A Longitudinal Investigation of the Co-Development and Bidirectional Relations Among Whole Number Arithmetic and Conceptual and Procedural Fraction Knowledge

Chang Xu[a], Sabrina Di Lonardo Burr[b], Lisa Jane Rodgers[a], Judith Wylie[a],

Jiwei Si[c], Bijuan Huang[c], Kaiyue Guo[c], Hongxia Li[c]

[a] School of Psychology, Queen's University Belfast, United Kingdom

[b] School of Psychology, University of Sheffield, United Kingdom

[c] School of Psychology, Shandong Normal University, China

This manuscript was accepted for publication in the *Journal of Educational Psychology* on November 18, 2024. This preprint is the peer-reviewed accepted version but has not yet been copyedited and may differ from the final version published in the journal.

Abstract

How do conceptual and procedural fraction knowledge influence the development of each other over time? Is their pattern of development a reflection of instruction? In the present study we conducted a four-wave longitudinal investigation of the co- and bidirectional development of whole number arithmetic, and conceptual and procedural fraction knowledge during a critical phase of fraction learning. Chinese students (N = 1,055, M_{age} = 9.8, SD = 0.7) educated through a linear curriculum completed whole number arithmetic, and conceptual and procedural fraction assessments during the first and second terms in Grade 4 and Grade 5. Cross-lagged panel analysis, controlling for students' non-verbal reasoning skills, revealed that conceptual and procedural fraction knowledge did not influence the development of one another prior to Grade 5. However, starting in Grade 5, a unidirectional pattern emerged, where conceptual fraction knowledge supported the development of procedural fraction knowledge. This unilateral conceptual-to-procedural pattern of development contrasts with findings from studies with students in the U.S., suggesting that educational experiences may shape the co-development of these two types of fraction knowledge. Furthermore, proficiency in whole number arithmetic predicted the development in both conceptual and procedural fraction knowledge, highlighting its important role alongside conceptual knowledge in supporting the acquisition of fraction procedures. Our findings emphasize the need to consider educational experiences and foster meaningful connections between concepts and procedures during fraction instruction while promoting mastery of whole number arithmetic to promote students' development of fraction knowledge.

Word count: 238 words

Key words: fractions; conceptual; procedural; whole number arithmetic; development

Educational Impact and Implications Statement

We investigated the co-development of whole number arithmetic and conceptual and procedural fraction knowledge during a critical period of fraction learning for Chinese students educated through a linear curriculum. In contrast to findings from North American studies, we found that as students progressed through their fraction learning and were introduced to more complex fraction concepts, whole number arithmetic and conceptual knowledge of fractions facilitated the development of fraction procedures, but fraction procedures did not lead to growth in conceptual knowledge. Our findings emphasize the need to consider educational experiences and foster meaningful connections between concepts and procedures during fraction instruction while promoting mastery of whole number arithmetic to promote students' development of fraction knowledge.

A Longitudinal Investigation of the Co-Development and Bidirectional Relations Among Whole Number Arithmetic and Conceptual and Procedural Fraction Knowledge

In mathematics learning, conceptual knowledge refers to the understanding, whether explicit or implicit, of the underlying principles and the interconnectedness among different components within a domain (Rittle-Johnson et al., 2001; Rittle-Johnson & Alibali, 1999), whereas procedural knowledge refers to the ability to execute a series of operational steps to accurately solve mathematical problems (Rittle-Johnson et al., 2001). Although it is a well-accepted fact that mathematical understanding requires knowledge of both concepts and procedures, how these two types of knowledge are related and the way in which they develop together remains a topic of debate. There appear to be three overarching views with respect to the development of these two types of knowledge: i) a conceptual-to-procedural view wherein conceptual knowledge is a precursor to procedural knowledge, with students using this knowledge to generate and grasp procedures (Halford, 1993; Hiebert & LeFevre, 1986); ii) a procedural-to-conceptual view, wherein procedural knowledge precedes conceptual knowledge, suggesting students gradually extract conceptual knowledge by applying and refining procedures (Karmiloff-Smith, 1992; Siegler & Stern, 1998); and iii) a bidirectional view, wherein the two types of knowledge are intertwined along a continuum, with an increase in one type of knowledge leading to development in the other, which in turn leads to subsequent advances in the first (Rittle-Johnson et al., 2001; Rittle-Johnson & Alibali, 1999). To evaluate these views, in the present study we conducted a four-wave longitudinal study with Chinese students, focusing on the development of whole number arithmetic and conceptual and procedural fraction knowledge during a critical period of fraction learning.

Conceptual and Procedural Knowledge of Fractions

In general, fraction knowledge is the “gatekeeper” for learning more advanced science and mathematics (Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2011) and yet challenges with fractions are a global phenomenon (Chan et al., 2007; Gabriel et al., 2013; Meert et al., 2010; Siegler & Lortie-Forgues, 2017). Indeed, in primary education fractions are among the most challenging concepts that students encounter in their mathematics learning (Boutlet, 1998; Davis et al., 1993) potentially because fractions encompass a multifaceted construct (Brousseau et al., 2004). To demonstrate fraction proficiency, students need to comprehend multiple, interrelated subconstructs of fractions including part-whole, quotient, measure, operator, and ratio (Charalambous & Pitta-Pantazi, 2007; Kieren, 1976). Building upon this multifaceted view, Behr et al. (1983) proposed a theoretical model in which these subconstructs are linked to fraction operations, equivalence, and problem solving. Because fraction learning encompasses numerous concepts and procedures introduced over several years of classroom instruction, the distinction between these two types of knowledge has received significant attention in research regarding the instruction and learning of fractions (Bempeni et al., 2018; Gabriel et al., 2023; Hallett et al., 2010; Moss & Case, 1999; Siegler & Lortie-Forgues, 2015). Notably, the correlations between conceptual and procedural knowledge of fractions have been found to vary from medium to strong (Hansen et al., 2015; Hecht et al., 2003; Jordan et al., 2013; Lenz & Wittmann, 2020; Schneider & Stern, 2010; see a meta-analysis in Lin & Powell, 2021) and it can be challenging to operationalize both types of knowledge and find assessments that solely rely on the use of one and not the other (Crooks & Alibali, 2014). Nevertheless, Lenz and colleagues have demonstrated that conceptual and procedural knowledge of fractions can indeed be empirically separated (Lenz et al., 2020). Thus, understanding how these two types of knowledge co-develop can provide further insights into students' overall development of fraction knowledge.

Conceptual Knowledge

Developing conceptual knowledge of fractions is complex in that there are multiple subconstructs of fractions (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieren, 1976). Children's initial understanding of fractions is often conceptual; prior to formal fraction instruction, they develop an informal understanding of common terms like "half" and "quarter" through everyday activities, such as sharing food or folding paper (Hunting & Sharpley, 1988; Viegut et al., 2023). This early knowledge is part of the part-whole subconstruct of fractions in which fractions represent a relation between part(s) of an equally divided whole and the total number of parts. For example, $\frac{2}{5}$ might refer to two parts out of five equal parts of a pie. Building upon part-whole knowledge, students need to learn to view fractions as a quotient wherein any fraction can represent the result of a division. For example, $\frac{2}{5}$ might represent two pies divided by five people or the amount each person receives when five people share two units of pie.

Next, students need to develop measurement knowledge of fractions, that is, the magnitude of fractions can be compared, ordered, and located on a continuous number line (Charalambous & Pitta-Pantazi, 2007; Smith et al., 2005). This type of knowledge, which is typically mastered after part-whole and quotient knowledge, also requires conceptual fraction knowledge. For example, when asked to order fractions according to their relative magnitude such as $\frac{2}{3}$, $\frac{5}{4}$, $\frac{4}{9}$, students with strong conceptual knowledge could use benchmarks such as $\frac{1}{2}$ and 1 to effectively order fractions, that is, given that $\frac{4}{9} < \frac{1}{2} < \frac{2}{3} < 1 < \frac{5}{4}$, we can conclude that $\frac{4}{9} < \frac{2}{3} < \frac{5}{4}$. The use of benchmarks when determining the relative magnitude of fractions is often more efficient than procedural methods like finding common denominators or using cross multiplication (Bray & Abreu-Sanchez, 2010). Thus, the development of strong conceptual knowledge of fractions is essential for tasks involving fraction comparison, ordering, and equivalence (Lamon, 2012).

After developing part-whole, quotient, and measurement knowledge, students are introduced to the operator and ratio subconstructs (Zhang, 2016). The operator construct is typically related to fraction multiplication and division, emphasising the enlargement and/or shrinking relations between things or within the same thing. The ratio construct reflects fractions as a comparative index rather than a number. Because students learn about ratios after learning about fractions, a definition involving ratio cannot be introduced during the early stages of fraction learning (Zhang, 2016).

Moving into more abstract representations, such as comprehending the density property of fractions, students can be asked to identify how many fractions there are between any two fractions, such as $\frac{1}{4}$ and $\frac{1}{2}$ (Jordan et al., 2017; Van Hoof, Verschaffel, et al., 2015). This understanding that there are infinite fractions between any two fractions hinges on the knowledge that any particular fraction magnitude can be represented in numerous ways using fractions with different numerators and denominators (i.e., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, and so forth; Behr et al., 1984; Kamii & Clark, 1995; Pedersen & Bjerre, 2021). Conceptual knowledge of fractions is foundational, fostering the development of accurate strategies adaptable to procedural problems (Perry, 1991). It is also essential for retaining knowledge during procedural learning of fractions (Hiebert & LeFevre, 1986). Thus, fraction instruction often emphasizes conceptual knowledge before procedural knowledge (Grouws & Cebulla, 2000).

Procedural Knowledge

While conceptual knowledge involves a more flexible understanding and is thus more generalizable beyond specific problem types (Rittle-Johnson et al., 2001), procedural knowledge is more tied to specific types of problems. In the context of fractions, procedural knowledge that goes beyond conceptual knowledge of the magnitude of fractions is necessary for later-learned fraction topics, such as fraction arithmetic. Whereas students can use conceptual knowledge to estimate fraction sums (i.e., rounding $\frac{8}{9} + \frac{11}{12}$ to the nearest whole number), fraction arithmetic requires students to apply a sequence of procedural steps to obtain the correct solution.

Early on, students may misapply whole number arithmetic strategies to fractions, leading to errors such as overgeneralization (e.g., adding both numerators and denominators) and the skipping of crucial procedural steps (e.g., finding a common denominator but failing to update the numerator to create an equivalent fraction; Braithwaite & Siegler, 2023). As students' procedural knowledge of fractions develops, such errors, potentially resulting from the interference of whole number concepts with fraction arithmetic (DeWolf & Vosniadou, 2015; Ni & Zhou, 2005), likely decrease (Braithwaite & Siegler, 2018). The process of adding or subtracting fractions with unlike denominators introduces additional complexity; students need to first find a common denominator, ideally by determining the least common multiple of the original denominators and then appropriately adjust the numerators before summing the fractions to get the correct result. Thus, mastering fraction arithmetic, which is particularly challenging for students (Lortie-Forgues et al., 2015), requires formal instruction and practice (Braithwaite & Siegler, 2023).

The Bi- and Co-development of Fraction Knowledge

Despite years of instruction, many students show little improvement in both conceptual and procedural knowledge of fractions, with some retaining only a partial understanding of fractions (Jordan et al., 2017). This partial understanding persists, as evidenced in studies involving secondary school students (Abreu-Mendoza et al., 2023; Van Hoof et al., 2013; Van Hoof, Vandewalle, et al., 2015) and adults (DeWolf & Vosniadou, 2015; Di Lonardo Burr et al., 2020; Obersteiner et al., 2013; Vamvakoussi et al., 2012), where misconceptions are observed at both the conceptual and procedural level. Therefore, consideration of the initial development of these two types of fraction knowledge in students is essential.

There have been few longitudinal studies that have explored the co-development of conceptual and procedural knowledge of fractions (Bailey et al., 2017; Hansen et al., 2017; Hecht & Vagi, 2010; Jordan et al., 2013; Rittle-Johnson et al., 2001). For example, Hecht and Vagi (2010) conducted a longitudinal study to examine the development of conceptual and procedural fraction knowledge among students from Grades 4 to 5. Students' whole number arithmetic predicted the development of both conceptual and procedural knowledge of fractions. They also found that students' conceptual knowledge of fractions predicted their subsequent procedural knowledge, and vice versa, supporting bidirectional relations between the two types of knowledge. Moreover, Bailey and colleagues (2017) examined the development of conceptual and procedural knowledge of fractions, following students from Grade 4 through Grade 6. Students' procedural knowledge was assessed by a fraction arithmetic task, and their conceptual knowledge was assessed by a fraction number line task. After controlling for a wide range of variables including nonverbal reasoning and whole number arithmetic skills, they found that, in Grade 4, fraction arithmetic did not predict the growth of fraction magnitude knowledge, and vice versa. However, after instruction on fraction arithmetic with unlike denominators was introduced in Grade 5, a reciprocal relation began to emerge, with each supporting the growth of the other. Furthermore, from Grade 5 to 6 Rittle-Johnson and colleagues found a bidirectional relation between conceptual and procedural knowledge of decimal fractions (Rittle-Johnson et al., 2001). The findings from these studies collectively show an iterative development of conceptual and procedural knowledge of fractions.

The Role of Whole Number Arithmetic in the Development of Fraction Knowledge

In addition to considering how conceptual and procedural fraction knowledge co-develop, it is also important to consider how this knowledge develops in relation to whole number skills. Connections between whole number arithmetic and fraction knowledge are evident in a variety of fraction tasks. For example, knowledge acquired from whole number arithmetic, such as determining common denominators or converting fractions to decimals, is essential for many fraction tasks such as comparing, ordering, and performing operations with fractions (Braithwaite & Siegler, 2023; Fazio et al., 2016; Rinne et al., 2017). Proficiency in whole number arithmetic not only facilitates these tasks but also reduces the mental load for students when approaching more complex problems such as fraction word problems (Ma & Kessel, 2022). These interconnections between whole number arithmetic and fraction knowledge highlight the importance of a strong foundation in basic whole number arithmetic for mastering more complex fraction concepts. Indeed, a large meta-analysis, with data from over 6,000 students, provides empirical support for the view that whole number arithmetic is linked to both conceptual and procedural fraction knowledge, independent of age, reasoning, and working memory capabilities (Lin & Powell, 2021). Notably, although research suggests that the development of conceptual and procedural knowledge of fractions build upon the foundation of whole number arithmetic (see Gabriel et al., 2023; Lortie-Forgues, 2015; Siegler & Lortie-Forgues, 2017 for reviews), the few longitudinal studies investigating the co-development of conceptual and procedural fraction knowledge have either excluded whole number arithmetic (Rittle-Johnson et al., 2001) or included it as a predictor or control variable (Bailey et al., 2017; Hansen et al., 2017; Hecht & Vagi, 2010; Jordan et al., 2013). Given the continuous development of whole number arithmetic throughout the later years of primary school, it is essential to also examine its co-development alongside fractions.

The Role of Instruction in Fraction Learning

As mentioned above, few studies have considered the longitudinal development of conceptual and procedural fraction knowledge, and none have considered the role of co-developing whole number arithmetic skills. Moreover, these existing longitudinal studies were conducted with students in the U.S. and the findings are situated within the context of instruction. Instructional choices may reflect cultural differences, with countries adopting distinct curriculum approaches that influence students' learning experiences. Thus, longitudinal research outside of the U.S. is needed to provide insights into the role of instruction in the co-development of conceptual and procedural fraction knowledge. Of particular interest may be investigating this co-development in high-achieving East-Asian countries, such as China, where the fraction curricula have substantial differences in comparison to the U.S. (Zhang & Siegler, 2022).

In the U.S. most schools follow the Common Core State Standards in Mathematics (CCSSM; Common Core State Standards Initiative, 2010). Within the CCSSM, fractions are first introduced in Grade 1 and their instruction is spread over six years (Grades 1 through 6) with instruction emphasizing part-whole and measurement models of fractions. In contrast, in China fractions are not introduced until Grade 3 (Ministry of Education, 2011), with instruction highlighting how whole-number operations extend to fraction operations. Whereas the introduction of fractions in the U.S. is focused on equal-share concepts and knowledge of common fractions (i.e., halves, thirds, quarters), in China the expectations are high when fractions are first introduced, with students expected to master how to read and write fractions; compare fraction magnitudes with picture representations; and compare, add, and subtract fractions with common denominators (Zhang & Siegler, 2022). Moreover, because in China fractions are introduced after whole-number multiplication and division, unlike in the U.S., fractions can be explained in relation to division (i.e., numerators divided by denominator; Zhang & Siegler, 2022). Overall, because there are fundamental differences in both the timing of fraction instruction as well as the way in which fractions are conceptually introduced (i.e., equal-share concepts in the U.S. versus fractions as an extension of division in China), longitudinal research from an East-Asian country can provide further insights into how whole number arithmetic, conceptual and procedural fraction knowledge co-develop.

In China, the curriculum features a limited number of topics each year, which are covered in a short duration and with fast-paced progression (Wang & McDougall, 2019). In alignment with classroom instruction, and in contrast to the U.S., textbooks present topics in more focused ways, such that examples and problems for a single topic (e.g., fraction division) are presented in one chapter in one grade as opposed to over multiple volumes and grades (Zhang & Siegler, 2022). The concept of fractions is initially grounded in division, particularly in dividing wholes into equal units (Guo, 2010; Xu et al., 2022; Zhang & Siegler, 2022). In Grade 3, students are taught to expand their understanding of integral units to include fractional units. These fractional units differ from integral units, representing the equal division of a whole unit (e.g., if a whole unit is divided into three equal parts, each part represents a fractional unit, such as $\frac{1}{3}$; Ma & Kessel, 2022). Once students develop sufficient knowledge of fractional units, they can represent fractions as composite units that can be iterated (Steffe, 2001; Steffe & Olive, 2010). For example, $\frac{2}{3}$ can be obtained by iterating the fractional unit $\frac{1}{3}$ twice. On this view, fraction arithmetic is fundamentally linked with manipulations of fractional units (Braithwaite & Siegler, 2021), because once students' knowledge expands from whole numbers to fractions, analogies between whole numbers and fractions become apparent, such as the arithmetic principles (e.g., distributive, commutative and associative properties) and procedures (e.g., requiring the same unit values when adding or subtracting numbers; Ma & Kessel, 2022).

Beyond making connections between whole numbers and fractions, prior to Grade 5, students also learn to solve simple fraction addition and subtraction problems with common denominators, typically with the support of pictorial representations. In contrast to simply memorizing procedures, by linking symbolic and pictorial representations of fractions the learning process becomes meaningful, thereby assisting pupils in developing a conceptual understanding of fraction arithmetic (Carpenter, 1986; Cramer et al., 2008; Silver, 1986). This link between these two fraction representations is evident in Chinese textbooks where examples and exercises include both pictorial and symbolic fraction arithmetic to encourage students to make connections between symbolic fractions and magnitude (Sun, 2019). Formal instruction in fraction arithmetic, including procedures to solve fraction arithmetic equations involving uncommon denominators, is deferred until Grade 5 (Ministry of Education, 2011). Grade 5 does not just focus on procedures, however. During this time students also expand their knowledge of fraction units to include improper and mixed fractions, fraction reductions and equivalency, and transformations between different types of rational numbers (i.e., decimals and fractions). These concepts, which focus on furthering understanding of the properties of fractions form the foundation for fraction arithmetic.

Present Research

In the present study, we investigated the co-development of conceptual and procedural fraction knowledge for Chinese students from Grade 4 to Grade 5. Students completed both whole-number arithmetic and fraction assessments at four time points, six months apart. We developed the fraction assessment based on existing literature that included both conceptual and procedural aspects of fraction knowledge (Hecht, 1998; Hecht et al., 2003). Based on the meta-analysis by Lin and Powell (2021), we defined tasks that did not explicitly require arithmetic operations as conceptual fraction knowledge tasks. We adopted a cross-lagged design to investigate the complex relations among whole number arithmetic, conceptual and procedural fraction knowledge, controlling for students' nonverbal reasoning. Cross-lagged panel models are particularly effective for longitudinal studies, allowing for the exploration of the potential causal relations among multiple variables (Bollen & Curran, 2006).

Method

Participants

The present study was approved by the Institutional Review board at Shandong Normal University. Participants included 1,055 monolingual Chinese students (546 boys; Time 1 M_{age} = 9.8 years; *SD* = 0.7), recruited from three public elementary schools (24 classrooms) in the northern part of China. Students were tested at four time points: In the first semester of Grade 4 (Time 1; December 2021), in the second semester of Grade 4 (Time 2; June 2022), in the first semester of Grade 5 (Time 3; December 2022), and in the second semester of Grade 5 (Time 4;

June 2023). Fathers and mothers reported their education level in the following categories: Elementary school or below (1.4% fathers, 2.2% mothers), junior high school (20.8% fathers, 23.4% mothers), high school or technical secondary school (27.8% fathers, 30.9% mothers), college (27.1% fathers, 25.7% mothers), undergraduate degree (21.6% fathers, 16.9% mothers), and postgraduate degree (1.3% fathers, 0.9% mothers). The median education level was a high school degree for fathers and between a high school degree and a college degree for mothers, representative of low-middle socioeconomic status (SES) in China.

Measures

Nonverbal Reasoning

At Time 1, students completed the Raven's progressive matrices (Raven, 1938) as an index of nonverbal reasoning. Students had to identify the missing piece in a sequence of six geometric figures presented in multiple-choice format. They were given 40 minutes to solve five sets of increasingly difficult problems, with 12 problems in each set. The score was based on the number of correct answers. In previous research, the task demonstrated excellent test-retest reliability ($r = .88$; Sheppard et al., 1968), and high internal reliability based on Chinese samples of students of a similar age (Cronbach's $\alpha = .86$; Xu, Li, et al., 2024).

Whole Number Arithmetic

Across the four time points, students completed a paper-and-pencil assessment of addition, subtraction, multiplication, and division problems from the modified version of the *Arithmetical Ability* subscale of the *Heidelberg Rechen Test* (Haffner et al., 2005), adapted by Wu and Li (2006). For each operation, 40 problems were presented in two columns with increasing difficulty. Students had one minute per operation to solve as many problems as they could, in order. For addition, the first column had problems with single- and double-digit addends; the second column had problems with single-, double-, and triple-digit addends. For subtraction, the first column had single- and double-digit minuends and subtrahends; the second column had problems with double- and triple-digit minuends, and single-, double- and triple-digit subtrahends. For multiplication, the first column had single-digit multiplicands and multipliers; the second column had problems with single- and double-digit multiplicands and multipliers. For division, the first column had single-and double-digit dividends and single-digit divisors; the second column had problems with double- and triple-digit dividends and single-digit divisors. Scoring for the whole number arithmetic task was based on the average number of problems correctly solved across the four operations. Internal reliability (Cronbach's α) was calculated by using the total scores from addition, subtraction, multiplication, and division at Time 1 (α = .87), Time 2 (α = .87), Time 3 (α = .87), and Time 4 (α = .88).

Fraction Knowledge

At each of the four time points, students completed a paper-and-pencil fraction assessment (adapted from Hecht, 1998; Hecht et al., 2003; see the full version of the assessment on the OSF). Students had 10 minutes to solve as many problems as possible, with the option to skip questions. The assessment consisted of 11 items tapping into conceptual fraction knowledge and 8 items tapping into procedural fraction knowledge.[1] Notably, measuring conceptual knowledge of fractions with a single assessment is challenging because conceptual knowledge is a complex and multifaceted construct. As such, our measure does not exhaustively cover all aspects of conceptual knowledge of fractions but rather focuses on aspects that closely align with the curriculum and types of problems students would encounter in their textbooks during this

[1] Due to an administrative error at Time 3, some students completed an incorrect version of one of the conceptual word problems (Question 6). As a result, this item was excluded from further analysis at all time points.

period of learning.

Conceptual Fraction knowledge. Conceptual fraction knowledge was assessed through 11 questions, covering a range of fraction concepts, including ordering (e.g., ordering the fractions $\frac{1}{4}, \frac{1}{8}, \frac{4}{7}, \frac{8}{9}$ from smallest to largest), magnitude (i.e., identifying $\frac{1}{2}$ of $\frac{1}{2}$), estimation (i.e., rounding $\frac{8}{9} + \frac{11}{12}$ to the nearest whole number), density (i.e., stating how many possible fractions exist between $\frac{1}{4}$ and $\frac{1}{2}$), and word problems (e.g., finding the fraction of pizza eaten if a pizza was cut into 4 equal pieces and one slice was eaten). Additionally, two problems required reasoning in the context of fractions: the first involved a fraction number line (When Ben was asked to locate $\frac{4}{7}$ on the number line, he incorrectly stated that it could not be placed on the line because both 4 and 7 are greater than 1. Why is Ben incorrect?). The second involved a depiction of an unequally-divided pictorial representation of a fraction. The accuracy of responses for these problems were coded by two researchers. Notably, these problems could have multiple valid answers; thus, for approximately 10% of the dataset (n = 100) both researchers independently assessed the accuracy of responses to determine inter-rater reliability. For both problems, inter-rater reliability was very high with Cohen's ks > .90. Any discrepancies in coding were discussed until agreement was reached among the researchers. Internal reliability (Cronbach's α) for conceptual fraction knowledge was calculated based on the accuracy of the individual items at each time point: Time 1 (α = .77), Time 2 (α = .82), Time 3 (α = .91), and Time 4 (α = .87).

Procedural Fraction knowledge. Procedural fraction knowledge included addition (n = 4) and subtraction (n = 4) problems. All fractions were proper fractions, with single-digit numerators and denominators (i.e., 1-9). Half of the problems had common denominators whereas the other half had uncommon denominators (see examples in Table 1). Half of the problems were presented in symbolic format whereas the other half were presented pictorially.

For all problems students were asked to provide a symbolic response (i.e., fraction notation). Internal reliability (Cronbach's α) for procedural fraction knowledge was calculated based on the accuracy of the individual items at each time point: Time 1 ($\alpha = .83$), Time 2 ($\alpha = .84$), Time 3 ($\alpha = .94$), and Time 4 ($\alpha = .94$).

Procedure

During school hours, trained experimenters conducted group testing in each classroom across the four time points. Students completed the whole number arithmetic tasks followed by the fraction tasks, with a five-minute break in between. At Time 1, nonverbal reasoning was assessed in a separate session.

Data Analysis

We first conducted repeated measures analysis of variance (ANOVA) to assess the development of students' whole number arithmetic skills and conceptual and procedural fraction knowledge across the four time points using SPSS. Following Field's (2013) recommendation, we applied Greenhouse-Geisser corrections to adjust for any degree of violations of the assumption of sphericity. Next, we tested a cross-lagged structural equation model to examine the co- and bi-direction development of whole number arithmetic skills, conceptual fraction knowledge, and procedural fraction knowledge across the four time points. Model fit was evaluated based on a combination of the comparative fit index (CFI $> .90$), root mean square error of approximation (RMSEA $< .06$), and standardized root mean square residual (SRMR $<.08$; Hu & Bentler, 1999; Kline, 2011).

Except for nonverbal reasoning at Time 1 (15%) and whole number arithmetic at Time 3 (14%) and Time 4 (14%), the percentage of missing cases was below 5% for all variables. To determine if there were differences between participants who completed all four waves of testing (*n* = 911) versus those who were missing at least one wave of data (*n* = 144), t-tests and χ^2-tests were conducted on two demographic variables (i.e., gender and age), nonverbal reasoning, and all mathematical variables (i.e., whole number arithmetic and fraction measures). After Bonferroni corrections for multiple comparisons, no significant differences were found for those with complete versus incomplete data (*p*s > .003), except for procedural fraction knowledge at Time 1; students with complete data had higher scores than those with incomplete data, $p < .001$. Based on our missing data analysis, we were confident that our data met the criteria for missing at random and thus the cross-lagged model was estimated by a full information maximum likelihood method (Enders, 2010). However, to be certain that missing data did not influence patterns of results, we conducted a sensitivity analysis, comparing the cross-lagged model from the whole dataset to the model from only students with complete data. The results were highly similar across the two models and thus all available information was used in all observations to estimate the model.

Transparency and Openness

We adhered to the *Journal Article Reporting Standards* (Kazak, 2018) as recommended in the American Psychological Association guidelines. We report where and how data were collected, provide justification for any data exclusions, report all manipulations, and describe all measures used in the study. The data presented here come from the Chinese Children Mathematical Affection and Cognition Project, a large, longitudinal project focusing on the development of mathematical affect (e.g., anxiety, motivation, attitude, engagement) and mathematical skills (e.g., whole number, fraction, and word problem-solving) from Grades 4 to 5. The present study addressed a unique set of theoretical questions that have not been reported elsewhere. Data were analyzed using SPSS Version 28 (IBM Corp, 2020) and M*plus* Version 8

(Muthén & Muthén, 1998). For visualization, data were transformed in R Version 4.2.1 (R Core Team, 2022) using the *dplyr* (Wickham et al., 2023) and *data.table* (Dowle & Scrinivasan, 2021) packages, and boxplots were produced with the *ggplot2* (Wickham, 2016) package. The anonymized data and code for the measures analyzed in the current study are available for download at the Open Science Framework (OSF): https://osf.io/yspb6/?view_only=588aad76e8f54930abc8863a3294e212. This study's design and its analysis were not preregistered.

Results

Descriptive Statistics

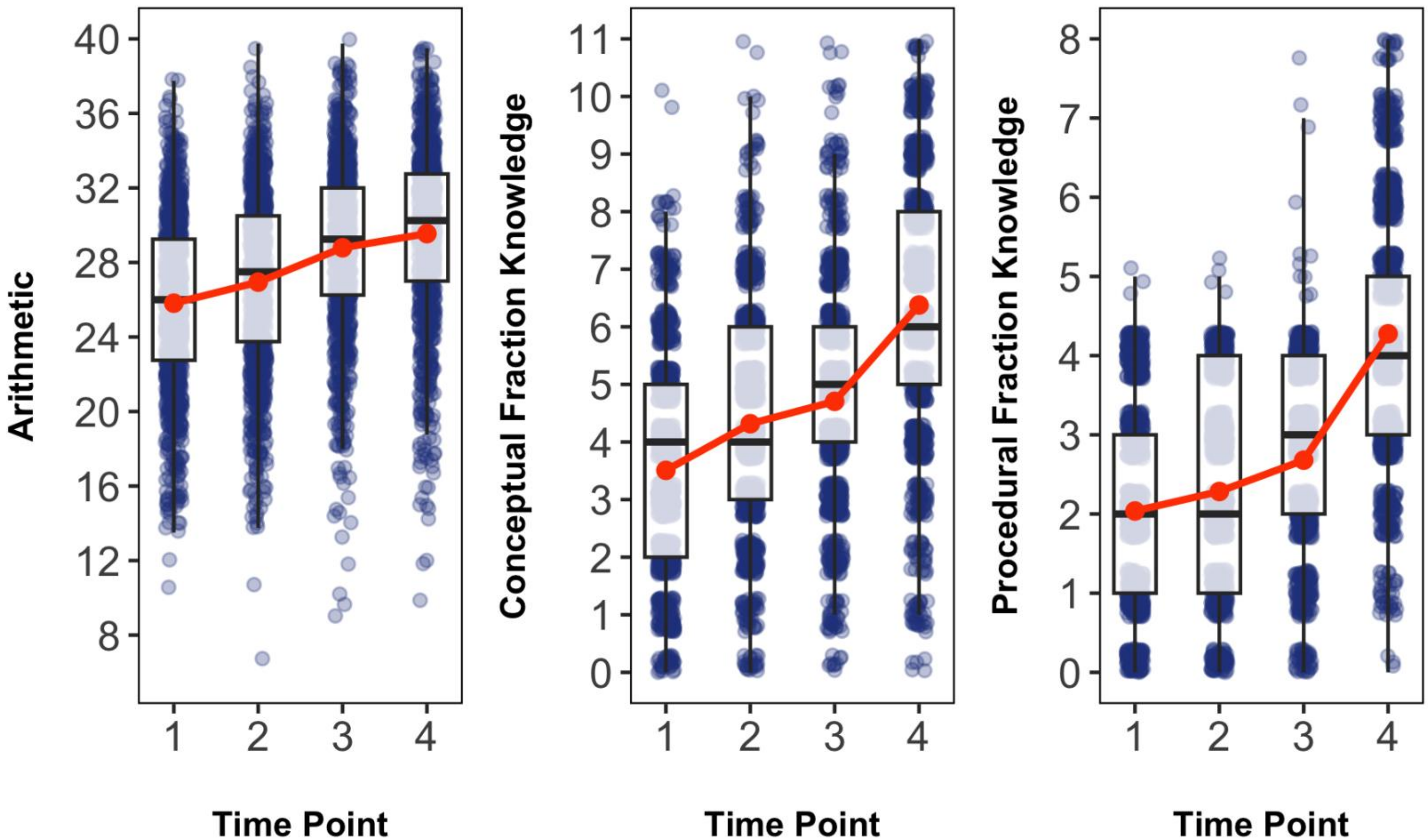
We defined outliers as cases with |z-scores| > 3.29 (Field, 2013). For whole number arithmetic, there were a few negative outliers at Time 1 (n = 1), Time 2 (n = 2), Time 3 (n = 5), and Time 4 (n = 4). For fraction knowledge, there were a few positive outliers for conceptual fraction knowledge at Time 1 (n = 2) and Time 2 (n = 2), and for procedural fraction knowledge at Time 3 (n = 3). Sensitivity analyses conducted with and without these outliers yielded the same patterns of results. Thus, all data were retained for the final analyses.

Box plots (see Figures 1) show the distribution of the data at each time point for whole number arithmetic, conceptual fraction knowledge, and procedural fraction knowledge, respectively. Descriptive statistics and correlations among variables are shown in Tables 1 and 2. Notably, procedural fraction knowledge had low mean scores, particularly at the first three time points. Prior to Grade 5 students had not yet received formal instruction on adding and subtracting fractions with uncommon denominators and thus these assessments were quite challenging. In contrast, many students were able to solve common denominator fraction arithmetic problems in both pictorial and symbolic form. Notably, scores were similar on both pictorial and symbolic fraction arithmetic problems.[2] No significant gender differences were found for any of the measures (i.e., nonverbal reasoning, whole number arithmetic, and fraction assessments) at any of the time points. Thus, we did not control for gender in the cross-lagged analysis.

[2] The slightly lower mean for pictorial compared to symbolic performance at T3 and T4 reflects more wrong operation errors in the pictorial task where the problems alternated in sign (i.e., addition, subtraction, addition, subtraction).

Figure 1

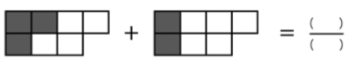
Box Plots and Scatterplots of Performance (Number of Items Correct) by Time



Note. The red dot, horizontal black bar within the box, and vertical black bar show the mean, median, and range, respectively.

Table 1

Descriptive Statistics for Each Measure at Each Time Point

Measure	Time 1			Time 2			Time 3			Time 4		
	M	*SD*	Skew	*M*	*SD*	Skew	*M*	*SD*	Skew	*M*	*SD*	Skew
Nonverbal Reasoning	31.1	8.1	-0.1	-	-	-	-	-	-	-	-	-
Whole Number Arithmetic	25.8	4.6	-0.2	26.9	4.7	-0.4	28.8	4.6	-0.7	29.5	4.7	-0.8
Conceptual Fraction	3.5	1.9	0.1	4.3	2.0	0.0	4.7	2.0	0.2	6.4	2.3	-0.3
Procedural Fraction	2.0	1.4	0.0	2.3	1.4	-0.3	2.7	1.3	-0.5	4.3	1.7	-0.2
Pictorial 	1.0	0.9	-0.0	1.1	0.8	-0.2	1.2	0.8	-0.2	1.6	0.9	0.5
Symbolic $\frac{1}{2} + \frac{2}{3} = \frac{(\quad)}{(\quad)}$	1.3	0.8	-0.6	1.3	0.8	-0.6	1.6	0.7	-1.2	2.8	1.0	-0.2

Note. All scores represent sum scores (i.e., total number of items correct).

Table 2

Correlations Among Nonverbal Reasoning, Whole Number Arithmetic, and Conceptual and Procedural Fraction Knowledge

Variable	T1				T2			T3			T4	
	1	2	3	4	5	6	7	8	9	10	11	12
1. T1 Reasoning	-											
2. T1 Arithmetic	.22	-										
3. T1 Conceptual	.36	.35	-									
4. T1 Procedural	.19	.39	.32	-								
5. T2 Arithmetic	.25	.88	.34	.33	-							
6. T2 Conceptual	.39	.33	.52	.26	.34	-						
7. T2 Procedural	.28	.29	.26	.39	.29	.36	-					
8. T3 Arithmetic	.28	.86	.35	.31	.88	.35	.29	-				
9. T3 Conceptual	.42	.37	.48	.29	.37	.55	.30	.41	-			
10. T3 Procedural	.26	.30	.22	.34	.30	.30	.42	.33	.34	-		
11. T4 Arithmetic	.30	.81	.34	.27	.83	.35	.24	.88	.40	.31	-	
12. T4 Conceptual	.34	.34	.38	.24	.33	.39	.28	.37	.51	.28	.42	-
13. T4 Procedural	.35	.25	.28	.19	.25	.28	.23	.32	.38	.29	.37	.41

Note. Shaded regions indicate within-grade correlations. All correlations were statistically significant at $p < .001$. T1 = Time 1. T2 = Time 2. T3 = Time 3. T4 = Time 4.

Development of Whole Number Arithmetic and Fraction Knowledge

The development of whole number arithmetic, conceptual fraction knowledge, and procedural fraction knowledge were analyzed in three repeated measures ANOVAs. Post hoc pairwise comparisons were conducted using the Bonferroni adjustment. Students improved over time on whole number arithmetic, $F(2.74, 2400.43) = 779.12$, $p < .001$, $\eta_p^2 = .73$, conceptual fraction knowledge, $F(2.84, 2720.58) = 721.81$, $p < .001$, $\eta_p^2 = .43$, and procedural fraction knowledge, $F(2.72, 2583.70) = 637.03$, $p < .001$, $\eta_p^2 = .40$. For all three measures, all pairwise comparisons were significant, *p*s < .001. All measures were also significantly correlated, *p*s < .001 (see Table 2). Thus, we proceeded with cross-lagged panel analyses to further investigate the co- and bidirectional development of these measures.

Cross-lagged Structural Equation Modelling

Cluster Effect of Classroom

Students were recruited from 24 different classrooms. To evaluate the potential effects of classroom, we tested an intercept-only multilevel model with classroom as a random effect. Across the four time points, intra-class correlation coefficients (ICCs), indicating the amount of variance accounted for by differences between classrooms, varied. Specifically, ICCs ranged from 5.0% to 13.9% for whole number arithmetic, 2.0% to 6.3% for conceptual fraction knowledge, and 1.0% to 19.4% for procedural fraction knowledge, with all *p*s < .05. These results suggest low to modest between-classroom variability for these measures. Accounting for the cluster effect of classroom in the cross-lagged models led to misspecification, likely due to the number of estimated parameters exceeding the number of clusters. However, the pattern of results was highly consistent with those from analyses that did not consider the cluster effect of classroom. Therefore, the most parsimonious model, which did not include the classroom cluster effect, was retained for subsequent analyses.

Model Specifications

We tested a series of nested panel models to assess the overall structure of the relations among whole number arithmetic, conceptual fraction knowledge, and procedural fraction knowledge across Times 1 through 4. For each of these variables at each time point, we controlled for nonverbal reasoning. Fit indices for all tested models are shown in Table 3.

We first tested an independent first-order autoregressive model (Model 1) including only the first-order autoregressive effects (i.e., the path coefficients between adjacent time points). This model showed a poor fit to the data. Following Geiser's (2013) recommendation, we modified the model by adding second-order autoregressive effects (i.e., the path coefficients between nonadjacent time points) for whole number arithmetic, conceptual fraction knowledge, and procedural fraction knowledge one at a time (Model 2A-C). The results showed that adding second-order autoregressive effects yielded better model fit (see Table 3). Finally, we added the cross-lagged paths to the model (Model 3), which led to further significant improvements in model fit (see Table 3). This model showed a great fit to the data. Thus, we present and interpret the results of Model 3. For readability, path coefficients are reported in Tables 4 to 6 but are not shown in Figure 2.

Model Interpretations

Whole number arithmetic contributed to the development of both conceptual and procedural knowledge across all four time points (see Figure 2). In contrast, neither the cross-lagged path coefficients from conceptual knowledge to whole number arithmetic nor procedural knowledge to whole number arithmetic were significant. These results suggest that improvements in fraction knowledge do not lead to significant improvements in whole number arithmetic.

Across the time points, procedural fraction knowledge did not significantly predict later conceptual knowledge, indicating that the development of conceptual knowledge was not dependent on prior procedural knowledge. Conceptual knowledge at Time 1 did not significantly predict procedural knowledge at Time 2. However, starting at Time 2, conceptual fraction knowledge predicted the growth in procedural fraction knowledge. That is, individual differences in procedural fraction knowledge at subsequent time points were, in part, accounted for by prior conceptual fraction knowledge. This relation is illustrated in Figure 2. Moreover, as shown in Table 6, a significant indirect path was observed from whole number arithmetic at Time 2 to procedural knowledge at Time 4 via conceptual knowledge at Time 3, showing that conceptual knowledge served as a mediator in the relation between whole number arithmetic and procedural fraction knowledge between Times 2 and 4.

We further examined whether the strengths of the significant cross-lagged path coefficients from conceptual to procedural fraction knowledge increased progressively from Times 2 to 4. Thus, we compared models in which these cross-lagged paths from Times 2 to 3 and Times 3 to 4 were either constrained to be equal or were freely estimated using the Satorra-Bentler scaled chi-square difference test (Satorra & Bentler, 2010). The unconstrained model comparing the cross-lagged paths from Time 2 to 3 and from Time 3 to 4 fit better than the constrained model, $\Delta\chi^2(1) = 9.97$, $p = .002$, suggesting the latter cross-lagged path coefficient was stronger than the former ($\beta_{\text{Time 3-4}} = .21 > \beta_{\text{Time 2-3}} = .09$). These results show that the predictive relation between conceptual knowledge and the development of procedural knowledge increased as time progressed.

Table 3

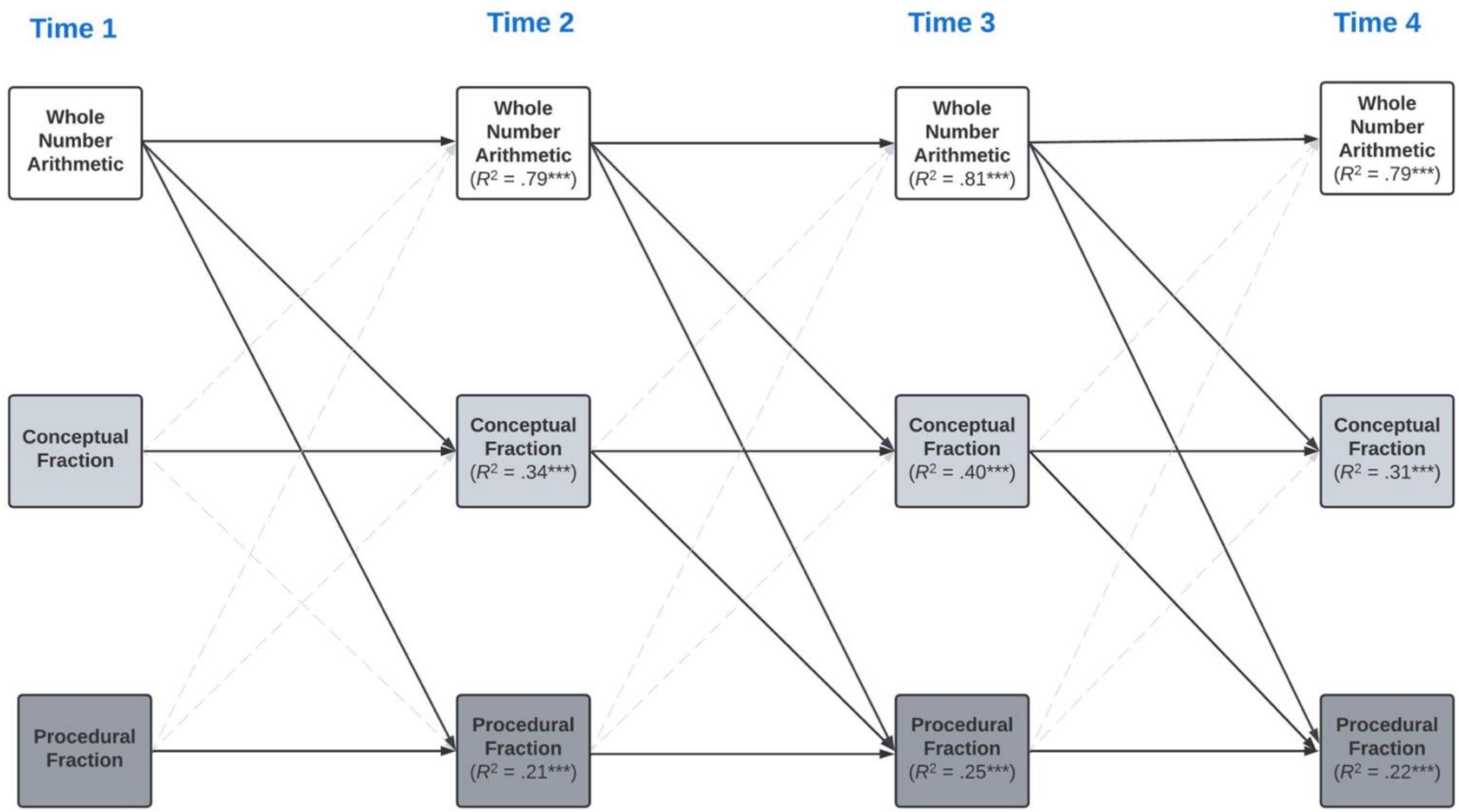
Fit Indices and Likelihood Ratio Tests comparing Nested Models of the Relations Among Whole Number Arithmetic, Conceptual Fraction Knowledge, and Procedural Fraction Knowledge

Fit Indices	Model 1	Model 2A	Model 2B	Model 2C	Model 3
df	45	42	39	36	18
χ^2	579.23	365.69	281.13	236.96	50.49
RMSEA	.106	.085	.077	.073	.041
90% CI	[.098, .114]	[.078, .094]	[.068, .085]	[.064, .082]	[.028, .055]
CFI	.919	.951	.963	.970	.995
TLI	.871	.916	.933	.939	.980
SRMR	.099	.098	.086	.078	.012
Likelihood Ratio Test	-	$\Delta\chi^2(3) = 213.54$, $p < .001$	$\Delta\chi^2(3) = 84.56$, $p < .001$	$\Delta\chi^2(3) = 44.17$, $p < .001$	$\Delta\chi^2(18) = 186.47$, $p < .001$

Note. Model 1 is a first-order autoregressive model. Model 2A is a second-order autoregressive model with arithmetic. Model 2B is a second-order autoregressive model with arithmetic and conceptual fraction knowledge. Model 2C is a second-order autoregressive model with arithmetic, conceptual fraction knowledge, and procedural fraction knowledge. Model 3 is a second-order autoregressive cross-lagged model.

Figure 2

Cross-Lagged Panel Model (N = 1055)



Note. Solid lines represent significant paths. Faded dashed line represents nonsignificant paths (see the full report of the path coefficients in Tables 4-7).

Table 4

Standardized Parameters for the Cross-Lagged and Autoregressive Paths for the Cross-Lagged Model

Paths	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI
Cross-lagged Paths	Time 1 – Time 2			Time 2 – Time 3			Time 3 – Time 4		
Conceptual to Procedural	.06(.03)	.075	[-.006, .119]	.09(.03)	.007	[.024, .153]	.21(.03)	<.001	[.139, .272]
Arithmetic to Procedural	.11(.03)	<.001	[.049, .170]	.11(.03)	<.001	[.052, .173]	.13(.03)	<.001	[.056, .194]
Procedural to Conceptual	.04(.03)	.127	[-.012, .099]	.05(.03)	.069	[-.004, .103]	.05(.03)	.102	[-.010, .110]
Arithmetic to Conceptual	.14(.03)	<.001	[.080, .192]	.14(.03)	<.001	[.084, .191]	.13(.03)	<.001	[.075, .194]
Conceptual to Arithmetic	.03(.02)	.080	[-.004, .063]	.01(.02)	.747	[-.028, .040]	.03(.02)	.125	[-.028, .040]
Procedural to Arithmetic	-.02(.02)	.230	[-.052, .012]	.00(.02)	.945	[-.033, .031]	-.01(.02)	.446	[-.033, .031]
First-order Autoregressive	Time 1 – Time 2			Time 2 – Time 3			Time 3 – Time 4		
Conceptual to Conceptual	.38(.03)	<.001	[.328, .437]	.30(.03)	<.001	[.243, .364]	.30(.04)	<.001	[.231, .372]
Procedural to Procedural	.30(.03)	<.001	[.239, .356]	.26(.03)	<.001	[.196, .318]	.10(.03)	.002	[.040, .169]
Arithmetic to Arithmetic	.87(.01)	<.001	[.852, .893]	.53(.03)	<.001	[.474, .593]	.56(.03)	<.001	[.495, .631]
Second-order Autoregressive	Time 1 – Time 3			Time 1 – Time 4			Time 2 – Time 4		
Conceptual to Conceptual	.20(.03)	<.001	[.144, .260]	.11(.03)	.001	[.043, ,167]	.07(.03)	.045	[.002, .132]
Procedural to Procedural	.15(.03)	<.001	[.092, .213]	.01(.03)	.727	[-.050, .072]	.03(.03)	.295	[-.029, .097]
Arithmetic to Arithmetic	.22(.04)	<.001	[.311, .432]	.12(.04)	.001	[.045, .185]	.22(.04)	<.001	[.146, .289]

Table 5

Standardized Parameters for the Covariances for the Cross-Lagged Model

Covariances	Time 1			Time 2			Time 3			Time 4		
	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI
Conceptual with Procedural	.32(.03)	<.001	[.269, .378]	.21(.03)	<.001	[.149, .267]	.14(.03)	<.001	[.073, .197]	.21(.03)	<.001	[.144, .278]
Conceptual with Arithmetic	.34(.03)	<.001	[.290, .397]	.07(.03)	.019	[.012, .137]	.10(.03)	.004	[.031, .165]	.16(.04)	<.001	[.089, .230]
Procedural with Arithmetic	.39(.03)	<.001	[.336, .439]	.07(.03)	.036	[.004, .130]	.09(.03)	.009	[.023, .157]	.21(.03)	<.001	[.147, .274]
Reasoning with Conceptual	.36(.03)	<.001	[.304, .416]	.22(.03)	<.001	[.158, .272]	.17(.03)	<.001	[.116, .233]	.11(.03)	.001	[.042, 168]
Reasoning with Procedural	.20(.03)	<.001	[.139, .263]	.18(.03)	<.001	[.114, .238]	.11(.03)	.002	[.040, .179]	.18(.03)	<.001	[.114, .248]
Reasoning with Arithmetic	.24(.03)	<.001	[.177, .299]	.04(.02)	.024	[.005, .072]	.06(.02)	.002	[.022, .094]	.04(.02)	.020	[.007, .081]

Table 6

Standardized Parameters for the Indirect Paths from Whole Number Arithmetic to Conceptual or Procedural Fraction Knowledge for the Cross-Lagged Model

Indirect Paths	Time 1 – Time 2 – Time 3			Time 2 – Time 3 – Time 4		
	β (SE)	*p*	95% CI	β (SE)	*p*	95% CI
Arithmetic to Conceptual to Conceptual	.04(.01)	<.001	[.022, .060]	.04(.01)	<.001	[.022, .060]
Arithmetic to Procedural to Conceptual	.01(.00)	.107	[-.001, .012]	.01(.00)	.139	[-.002, .013]
Arithmetic to Arithmetic to Conceptual	.12(.02)	<.001	[.073, .167]	.07(.02)	<.001	[.039, .105]
Arithmetic to Conceptual to Procedural	.01(.01)	.020	[.002, .022]	.03(.01)	<.001	[.014, .043]
Arithmetic to Procedural to Procedural	.03(.01)	.001	[.011, .045]	.01(.01)	.018	[.002, .021]
Arithmetic to Arithmetic to Procedural	.10(.03)	<.001	[.045, .151]	.07(.02)	<.001	[.031, .101]

Discussion

The development of conceptual and procedural knowledge has received considerable attention in mathematical education over the past several decades, both from studies of fraction learning (e.g., Bailey et al., 2015; Hallett et al., 2010, 2012) and in mathematics more generally (e.g., Halford, 1993; Hiebert & LeFevre, 1986; Karmiloff-Smith, 1992; Rittle-Johnson, 2017; Rittle-Johnson & Alibali, 1999). Longitudinal studies investigating the co-development of these two types of fraction knowledge have focused on students educated in the U.S. These studies have found an iterative relation between conceptual and procedural fraction knowledge, with one type of knowledge supporting the development of the other (Bailey et al., 2017; Hecht & Vagi, 2010; Rittle-Johnson et al., 2001). However, the development of fraction knowledge is inherently influenced by the mathematical curriculum to which students are exposed and thus it remains unclear whether this pattern of development is universal or specific to the U.S. For example, studies have suggested that important differences in curricula and textbooks likely contribute to the superior math learning, including fractions, that is seen in East-Asian countries compared to the U.S. (Zhang & Siegler, 2022). Thus, in the present study we examined the co-development of conceptual fraction knowledge, procedural fraction knowledge, and whole number arithmetic skills from Grade 4 to Grade 5 for Chinese students.

The Bi- and Co-development of Fraction Knowledge

Using cross-lagged panel analysis, we found that conceptual and procedural fraction knowledge did not initially predict each other's development in Grade 4. This finding is consistent with that of Bailey et al. (2017), who observed little transfer between conceptual and procedural fraction knowledge among U.S. students from Grades 4 to 5. Bailey et al. suggested that this limited transfer of knowledge during the initial phase of fraction learning may arise from a weak understanding of both fraction concepts and procedures. Consistent with this perspective, Chinese students demonstrated weak conceptual and procedural fraction knowledge in Grade 4, with accuracy rates of approximately 30% and 20%, respectively. In alignment with curriculum expectations, students had particularly poor performance on concepts and procedures for which formal instruction is not received prior to Grade 5 (i.e., symbolic fraction magnitude comparison, ordering a mix of proper and improper fractions, fraction arithmetic with uncommon denominators).

Starting from Grade 5 (i.e., Time 2 to Time 3 and Time 3 to Time 4), we observed a unidirectional development, with conceptual knowledge driving the development of procedural knowledge. This finding aligns with four decades of research wherein conceptual knowledge of fractions has been found to impact the learning of fraction arithmetic procedures (Braithwaite & Siegler, 2021; Byrnes & Wasik, 1991; Hiebert & Wearne, 1986; Rittle-Johnson et al., 2001). The development of conceptual fraction knowledge is believed to lay the foundation for developing accurate strategies for procedural fraction problems (Halford, 1993; Hiebert & LeFevre, 1986; Lamon, 2012; Perry, 1991) with conceptual knowledge acting as a support system to make the learning process of procedural knowledge more meaningful (Silver, 1986). For example, understanding fractional units helps students comprehend why they need to find a common denominator (to ensure they share the same fractional units) when adding or subtracting two fractions (Ma & Kessel, 2022). Moreover, students with stronger conceptual knowledge of fractions are more equipped to identify errors in procedural tasks than their peers who merely follow algorithmic steps without understanding the underlying fraction concepts (Hecht, 1998; Tian & Siegler, 2017). In the present study, we further found that the predictive relation between conceptual fraction knowledge and the development of procedural fraction knowledge strengthened over time. We speculate that improvements in both types of knowledge over time may reflect a more fluid transfer of knowledge as students gain more experience and receive more instructions with fractions. With time, students were able to correctly answer more problems, such that at Time 4 Chinese students had accuracy rates of approximately 64% and 54% for conceptual and procedural fraction knowledge, respectively.

Contrary to conceptual fraction knowledge, and inconsistent with findings from studies with U.S. students (Bailey et al., 2017; Hecht & Vagi, 2010; Rittle-Johnson et al., 2001), procedural fraction knowledge did not significantly predict the development of conceptual fraction knowledge. Notably, the acquisition of procedural knowledge does not necessarily depend on conceptual knowledge (Silver, 1986), with Braithwaite et al. (2018) suggesting that the development of fraction arithmetic may be unconstrained by conceptual understanding. For example, when students need to add or subtract fractions, they can reach an accurate solution by simply memorizing the procedural steps without developing an understanding of the underlying conceptual principles. However, a lack of conceptual understanding of fraction arithmetic may lead to overgeneralizations in both whole number and wrong fraction operation errors (Braithwaite et al., 2017). Likewise, relying solely on procedural steps is unlikely to assist students in developing their overall fraction knowledge. For example, in a study with students in Grades 4 and 5 (ages 8 to 9) in the U.K., clusters which varied in terms of relative success with the two types of fraction knowledge were identified such that students with higher conceptual and lower procedural knowledge outperformed those with higher procedural and lower conceptual knowledge suggesting that conceptual approaches are more successful in supporting mathematics learning (Hallett et al., 2010). Interestingly, in a study with students in Grades 6 and 8, conceptual fraction knowledge uniquely predicted mathematics achievement scores for U.S.

and Chinese students, whereas procedural fraction knowledge only predicted U.S. students' achievement scores (Torbeyns et al., 2015). Thus, the co-development of conceptual and procedural fraction knowledge and their relation to both general fraction performance and mathematics achievement may reflect both individual differences as well as differences in educational experiences.

The Role of Whole Number Arithmetic in the Development of Fraction Knowledge

The positive relation between previously acquired whole number arithmetic fluency and conceptual and procedural fraction knowledge is well established (Hecht, 1998; Hecht et al., 2003; Hecht & Vagi, 2010) with both types of fraction knowledge building upon strong foundational whole number arithmetic skills (Braithwaite & Siegler, 2023; Hiebert, 1988; Siegler & Lortie-Forgues, 2014). However, how whole number arithmetic skills develop in tandem to fraction knowledge is under researched. To our knowledge, this study is the first to simultaneously consider the co- and bidirectional development among whole number arithmetic, and conceptual and procedural fraction knowledge.

The finding of a unidirectional predictive link from whole number arithmetic to fraction knowledge expands upon the significant correlations observed between whole number arithmetic and both conceptual and procedural fraction knowledge, as reported in a meta-analysis by Lin and Powell (2021). The shared link between whole number arithmetic and fraction knowledge may reflect a stronger understanding of integrated whole and rational number knowledge (Braithwaite & Siegler, 2023; Siegler et al., 2011; Siegler, 2016; Xu et al., 2023; Xu, Di Lonardo Burr, et al., 2024). Possibly, however, the unidirectional relation between whole number arithmetic and fraction knowledge reflects that students at this age are still in the process of integrating both whole and rational number knowledge. While research with Chinese students shows that the highest speeds of mental operations are found in Grade 3 and 4, a period of intense practice and training (Zhang & Zhou, 2003; Zhang et al., 2002), students are still only in the early phases of rational number learning, with instruction continuing until Grade 6 (Zhang & Siegler, 2022). Thus, we speculate that the influence of whole number arithmetic on both conceptual and procedural fraction knowledge may instead reflect the mathematical hierarchy, wherein whole number arithmetic forms the foundation for acquiring fraction knowledge (Siegler et al., 2011).

Some researchers emphasize the distinctions between whole number and fraction knowledge (e.g., Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004) or suggest that whole number arithmetic instruction featuring a limited variety of unit types may contribute to students' rigid concept of number (Behr et al., 1993). Behr et al. (1993) suggest that more practice with whole number arithmetic situations involving units of units and a variety of unit types will create a cognitive bridge for learning concepts and operations with fractions. Indeed, in China arithmetic instruction emphasizes unit conversions as well as composing and decomposing approaches which may promote flexible thinking and reasoning in students' number concept (Sun et al., 2019). This experience with representing and manipulating quantities may lead to strong foundational skills which in turn support fraction learning.

Our view and findings closely align with researchers who emphasize the similarities between whole number and fraction knowledge (Sidney, 2020; Sidney et al., 2022; Sun, 2019). In particular, Ma and Kessel (2022) highlighted the parallels between whole number and fraction arithmetic, demonstrating how the procedures for addition and subtraction of fractions closely resemble those for whole numbers. When adding two numbers, a third number is formed containing units from both. This principle applies to both whole numbers and fractions, with only numbers that have identical unit values being suitable for the arithmetic operation (Ma & Kessel, 2022). Thus, whole numbers and fractions are inherently interconnected, with strong skills in whole number arithmetic supporting the development of fraction knowledge.

The Role of Instruction and Educational Implications

The co-development of conceptual and procedural fraction knowledge may, in part, be related to differences in instruction. For example, with students in the U.K., Hallett et al. (2010) found differences in the relative distribution of clusters across schools and suggested that these differences might reflect differing emphasis on concepts over procedures in some classrooms. The findings of the present study are consistent with the Chinese curriculum wherein prior to formal instruction on fraction procedures, conceptual foundations for these procedures are emphasized by highlighting the link between conceptual and procedural knowledge. For example, Chinese students are introduced to fractional units before formal instruction on fraction procedures. They are guided to extend their knowledge to include counting and arithmetic operations using unit fractions (Sun, 2019). This process lays the foundation for learning operations involving fractions, as conceptual knowledge may enhance students' efficiency in learning procedures (Braithwaite & Siegler, 2021; Carpenter, 1986).

Differences in co-development of conceptual and procedural fraction knowledge may also reflect differences in teacher knowledge. For example, in a study comparing procedural and conceptual knowledge of fraction operations for Chinese and American preservice teachers, Chinese preservice teachers had better conceptual and procedural fraction knowledge, with the difference between conceptual knowledge greater than that of procedural knowledge (Lin et al., 2013). These differences have also been found in experienced teachers wherein Chinese teachers succeeded not only in correctly dividing fractions (e.g., $1\frac{3}{4} \div \frac{1}{2}$) but were also able to demonstrate a conceptual understanding; in contrast none of the American teachers demonstrated an understanding of the algorithm and less than half succeeded in their calculations (Ma, 1999). Looking at the Trends in International Mathematics and Science Study (Mullis et al., 2020) mathematics performance in Grades 4 and 8 was found to be strongly related to both cultural and instructional differences, suggesting that both of these factors may play a critical role in mathematical development. In the present study, we speculate that teachers' strong conceptual fraction knowledge coupled with an emphasis on connecting procedures to conceptual knowledge may help to explain the unidirectional pattern of fraction knowledge development.

Beyond connecting fraction concepts and procedures, strong foundational skills in whole number arithmetic are necessary to accurately identify common denominators, combine numerators, reduce fractions to their simplest form, or convert fractions to decimals for a wide range of fraction tasks (Braithwaite & Siegler, 2023; Fazio et al., 2016; Rinne et al., 2017). Thus, students who excel in whole number arithmetic are better equipped to solve fraction arithmetic problems more efficiently compared to those with weaker skills. With respect to classroom instruction, exploiting the conceptual similarities between whole number and fraction arithmetic principles may further support students' knowledge of fraction arithmetic. For example, educators can make use of analogies that emphasize the similar goal structure between whole number and fraction arithmetic, such as "whole number/fractional unit" and "like units" (Ma & Kessel, 2022; Sidney, 2020). In China, such mastery of whole number arithmetic is an important component of the early primary mathematics curriculum in China (Ministry of Education, 2011). As such, in contrast to the U.S., fractions are not introduced until after students have developed strong whole number arithmetic skills (Zhang & Siegler, 2022). Thus, our finding that whole number arithmetic predicted the change in conceptual and procedural fraction knowledge over time is consistent with the educational experiences of these students. Moreover, in addition to considering educational experiences, our research highlights the need for researchers to include the measurement of whole number arithmetic skills alongside fraction knowledge, rather than treating them as control variables.

In summary, an emphasis on conceptual understanding in fraction instruction and strong foundational whole number arithmetic skills may help to explain the results of the present study as well as the superior fraction performance often seen in Chinese students. Emphasis on conceptual understanding may extend beyond just fraction learning. For example, in a study with Grade 2 students in the U.S., Rittle-Johnson and colleagues found that, within a single lesson on mathematical equivalence, spending more time on conceptual instruction led to better retention of conceptual and procedural knowledge than time spent teaching a procedure (Rittle-Johnson et al., 2016). Moving into more advanced mathematics, many upper secondary school students showed good procedural knowledge for mathematical functions but modest conceptual knowledge whereas all students who showed good conceptual knowledge also showed good procedural knowledge; procedural knowledge alone was insufficient for students to be able to apply functions (Lauritzen, 2012). Interview responses suggested that these findings reflected instructional practices with students reporting that instruction had predominately focused on procedures without links to abstract concepts. Kilpatrick et al. (2001) suggest that conceptual understanding and procedural fluency are interconnected, with conceptual knowledge leading to more effective learning through reduction in susceptible errors and procedural fluency building upon strong conceptual understanding when students are faced with higher, more complex mathematical concepts. Overall, both conceptual and procedural knowledge are necessary not only for fractions but for many aspects of mathematics, with conceptual knowledge potentially having a greater impact on procedural knowledge than vice versa.

Limitations and Future Research

In the present study, students were given ten minutes to complete fraction assessments. While this time frame is consistent with the types of assessments commonly administered to students in Chinese classrooms, it is possible that, given their relative novelty to fraction concepts, particularly in Grade 4, the allotted time might have been insufficient for them to carefully consider different strategies. This limitation could have potentially contributed to the observed low performance. Previous research indicates that time constraints may impact strategy, thereby affecting performance outcomes (see a review in Caviola et al., 2017). Thus, future research should consider providing students with more time to complete both conceptual and procedural fraction knowledge assessments at their own pace.

Also, students in the present study were tested in groups to accommodate for the large number of students being tested at four time points. With this approach, we were only able to control for one domain-general cognitive skill (i.e., non-verbal reasoning). However, it is important to acknowledge that the observed relations between the development of whole number arithmetic and fraction knowledge may be explained by shared variance in other domain-general skills, such as working memory, language and executive functions (see a review by De Smedt, 2022). Therefore, future studies should include measures of working memory, language, and executive functions to provide a more comprehensive understanding of the development of mathematics (Cragg & Gilmore, 2014; Peng et al., 2016, 2020).

Our study focused on the development of fraction knowledge during the early stages of fraction learning. Our findings suggest that there is still room for improvement among the students, especially in procedural fraction knowledge. Considering that students in our sample had only been introduced to fraction procedures for a few months, the limited transfer from procedural to conceptual knowledge may reflect insufficient practice with fraction procedures. Notably, by Grade 6, Chinese students demonstrate strong performance in fraction arithmetic (Bailey et al., 2015; Xu, Di Lonardo Burr, et al., 2024). Therefore, it is possible that if we had continued to follow these students for another year, we might have observed bidirectional relations between conceptual and procedural knowledge of fractions. In the future, researchers should consider following students from Grades 4 to 6 to capture the development of conceptual and procedural fraction knowledge more comprehensively.

Our study made clear distinctions between conceptual versus procedural fraction knowledge based on operationalizations from the literature (e.g., Lin & Powell, 2021). Nevertheless, it is challenging to separate and measure these two types of knowledge in isolation (Crooks & Alibali, 2014). For example, students may rely on some fraction concepts to solve procedural problems and vice versa. With respect to the present study, it is possible that our pictorial fraction arithmetic task tapped into aspects of both conceptual (i.e., fraction mapping) and procedural (i.e., fraction arithmetic) fraction knowledge. While the similar scores on both pictorial and symbolic arithmetic suggests that this potential overlap in concepts did not drive the results of our study, nevertheless, future research gathering information about how students approach problems (e.g., strategy reports, showing their work, providing explanations) may provide more insights into how these two types of knowledge, together, support fraction understanding. Moreover, the operationalizations of conceptual versus procedural tasks are not always consistent in the literature. In particular, the conceptual classification of some of the tasks in the present study (i.e., estimation, word problems) contrasts with other longitudinal studies (i.e., Hecht & Vagi, 2010; Rittle-Johnson et al., 2001). Thus, it is possible that differences in directional findings may be explained by how tasks are classified across studies. In the future, to better understand fraction knowledge development from a cross-cultural perspective, there is a need for researchers to reach a consensus on how the two types of fraction knowledge are operationalized.

Our measure of conceptual fraction knowledge only tapped into some subconstructs of fractions. Conceptual knowledge of fractions is a complex and multifaceted construct. While our measure was chosen based on students' grade level and expected exposure to various fraction concepts, it did not exhaustively cover all aspects of conceptual knowledge. Moreover, our study did not capture other important aspects of fraction knowledge, such as flexibility between different symbolic representations of rational number (Pittalis, 2024; Schiller & Siegler, 2023). Thus, in future research, a more comprehensive measure of conceptual knowledge combined with measures that tap into other aspects of rational number knowledge (e.g., decimals, percentages) may provide a more complete picture of how conceptual and procedural rational number knowledge co-develop.

Conclusion

The co-development of conceptual and procedural fraction knowledge is not well understood because there are limited longitudinal studies. Moreover, while the role of instruction may be an underlying factor driving previous findings, studies have only looked at students educated in the U.S. The present study substantially builds upon previous research by longitudinally investigating the co-development of conceptual and procedural fraction knowledge amongst students educated in China, where the centralized curriculum is more condensed, less repetitive, and whole-number multiplicative knowledge is a prerequisite for fraction learning (Zhang & Siegler, 2022). We show that in the early stages of fraction learning, conceptual and procedural fraction knowledge do not support the development of one another. However, as students progress and are introduced to more advanced fraction concepts, conceptual fraction knowledge supports the development of procedural fraction knowledge. This unilateral conceptual-to-procedural pattern of development is inconsistent with findings from studies in the U.S., emphasizing the importance of considering the role of instruction in the development of these two types of fraction knowledge. Moreover, we consider the role of whole number arithmetic fluency in this co-development, treating it not as a control variable or single time-point predictor, but rather a skill that students continue to develop in Grades 4 and 5. We show that proficiency in whole number arithmetic emerges as a crucial precursor, with strong skills supporting the development of both conceptual and procedural fraction knowledge over time. These findings highlight the important roles of whole number arithmetic and conceptual fraction knowledge in laying the groundwork for students to acquire fraction procedures.

References

Abreu-Mendoza, R. A., Powell, A. B., Renninger, K. A., Rivera, L. M., Vulic, J., Weimar, S., & Rosenberg-Lee, M. (2023). Middle-schoolers' misconceptions in discretized nonsymbolic proportional reasoning explain fraction biases better than their continuous reasoning: Evidence from correlation and cluster analyses. *Cognitive Psychology*, *143*, 101575. https://doi.org/10.1016/j.cogpsych.2023.101575

Bailey, D. H., Hansen, N., & Jordan, N. C. (2017). The codevelopment of children's fraction arithmetic skill and fraction magnitude understanding. *Journal of Educational Psychology*, *109*(4), 509–519. https://doi.org/10.1037/edu0000152

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, *113*(3), 447–455. https://doi.org/10.1016/j.jecp.2012.06.004

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., Gersten, R., & Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68–83. https://doi.org/10.1016/j.jecp.2014.08.006

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis-emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13–47). Lawrence Erlbaum Associates.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 91–125). Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*(5), 323–341. https://doi.org/10.5951/jresematheduc.15.5.0323

Bempeni, M., Poulopoulou, S., Tsiplaki, I., & Vamvakoussi, X. (2018). Individual differences in fractions' conceptual and procedural knowledge: What about older students? In M. Bergqvist, C. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (pp. 147–154). PME.

Bollen, K. A., & Curran, P. J. (2006). *Latent curve models: A structural equation approach.* Wiley.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253. https://doi.org/10.1016/j.cedpsych.2012.07.001

Boutlet, G. (1998). Didactical implications of children's difficulties in learning the fraction concept. *Focus on Learning Problems in Mathematics*, *20*(4), 19–34.

Braithwaite, D. W., Pyke, A. A., & Siegler, R. S. (2017). A computational model of fraction arithmetic. *Psychological Review*, *124*(5), 603–625. https://doi.org/10.1037/rev0000072

Braithwaite, D. W., & Siegler, R. S. (2018). Developmental changes in the whole number bias. *Developmental Science*, *21*(2), e12541. https://doi.org/10.1111/desc.12541

Braithwaite, D. W., & Siegler, R. S. (2021). Putting fractions together. *Journal of Educational Psychology*, *113*(3), 556–571. https://doi.org/10.1037/edu0000477

Braithwaite, D. W., & Siegler, R. S. (2023). A unified model of arithmetic with whole numbers, fractions, and decimals. *Psychological Review*. https://doi.org/10.1037/rev0000440

Braithwaite, D. W., Tian, J., & Siegler, R. S. (2018). Do children understand fraction addition? *Developmental Science*, *21*(4), e12601. https://doi.org/10.1111/desc.12601

Bray, W. S., & Abreu-Sanchez, L. (2010). Using number sense to compare fractions. *Teaching Children Mathematics*, *17*(2), 90–97. https://doi.org/10.5951/TCM.17.2.0090

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum. *The Journal of Mathematical Behavior*, *23*(1), 1–20. https://doi.org/10.1016/j.jmathb.2003.12.001

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*(5), 777–786. https://doi.org/10.1037/0012-1649.27.5.777

Carpenter, T. P. (1986). Conceptual knowledge as a foundation for procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 113–132). Lawrence Erlbaum Associates, Inc.

Caviola, S., Carey, E., Mammarella, I. C., & Szucs, D. (2017). Stress, time pressure, strategy selection and math anxiety in mathematics: A review of the literature. *Frontiers in Psychology*, *8*, 1488. https://doi.org/10.3389/fpsyg.2017.01488

Chan, W.-H., Leu, Y.-C., & Chen, C.-M. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth gaders in Taiwan. *The Journal of Experimental Education*, *76*(1), 26–57. https://doi.org/10.3200/JEXE.76.1.26-58

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, *64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Cragg, L., & Gilmore, C. (2014). Skills underlying mathematics: The role of executive function in the development of mathematics proficiency. *Trends in Neuroscience and Education*, *3*(2), 63–68. https://doi.org/10.1016/j.tine.2013.12.001

Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. *National Council of Teachers of Mathematics*, *13*(8), 490–496. https://doi.org/10.5951/MTMS.13.8.0490

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review*, *34*(4), 344–377. https://doi.org/10.1016/j.dr.2014.10.001

Davis, G., Hunting, R. P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior*, *12*(1), 63–76.

De Smedt, B. (2022). Individual differences in mathematical cognition: A Bert's eye view. *Current Opinion in Behavioral Sciences*, *46*, 101175. https://doi.org/10.1016/j.cobeha.2022.101175

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, *37*, 39–49. https://doi.org/10.1016/j.learninstruc.2014.07.002

Di Lonardo Burr, S. M., Douglas, H., Vorobeva, M., & Muldner, K. (2020). Refuting misconceptions: Computer tutors for fraction arithmetic. *Journal of Numerical Cognition*, *6*(3), 355–377. https://doi.org/10.5964/jnc.v6i3.310

Dowle, M., & Scrinivasan, A. (2021). _*data.table: Extension of 'data.frame'*_ [Computer software]. https://CRAN.R-project .org/package=data.table

Enders, C. K. (2010). *Applied missing data analysis*. Guilford Press.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1–16. https://doi.org/10.1037/xlm0000153

Field, A. (2013). *Discovering Statistics using IBM SPSS Statistics*. Sage.

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology*, *4*, 715. https://doi.org/10.3389/fpsyg.2013.00715

Gabriel, F., Van Hoof, J., Gómez, D. M., & Van Dooren, W. (2023). Obstacles in the development of the understanding of fractions. In K. M. Robinson, A. K. Dubé, & D. Kotsopoulos (Eds.), *Mathematical Cognition and Understanding* (pp. 209–225). Springer International Publishing. https://doi.org/10.1007/978-3-031-29195-1_11

Geiser, C. (2013). *Data analysis with Mplus*. The Guilford Press.

Grouws, D. A., & Cebulla, K. J. (2000). *Improving student achievement in mathematics*. International Academy of Education.

Guo, S. (2010). *Chinese history of science and technology*. Science Press.

Haffner, J., Baro, K., & Resch, F. (2005). *Heidelberger Rechentest (HRT 1-4) Erfassung mathematischer Basiskompetenzen im Grundschulalter [The Heidelberg Mathematics Test (HRT 1-4): Assessing Mathematics at Primary School age]*. Hogrefe.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Erlbaum.

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, *102*(2), 395–406. https://doi.org/10.1037/a0017486

Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of Experimental Child Psychology*, *113*(4), 469–486. https://doi.org/10.1016/j.jecp.2012.07.009

Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development*, *35*, 34–49. https://doi.org/10.1016/j.cogdev.2015.02.001

Hansen, N., Jordan, N. C., & Rodrigues, J. (2017). Identifying learning difficulties with fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*, *50*, 45–59. https://doi.org/10.1016/j.cedpsych.2015.11.002

Hecht, S. A. (1998). Toward an information-processing account of individual differences in fraction skills. *Journal of Educational Psychology*, *90*(3), 545–559. https://doi.org/10.1037/0022-0663.90.3.545

Hecht, S. A., Close, L., & Santisi, M. (2003). Sources of individual differences in fraction skills. *Journal of Experimental Child Psychology*, *86*(4), 277–302. https://doi.org/10.1016/j.jecp.2003.08.003

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology*, *102*(4), 843–859. https://doi.org/10.1037/a0019824

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*(3), 333–355. https://doi.org/10.1007/BF00312451

Hiebert, J., & LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 3–20). Lawrence Erlbaum Associates, Inc.

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. Lawrence Erlbaum.

Hu, L., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling: A Multidisciplinary Journal*, *6*(1), 1–55. https://doi.org/10.1080/10705519909540118

Hunting, R. P., & Sharpley, C. F. (1988). Fraction knowledge in preschool children. *Journal for Research in Mathematics Education*, *19*(2), 175. https://doi.org/10.2307/749411

IBM Corp. (2020). *IBM SPSS Statistics for Windows (Version 27.0) [Computer software]* [Computer software]. IBM Corp.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, *116*(1), 45–58. https://doi.org/10.1016/j.jecp.2013.02.001

Jordan, N. C., Resnick, I., Rodrigues, J., Hansen, N., & Dyson, N. (2017). Delaware longitudinal study of fraction learning: Implications for helping children with mathematics difficulties. *Journal of Learning Disabilities*, *50*(6), 621–630. https://doi.org/10.1177/0022219416662033

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior*, *14*(4), 365–378. https://doi.org/10.1016/0732-3123(95)90035-7

Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. MIT Press.

Kazak, A. E. (2018). Editorial: Journal article reporting standards. *American Psychologist*, *73*(1), 1–2. https://doi.org/10.1037/amp0000263

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop ERIC/SMEAC* (pp. 101–144).

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Kline, R. B. (2011). *Principles and practice of structural equation modeling* (4th ed.). Guilford publications.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Routledge.

Lauritzen, P. (2012). *Conceptual and procedural knowledge of mathematical functions* [Doctoral Dissertation]. University of Eastern Finland.

Lenz, K., Dreher, A., Holzäpfel, L., & Wittmann, G. (2020). Are conceptual knowledge and procedural knowledge empirically separable? The case of fractions. *British Journal of Educational Psychology*, *90*(3), 809–829. https://doi.org/10.1111/bjep.12333

Lenz, K., & Wittmann, G. (2020). Individual differences in conceptual and procedural fraction knowledge: What makes the difference and what does it look like? *International*

Electronic Journal of Mathematics Education, *16*(1), em0615. https://doi.org/10.29333/iejme/9282

Lin, C., Becker, J., Byun, M., Yang, D., & Huang, T. (2013). Preservice teachers' conceptual and procedural knowledge of fraction operations: A comparative study of the United States and Taiwan. *School Science and Mathematics*, *113*(1), 41–51. https://doi.org/10.1111/j.1949-8594.2012.00173.x

Lin, X., & Powell, S. R. (2021). Examining the relation between whole numbers and fractions: A meta-analytic structural equation modeling approach. *Contemporary Educational Psychology*, *67*, 102017. https://doi.org/10.1016/j.cedpsych.2021.102017

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Routledge. https://doi.org/10.4324/9780203856345

Ma, L., & Kessel, C. (2022). The theory of school arithmetic: Fractions. *Asian Journal for Mathematics Education*, *1*(3), 265–284. https://doi.org/10.1177/27527263221107162

Meert, G., Grégoire, J., & Noël, M.-P. (2010). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology*, *107*(3), 244–259. https://doi.org/10.1016/j.jecp.2010.04.008

Ministry of Education. (2011). *Curriculum standards for school mathematics of nine-year compulsory education*. Beijing Normal University Press.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122. https://doi.org/10.2307/749607

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 International Results in Mathematics and Science.* https://timssandpirls.bc.edu/timss2019/international-results/

Muthén, L. K., & Muthén, B. O. (1998). *Mplus user's guide*. Muthén & Muthén.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52. https://doi.org/10.1207/s15326985ep4001_3

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, *28*, 64–72. https://doi.org/10.1016/j.learninstruc.2013.05.003

Pedersen, P. L., & Bjerre, M. (2021). Two conceptions of fraction equivalence. *Educational Studies in Mathematics*, *107*(1), 135–157. https://doi.org/10.1007/s10649-021-10030-7

Peng, P., Lin, X., Ünal, Z. E., Lee, K., Namkung, J., Chow, J., & Sales, A. (2020). Examining the mutual relations between language and mathematics: A meta-analysis. *Psychological Bulletin*. https://doi.org/10.1037/bul0000231

Peng, P., Namkung, J., Barnes, M., & Sun, C. (2016). A meta-analysis of mathematics and working memory: Moderating effects of working memory domain, type of mathematics skill, and sample characteristics. *Journal of Educational Psychology*, *108*(4), 455–473. https://doi.org/10.1037/edu0000079

Perry, M. (1991). Learning and transfer: Instructional conditions and conceptual change. *Cognitive Development*, *6*(4), 449–468. https://doi.org/10.1016/0885-2014(91)90049-J

Pittalis, M. (2024). An empirically validated rational number sense framework. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-024-00484-2

R Core Team. (2022). *R: A Language and Environment for Statistical Computing* [Computer software]. R Foundation for Statistical Computing. https://www.R-project.org

Raven, J. C. (1938). *Raven's progressive matrices (1938): Sets A, B, C, D, E.* Australian Council for Educational Research.

Rinne, L. F., Ye, A., & Jordan, N. C. (2017). Development of fraction comparison strategies: A latent transition analysis. *Developmental Psychology*, *53*(4), 713–730. https://doi.org/10.1037/dev0000275

Rittle-Johnson, B. (2017). Developing mathematics knowledge. *Child Development Perspectives*, *11*(3), 184–190. https://doi.org/10.1111/cdep.12229

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*(1), 175–189. https://doi.org/10.1037/0022-0663.91.1.175

Rittle-Johnson, B., Fyfe, E. R., & Loehr, A. M. (2016). Improving conceptual and procedural knowledge: The impact of instructional content within a mathematics lesson. *British Journal of Educational Psychology*, *86*(4), 576–591. https://doi.org/10.1111/bjep.12124

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346–362. https://doi.org/10.1037//0022-0663.93.2.346

Satorra, A., & Bentler, P. M. (2010). Ensuring positiveness of the scaled difference Chi-square test statistic. *Psychometrika*, *75*(2), 243–248. https://doi.org/10.1007/s11336-009-9135-y

Schiller, L. K., & Siegler, R. S. (2023). Integrated knowledge of rational number notations predicts children's math achievement and understanding of numerical magnitudes. *Cognitive Development*, *68*, 101380. https://doi.org/10.1016/j.cogdev.2023.101380

Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *Developmental Psychology*, *46*(1), 178–192. https://doi.org/10.1037/a0016701

Sheppard, C., Fiorentino, D., Collins, L., & Merlis, S. (1968). Ravens progressive matrices (1938): Normative data on male narcotic addicts. *Psychological Reports*, *23*(2), 343–348. https://doi.org/10.2466/pr0.1968.23.2.343

Sidney, P. G. (2020). Children's learning from implicit analogies during instruction: Evidence from fraction division. *Cognitive Development*, *56*, 100956. https://doi.org/10.1016/j.cogdev.2020.100956

Sidney, P. G., Shirah, J. F., Zahrn, L., & Thompson, C. A. (2022). Diagrams support spontaneous transfer across whole number and fraction concepts. *Contemporary Educational Psychology*, *69*, 102066. https://doi.org/10.1016/j.cedpsych.2022.102066

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science*, *19*(3), 341–361. https://doi.org/10.1111/desc.12395

Siegler, R. S., & Lortie-Forgues, H. (2014). An integrative theory of numerical development. *Child Development Perspectives*, *8*(3), 144–150. https://doi.org/10.1111/cdep.12077

Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology*, *107*(3), 909–918. https://doi.org/10.1037/edu0000025

Siegler, R. S., & Lortie-Forgues, H. (2017). Hard lessons: Why rational number arithmetic is so difficult for so many people. *Current Directions in Psychological Science*, *26*(4), 346–351. https://doi.org/10.1177/0963721417700129

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, *127*(4), 377–397. https://doi.org/10.1037/0096-3445.127.4.377

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Silver, E. A. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 181–198). Lawrence Erlbaum Associates, Inc.

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, *51*(2), 101–140. https://doi.org/10.1016/j.cogpsych.2005.03.001

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015

Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, *20*(3), 267–307. https://doi.org/10.1016/S0732-3123(02)00075-5

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer Science & Business Media.

Sun, X. H. (2019). Bridging whole numbers and fractions: Problem variations in Chinese mathematics textbook examples. *ZDM*, *51*(1), 109–123. https://doi.org/10.1007/s11858-018-01013-9

Sun, X. H., Xin, Y. P., & Huang, R. (2019). A complementary survey on the current state of teaching and learning of Whole Number Arithmetic and connections to later mathematical content. *ZDM*, *51*(1), 1–12. https://doi.org/10.1007/s11858-019-01041-z

Tian, J., & Siegler, R. S. (2017). Fractions learning in children with mathematics difficulties. *Journal of Learning Disabilities*, *50*(6), 614–620. https://doi.org/10.1177/0022219416662032

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13. https://doi.org/10.1016/j.learninstruc.2014.03.002

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior*, *31*(3), 344–355. https://doi.org/10.1016/j.jmathb.2012.02.001

Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education*, *15*(2), 154–164. https://doi.org/10.1080/14794802.2013.797747

Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2015). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction*, *37*, 30–38. https://doi.org/10.1016/j.learninstruc.2014.03.004

Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics*, *90*(1), 39–56. https://doi.org/10.1007/s10649-015-9613-3

Viegut, A. A., Resnick, I., Miller-Cotto, D., Newcombe, N. S., & Jordan, N. C. (2023). Tracking informal fraction knowledge and its correlates across first grade. *Developmental Psychology*, *59*(10), 1739–1756. https://doi.org/10.1037/dev0001581

Wang, Z., & McDougall, D. (2019). Curriculum matters: What we teach and what students gain. *International Journal of Science and Mathematics Education*, *17*(6), 1129–1149. https://doi.org/10.1007/s10763-018-9915-x

Wickham, H. (2016). *Ggplot2*. Springer International Publishing. https://doi.org/10.1007/978-3-319-24277-4

Wickham, H., François, R., Henry, L., Müller, K., & Vaughan, D. (2023). *Dplyr: A Grammar of Data Manipulation. R package version 1.1.4* [Computer software]. https://dplyr.tidyverse.org

Wu, H., & Li, L. (2006). Norm establishment for Chinese rating scale of pupil's mathematics abilities. *Chinese Journal of Clinical Rehabilitation*, *10*(30), 168–170.

Xu, C., Di Lonardo Burr, S., & LeFevre, J.-A. (2023). The hierarchical relations among mathematical competencies: From fundamental numeracy to complex mathematical skills. *Canadian Journal of Experimental Psychology / Revue Canadienne de Psychologie Expérimentale*. https://doi.org/10.1037/cep0000311

Xu, C., Di Lonardo Burr, S., Li, H., Liu, C., & Si, J. (2024). From whole numbers to fractions to word problems: Hierarchical relations in mathematics knowledge for Chinese Grade 6 students. *Journal of Experimental Child Psychology*, *242*, 105884. https://doi.org/10.1016/j.jecp.2024.105884

Xu, C., Li, H., Di Lonardo Burr, S., Si, J., LeFevre, J.-A., & Huang, B. (2022). Divide and conquer: Relations among arithmetic operations and emerging knowledge of fraction notation for Chinese students in Grade 4. *Journal of Experimental Child Psychology*, *217*, 105371. https://doi.org/10.1016/j.jecp.2021.105371

Xu, C., Li, H., Di Lonardo Burr, S., Si, J., LeFevre, J.-A., & Zhuo, X. (2024). We cannot ignore the signs: The development of equivalence and arithmetic for students from grades 3 to 4. *Journal of Cognition and Development*, *25*(1), 46–65. https://doi.org/10.1080/15248372.2023.2245507

Zhang, D., & Siegler, R. S. (2022). Curriculum standards and textbook coverage of fractions in high-achieving East-Asian countries and the United States. *Current Opinion in Behavioral Sciences*, *47*, 101184. https://doi.org/10.1016/j.cobeha.2022.101184

Zhang, H., & Zhou, Y. (2003). The teaching of mathematics in Chinese elementary schools. *International Journal of Psychology*, *38*(5), 286–298. https://doi.org/10.1080/00207590344000097

Zhang, Q. (2016). Teaching of fraction in elementary schools based on understanding of different sub-constructs. *Proceedings of the 13th International Congress on Mathematical Education*.

Zhang, Q., Lin, C.-D., Zhao, D.-M., & Wang, X.-L. (2002). The developmental study of the speed and span of mental addition in pupils. *Psychological Development and Education*, *18*(1), 16–21. https://doi.org/10.3969/j.issn.1001-4918.2002.01.004