zbl 07782631 MR 4681145

[11] D. Cristofaro-Gardiner, M. Hutchings, and V. G. B. Ramos, The asymptotics of ECH capacities. *Invent. Math.* **199** (2015), no. 1, 187–214 Zbl 1315.53091 MR 3294959

[12] M. Entov and L. Polterovich, Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.* (2003), no. 30, 1635–1676 Zbl 1047.53055 MR 1979584

[13] A. Fathi, Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 1, 45–93 Zbl 0443.58011 MR 0584082

[14] V. Humilière, Géométrie symplectique C^0 et sélecteurs d'action. 2017, available at https://webusers.imj-prg.fr/~vincent.humiliere/HDR.pdf, visited on 1 May 2025

[15] V. Humilière, R. Leclercq, and S. Seyfaddini, Coisotropic rigidity and C^0-symplectic geometry. *Duke Math. J.* **164** (2015), no. 4, 767–799 Zbl 1327.53109 MR 3322310

[16] M. Hutchings, Quantitative embedded contact homology. *J. Differential Geom.* **88** (2011), no. 2, 231–266 Zbl 1238.53061 MR 2838266

[17] R. Leclercq and F. Zapolsky, Spectral invariants for monotone Lagrangians. *J. Topol. Anal.* **10** (2018), no. 3, 627–700 Zbl 1398.57037 MR 3850107

[18] R. Lipshitz, A cylindrical reformulation of Heegaard Floer homology. *Geom. Topol.* **10** (2006), 955–1096 Zbl 1130.57035 MR 2240908

[19] C. Y. Mak and I. Smith, Non-displaceable Lagrangian links in four-manifolds. *Geom. Funct. Anal.* **31** (2021), no. 2, 438–481 Zbl 1479.57068 MR 4268306

[20] Y.-G. Oh and S. Müller, The group of Hamiltonian homeomorphisms and C^0-symplectic topology. *J. Symplectic Geom.* **5** (2007), no. 2, 167–219 Zbl 1144.37033 MR 2377251

[21] P. Ozsváth and Z. Szabó, Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)* **159** (2004), no. 3, 1027–1158 Zbl 1073.57009 MR 2113019

[22] P. Ozsváth and Z. Szabó, Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.* **8** (2008), no. 2, 615–692 Zbl 1144.57011 MR 2443092

[23] T. Perutz, Lagrangian matching invariants for fibred four-manifolds. I. *Geom. Topol.* **11** (2007), 759–828 Zbl 1143.53079 MR 2302502

[24] T. Perutz, Hamiltonian handleslides for Heegaard Floer homology. In *Proceedings of Gökova Geometry-Topology Conference 2007*, pp. 15–35, Gökova Geometry/Topology Conference (GGT), Gökova, 2008 Zbl 1193.30059 MR 2509747

[25] L. Polterovich and E. Shelukhin, Lagrangian configurations and Hamiltonian maps. *Compos. Math.* **159** (2023), no. 12, 2483–2520 Zbl 1531.53091 MR 4651502

[26] S. Seyfaddini, C^0-limits of Hamiltonian paths and the Oh–Schwarz spectral invariants. *Int. Math. Res. Not. IMRN* **2013** (2013), no. 21, 4920–4960 Zbl 1378.53101 MR 3123670

[27] N. Sheridan, On the Fukaya category of a Fano hypersurface in projective space. *Publ. Math. Inst. Hautes Études Sci.* **124** (2016), 165–317 Zbl 1453.53079 MR 3578916

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