MULTIPLICATIVE RELATIONS AMONG DIFFERENCES OF SINGULAR MODULI

VAHAGN ASLANYAN, SEBASTIAN ETEROVIĆ, AND GUY FOWLER

To Jonathan Pila

ABSTRACT. Let $n \in \mathbb{Z}_{>0}$. We prove that there exist a finite set V and finitely many algebraic curves $T_1, \ldots, T_k$ with the following property: if $(x_1, \ldots, x_n, y)$ is an $(n+1)$-tuple of pairwise distinct singular moduli such that $\prod_{i=1}^n (x_i - y)^{a_i} = 1$ for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$, then $(x_1, \ldots, x_n, y) \in V \cup T_1 \cup \ldots \cup T_k$. Further, the curves $T_1, \ldots, T_k$ may be determined explicitly for a given n.

1. Introduction

Let $\mathbb{H}$ denote the complex upper half plane. The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by fractional linear transformations. The modular j-function $j\colon \mathbb{H} \to \mathbb{C}$ is the unique holomorphic function $\mathbb{H} \to \mathbb{C}$ which is invariant under this action of $\mathrm{SL}_2(\mathbb{Z})$, has a simple pole at $i\infty$, and satisfies $j(i) = 1728$ and $j(\rho) = 0$, where $\rho = \exp(2\pi i/3)$.

A singular modulus is a complex number $j(\tau)$ for some $\tau \in \mathbb{H}$ such that $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$. For example, 0 and 1728 are both singular moduli. Equivalently, a singular modulus is the j-invariant of an elliptic curve with complex multiplication. Singular moduli are algebraic integers and generate the ring class fields of imaginary quadratic fields. By Schneider's theorem [32, IIc], if $\tau \in \mathbb{H}$ is such that both $\tau, j(\tau) \in \overline{\mathbb{Q}}$, then $j(\tau)$ is a singular modulus.

In this paper, we consider multiplicative relations among differences $x - y$ of singular moduli x, y. Since 0 is a singular modulus, every singular modulus is equal to the difference of two singular moduli. Our aim is to generalise the following theorem.

Theorem 1.1. *Let $n \in \mathbb{Z}_{>0}$. Let y be a singular modulus. Then there exist only finitely many n-tuples $(x_1, \ldots, x_n)$ of pairwise distinct singular moduli $x_1, \ldots, x_n$ such that $y \notin \{x_1, \ldots, x_n\}$ and there exist $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ for which*

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1.$$

Date: 9th December 2024.

2020 *Mathematics Subject Classification.* 11G18, 11G15, 03C64.

VA was supported by Leverhulme Trust Early Career Fellowship ECF-2022-082. SE was supported by EPSRC fellowship EP/T018461/1. GF has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 945714).

Theorem 1.1 was proved by Pila and Tsimerman [27] for $y = 0$ and by the third author [13] for y not in the real interval $(0, 1728)$. In Section 3, we prove the remaining case where y is in the real interval $(0, 1728)$.

This paper addresses the case where y is allowed to vary over all singular moduli. That is, we consider $(n+1)$-tuples $(x_1, \ldots, x_n, y)$ of pairwise distinct singular moduli $x_1, \ldots, x_n, y$ such that

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1 \text{ for some } a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}. \tag{1.1}$$

In this setting, one must account for the following situation.

Definition 1.2 ([6, p. 1052])**.** A function $f \colon \mathbb{H} \to \mathbb{C}$ is called a j-map if either there exists a singular modulus x such that $f(z) = x$ for every $z \in \mathbb{H}$, or there exists $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $f(z) = j(gz)$ for every $z \in \mathbb{H}$. Here $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathbb{H}$ by fractional linear transformations.

Definition 1.3. Let $n \in \mathbb{Z}_{>0}$. Let $f_1, \ldots, f_n, f$ be pairwise distinct j-maps, at least one of which is non-constant. The set

$$\Big\{ (f_1(z), \ldots, f_n(z), f(z)) : z \in \mathbb{H} \Big\}$$

is called a multiplicative special curve in $\mathbb{C}^{n+1}$ if there exist $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ such that, for all $z \in \mathbb{H}$,

$$\prod_{i=1}^{n} (f_i(z) - f(z))^{a_i} = 1.$$

Note that a multiplicative special curve is always an algebraic curve (see Proposition 5.3). Clearly, any multiplicative special curve contains infinitely many $(n+1)$-tuples $(x_1, \ldots, x_n, y)$ of pairwise distinct singular moduli satisfying (1.1). If $N \in \mathbb{Z}_{>0}$ is not a perfect square, then the modular polynomial $\Phi_N \in \mathbb{Z}[X, Y]$ gives rise to a multiplicative special curve, as we explain in Section 1.1. Thus one cannot hope to show, for an arbitrary $n \in \mathbb{Z}_{>0}$, that there exist only finitely many such $(n + 1)$-tuples $(x_1, \ldots, x_n, y)$.

Instead, we prove that the multiplicative special curves arising from the modular polynomials are the only multiplicative special curves. In particular, for a given n, there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ and these may be determined effectively.

Theorem 1.4. *Let $n \in \mathbb{Z}_{>0}$. Then there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ and these may be determined effectively. If $n \leq 5$, then there are no multiplicative special curves in $\mathbb{C}^{n+1}$.*

We then prove that, for every $n \in \mathbb{Z}_{>0}$, the finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ account for all but finitely many of the $(n+1)$-tuples $(x_1, \ldots, x_n, y)$ of pairwise distinct singular moduli satisfying (1.1).

Theorem 1.5. *Let $n \in \mathbb{Z}_{>0}$. Then there exist only finitely many $(n + 1)$-tuples $(x_1, \ldots, x_n, y)$ of pairwise distinct singular moduli $x_1, \ldots, x_n, y$ such that*

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ *and* $(x_1, \ldots, x_n, y)$ *does not belong to one of the finitely many multiplicative special curves in* $\mathbb{C}^{n+1}$*.*

Since there are no multiplicative special curves in $\mathbb{C}^{n+1}$ for $n \leq 5$, one immediately obtains the following corollary.

Corollary 1.6. *Let* $n \in \{1, \ldots, 5\}$*. There exist only finitely many* $(n+1)$*-tuples* $(x_1, \ldots, x_n, y)$ *of pairwise distinct singular moduli* $x_1, \ldots, x_n, y$ *such that*

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$*.*

The proof of Theorem 1.5 uses o-minimality and is ineffective. Recently, Li [21] has proved that the difference of two singular moduli is never a unit (in the ring of algebraic integers). Hence, there are no distinct singular moduli x, y such that $(x - y)^a = 1$ for some $a \in \mathbb{Z} \setminus \{0\}$.

1.1. **Modular polynomials and multiplicative special curves.** For background on modular polynomials, see [10, §11]. For $N \in \mathbb{Z}_{>0}$, let

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad = N, a > 0, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}.$$

There exists [10, (11.15)] a polynomial $\Phi_N \in \mathbb{Z}[X, Y]$ with the property that

$$\Phi_N(X, j(z)) = \prod_{g \in C(N)} (X - j(gz))$$

for all $z \in \mathbb{H}$. The polynomial Φ_N is called the Nth modular polynomial.

For $N > 1$, let $F_N \in \mathbb{Z}[X]$ be defined by $F_N(X) = \Phi_N(X, X)$. Then F_N is a non-constant polynomial (the explicit formula in [10, Proposition 13.8] in fact implies that $\deg F_N \geq 2N$). The roots of F_N are all singular moduli (see Corollary 2.4). If N is not a perfect square, then, by [10, Theorem 11.18], the polynomial F_N has leading coefficient ± 1.

Suppose then that $N \in \mathbb{Z}_{>1}$ is such that the leading coefficient of F_N is ± 1 (e.g. take N not a perfect square). Write $\alpha_1, \ldots, \alpha_k$ for the distinct roots of F_N and a_i for their multiplicities. Write $g_1, \ldots, g_l$ for the elements of $C(N)$. Since

$$F_N(j(z)) = \prod_{i=1}^{l} (j(z) - j(g_i z))$$

for all $z \in \mathbb{H}$, one thus obtains (doubling the exponents to eliminate a potential factor of -1) that

$$\prod_{i=1}^{k} (j(z) - \alpha_i)^{2a_i} = \prod_{i=1}^{l} (j(z) - j(g_i z))^2$$

for all $z \in \mathbb{H}$, and hence, for all $z \in \mathbb{H}$,

$$\prod_{i=1}^{k} (j(z) - \alpha_i)^{2a_i} \prod_{i=1}^{l} (j(z) - j(g_i z))^{-2} = 1. \tag{1.2}$$

In particular, the set

$$\Big\{(\alpha_1, \dots, \alpha_k, j(g_1 z), \dots, , j(g_l z), j(z)) : z \in \mathbb{H}\Big\}$$

is a multiplicative special curve in $\mathbb{C}^{k+l+1}$.

Further examples of multiplicative special curves may be generated by multiplying together integer powers of relations of the form (1.2) coming from different F_N. In this case, one must also consider polynomials F_N with leading coefficient not equal to ± 1, because these leading coefficients may cancel with one another. For example, -2 is the leading coefficient of both F_4 and F_{16}. Theorem 4.1 will show that all the multiplicative special curves arise from the polynomials F_N in this way.

Example 1.7. The modular polynomial Φ_2 is

$$\begin{aligned}\Phi_2(X,Y) = &- X^2Y^2 + X^3 + Y^3 + 1488(X^2Y + XY^2)\\ &- 162000(X^2 + Y^2) + 40773375XY\\ &+ 8748000000(X+Y) - 157464000000000.\end{aligned}$$

Thus,

$$\begin{aligned}F_2(X) =&\Phi_2(X,X)\\ =&- X^4 + 2978X^3 + 40449375X^2 + 17496000000X\\ &- 157464000000000\\ =&- (X-1728)(X+3375)^2(X-8000).\end{aligned}$$

Note that 1728, -3375, 8000 are singular moduli of discriminant -4, -7, -8 respectively. The discriminant of a singular modulus $j(\tau)$ is $b^2 - 4ac$, where $a, b, c \in \mathbb{Z}$, not all zero, are such that $a\tau^2 + b\tau + c = 0$ and $\gcd(a,b,c) = 1$.

Observe that

$$C(2) = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

Thus,

$$\Phi_2(X, j(z)) = \Big(X - j(2z)\Big)\Big(X - j\Big(\frac{z}{2}\Big)\Big)\Big(X - j\Big(\frac{z+1}{2}\Big)\Big).$$

Hence, for all $z \in \mathbb{H}$,

$$\begin{aligned}&- (j(z) - 1728)(j(z)+3375)^2(j(z) - 8000)\\ &= \Big(j(z) - j(2z)\Big)\Big(j(z) - j\Big(\frac{z}{2}\Big)\Big)\Big(j(z) - j\Big(\frac{z+1}{2}\Big)\Big).\end{aligned} \tag{1.3}$$

The set

$$\Big\{\Big(1728, -3375, 8000, j(2z), j\Big(\frac{z}{2}\Big), j\Big(\frac{z+1}{2}\Big), j(z)\Big) : z \in \mathbb{H}\Big\}$$

is thus a multiplicative special curve in $\mathbb{C}^7$.

For an example of a 7-tuple of singular moduli lying on this curve, take

$$z = \frac{-1 + \sqrt{163}i}{2}.$$

Then

$$j(z) = -262537412640768000 = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3,$$

which we denote by k, is a singular modulus of discriminant -163. In this case, $j(2z), j(z/2), j((z+1)/2)$ are the three singular moduli of discriminant -652. These are respectively the roots $r, s, \bar{s}$ of the irreducible polynomial

$$\begin{aligned} &X^3 - 68925893036109279891085639286946000X^2 \\ &+ 102561728837719322645921325412908000000X \\ &- 18095625621665522953693950872675200892692248000000000, \end{aligned}$$

where $r \in \mathbb{R}$ and $s, \bar{s}$ are complex conjugate with $s \in \mathbb{H}$. In this case, (1.3) yields that

$$-(k-1728)(k+3375)^2(k-8000) = (k-r)(k-s)(k-\bar{s}). \tag{1.4}$$

The prime factorisation of the two sides of (1.4) is given by

$$\begin{aligned} &- 2^{12} \cdot 3^{22} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 37 \cdot 101 \cdot 103^2 \cdot 127^2 \\ &\cdot 157 \cdot 163 \cdot 229^2 \cdot 277 \cdot 283^2 \cdot 317. \end{aligned}$$

1.2. **Multiplicative properties of differences of singular moduli.** The study of the multiplicative properties of differences of singular moduli goes back at least as far as Berwick [3], who in 1927 determined the factorisations of x and $x - 1728$ for all singular moduli x such that $[\mathbb{Q}(x) : \mathbb{Q}] \leq 3$.

The differences of singular moduli are highly divisible numbers, in the sense that they tend to have relatively small prime factors. For example,

$$j\Big(\frac{-1+\sqrt{163}i}{2}\Big) - j\Big(\frac{-1+\sqrt{67}i}{2}\Big) = -2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.$$

Example 1.7 gives another illustration of this tendency. This observation led Gross and Zagier [15] to prove a formula for the prime ideal factorisations of differences of singular moduli, subject to some restrictions on the discriminants of the singular moduli considered. A version of their result for arbitrary discriminants has since been proved by Lauter and Viray [20].

Recent work on multiplicative relations among singular moduli, for example the proof of Theorem 1.1 by Pila and Tsimerman [27] and the third author [13], has been motivated by connections to the Zilber–Pink conjecture on atypical intersections.

Effective results on multiplicative relations among singular moduli in low dimensions have also been studied extensively [4, 7, 12, 14, 31] as special cases of the André–Oort conjecture for $\mathbb{C}^n$, which was proved ineffectively by Pila [24]. In particular, for $n \leq 3$, explicit bounds on multiplicatively dependent n-tuples of pairwise distinct singular moduli are known [4, 31].

For differences of singular moduli, the most general effective result we are aware of is Li's result [21] that the difference of two singular moduli is never an algebraic unit. Li's result generalised Bilu, Habegger, and Kühne's theorem [5] that no singular modulus is a unit. Work on this topic was prompted by a question of Masser, answered affirmatively by Habegger [16], as to whether only finitely many singular moduli are algebraic units.

1.3. **Structure of this paper.** In Section 2, we give some of the basic results we will need for this paper. Section 3 completes the proof of Theorem 1.1. The proof of Theorem 1.4 is in Section 4. Section 5 contains the

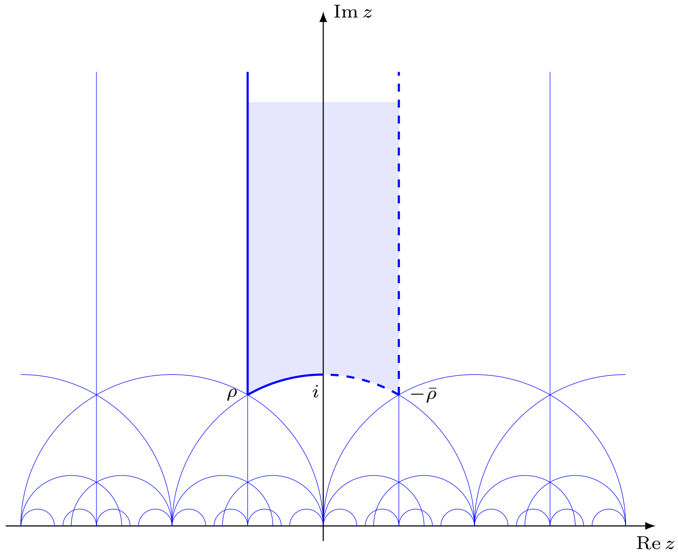

FIGURE 1. The fundamental domain $\mathfrak{F}_j$

functional transcendence results which are required for the proof of Theorem 1.5, which is then carried out in Section 6. Finally, the connection to the Zilber–Pink conjecture is considered in Section 7.

ACKNOWLEDGEMENTS. The authors would like to thank Gabriel Dill for helpful comments.

2. PRELIMINARIES

2.1. **The fundamental domain.** The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices corresponding to the transformations $T\colon z \mapsto z+1$ and $S\colon z \to -1/z$. Let $\mathfrak{F}_j$ be the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ given by

$$\left\{z \in \mathbb{H} : \mathrm{Re}\, z \in \left[-\frac{1}{2}, \frac{1}{2}\right), |z| \geq 1, \text{ and if } |z| = 1, \text{ then } \mathrm{Re}\, z \in \left[-\frac{1}{2}, 0\right]\right\}.$$

This is a hyperbolic triangle with corners at $\rho, -\bar{\rho}, i\infty$. The j-function restricts to a bijection $j\colon \mathfrak{F}_j \to \mathbb{C}$.

The j-function has a series expansion

$$j(z) = e^{-2\pi i z} + 744 + \sum_{n=1}^{\infty} c(n) e^{2n\pi i z},$$

where the coefficients $c(n) \in \mathbb{Z}$. It follows immediately that the j-function is real valued on $\mathfrak{F}_j$ only along the boundary of $\mathfrak{F}_j$ and on the imaginary axis. Further, the image under j of the set $\{z \in \mathfrak{F}_j : |z| = 1\}$ is the real interval $[0, 1728]$.

Proposition 2.1. *Let $z_0 \in \mathfrak{F}_j$. If $\operatorname{Re} z_0 \neq -1/2$, then the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z_0 is equal to*

$$\{z_0 + k : k \in \mathbb{Z}\} \cup \left\{\frac{-1}{z_0} + k : k \in \mathbb{Z}\right\} \\ \cup \left\{w \in \mathbb{H} : w \in \operatorname{Orbit}(z_0) \text{ and } \operatorname{Im} w < \operatorname{Im} \frac{-1}{z_0}\right\}.$$

If $\operatorname{Re} z_0 = -1/2$, then the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z_0 is equal to

$$\{z_0 + k : k \in \mathbb{Z}\} \cup \left\{\frac{-1}{z_0} + k : k \in \mathbb{Z}\right\} \cup \left\{\frac{-1}{z_0+1} + k : k \in \mathbb{Z}\right\} \\ \cup \left\{w \in \mathbb{H} : w \in \operatorname{Orbit}(z_0) \text{ and } \operatorname{Im} w < \operatorname{Im} \frac{-1}{z_0}\right\}.$$

Proof. First, we claim that the following algorithm applied to a point $z \in \mathbb{H}$ will output the unique point in $\mathfrak{F}_j \cap \operatorname{Orbit}(z)$.

(1) If $z \in \mathfrak{F}_j$, then output z. Otherwise proceed to step (2).
(2) Replace z with $z + k$, where $k \in \mathbb{Z}$ is such that $\operatorname{Re}(z + k) \in [-1/2, 1/2)$.
(3) If $z \in \mathfrak{F}_j$, then output z. Otherwise proceed to step (4).
(4) Replace z with $-1/z$. Return to step (1).

Clearly, if this algorithm terminates, then it outputs the unique point in $\mathfrak{F}_j$ in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit as the initial input. We claim that this algorithm always terminates. To prove this, note that $S\colon z \mapsto -1/z$ sends z to

$$-\frac{\operatorname{Re} z}{|z|^2} + \frac{\operatorname{Im} z}{|z|^2} i.$$

In particular, if $|z| < 1$, then the imaginary part of $-1/z$ is strictly larger than $\operatorname{Im} z$. Now every application of step (4) is performed on some z with $|z| \leq 1$. And if $|z| = 1$, then applying (4) immediately yields a point in $\mathfrak{F}_j$.

Hence, it suffices to prove that, given $z \in \mathbb{H}$ with $\operatorname{Re} z \in [-1/2, 1/2)$, there are only finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\operatorname{Re} \gamma z \in [-1/2, 1/2)$ and $\operatorname{Im} \gamma z > \operatorname{Im} z$. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\operatorname{Im} \gamma z = \operatorname{Im} \frac{az + b}{cz + d} = \frac{1}{|cz + d|^2} \operatorname{Im} z,$$

and so $\operatorname{Im} \gamma z > \operatorname{Im} z$ implies that

$$|cz + d|^2 < 1.$$

Hence,

$$(c \operatorname{Re} z + d)^2 + (c \operatorname{Im} z)^2 < 1.$$

Hence, there are only finitely many possibilities for c, and for each such c, only finitely many possibilities for d. So we may assume that c, d are fixed.

We now show that, for the pair (c, d), there exists a unique pair (a, b) such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$\operatorname{Re}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Suppose a, b, a', b' are such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

So $ad - bc = 1$ and $a'd - b'c = 1$. Hence, by Bézout's Lemma, there exists $k \in \mathbb{Z}$ such that

$$(a', b') = (a + kc, b + kd).$$

Thus,

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z = \frac{(a + kc)z + (b + kd)}{cz + d} = \frac{az + b}{cz + d} + k.$$

In particular, there is a unique $k \in \mathbb{Z}$ (and hence a unique pair (a', b')) such that

$$\operatorname{Re}\left(\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Thus, the algorithm always terminates. We may now complete the proof of the proposition. Let $z_0 \in \mathfrak{F}_j$. The proposition amounts to classifying all the $w \in \operatorname{Orbit}(z_0)$ such that

$$\operatorname{Im} w \geq \operatorname{Im} \frac{-1}{z_0}.$$

Suppose first that $\operatorname{Re} z_0 \neq -1/2$. Let $w_0 \in \operatorname{Orbit}(z_0)$ be such that

$$w_0 \notin \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{\frac{-1}{z_0} + k : k \in \mathbb{Z}\right\}.$$

We claim that $\operatorname{Im} w_0 < \operatorname{Im}(-1/z_0)$. Applying the above algorithm to w_0, we must obtain z_0 after finitely many steps. The last transformation applied is either $z \mapsto -1/z$ or $z \mapsto z + k$ for some $k \in \mathbb{Z} \setminus \{0\}$. Recall that $z \mapsto -1/z$ is its own inverse.

If the last transformation applied is $z \mapsto -1/z$, then the algorithm applied to w_0 must pass through $-1/z_0$. By assumption on w_0, this must happen after an application of $z \mapsto -1/z$, which must have strictly increased the imaginary part.

If the last transformation applied is $z \mapsto z + k$ with $k \neq 0$, then the transformation prior to that must have been $z \mapsto -1/z$. We thus must have that

$$\operatorname{Im} w_0 \leq \operatorname{Im} \frac{-1}{z_0 - k} = \frac{1}{|z_0 - k|^2} \operatorname{Im} z_0.$$

Then

$$|z_0 - k| > |z_0| \geq 1,$$

since $\operatorname{Re} z_0 \in (-1/2, 1/2)$ and $k \neq 0$. Hence,

$$\operatorname{Im} w_0 < \frac{1}{|z_0|^2} \operatorname{Im} z_0 = \operatorname{Im} \frac{-1}{z_0} \leq \operatorname{Im} z_0.$$

Now let $z_0 \in \mathfrak{F}_j$ be such that $\operatorname{Re} z_0 = -1/2$. Let $w_0 \in \operatorname{Orbit}(z_0)$ be such that

$$w_0 \notin \{z_0 + k : k \in \mathbb{Z}\} \cup \left\{\frac{-1}{z_0} + k : k \in \mathbb{Z}\right\} \cup \left\{\frac{-1}{z_0 + 1} + k : k \in \mathbb{Z}\right\}.$$

We claim that $\operatorname{Im} w_0 < \operatorname{Im}(-1/z_0)$. To show this, we repeat the above argument. The only place where $\operatorname{Re} z \neq -1/2$ was used was to obtain the inequality

$$|z_0 - k| > |z_0|.$$

If $k \neq -1$, then this inequality still holds and the above argument works. So assume $k = -1$. Then, by the assumption that

$$w_0 \notin \Big\{ \frac{-1}{z_0+1} + k : k \in \mathbb{Z} \Big\},$$

there must have been an application of $z \mapsto -1/z$ prior to passing through $-1/(z_0+1)$ and this application must have strictly increased the imaginary part. Hence,

$$\operatorname{Im} w_0 < \frac{1}{|z_0+1|^2} \operatorname{Im} z_0 = \operatorname{Im} \frac{-1}{z_0} \leq \operatorname{Im} z_0.$$

This completes the proof. □

2.2. **Singular moduli.** Let $\tau \in \mathbb{H}$ be such that $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$. So

$$a\tau^2 + b\tau + c = 0$$

for some $(a, b, c) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$ with $\gcd(a, b, c) = 1$. The discriminant of the singular modulus $j(\tau)$ is defined to be

$$b^2 - 4ac.$$

This depends only on the value of $j(\tau)$ and not on the choice of τ. For a singular modulus x, write $\Delta(x)$ for the discriminant of x. The singular moduli of a given discriminant Δ form a complete set of $\mathbb{Q}$-conjugates [10, Proposition 13.2].

Lemma 2.2. *Suppose that* $z \in \mathbb{H}$ *is such that* $gz = z$ *for some* $g \in \mathrm{M}_2(\mathbb{Z})$ *such that* $\det g > 0$ *and* $\lambda g \neq \mathrm{Id}_2$ *for every* $\lambda \in \mathbb{Q}^\times$. *Then* $j(z)$ *is a singular modulus and* $|\Delta(j(z))| \leq 4 \det g$.

Proof. Let $N = \det g$. Write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So $ad - bc = N$. Since z is a fixed point, we have that

$$z = \frac{az+b}{cz+d}.$$

Thus,

$$cz^2 + (d-a)z - b = 0.$$

If $b = c = d - a = 0$, then $N = a^2$ and $g = a\mathrm{Id}_2$, which is excluded. So some coefficient of this quadratic equation is non-zero. Hence, $j(z)$ is a singular modulus. Let $h = \gcd(c, -b, (d-a))$. Then

$$\Delta(j(z)) = \Big(\frac{d-a}{h}\Big)^2 + \frac{4bc}{h^2}.$$

Since $bc = ad - N$, we have that

$$\Delta(j(z)) = \frac{1}{h^2}((a+d)^2 - 4N).$$

In particular, $|\Delta(j(z))| \leq 4N$. □

Proposition 2.3. *Suppose that $z \in \mathbb{H}$ is such that*

$$j(z) = j(gz)$$

for some $g \in \mathrm{M}_2(\mathbb{Z})$ such that $\det g > 0$ and $\lambda g \notin \mathrm{SL}_2(\mathbb{Z})$ for every $\lambda \in \mathbb{Q}^\times$. Then $j(z)$ is a singular modulus and $|\Delta(j(z))| \leq 4 \det g$.

Proof. Since

$$j(z) = j(gz),$$

there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma z = gz.$$

In particular, z is a fixed point for the action of the integer matrix $\gamma^{-1}g$ on $\mathbb{H}$. Apply Lemma 2.2 to $\gamma^{-1}g$. □

Corollary 2.4. *Let $x \in \mathbb{C}$ be a root of the polynomial F_N for some $N \in \mathbb{Z}_{>1}$. Then x is a singular modulus and $|\Delta(x)| \leq 4N$. Further, if $N \in \mathbb{Z}_{>1}$, then every singular modulus of discriminant $-4N$ is a root of F_N.*

Proof. Suppose $N \in \mathbb{Z}_{>1}$ and $x \in \mathbb{C}$ are such that $F_N(x) = 0$. Recall that

$$F_N(j(z)) = \prod_{g \in C(N)} (j(z) - j(gz)).$$

The j-function is surjective, so there exists $z_0 \in \mathbb{H}$ and $g \in \mathbb{C}(N)$ such that $j(z_0) = j(gz_0) = x$. Then, by Proposition 2.3, we have that x is a singular modulus and $|\Delta(x)| \leq 4N$.

For the second part, note that for $N \in \mathbb{Z}_{>1}$,

$$\begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \in C(N).$$

Since the j-function is invariant under $z \mapsto -1/z$, we have that

$$j(\sqrt{N}i) = j\Big(\frac{1}{N}\sqrt{N}i\Big).$$

So $F_N(j(\sqrt{N}i)) = 0$. Clearly, $j(\sqrt{N}i)$ is a singular modulus of discriminant $-4N$. Recall that the singular moduli of discriminant $-4N$ are all conjugate over $\mathbb{Q}$. Thus, every singular modulus of discriminant $-4N$ is a root of F_N, since F_N has coefficients in $\mathbb{Z}$. □

Corollary 2.5. *Let $N_1, \ldots, N_k \in \mathbb{Z}_{>1}$ be pairwise distinct. Let $b_1, \ldots, b_k \in \mathbb{Z} \setminus \{0\}$. Then*

$$\prod_{i=1}^{k} F_{N_i}(X)^{b_i}$$

is a non-constant rational function of X.

Proof. Each $F_{N_i}(X)$ is a non-constant polynomial in X. Hence, the rational function

$$\prod_{i=1}^{k} F_{N_i}(X)^{b_i}$$

is not constantly zero. Without loss of generality, we may assume that $N_k > N_1, \ldots, N_{k-1}$ and $b_k > 0$. By Corollary 2.4, the polynomial $F_{N_i}(X)$

vanishes at a singular modulus of discriminant $-4N_k$ if and only if $i = k$. Thus the rational function

$$\prod_{i=1}^{k} F_{N_i}(X)^{b_i}$$

vanishes at every singular modulus of discriminant $-4N_k$; in particular, the function is non-constant. □

Proposition 2.6. *Suppose that* $g_1, g_2 \in \mathrm{M}_2(\mathbb{Z})$ *are such that* $\det g_1, \det g_2 > 0$ *and* $g_1 \neq \lambda\gamma g_2$ *for every* $\lambda \in \mathbb{Q}^\times$ *and* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*. If* $z \in \mathbb{H}$ *is such that* $j(g_1 z) = j(g_2 z)$*, then* $j(z)$ *is a singular modulus and* $|\Delta(j(z))| \leq 4\max\{\det(g_1), \det(g_2)\}^2$*.*

Proof. Since $j(g_1 z) = j(g_2 z)$, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma g_1 z = g_2 z$. Hence, z is a fixed point for the action of $g = g_2^{-1}\gamma g_1 \in \mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}$. Multiplying the entries of g by $\det(g_2)$, we may assume that $g \in \mathrm{M}_2(\mathbb{Z})$ and $\det g \leq \max\{\det(g_1), \det(g_2)\}^2$. Since $g_1 \neq \lambda\gamma' g_2$ for every $\lambda \in \mathbb{Q}^\times$ and $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$, g is not a rational scalar multiple of Id_2. The desired result thus follows from Lemma 2.2. □

Since the restriction $j\colon \mathfrak{F}_j \to \mathbb{C}$ of the j-function to the fundamental domain is bijective, the map

$$(a, b, c) \mapsto j\Big(\frac{-b + |\Delta|^{1/2} i}{2a}\Big)$$

is a bijection between the set

$$T_\Delta = \Big\{(a, b, c) \in \mathbb{Z}^3 : \Delta = b^2 - 4ac,\ \gcd(a, b, c) = 1, \\ \text{and either } -a < b \leq a < c \text{ or } 0 \leq b \leq a = c\Big\}$$

and the singular moduli of discriminant Δ. Observe that, for each discriminant Δ, there is a unique triple $(a, b, c) \in T_\Delta$ with $a = 1$. This triple is given by $(1, k, (k^2 - \Delta)/4)$, where $k = 0$ if Δ is even and $k = 1$ if Δ is odd. The corresponding singular modulus has preimage

$$\frac{-k + |\Delta|^{1/2} i}{2} \in \mathfrak{F}_j,$$

which has imaginary part strictly greater than the preimage of any other singular modulus of discriminant Δ and of any singular modulus of discriminant Δ' with $|\Delta'| < |\Delta|$.

Proposition 2.7. *For every* $\epsilon > 0$*, there exist an ineffective constant* $c_1(\epsilon) > 0$ *and an effective constant* $c_2(\epsilon) > 0$*, such that if* x *is a singular modulus of discriminant* Δ*, then*

$$[\mathbb{Q}(x) : \mathbb{Q}] \geq c_1(\epsilon)|\Delta|^{1/2-\epsilon}$$

and

$$[\mathbb{Q}(x) : \mathbb{Q}] \leq c_2(\epsilon)|\Delta|^{1/2+\epsilon}.$$

Proof. The ineffective lower bound is due to Siegel [33]. The upper bound is [23, Proposition 2.2]. □

For $\alpha \in \overline{\mathbb{Q}}$, write $H(\alpha)$ for the absolute multiplicative height of α and $h(\alpha)$ for the absolute logarithmic height (see e.g. [8, §1.5]).

Proposition 2.8 ([24, Proposition 5.7])**.** *Let x be a singular modulus of discriminant Δ. Let $\tau \in \mathfrak{F}_j$ be such that $j(\tau) = x$. Then*

$$H(\operatorname{Re}\tau), H(\operatorname{Im}\tau) \leq 2|\Delta|.$$

Proposition 2.9 ([17, Lemma 4.3])**.** *For every $\epsilon > 0$, there exists an ineffective constant $c(\epsilon) > 0$ such that if x is a singular modulus of discriminant Δ, then*

$$h(x) \leq c(\epsilon)|\Delta|^{\epsilon}.$$

2.3. **Properties of j-maps.** Let f be a non-constant j-map. Then [6, Proposition 7.1] there exist $r, s \in \mathbb{Q}$ such that $r > 0$ and $0 \leq s < 1$ such that $f(z) = j(rz + s)$ for all $z \in \mathbb{H}$. Two non-constant j-maps are equal if and only if the corresponding pairs (r, s) are equal.

Recall that, for $N \in \mathbb{Z}_{>0}$, we define

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad = N, a > 0, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}.$$

Proposition 2.10. *Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$. Then there exist a unique $N \in \mathbb{Z}_{>0}$ and a unique $g' \in C(N)$ such that $j(gz) = j(g'z)$ for all $z \in \mathbb{H}$.*

Proof. Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$. Then there exist $r, s \in \mathbb{Q}$ with $r > 0$ and $0 \leq s < 1$ such that $f(z) = j(rz + s)$ for all $z \in \mathbb{H}$. Further, the pair (r, s) is unique. Let $\lambda \in \mathbb{Q}_{>0}$ be such that

$$\lambda \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

for some $a, b, d \in \mathbb{Z}$ with $\gcd(a, b, d) = 1$. Since $0 \leq s < 1$, we have that $0 \leq b < d$. Let $N = ad$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in C(N),$$

and

$$j(gz) = j\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z \Big)$$

for all $z \in \mathbb{H}$.

Suppose

$$\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in C(M)$$

were also such that

$$j\Big(\frac{a}{d}z + \frac{b}{d}\Big) = j\Big(\frac{a'}{d'}z + \frac{b'}{d'}\Big)$$

for all $z \in \mathbb{H}$. Then, by the uniqueness of the representation of a j-map in terms of r and s, we would have that

$$\frac{a}{d} = \frac{a'}{d'} \text{ and } \frac{b}{d} = \frac{b'}{d'}.$$

Hence,

$$\frac{a}{a'} = \frac{d}{d'},$$

and either $b = b' = 0$ or

$$\frac{b}{b'} = \frac{d}{d'}.$$

So one matrix is just the rescaling of the other. Since the entries of each of the two matrices are coprime integers and $a, a' > 0$, the two matrices are in fact identical and $M = N$. □

Proposition 2.11. *Let $N \in \mathbb{Z}_{>0}$.*

1. *For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $g \in C(N)$, there exists $h \in C(N)$ such that $j(g\gamma z) = j(hz)$.*
2. *For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $g, h \in C(N)$, if $g \neq h$, then $j(g\gamma z) \neq j(h\gamma z)$.*
3. *For every $g, h \in C(N)$, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $j(g\gamma z) = j(hz)$.*

Proof. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $g \in C(N)$. The entries of g are coprime integers and $\det g = N$. Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the entries of $g\gamma$ are coprime integers and $\det g\gamma = N$. Write $g\gamma$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d) = 1$ and $N = ad - bc$. Let $\mu = \gcd(a, c)$ and $m, n \in \mathbb{Z}$ be such that $\mu = ma + nc$. Then

$$\underbrace{\begin{pmatrix} m & n \\ -c/\mu & a/\mu \end{pmatrix}}_{\in \mathrm{SL}_2(\mathbb{Z})} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu & mb + nd \\ 0 & -bc/\mu + ad/\mu \end{pmatrix},$$

which is upper triangular with coprime integer entries and has determinant N.

Let $p = mb + nd$ and $q = N/\mu$. Then

$$j(g\gamma z) = j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = j\left(\begin{pmatrix} \mu & p \\ 0 & q \end{pmatrix} z\right) = j\left(\begin{pmatrix} \mu & p + kq \\ 0 & q \end{pmatrix} z\right)$$

for all $z \in \mathbb{H}$, where $k \in \mathbb{Z}$ is the unique integer such that $0 \leq p + kq < q$. This proves (1), since

$$\begin{pmatrix} \mu & p + kq \\ 0 & q \end{pmatrix} \in C(N).$$

Given Proposition 2.10, (2) follows immediately by making the change of variables $w = \gamma^{-1} z$.

Now we prove (3). Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}.$$

Let

$$\sigma_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that $\sigma_N \in C(N)$. By [10, Lemma 11.11], the map

$$g \mapsto \sigma_N^{-1} \mathrm{SL}_2(\mathbb{Z}) g \cap \mathrm{SL}_2(\mathbb{Z})$$

gives a bijection between the elements $g \in C(N)$ and the right cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. In particular, for every $g \in C(N)$, the set $\sigma_N^{-1}\mathrm{SL}_2(\mathbb{Z})g \cap \mathrm{SL}_2(\mathbb{Z})$ is non-empty.

Let $g, h \in C(N)$. Let $\gamma_1 \in \sigma_N^{-1}\mathrm{SL}_2(\mathbb{Z})h \cap \mathrm{SL}_2(\mathbb{Z})$ and $\gamma_2 \in \sigma_N^{-1}\mathrm{SL}_2(\mathbb{Z})g \cap \mathrm{SL}_2(\mathbb{Z})$. Let $\gamma_{1,1}, \gamma_{2,1} \in \mathrm{SL}_2(\mathbb{Z})$ be such that

$$\gamma_1 = \sigma_N^{-1}\gamma_{1,1}h$$

and

$$\gamma_2 = \sigma_N^{-1}\gamma_{2,1}g.$$

Then

$$\begin{aligned} j(g\gamma_2^{-1}\gamma_1 z) &= j(gg^{-1}\gamma_{2,1}^{-1}\sigma_N\sigma_N^{-1}\gamma_{1,1}hz) \\ &= j(\gamma_{2,1}^{-1}\gamma_{1,1}hz) \\ &= j(hz) \end{aligned}$$

for all $z \in \mathbb{H}$. So we may take $\gamma = \gamma_2^{-1}\gamma_1$ in (3). □

3. Completing the proof of Theorem 1.1

Definition 3.1. Let $n \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_n \in \mathbb{C}^\times$ be pairwise distinct. The set $\{x_1, \ldots, x_n\}$ is multiplicatively dependent if there exist $a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$\prod_{i=1}^{n} x_i^{a_i} = 1.$$

The set $\{x_1, \ldots, x_n\}$ is minimally multiplicatively dependent if $\{x_1, \ldots, x_n\}$ is multiplicatively dependent and no non-empty proper subset of $\{x_1, \ldots, x_n\}$ is multiplicatively dependent.

Theorem 3.2. *Let $y \in \mathbb{C}$ be such that $y \notin (0, 1728)$. Let $n \in \mathbb{Z}_{>0}$. Then there exist only finitely many n-tuples $(x_1, \ldots, x_n)$ of singular moduli $x_1, \ldots, x_n$ such that $x_1, \ldots, x_n, y$ are pairwise distinct and $\{x_1 - y, \ldots, x_n - y\}$ is minimally multiplicatively dependent.*

We do not need to assume that y is a singular modulus in Theorem 3.2, because the same proof works for all y outside the real interval $(0, 1728)$.

Proof. Let $f(z) = j(z) - y$. Then the only zero of f in $\mathfrak{F}_j$ is at the unique $\tau \in \mathfrak{F}_j$ such that $j(\tau) = y$. Since $y \notin (0, 1728)$, this point τ does not lie on the arc of the circle $|z| = 1$ strictly between i and ρ. So

$$\operatorname{Im} \frac{-1}{\tau} < \operatorname{Im} \tau,$$

and, by Proposition 2.1, $f(\tau + s) \neq 0$ for all $s \in (0, 1)$. Thus, f satisfies the "divisor condition" of [13, Definition 1.3], and hence [13, Theorem 1.6] implies the desired result. □

Theorem 3.3. *Let y be a singular modulus. Let $n \in \mathbb{Z}_{>0}$. There exist only finitely many n-tuples $(x_1, \ldots, x_n)$ such that $x_1, \ldots, x_n, y$ are pairwise distinct singular moduli and $\{x_1 - y, \ldots, x_n - y\}$ is minimally multiplicatively dependent.*

Proof. By Theorem 3.2, we may assume that $y \in (0, 1728)$. Let $\Delta = \Delta(y)$. Note that $|\Delta| > 4$, since $0, 1728$ are the only singular moduli with discriminant in the set $\{-3, -4\}$. In particular, y has the $\mathbb{Q}$-conjugate

$$y' = j\Big(\frac{-k + |\Delta|^{1/2} i}{2}\Big),$$

where $k = 0$ if Δ is even and $k = 1$ if Δ is odd. Since

$$\frac{|\Delta|^{1/2}}{2} > 1,$$

we have that $y' \notin (0, 1728)$. Thus, Theorem 3.2 holds for y', and so Theorem 3.2 for y follows since y, y' are conjugate over $\mathbb{Q}$. □

Theorem 1.1 seems stronger than Theorem 3.3, since the former does not require the multiplicative dependence to be minimal, only that all the exponents are non-zero. In fact, we may deduce Theorem 1.1 from Theorem 3.3 by the following formal argument.

Proposition 3.4. *Let $\mathcal{S} \subset \mathbb{C}^\times$. Let $n \in \mathbb{Z}_{>0}$. Suppose, for every $k \in \{1, \ldots, n\}$ there are only finitely many k-tuples $(s_1, \ldots, s_k) \in \mathcal{S}^k$ such that $s_1, \ldots, s_k$ are pairwise distinct and $\{s_1, \ldots, s_k\}$ is minimally multiplicatively dependent. Then there are only finitely many n-tuples $(s_1, \ldots, s_n) \in \mathcal{S}^n$ such that $s_1, \ldots, s_n$ are pairwise distinct and*

$$\prod_{i=1}^{n} s_i^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$.

Proof. Let $(s_1, \ldots, s_n) \in \mathcal{S}^n$ be such that $s_1, \ldots, s_n$ are pairwise distinct and

$$\prod_{i=1}^{n} s_i^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$. The set $\{s_1, \ldots, s_n\}$ is thus multiplicatively dependent. For each $i \in \{1, \ldots, n\}$, there exists [13, Lemma 5.9] a minimally multiplicatively dependent subset $S_i \subset \mathcal{S}$ such that $s_i \in S_i$. In particular, $s_1, \ldots, s_n$ all belong to the set consisting of, for each $k \in \{1, \ldots, n\}$, all the coordinates of tuples $(s'_1, \ldots, s'_k) \in \mathcal{S}^k$ such that $s'_1, \ldots, s'_k$ are pairwise distinct and the set $\{s'_1, \ldots, s'_k\}$ is minimally multiplicatively dependent. By assumption, this set is finite and hence there are only finitely many possibilities for $(s_1, \ldots, s_n)$. □

Proof of Theorem 1.1. Apply Proposition 3.4 to Theorem 3.3 with

$$\mathcal{S} = \{x - y : x \text{ is a singular modulus and } x \neq y\}.$$

□

4. Multiplicative special curves

In this section, we prove Theorem 1.4. To do this, we first prove the following result.

Theorem 4.1. *Let $n \in \mathbb{Z}_{>0}$. Suppose that $T \subset \mathbb{C}^{n+1}$ is a multiplicative special curve. Then there exist $k \in \{1, \ldots, n\}$, $b_1, \ldots, b_k \in \mathbb{Z} \setminus \{0\}$, and pairwise distinct $N_1, \ldots, N_k \in \mathbb{Z}_{>1}$ such that, after reordering the first n coordinates,*

$$T = \{(\alpha_1, \ldots, \alpha_m, j(g_1 z), \ldots, j(g_l z), j(z)) : z \in \mathbb{H}\},$$

where

(1) *$\alpha_1, \ldots, \alpha_m$ are pairwise distinct and such that*

$$\{\alpha_1, \ldots, \alpha_m\} = \{\alpha \in \mathbb{C} : \alpha \text{ is either a zero or a pole of } \prod_{i=1}^{k} F_{N_i}(X)^{b_i}\};$$

(2) *$g_1, \ldots, g_l \in \mathrm{GL}_2^+(\mathbb{Q})$ are pairwise distinct and such that*

$$\{g_1, \ldots, g_l\} = \bigcup_{i=1}^{k} C(N_i).$$

This follows immediately from the following result, which we will prove in Section 4.2. Throughout this paper, by a change of variables we mean replacing a variable z by gz for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

Theorem 4.2. *Let $n \in \mathbb{Z}_{>0}$. Let $f_1, \ldots, f_n, f$ be pairwise distinct j-maps, at least one of which is non-constant. Suppose that $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{C}^\times$ are such that*

$$\prod_{i=1}^{n} (f_i(z) - f(z))^{a_i} = c \tag{4.1}$$

for all $z \in \mathbb{H}$. Then, after a change of variables, $f(z) = j(z)$ and there exist $k \in \{1, \ldots, n\}$, $N_1, \ldots, N_k \in \mathbb{Z}_{>1}$ pairwise distinct, and $b_1, \ldots, b_k \in \mathbb{Z} \setminus \{0\}$ such that

$$\{f_i : f_i \text{ is non-constant}\} = \{j(gz) : g \in C(N_i),\ i = 1, \ldots, k\},$$

and, for all $z \in \mathbb{H}$,

$$\prod_{\substack{i \in \{1,\ldots,n\} \text{ s.t.} \\ f_i \text{ non-constant}}} (f_i(z) - f(z))^{a_i} = \prod_{i=1}^{k} \Big(\prod_{g \in C(N_i)} (j(gz) - j(z)) \Big)^{b_i}$$

and

$$\prod_{\substack{i \in \{1,\ldots,n\} \text{ s.t.} \\ f_i \text{ constant}}} (f_i(z) - f(z))^{a_i} = c \prod_{i=1}^{k} F_{N_i}(j(z))^{-b_i}.$$

4.1. **Functional independence modulo constants.** Before proving Theorem 4.2, we first prove some propositions using ideas from [13, §2]. These will allow us to show that if

$$\{(f_1(z), \ldots, f_n(z), f(z)) : z \in \mathbb{H}\} \tag{4.2}$$

is a multiplicative special curve, then we must be in the situation that some f_i is non-constant, f is non-constant, and some f_i is constant.

Definition 4.3. Functions $f_1, \ldots, f_n \colon \mathbb{H} \to \mathbb{C}$ are called multiplicatively independent modulo constants if, whenever $a_1, \ldots, a_n \in \mathbb{Z}$ are not all zero, the function $F \colon \mathbb{H} \to \mathbb{C}$ defined by

$$F(z) = \prod_{i=1}^{n} f_i(z)^{a_i}$$

is non-constant.

Proposition 4.4. *Let $n \in \mathbb{Z}_{>0}$. Let f be a non-constant j-map. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be pairwise distinct. Then the functions $h_i(z) = f(z) - \alpha_i$ are multiplicatively independent modulo constants.*

Proof. By changing variables, we may assume that $f(z) = j(z)$. The result is then immediate since j is a transcendental function. □

Thus, a multiplicative special curve as in (4.2) must have at least one of $f_1, \ldots, f_n$ non-constant.

Proposition 4.5. *Let $n \in \mathbb{Z}_{>0}$. Let $f_1, \ldots, f_n$ be pairwise distinct non-constant j-maps. Let α be a singular modulus. Then the functions $h_i(z) = f_i(z) - \alpha$ are multiplicatively independent modulo constants.*

Proof. Suppose, for contradiction, that $c \in \mathbb{C}^\times$ and $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ are such that

$$\prod_{i=1}^{n} (f_i(z) - \alpha)^{a_i} = c \tag{4.3}$$

for all $z \in \mathbb{H}$. If $[\mathbb{Q}(z) : \mathbb{Q}] = 2$, then $f_i(z)$ is a singular modulus and so $f_i(z) \in \overline{\mathbb{Q}}$. Hence, $c \in \overline{\mathbb{Q}}$. Let $K = \mathbb{Q}(\alpha, c)$.

We may write $f_i(z) = j(r_i z + s_i)$ for some $r_i, s_i \in \mathbb{Q}$ with $r_i > 0$, $s_i \in [0, 1)$, and the pairs (r_i, s_i) all distinct. Re-indexing and making a change of variables, we may assume that $f_1(z) = j(z)$ and $r_i \geq 1$ for $i \geq 2$.

For $k \in \mathbb{Z}_{>0}$, let $z_k = \sqrt{-k}$. Then $j(z_k)$ is a singular modulus of discriminant $-4k$ and every preimage under j of every singular modulus of discriminant $-4k$ has imaginary part $\leq \sqrt{k}$ with equality precisely at the preimages of $j(z_k)$ itself which have the form $z_k + l$ for $l \in \mathbb{Z}$.

For $i > 1$, we thus have that $f_i(z_k)$ is a singular modulus with discriminant not equal to $-4k$. Also, the $f_i(z_k)$ are all pairwise distinct, by Proposition 2.1, and, if k is large enough, not equal to α. One thus has that

$$(j(z_k) - \alpha)^{a_1} \prod_{i=2}^{n} (x_i - \alpha)^{a_i} = c$$

for some singular moduli $x_2, \ldots, x_n$ of discriminants not equal to $4k$.

For all k large enough, Proposition 2.7 implies that the tuple

$$(j(z_k), x_2, \ldots, x_n)$$

has some Galois conjugate over K of the form

$$(\beta, x_2', \ldots, x_n'),$$

where $\beta \neq j(z_k)$. Note that

$$(\beta - \alpha)^{a_1} \prod_{i=2}^{n} (x_i' - \alpha)^{a_i} = c.$$

Thus,

$$(j(z_k) - \alpha)^{a_1} \prod_{i=2}^{n} (x_i - \alpha)^{a_i} = (\beta - \alpha)^{a_1} \prod_{i=2}^{n} (x_i' - \alpha)^{a_i}.$$

The only singular moduli of discriminant $-4k$ in this relation are $j(z_k)$ and β, and they are distinct. Hence, at least the terms $(j(z_k) - \alpha)$ and $(\beta - \alpha)$ in the above relation do not cancel.

Grouping the terms where $x_i = x_k'$, which we then cancel if $a_i = a_k$, we obtain, for some $m \in \{2, \ldots, 2n\}$, an m-tuple

$$(j(z_k), \beta, y_1, \ldots, y_{m-2})$$

of singular moduli such that $j(z_k), \beta, y_1, \ldots, y_{m-2}, \alpha$ are pairwise distinct and

$$(j(z_k) - \alpha)^{e_1} (\beta - \alpha)^{e_2} \prod_{i=1}^{m-2} (y_i - \alpha)^{e_{i+2}} = 1$$

for some $e_1, \ldots, e_m \in \mathbb{Z} \setminus \{0\}$. Further, the tuples that arise in this way for different k are all distinct, since the $j(z_k)$ are all distinct.

By the pigeonhole principle, there is thus some $m \in \{2, \ldots, 2n\}$ for which there exist infinitely many m-tuples $(w_1, \ldots, w_m)$ of singular moduli such that $w_1, \ldots, w_m, \alpha$ are pairwise distinct and

$$\prod_{i=1}^{m} (w_i - \alpha)^{b_i} = 1$$

for some $b_1, \ldots, b_m \in \mathbb{Z} \setminus \{0\}$. This contradicts Theorem 1.1 and so we are done. □

Hence a multiplicative special curve as in (4.2) must have f non-constant.

Proposition 4.6. *Let $n \in \mathbb{Z}_{>0}$. Let $f_1, \ldots, f_n, f$ be pairwise distinct, non-constant j-maps. Then the functions $h_1, \ldots, h_n \colon \mathbb{H} \to \mathbb{C}$ defined by $h_i(z) = f_i(z) - f(z)$ are multiplicatively independent modulo constants.*

Proof. We will find some $z \in \mathbb{H}$ where precisely one of the functions h_i vanishes. By a change of variables, we may assume that $f(z) = j(z)$ and $f_i(z) = j(r_i z + s_i)$ for some $r_i, s_i \in \mathbb{Q}$ such that $r_i > 0$ and $0 \leq s_i < 1$. Note that $(r_i, s_i) \neq (1, 0)$ since $f_i \neq f$. We may and do assume that the pairs (r_i, s_i) are strictly increasing when ordered lexicographically.

Suppose first that $r_1 \geq 1$. Let

$$z_0 = -\frac{s_1}{2r_1} + \frac{\sqrt{4r_1 - s_1^2}}{2r_1} i,$$

so that

$$\frac{-1}{r_1 z_0 + s_1} = z_0.$$

Observe that $|r_1 z_0 + s_1| = \sqrt{r_1} \geq 1$. If $r_1 > 1$, then $r_1 z_0 + s_1 \in \mathfrak{F}_j$. If $r_1 = 1$, then $s_1 > 0$ and z_0 is on the left hand side of the lower boundary of $\mathfrak{F}_j$ and $r_1 z_0 + s_1$ is the reflection of z_0 in the imaginary axis.

Since $j(-1/z) = j(z)$, we have that $f_1(z_0) = f(z_0)$. If $r_i > r_1$, then, by Proposition 2.1, $\operatorname{Im} r_i z_0 + s_i > \operatorname{Im} r_1 z_0 + s_1$ and hence $j(r_i z_0 + s_i) \neq j(r_1 z_0 + s_1)$. If $r_i = r_1$ for $i \geq 2$, then $s_1 < s_i < 1$ and $j(r_i z_0 + s_i) \neq j(r_1 z_0 + s_1)$ by Proposition 2.1 again. Thus, $f_i(z_0) = f(z_0)$ if and only if $i = 1$ and we are done.

Now suppose that $r_1 < 1$. Let $k \in \mathbb{Z}$ be such that $0 \leq k r_1 - s_1 < r_1$. Let

$$z_1 = -\frac{k}{2} - \frac{s_1}{2r_1} + \frac{\sqrt{4r_1 - (kr_1 - s_1)^2}}{2r_1} i,$$

so that

$$\frac{-1}{z_1 + k} = r_1 z_1 + s_1.$$

Hence, $f_1(z_1) = f(z_1)$. Observe also that $|z_1 + k| = 1/\sqrt{r_1}$ and $z_1 + k \in \mathfrak{F}_j \setminus \partial \mathfrak{F}_j$ and so $r_1 z_1 + s_1 \in S\mathfrak{F}_j \setminus \partial(S\mathfrak{F}_j)$, where S denotes the transformation $z \mapsto -1/z$. Thus, by Proposition 2.1, the points in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z_1 with imaginary part $\geq \operatorname{Im} r_1 z_1 + s_1$ are the elements of

$$\{z_1 + m : m \in \mathbb{Z}\} \cup \{r_1 z_1 + s_1 + l : l \in \mathbb{Z}\}.$$

In particular, $r_i z_1 + s_i$ is not in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z_1 if $i \geq 2$, since $(r_i, s_i) \neq (1, 0)$. Hence, $f_i(z_1) = f(z_1)$ if and only if $i = 1$. The proof is thus complete. □

Therefore a multiplicative special curve as in (4.2) must have some f_i constant.

4.2. **The shape of multiplicative special curves.**

Proof of Theorem 4.2. Let $n \in \mathbb{Z}_{>0}$. Let $f_1, \ldots, f_n, f$ be pairwise distinct j-maps, at least one of which is non-constant. Suppose that $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{C}^{\times}$ are such that

$$\prod_{i=1}^{n} (f_i(z) - f(z))^{a_i} = c \tag{4.4}$$

for all $z \in \mathbb{H}$.

By Proposition 4.5, the j-map f must be non-constant. Thus, by Proposition 4.4, at least one of the j-maps $f_1, \ldots, f_n$ must be non-constant. By Proposition 4.6, at least one of the j-maps $f_1, \ldots, f_n$ is constant. After relabelling, we thus have that

$$\prod_{i \in I_1} (f_i - f)^{a_i} \prod_{i \in I_2} (f_i - f)^{a_i} = c$$

for all $z \in \mathbb{H}$, where the j-map f is non-constant and I_1, I_2 are non-empty index sets such that $I_1 \cup I_2 = \{1, \ldots, n\}$ and the j-map f_i is constant if $i \in I_1$ and non-constant if $i \in I_2$.

By a change of variables, we may write $f(z) = j(z)$. For $i \in I_1$, let α_i be the singular modulus such that $f_i = \alpha_i$. Note that the α_i must be pairwise distinct, since the f_i are. For $i \in I_2$, there is, by Proposition 2.10, a unique $N_i \in \mathbb{Z}_{>0}$ and $g_i \in C(N_i)$ such that $f_i(z) = j(g_i z)$ and $N_i > 1$ since $f_i(z) \neq j(z)$. Rearrange to obtain that

$$c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_2} (j(z) - j(g_i z))^{-a_i} \tag{4.5}$$

for all $z \in \mathbb{H}$, where

$$c' = \frac{(-1)^{a_1 + \ldots + a_n}}{c}.$$

We will show that the right hand side of (4.5) must be a product of powers of functions

$$\prod_{g \in C(N_i)} (j(z) - j(gz)).$$

Rewrite the right hand side of (4.5) by grouping factors with the same N_i to obtain that

$$c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(gz))^{a_i(g)} \tag{4.6}$$

for all $z \in \mathbb{H}$, where I_3 is a new index set and, for each $i \in I_3$, $S_i \subset C(M_i)$ is non-empty and the $M_i \in \mathbb{Z}_{>1}$ are pairwise distinct and the $a_i(g) \in \mathbb{Z} \setminus \{0\}$. We will show that, for each $i \in I_3$, we have that $S_i = C(M_i)$ and the $a_i(g)$ are equal for every $g \in C(M_i)$.

Suppose then that there is $i_0 \in I_3$ with the property that there exists $g_0 \in S_{i_0}$ and $h_0 \in C(M_{i_0})$ such that either $h_0 \notin S_{i_0}$ or $h_0 \in S_{i_0}$ but $a_{i_0}(h_0) \neq a_{i_0}(g_0)$. By Proposition 2.11, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $j(g_0 \gamma z) = j(h_0 z)$.

Since the function $j(z)$ is invariant under the map $z \mapsto \gamma z$, we obtain from (4.6) that

$$\prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(gz))^{a_i(g)} = \prod_{i \in I_3} \prod_{g \in S_i} (j(z) - j(g\gamma z))^{a_i(g)} \tag{4.7}$$

for all $z \in \mathbb{H}$. Now, by Proposition 2.11, the factor $j(z) - j(h_0 z)$ appears on the right hand side of (4.7) with exponent $a_{i_0}(g_0)$, and either does not appear on the left hand side (if $h_0 \notin S_{i_0}$) or appears on the left hand side with exponent equal to $a_{i_0}(h_0)$, which is not equal to $a_{i_0}(g_0)$, otherwise.

The equation (4.7) thus implies that there exists $l \in \mathbb{Z}_{>0}$ and non-constant j-maps $f_1, \ldots, f_l, f$ with $f(z) = j(z)$ and $f_1(z) = j(h_0 z)$ such that the functions v_i for $i = 1, \ldots, l$ defined by $v_i(z) = f_i(z) - f(z)$ are multiplicatively dependent modulo constants. This though contradicts Proposition 4.6.

Therefore, in (4.6), we must have, for each $i \in I_3$, that $S_i = C(M_i)$ and that the $a_i(g)$ are equal for every $g \in C(M_i)$. The right hand side of (4.6) may thus be rewritten to obtain that

$$c' \prod_{i \in I_1} (j(z) - \alpha_i)^{a_i} = \prod_{i \in I_3} \prod_{g \in C(M_i)} (j(z) - j(gz))^{b_i} \tag{4.8}$$

for all $z \in \mathbb{H}$, for some $b_i \in \mathbb{Z} \setminus \{0\}$, and pairwise distinct $M_i \in \mathbb{Z}_{>1}$. The right hand side is thus equal to the function

$$\prod_{i \in I_3} F_{M_i}(j(z))^{b_i},$$

the zeros and poles of which are thus equal to the α_i on the left hand side of (4.8). □

4.3. **Determining the multiplicative special curves.** We now complete the proof of Theorem 1.4. Let $n \in \mathbb{Z}_{>0}$. We will show that there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ and these may be determined effectively.

Proof of Theorem 1.4. Suppose that

$$T = \{(f_1(z), \dots, f_n(z), f(z)) : z \in \mathbb{H}\}$$

is a multiplicative special curve in $\mathbb{C}^{n+1}$. Then, by Theorem 4.1, we may reorder the first n coordinates of T in such a way that

$$T = \{(\alpha_1, \dots, \alpha_m, j(g_1 z), \dots, j(g_l z), j(z)) : z \in \mathbb{H}\},$$

where

(1) $\alpha_1, \dots, \alpha_m$ are pairwise distinct and such that

$$\{\alpha_1, \dots, \alpha_m\} = \{\alpha \in \mathbb{C} : \alpha \text{ is either a zero or a pole of } \prod_{i=1}^{k} F_{N_i}(X)^{b_i}\};$$

(2) $g_1, \dots, g_l \in \mathrm{GL}_2^+(\mathbb{Q})$ are pairwise distinct and such that

$$\{g_1, \dots, g_l\} = \bigcup_{i=1}^{k} C(N_i);$$

for some $k \in \mathbb{Z}_{>0}$, $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$, and pairwise distinct $N_1, \dots, N_k \in \mathbb{Z}_{>1}$. In particular,

$$m + l = n.$$

Also,

$$l = \sum_{i=1}^{k} \#C(N_i).$$

Since ([19, p. 53])

$$\#C(N_i) = N_i \prod_{p|N_i} \left(1 + \frac{1}{p}\right),$$

we have that $\#C(N_i) \geq N_i + 1$.

Corollary 2.5 implies that

$$\prod_{i=1}^{k} F_{N_i}(X)^{b_i}$$

is non-constant. Hence, $m \geq 1$. Thus we must have that

$$\sum_{i=1}^{k} \#C(N_i) \leq n - 1.$$

So $\max\{N_1, \dots, N_k\} \leq n - 2$. Since $N_1, \dots, N_k$ are pairwise distinct and ≥ 2, we must have that

$$\sum_{i=2}^{k+1} (i + 1) \leq n - 1.$$

Thus

$$\frac{1}{2}k(k+5) \leq n-1,$$

and hence

$$k \leq \frac{1}{2}(\sqrt{8n+17}-5).$$

In particular, there are only finitely many possibilities for $k, N_1, \ldots, N_k$ and these may be computed.

Let $k, N_1, \ldots, N_k$ be such a possible choice for a multiplicative special curve in $\mathbb{C}^{n+1}$. Compute

$$l = \sum_{i=1}^{k} \#C(N_i).$$

The corresponding polynomials F_{N_i} may also be computed [10, §13B]. Let $\beta_1, \ldots, \beta_r$ be pairwise distinct and such that

$$\{\beta_1, \ldots, \beta_r\} = \{\beta \in \mathbb{C} : F_{N_i}(\beta) = 0 \text{ for some } i = 1, \ldots, k\}.$$

Write $e_{i,u}$ for the multiplicity of β_u as a root of F_{N_i}. Let d_i be the leading coefficient of F_{N_i}. Note that $d_i \in \mathbb{Z} \setminus \{0\}$. Let $p_1, \ldots, p_t$ be a complete list of the prime factors of $d_1, \ldots, d_k$. Let $f_{i,v}$ be the exponent of p_v occurring in the prime factorisation of d_i.

The choice $k, N_1, \ldots, N_k$ then gives rise to a multiplicative special curve in $\mathbb{C}^{n+1}$ if and only if there exist $b_1, \ldots, b_k \in \mathbb{Z} \setminus \{0\}$ such that

$$\sum_{i=1}^{k} b_i f_{i,v} = 0$$

for every $v \in \{1, \ldots, t\}$ and

$$\sum_{i=1}^{k} b_i e_{i,u} = 0$$

for exactly $n - l$ choices of $u \in \{1, \ldots, r\}$. This condition may be checked effectively. Consequently, there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ and these may be determined effectively.

Now suppose that $n \leq 5$. Then

$$k \leq \frac{1}{2}(\sqrt{57}-5) < \frac{3}{2}.$$

So $k = 1$ is the only possibility. And

$$N_1 \leq 3.$$

So the only possible multiplicative special curves in $\mathbb{C}^{n+1}$ arise with $k = 1$ and $N_1 \in \{2, 3\}$. If $N_1 = 2$, then $l = 3$ and so one needs $m \leq 2$, which is impossible since F_2 has three distinct roots (see Example 1.7). If $N_1 = 3$, then $l = 4$ and so one needs $m \leq 1$, but the polynomial

$$F_3(X) = -X(X-8000)^2(X+32768)^2(X-54000)$$

has four distinct roots. Thus, there are no multiplicative special curves in $\mathbb{C}^{n+1}$ for $n \in \{1, \ldots, 5\}$. □

5. Weakly special subvarieties and Ax–Schanuel

5.1. **Weakly special subvarieties.** For the proof of Theorem 1.5, we will need the notion of (weakly) special subvarieties. Varieties and subvarieties are always irreducible over $\mathbb{C}$.

Definition 5.1. Let $m, n \in \mathbb{Z}_{>0}$.

(1) A weakly special subvariety of $\mathbb{C}^m$ is an irreducible component of a subvariety of $\mathbb{C}^m$ defined by equations of the form $\Phi_N(x_i, x_k) = 0$ and $x_l = c$ for $N \in \mathbb{Z}_{>0}$ and $c \in \mathbb{C}$.
(2) A special point of $\mathbb{C}^m$ is a point $(x_1, \ldots, x_m) \in \mathbb{C}^m$ such that $x_1, \ldots, x_m$ are singular moduli.
(3) A special subvariety of $\mathbb{C}^m$ is a weakly special subvariety of $\mathbb{C}^m$ which contains a special point of $\mathbb{C}^m$. Equivalently, a weakly special subvariety for which any constant coordinates are singular moduli.
(4) A weakly special subvariety of $(\mathbb{C}^\times)^n$ is a coset of a subtorus (i.e. a coset of an irreducible algebraic subgroup of $(\mathbb{C}^\times)^n$).
(5) A special point of $(\mathbb{C}^\times)^n$ is a point $(\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^n$ such that $\zeta_1, \ldots, \zeta_n$ are roots of unity.
(6) A special subvariety of $(\mathbb{C}^\times)^n$ is a weakly special subvariety of $(\mathbb{C}^\times)^n$ which contains a special point of $(\mathbb{C}^\times)^n$.
(7) A (weakly) special subvariety of $\mathbb{C}^m \times (\mathbb{C}^\times)^n$ is a product $M \times T$, where $M \subset \mathbb{C}^m$ is a (weakly) special subvariety of $\mathbb{C}^m$ and $T \subset (\mathbb{C}^\times)^n$ is a (weakly) special subvariety of $(\mathbb{C}^\times)^n$.

Note that a weakly special subvariety $T \subset (\mathbb{C}^\times)^n$ is defined by equations of the form

$$t_1^{a_1} \cdots t_n^{a_n} = c$$

for some $c \in \mathbb{C}^\times$ and $a_1, \ldots, a_n \in \mathbb{Z}$ not all zero. Also, T is a special subvariety if and only if T may be defined by equations of this kind with the additional property that every such c is a root of unity. See, for example, [34, Remark 1.0.1].

It follows from the above description that special subvarieties of $\mathbb{C}^m$ and $(\mathbb{C}^\times)^n$ are defined over $\overline{\mathbb{Q}}$. Special subvarieties of $\mathbb{C}^m$ have the following useful properties.

Proposition 5.2 ([6, Proposition 2.1])**.** *Let $m \in \mathbb{Z}_{>0}$. Let $M \subset \mathbb{C}^m$ be a positive-dimensional special subvariety. Then M contains a Zariski-dense union of special subvarieties of $\mathbb{C}^m$ of dimension* 1.

Proposition 5.3 ([6, Proposition 2.3])**.** *Let $m \in \mathbb{Z}_{>0}$. Let $M \subset \mathbb{C}^m$. Then M is a special subvariety of dimension* 1 *if and only if there exist j-maps $f_1, \ldots, f_n$, at least one of which is non-constant, such that*

$$M = \{(f_1(z), \ldots, f_n(z)) : z \in \mathbb{H}\}.$$

In particular, a multiplicative special curve in $\mathbb{C}^{n+1}$ is a special subvariety of $\mathbb{C}^{n+1}$ of dimension 1 which is contained in a coset of a torus.

5.2. **Ax–Schanuel.** Now we state the functional transcendence result of Pila and Tsimerman [27] which we will use in the proof of Theorem 1.5. This result is a consequence of Ax's theorem [2] for the exponential function and Pila and Tsimerman's [26] Ax–Schanuel theorem for the j-function.

Let $m, n \in \mathbb{Z}_{>0}$. Let

$$X = \mathbb{C}^m \times (\mathbb{C}^\times)^n$$

and

$$U = \mathbb{H}^m \times \mathbb{C}^n$$

Define $e \colon \mathbb{C} \to (\mathbb{C}^\times)$ by $e(t) = \exp(2\pi i t)$. Define $\pi \colon U \to X$ by

$$\pi(z_1, \ldots, z_m, t_1, \ldots, t_n) = (j(z_1), \ldots, j(z_m), e(t_1), \ldots, e(t_n)).$$

We make the following definitions.

Definition 5.4. An algebraic subvariety of U is a complex-analytically irreducible component of $U \cap W$ for some algebraic subvariety $W \subset \mathbb{C}^m \times \mathbb{C}^n$.

Definition 5.5. A (weakly) special subvariety of U is a complex-analytically irreducible component of $\pi^{-1}(W)$ for W a (weakly) special subvariety of X (as defined in Definition 5.1).

The result of Pila and Tsimerman we need is the following statement.

Theorem 5.6 (Weak Complex Ax [27, Theorem 3.3])**.** *Let $U' \subset U$ be a weakly special subvariety. Let $X' = \pi(U')$. Let $V \subset X'$ and $W \subset U'$ be algebraic subvarieties and $A \subset W \cap \pi^{-1}(V)$ a complex-analytically irreducible component. Then*

$$\dim A = \dim V + \dim W - \dim X',$$

unless A is contained in a proper weakly special subvariety of U'.

6. The proof of Theorem 1.5

We will prove Theorem 1.5 by applying the so-called Pila–Zannier strategy of o-minimal point counting. This strategy was proposed by Zannier and was first used by Pila and Zannier [29] to give a new proof of the Manin–Mumford conjecture. The approach used here is similar to that employed in [13, 27]. For background on o-minimality and on the Pila–Zannier method, see Pila's book [25].

6.1. **The counting theorem for semirational points.** We will use an extension, due to Habegger and Pila [18, Corollary 7.2], of the Pila–Wilkie o-minimal counting theorem [28]. We will always work in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$; see [25, p. 77] for details of this structure. Definable will mean definable with parameters in $\mathbb{R}_{\mathrm{an,exp}}$. Complex numbers, when considered as elements of definable sets, will be identified with their real and imaginary parts. Throughout this section, constants $c = c(\ldots)$ will be positive and have only the indicated dependencies.

To state Habegger and Pila's result, we need to define the k-height of a real number. Let $k \in \mathbb{Z}_{>0}$. For $y \in \mathbb{R}$, define the k-height of y by

$$H_k(y) = \min\{\max\{|a_0|, \ldots, |a_k|\} : a_0, \ldots, a_k \text{ are coprime integers, not all zero, such that } a_k y^k + \ldots + a_0 = 0\},$$

with the convention that $\min \emptyset = \infty$. Note that $y \in \mathbb{R}$ thus has $H_k(y) < \infty$ if and only if $[\mathbb{Q}(y) : \mathbb{Q}] \leq k$. For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define

$$H_k(y) = \max\{H_k(y_1), \ldots, H_k(y_n)\}.$$

The k-height is related to the multiplicative height in the following way.

Proposition 6.1. *Let $d \in \mathbb{Z}_{>0}$. There exists a constant $c(d) > 0$ with the property that if $\alpha \in \overline{\mathbb{Q}}$ is such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d$, then*

$$H_d(\alpha) \leq c(d) H(\alpha)^d.$$

Proof. Let

$$c(d) = \binom{d}{\lfloor d/2 \rfloor}.$$

Suppose that

$$f(t) = a_d t^d + \ldots + a_0$$

is a minimal polynomial over $\mathbb{Z}$ of some $\alpha \in \overline{\mathbb{Q}}$. Then

$$\begin{aligned} H_d(\alpha) &\leq \max\{|a_0|, \ldots, |a_d|\} \\ &\leq c(d) M(f) \\ &= c(d) H(\alpha)^d, \end{aligned}$$

where the second inequality is [8, Lemma 1.6.7] and the final equality is [8, Proposition 1.6.6]. Here $M(f)$ denotes the Mahler measure of f. □

Habegger and Pila's point counting result is the following.

Theorem 6.2 ([18, Corollary 7.2])**.** *Let $F \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be a definable family parametrised by $\mathbb{R}^l$. Let $\epsilon > 0$ and $k \in \mathbb{Z}_{>0}$. Let $\pi_1 \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\pi_2 \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection maps. Then there exists a constant $c = c(F, k, \epsilon) > 0$ with the following property.*

Let $x \in \mathbb{R}^l$. Write $F_x \subset \mathbb{R}^m \times \mathbb{R}^n$ for the fibre of F over x. If $T \geq 1$ and

$$\Sigma \subset \{(y, z) \in F_x : H_k(y) \leq T\}$$

is such that $\#\pi_2(\Sigma) > cT^\epsilon$, then there exists a continuous, definable function $\beta \colon [0, 1] \to F_x$ such that:

1. *The composition $\pi_1 \circ \beta \colon [0, 1] \to \mathbb{R}^m$ is semialgebraic and its restriction to $(0, 1)$ is real analytic.*
2. *The composition $\pi_2 \circ \beta \colon [0, 1] \to \mathbb{R}^n$ is non-constant.*
3. *$\pi_2(\beta(0)) \in \pi_2(\Sigma)$.*
4. *The restriction of β to $(0, 1)$ is real analytic.*

The constant $c = c(F, k, \epsilon)$ here is not effective. For (4), we use the fact that $\mathbb{R}_{\text{an,exp}}$ admits analytic cell decomposition [11, Theorem 8.8].

We will also require the following bound on the size of the exponents in a multiplicative dependency.

Proposition 6.3. *Let $n \in \mathbb{Z}_{>0}$. There exist constants $c_1(n), c_2(n) > 0$ with the following property. Let L be a number field and $d = [L : \mathbb{Q}]$. If $\alpha_1, \ldots, \alpha_n \in L^\times$ are pairwise distinct and such that*

$$\prod_{i=1}^{n} \alpha_i^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$, then there exist $b_1, \ldots, b_n \in \mathbb{Z} \setminus \{0\}$ such that

$$\prod_{i=1}^{n} \alpha_i^{b_i} = 1$$

and, for every $i \in \{1, \ldots, n\}$,

$$|b_i| \leq c_1(n)(d \max\{1, h(\alpha_1), \ldots, h(\alpha_n)\})^{c_2(n)}.$$

This will follow from the following bound, which covers the case where the multiplicative dependency is minimal.

Lemma 6.4. *Let* $n \in \mathbb{Z}_{>0}$. *There exists an explicit constant* $c(n) > 0$ *with the following property. Let* L *be a number field and* $d = [L : \mathbb{Q}]$. *If* $\alpha_1, \ldots, \alpha_n \in L^\times$ *are pairwise distinct and the set* $\{\alpha_1, \ldots, \alpha_n\}$ *is minimally multiplicatively dependent, then there exist* $b_1, \ldots, b_n \in \mathbb{Z} \setminus \{0\}$ *such that*

$$\prod_{i=1}^{n} \alpha_i^{b_i} = 1$$

and

$$|b_i| \leq c(n) d^{n+1}(1 + \log d) \prod_{\substack{k=1 \\ k \neq i}}^{n} h(\alpha_k).$$

Proof. If $n = 1$, then α_1 is a root of unity of degree $\leq d$. Hence, α_1 is a primitive Nth root of unity for some N with $\phi(N) \leq d$, where ϕ denotes Euler's totient function. The desired result then follows from the elementary bound

$$\phi(N) \geq \sqrt{\frac{N}{2}}.$$

For $n \geq 2$, this is a result of Yu [22, Corollary 3.2]. The version stated in [22] has $d^n \log d$ in place of the $d^{n+1}(1 + \log d)$ here; the slight weakening here allows one to state a uniform result for all $d, n \geq 1$ which still suffices for our purposes. □

Proposition 6.3 follows from Lemma 6.4 via the following elementary lemma.

Lemma 6.5. *Let* $n \in \mathbb{Z}_{>1}$. *Let* $v, w \in \mathbb{Z}^n$. *Suppose that, for some* $k \in \{1, \ldots, n-1\}$, *we have that*

$$v = (v_1, \ldots, v_k, 0, \ldots, 0),$$

where $v_1, \ldots, v_k \neq 0$, *and that*

$$w = (w_1, \ldots, w_n)$$

with $w_{k+1} \neq 0$. *Let*

$$\lambda = 1 + \max\{|v_1|, \ldots, |v_k|, |w_1|, \ldots, |w_n|\}.$$

Let

$$u = v + \lambda w,$$

and write

$$u = (u_1, \ldots, u_n).$$

Then $u_1, \ldots, u_{k+1} \neq 0$ *and*

$$|u_i| \leq 2\lambda^2 \text{ for } i = 1, \ldots, n.$$

Proof. For $i = 1, \ldots, k$, if $w_i \neq 0$, then $|w_i| \geq 1$ and so $\lambda |w_i| > |v_i|$. The result follows immediately since $m^2 + m \leq 2m^2$ for all $m \in \mathbb{Z}$. □

Proof of Proposition 6.3. Let L be a number field and $d = [L : \mathbb{Q}]$. Suppose that $\alpha_1, \ldots, \alpha_n \in L^\times$ are pairwise distinct and such that

$$\prod_{i=1}^{n} \alpha_i^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$. The set

$$\mathcal{S} = \{\alpha_1, \ldots, \alpha_n\}$$

is thus multiplicatively dependent, but not necessarily minimally multiplicatively dependent. For each $i = 1, \ldots, n$ though, there exists $S_i \subset \mathcal{S}$ such that $\alpha_i \in S_i$ and S_i is minimally multiplicatively dependent, see e.g. [13, Lemma 5.9]. We will apply Lemma 6.4 to each set S_i.

By Lemma 6.4, there exist constants $c_1(n), c_2(n) > 0$ such that, for each $i = 1, \ldots, n$, there exist $b_{i,k} \in \mathbb{Z} \setminus \{0\}$ for $k \in S_i$ with

$$\prod_{k \in S_i} \alpha_k^{b_{i,k}} = 1$$

and

$$|b_{i,k}| \leq c_1(n)(d \max\{1, h(\alpha_1), \ldots, h(\alpha_n)\})^{c_2(n)}.$$

Now let $v_i \in \mathbb{Z}^n$ be the vector with kth coordinate $v_{i,k}$ equal to $b_{i,k}$ if $k \in S_i$ and 0 otherwise. Hence, for $i = 1, \ldots, n$, we have that $v_{i,i} \neq 0$ and

$$\prod_{k=1}^{n} \alpha_k^{v_{i,k}} = 1.$$

Let

$$\mu = 1 + \max\{|v_{i,k}| : i, k \in \{1, \ldots, n\}\}.$$

Apply Lemma 6.5 inductively to $v_1, \ldots, v_n$ to obtain a vector

$$w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$$

which is a $\mathbb{Z}$-linear combination of $v_1, \ldots, v_n$ and such that $w_1, \ldots, w_n \neq 0$ and

$$|w_i| \leq c_3(n)\mu^{c_4(n)} \text{ for } i = 1, \ldots, n$$

for some constants $c_3(n), c_4(n) > 0$. In particular, there are constants $c_5(n), c_6(n) > 0$ such that

$$|w_i| \leq c_5(n)(d \max\{1, h(\alpha_1), \ldots, h(\alpha_n)\})^{c_6(n)} \text{ for } i = 1, \ldots, n.$$

Since w is a $\mathbb{Z}$-linear combination of $v_1, \ldots, v_n$, one has that

$$\prod_{i=1}^{n} \alpha_i^{w_i} = 1,$$

as required. □

6.2. **Completing the proof of Theorem 1.5.** Now we come to the proof of Theorem 1.5. Fix $n \in \mathbb{Z}_{>0}$. In the proof, $c_1, c_2, \ldots$ will denote positive constants which depend only on n. Any other dependencies among constants will be explicitly indicated.

By Theorem 1.4, there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$. Since every multiplicative special curve is defined over $\overline{\mathbb{Q}}$, we may fix some number field K over which all the multiplicative special curves in $\mathbb{C}^{n+1}$ are defined.

Define the complexity Δ of an $(n+1)$-tuple $(x_1, \ldots, x_n, y)$ of singular moduli $x_1, \ldots, x_n, y$ by

$$\Delta = \max\{|\Delta(x_1)|, \ldots, |\Delta(x_n)|, |\Delta(y)|\}.$$

Recall that $e\colon \mathbb{C} \to \mathbb{C}^\times$ is given by $e(z) = \exp(2\pi i z)$. Let $\mathfrak{F}_e = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1\}$, so that e restricted to $\mathfrak{F}_e$ is a bijection. The restrictions $j\colon \mathfrak{F}_j \to \mathbb{C}$ and $e\colon \mathfrak{F}_e \to \mathbb{C}^\times$ are definable. Let

$$Y = \Big\{(z_1, \ldots, z_n, z, w_1, \ldots, w_n, w, u_1, \ldots, u_n, r_1, \ldots, r_n, s) \in \mathfrak{F}_j^{2(n+1)} \times \mathfrak{F}_e^n \times \mathbb{R}^{n+1} : \sum_{i=1}^n r_i u_i = s, w = z, \text{ and } w_i = z_i \text{ and } e(u_i) = j(z_i) - j(z) \text{ for } i = 1, \ldots, n\Big\}$$

and

$$Z = \Big\{(z_1, \ldots, z_n, z, r_1, \ldots, r_n, s) \in \mathfrak{F}_j^{n+1} \times \mathbb{R}^{n+1} : \exists (u_1, \ldots, u_n) \in \mathfrak{F}_e^n \text{ such that } (z_1, \ldots, z_n, z, z_1, \ldots, z_n, z, u_1, \ldots, u_n, r_1, \ldots, r_n, s) \in Y\Big\}.$$

The sets Y, Z are both definable.

Suppose that $(x_1, \ldots, x_n, y)$ is an $(n+1)$-tuple of pairwise distinct singular moduli $x_1, \ldots, x_n, y$ such that

$$\prod_{i=1}^n (x_i - y)^{a_i} = 1$$

for some $a_i \in \mathbb{Z} \setminus \{0\}$. Let Δ be the complexity of this tuple. Let

$$d = [\mathbb{Q}(x_1, \ldots, x_n, y) : \mathbb{Q}].$$

By Proposition 6.3, we may assume that

$$|a_i| \leq c_1 (d \max\{1, h(x_1), \ldots, h(x_n), h(y)\})^{c_2}$$

for $i \in \{1, \ldots, n\}$. Then apply Proposition 2.7 (with $\epsilon = 1/4$ say) to give an upper bound on d and Proposition 2.9 to bound the logarithmic heights of the singular moduli. One thereby obtains that, for $i \in \{1, \ldots, n\}$,

$$|a_i| \leq c_3 \Delta^{c_4}.$$

Let

$$(\tau_1, \ldots, \tau_n, \tau, \nu_1, \ldots, \nu_n) \in \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n$$

be the preimage of

$$(x_1, \ldots, x_n, y, x_1 - y, \ldots, x_n - y)$$

with respect to the map $(j, e)\colon \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^n \to \mathbb{C}^{n+1} \times (\mathbb{C}^\times)^n$. Note that $\tau_1, \ldots, \tau_n, \tau$ are all quadratic, since they are the preimages for j of singular moduli. By Proposition 2.8, the real and imaginary parts of $\tau_1, \ldots, \tau_n, \tau$ all have multiplicative height $\leq 2\Delta$. Observe also that

$$\sum_{i=1}^{n} a_i \nu_i \in \mathbb{Z},$$

since

$$\prod_{i=1}^{n} e(\nu_i)^{a_i} = 1.$$

Let $b = \sum_{i=1}^{n} a_i \nu_i$. Then

$$|b| \leq \sum_{i=1}^{n} |a_i|,$$

since $\nu_1, \ldots, \nu_n$ all have real part in the interval $[0, 1)$. In particular,

$$|b| \leq c_5 \Delta^{c_6}.$$

The tuple $(x_1, \ldots, x_n, y)$ thus gives rise to the point

$$(\tau_1, \ldots, \tau_n, a_1, \ldots, a_n, b) \in Z,$$

which is quadratic in the τ_i, τ coordinates and integral in the a_i, b coordinates. Further, the 2-height of this point is $\leq c_7 \Delta^{c_8}$ by Proposition 6.1.

Every Galois conjugate $(x_1', \ldots, x_n', y')$ of $(x_1, \ldots, x_n, y)$ over K satisfies the multiplicative relation

$$\prod_{i=1}^{n} (x_i' - y')^{a_i} = 1,$$

where $a_1, \ldots, a_n$ are the same integers as before. The conjugate $(x_1', \ldots, x_n', y')$ thus gives rise, in the same way as $(x_1, \ldots, x_n, y)$ did, to a point

$$(\tau_1', \ldots, \tau_n', \tau', a_1, \ldots, a_n, b') \in Z,$$

where the τ_i', τ' are quadratic and of multiplicative height $\leq 2\Delta$, the a_i are the same integers as before, and b' is an integer such that $|b'| \leq c_5 \Delta^{c_6}$. Note that b' is not necessarily the same as b. In particular, the point $(\tau_1', \ldots, \tau_n', \tau', a_1, \ldots, a_n, b')$ also has 2-height $\leq c_7 \Delta^{c_8}$. Further, the corresponding points of Z arising from distinct K-conjugates of $(x_1, \ldots, x_n, y)$ are always distinct in the $\mathfrak{F}_j^{n+1}$ coordinates. By Proposition 2.7 with $\epsilon = 1/4$, there are at least $c_9 \Delta^{1/4}$ distinct K-conjugates of $(x_1, \ldots, x_n, y)$, each of which gives rise to a distinct point of Z in the above way.

View Y as a definable family of sets fibred over the $(r_1, \ldots, r_n)$ coordinates. Each of the points

$$(\tau_1', \ldots, \tau_n', \tau', a_1, \ldots, a_n, b') \in Z$$

described above is the projection of a point

$$(\tau_1', \ldots, \tau_n', \tau', \tau_1', \ldots, \tau_n', \tau', \nu_1', \ldots, \nu_n', a_1, \ldots, a_n, b') \in Y.$$

Note that the Y-points arising in this way from distinct conjugates over K of $(x_1, \ldots, x_n, y)$ are distinct in their $(\tau_1', \ldots, \tau_n', \tau')$ coordinates.

These points $(\tau_1', \ldots, \tau_n', \tau', \tau_1', \ldots, \tau_n', \tau', \nu_1', \ldots, \nu_n', b')$ all lie on the fibre $Y_{(a_1, \ldots, a_n)}$ of Y over $(a_1, \ldots, a_n)$. Let

$$\pi_1 \colon Y_{(a_1, \ldots, a_n)} \to \mathfrak{F}_j^{n+1} \times \mathbb{R} \text{ and } \pi_2 \colon Y_{(a_1, \ldots, a_n)} \to \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^{n} \times \mathbb{R}$$

be the projection maps

$$(z_1, \ldots, z_n, z, w_1, \ldots, w_n, w, u_1, \ldots, u_n, s) \mapsto (z_1, \ldots, z_n, z, s)$$

and

$$(z_1, \ldots, z_n, z, w_1, \ldots, w_n, w, u_1, \ldots, u_n, s) \mapsto (w_1, \ldots, w_n, w, u_1, \ldots, u_n)$$

respectively. Observe that π_2 is injective. Let $\Sigma \subset Y_{(a_1,\ldots,a_n)}$ be the set consisting of all the points arising in the way described above from the K-conjugates of $(x_1, \ldots, x_n, y)$. Then $\pi_1(\Sigma)$ contains only algebraic points of degree at most 2 and which have 2-height $\leq c_7\Delta^{c_8}$. Also, $\#\pi_2(\Sigma) > c_9\Delta^{1/4}$.

Now let $C_{HP} > 0$ be the constant given by Theorem 6.2 applied to Y with $k = 2$ and $\epsilon = (8c_8)^{-1}$. Note that c_8 depends only on n, which is fixed, and C_{HP} depends only on Y, k, ϵ, which are all fixed. In particular, C_{HP} is independent of $(x_1, \ldots, x_n, y)$ and $a_1, \ldots, a_n$. Let $T = c_7\Delta^{c_8}$. Then $\pi_1(\Sigma)$ contains only algebraic points of degree ≤ 2 and 2-height $\leq T$ and

$$\#\pi_2(\Sigma) > c_9\Delta^{1/4}.$$

In particular, if $\Delta > (c_7C_{HP}/c_9)^8$, then

$$\#\pi_2(\Sigma) > C_{HP}T^{\epsilon}$$

(here we assume without loss of generality that $c_7, c_8 \geq 1$, so $c_7^{\epsilon} \leq c_7$).

Suppose then that $\Delta > (c_7C_{HP}/c_9)^8$. Then Theorem 6.2 implies that there exists a continuous, definable function $\beta\colon [0,1] \to Y_{(a_1,\ldots,a_n)}$ with the following properties:

1. The composition $\pi_1 \circ \beta\colon [0,1] \to \mathfrak{F}_j^{n+1} \times \mathbb{R}$ is semialgebraic and its restriction to $(0,1)$ is real analytic.
2. The composition $\pi_2 \circ \beta\colon [0,1] \to \mathfrak{F}_j^{n+1} \times \mathfrak{F}_e^{n}$ is non-constant.
3. $\pi_2(\beta(0)) \in \pi_2(\Sigma)$.
4. The restriction of β to $(0,1)$ is real analytic.

Note that (2) implies that $\pi_1 \circ \beta$ composed with projection to the $\mathfrak{F}_j^{n+1}$ coordinates is non-constant. Since π_2 is injective, we have that $\beta(0) \in \Sigma$, i.e. $\beta(0)$ is a point of $Y_{(a_1,\ldots,a_n)}$ arising from a K-conjugate of $(x_1, \ldots, x_n, y)$.

Let $Z_{(a_1,\ldots,a_n)}$ be the fibre of Z over $(a_1, \ldots, a_n)$. Then $(\pi_1 \circ \beta)([0,1]) \subset Z_{(a_1,\ldots,a_n)}$. Let

$$U = \{(z_1, \ldots, z_n, z, t) \in \mathbb{H}^{n+1} \times \mathbb{C} : \prod_{i=1}^{n}(j(z_i) - j(z))^{a_i} = e(t)\}.$$

Note that $Z_{(a_1,\ldots,a_n)} \subset U$. Hence, by [1, Proposition 1], there exists an open neighbourhood $W \subset \mathbb{H}^{n+1} \times \mathbb{C}$ of $\pi_1(\beta(0))$ such that

$$W \cap (\pi_1 \circ \beta)([0,1]) \subset V \subset U,$$

where V is a finite union of irreducible Nash subsets (see [1, §2.2]) of W, all of which contain $\pi_1(\beta(0))$.

We now use the characterisation of Nash subsets in [1, Remark 4]. Since $(\pi_1 \circ \beta)([0,1])$ has non-constant projection to $\mathfrak{F}_j^{n+1}$, by real analytic continuation there must exist some complex-analytically irreducible component of V with non-constant projection to $\mathfrak{F}_j^{n+1}$. Every complex-analytically irreducible component of V contains $\pi_1(\beta(0))$. Therefore, there exists a complex algebraic subvariety $A \subset \mathbb{C}^{n+2}$ and a complex-analytically irreducible component $B \subset (\mathbb{H}^{n+1} \times \mathbb{C}) \cap A$ such that B has non-constant projection to $\mathbb{H}^{n+1}$ and $\pi_1(\beta(0)) \in B \subset U$.

By the Ax–Schanuel result of Theorem 5.6, there thus exist weakly special subvarieties $W_1 \subset \mathbb{H}^{n+1}$ and $W_2 \subset \mathbb{C}$ such that

$$B \subset W_1 \times W_2 \subset U.$$

Since the projection $U \to \mathbb{H}^{n+1}$ has discrete fibres, the weakly special subvariety W_2 must just be a point. Hence, W_2 is equal to the projection of $\pi_1(\beta(0))$, which is $\{b'\}$ for some $b' \in \mathbb{Z}$.

The weakly special subvariety W_1 must be positive-dimensional, since B has non-constant projection to $\mathbb{H}^{n+1}$. Also, W_1 contains the preimage of some K-conjugate of $(x_1, \ldots, x_n, y)$, and so any constant coordinates of W_1 must be quadratic. Finally, note that

$$W_1 \subset \{(z_1, \ldots, z_n, z) : \prod_{i=1}^{n} (j(z_i) - j(z))^{a_i} = 1\}.$$

By setting all free variables of W_1 except one equal to the corresponding coordinates of the preimage of the K-conjugate of $(x_1, \ldots, x_n, y)$ which W_1 contains, we may and do assume that $\dim W_1 = 1$. Now take the image of W_1 under j. One thus obtains j-maps $f_1, \ldots, f_n, f$, which are not all constant, and satisfy

$$\prod_{i=1}^{n} (f_i(z) - f(z))^{a_i} = 1 \text{ for all } z \in \mathbb{H}. \tag{6.1}$$

Since $j(W_1)$ contains some K-conjugate $(x'_1, \ldots, x'_n, y')$ of $(x_1, \ldots, x_n, y)$, the j-maps $f_1, \ldots, f_n, f$ must be pairwise distinct. In particular, $j(W_1)$ is a multiplicative special curve in $\mathbb{C}^{n+1}$ which contains $(x'_1, \ldots, x'_n, y')$.

Thus, $j(W_1)$ is one of the finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ given by Theorem 1.4. In particular, $j(W_1)$ is defined over K. Thus, $(x_1, \ldots, x_n, y) \in j(W_1)$, since $j(W_1)$ contains a K-conjugate of $(x_1, \ldots, x_n, y)$.

We have therefore shown that, for $(x_1, \ldots, x_n, y)$ an $(n+1)$-tuple of pairwise distinct singular moduli $x_1, \ldots, x_n, y$ of complexity Δ such that

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$, if $\Delta > (c_7 C_{HP}/c_9)^8$, then $(x_1, \ldots, x_n, y)$ belongs to one of the finitely many multiplicative special curves in $\mathbb{C}^{n+1}$. Hence, the complexity of every such $(n+1)$-tuple which does not lie on a multiplicative special curve in $\mathbb{C}^{n+1}$ is $\leq (c_7 C_{HP}/c_9)^8$. In particular, there are only finitely many such $(n+1)$-tuples. This completes the proof of Theorem 1.5. Corollary 1.6 follows immediately.

7. The Zilber–Pink connection

Let $m, n \in \mathbb{Z}_{>0}$. Let

$$X_{m,n} = \mathbb{C}^m \times (\mathbb{C}^\times)^n.$$

Recall the definition of a special subvariety of $X_{m,n}$ from Definition 5.1.

Definition 7.1. Let $V \subset X_{m,n}$ be a subvariety. A subvariety $W \subset V$ is called an atypical component of V in $X_{m,n}$ if there exists a special subvariety $T \subset X_{m,n}$ such that W is an irreducible component of $V \cap T$ and

$$\dim W > \dim V + \dim T - \dim X_{m,n}.$$

An atypical component W of V in $X_{m,n}$ is a maximal atypical component of V in $X_{m,n}$ if there does not exist any atypical component W' of V in $X_{m,n}$ such that $W \subsetneq W'$.

The Zilber–Pink conjecture was formulated independently in different contexts by Zilber [35], Pink [30], and Bombieri, Masser, and Zannier [9]. The conjecture is wide open; see [25, Part IV] for more details. In our context, the Zilber–Pink conjecture is the following statement.

Conjecture 7.2 (Zilber–Pink conjecture)**.** *Let $V \subset X_{m,n}$ be a subvariety. Then there are only finitely many maximal atypical components of V in $X_{m,n}$.*

In the remainder of this section, we show that, in light of Theorem 1.4, Theorem 1.5 would follow from Conjecture 7.2. For $n \in \mathbb{Z}_{>0}$, we define $V_n \subset X_{n+1,n}$ by

$$V_n = \{(w_1, \ldots, w_n, w, t_1, \ldots, t_n) \in X_{n+1,n} : t_i = w_i - w \text{ for } i = 1, \ldots, n\}.$$

Note that $\dim X_{n+1,n} = 2n + 1$ and $\dim V_n = n + 1$.

Proposition 7.3. *Let $n \in \mathbb{Z}_{>0}$. Suppose that $x_1, \ldots, x_n, y$ are singular moduli such that $x_i \neq y$ for $i \in \{1, \ldots, n\}$ and*

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \ldots, a_n \in \mathbb{Z}$ which are not all zero. Then

$$\{(x_1, \ldots, x_n, y, x_1 - y, \ldots, x_n - y)\}$$

is an atypical component of V_n in $X_{n+1,n}$.

Proof. Let

$$\sigma = (x_1, \ldots, x_n, y, x_1 - y, \ldots, x_n - y).$$

Observe that $\sigma \in V_n$. Since $x_1, \ldots, x_n, y$ are singular moduli, the set

$$\{(x_1, \ldots, x_n, y)\}$$

is a special subvariety of $\mathbb{C}^{n+1}$ of dimension 0. Write M for this special subvariety. Since $x_i \neq y$ and

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1,$$

the point

$$(x_1 - y, \ldots, x_n - y)$$

is contained in a special subvariety $T \subset (\mathbb{C}^{\times})^n$ of dimension at most $n - 1$. Hence, $M \times T$ is a special subvariety of $X_{n+1,n}$ of dimension $\leq n - 1$. Thus,

$$\dim V_n + \dim(M \times T) - \dim X_{n+1,n} \leq (n + 1) + (n - 1) - (2n + 1) < 0.$$

Thus, $\{\sigma\} \subset V_n \cap (M \times T)$ is an atypical component of V_n in $X_{n+1,n}$. □

Proposition 7.4. *Let $n \in \mathbb{Z}_{>0}$. Suppose that $x_1, \dots, x_n, y$ are pairwise distinct singular moduli such that the set $\{x_1 - y, \dots, x_n - y\}$ is minimally multiplicatively dependent. Then either $(x_1, \dots, x_n, y)$ lies on a multiplicative special curve in $\mathbb{C}^{n+1}$ or*

$$\{(x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y)\}$$

is a maximal atypical component of V_n in $X_{n+1,n}$.

Proof. Suppose that $x_1, \dots, x_n, y$ are pairwise distinct singular moduli such that the set $\{x_1 - y, \dots, x_n - y\}$ is minimally multiplicatively dependent. Let

$$\sigma = (x_1, \dots, x_n, y, x_1 - y, \dots, x_n - y).$$

Then, by Proposition 7.3, $\{\sigma\}$ is an atypical component of V_n in $X_{n+1,n}$.

Suppose then that $\{\sigma\}$ is not a maximal atypical component of V_n in $X_{n+1,n}$. Then there exist special subvarieties $M \subset \mathbb{C}^{n+1}$ and $T \subset (\mathbb{C}^\times)^n$ and an irreducible component $W \subset V_n \cap (M \times T)$ such that $\dim W > 0$, $\sigma \in W$, and

$$\dim W > \dim V_n + \dim(M \times T) - \dim X_{n+1,n}.$$

In order to intersect V_n atypically, both M, T must be proper subvarieties.

If T was defined by two independent multiplicative conditions, then two independent multiplicative relations would hold on the set

$$\{x_1 - y, \dots, x_n - y\},$$

and thus some proper subset would be multiplicatively dependent, a contradiction. So T must be defined by one independent multiplicative condition.

Hence, for $M \times T$ to intersect V_n atypically, one must have that

$$(\alpha_1 - \beta, \dots, \alpha_n - \beta) \in T$$

for every $(\alpha_1, \dots, \alpha_n, \beta) \in M$. Thus, by Proposition 5.3, if $M_0 \subset M$ is a special subvariety of $\mathbb{C}^{n+1}$ such that $\dim M_0 = 1$ and no two coordinates of M_0 are identically equal, then M_0 is a multiplicative special curve in $\mathbb{C}^{n+1}$.

Suppose that $\dim M > 1$. Since $(x_1, \dots, x_n, y) \in M$ and $x_1, \dots, x_n, y$ are pairwise distinct, the locus in M where some two coordinates are equal is a Zariski-closed proper subset of M. Thus, by Proposition 5.2, M must contain infinitely many multiplicative special curves in $\mathbb{C}^{n+1}$. However, there are only finitely many multiplicative special curves in $\mathbb{C}^{n+1}$ by Theorem 1.4. So we must have that $\dim M = 1$, and so M itself is a multiplicative special curve in $\mathbb{C}^{n+1}$. Since $(x_1, \dots, x_n, y) \in M$, the proof is complete. □

Proposition 7.5. *Assume Conjecture 7.2. Then Theorem 1.4 implies Theorem 1.5.*

Proof. Fix $n \in \mathbb{Z}_{>0}$. We will show that there exists a constant $C > 0$ with the following property. Suppose that $(x_1, \dots, x_n, y)$ is an $(n+1)$-tuple of pairwise distinct singular moduli $x_1, \dots, x_n, y$ such that

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$. Then either $|\Delta(y)| \leq C$ or $(x_1, \dots, x_n, y)$ lies on a multiplicative special curve in $\mathbb{C}^{n+1}$. By Theorem 1.1, this suffices to prove Theorem 1.5.

Suppose then that $(x_1, \dots, x_n, y)$ is an $(n+1)$-tuple of pairwise distinct singular moduli $x_1, \dots, x_n, y$ such that

$$\prod_{i=1}^{n} (x_i - y)^{a_i} = 1$$

for some $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$. For every $k \in \{1, \dots, n\}$, there exists, by [13, Lemma 5.9], a set $I_k \subset \{1, \dots, n\}$ such that $k \in I_k$ and the set

$$\{x_i - y : i \in I_k\}$$

is minimally multiplicatively dependent. If, writing $I_k = \{i_1, \dots, i_{m_k}\}$, we have that

$$\{(x_{i_1}, \dots, x_{i_{m_k}}, y, x_{i_1} - y, \dots, x_{i_{m_k}} - y)\}$$

is a maximal atypical component of V_k in $X_{k+1,k}$ for some k, then, by Conjecture 7.2 applied to V_k, there are only finitely many possibilities for y and we are done. Hence, by Proposition 7.4, we may assume that $(x_{i_1}, \dots, x_{i_{m_k}}, y)$ lies on a multiplicative special curve M_k in $\mathbb{C}^{m_k+1}$ for every k.

Suppose that $I_k \cap I_{k'} \neq \emptyset$ for some $k \neq k'$. Let $i \in I_k \cap I_{k'}$. If the corresponding coordinates (i.e. the coordinate where x_i appears) of both M_k and $M_{k'}$ are fixed coordinates, then clearly these fixed coordinates must both be equal to x_i.

Suppose that precisely one of the corresponding coordinates of M_k and $M_{k'}$ is a fixed coordinate. Without loss of generality, assume that M_k has the constant coordinate. Then x_i must be equal to the constant coordinate of M_k. Since M_k is a multiplicative special curve in $\mathbb{C}^{m_k+1}$, this means that x_i is a singular modulus of bounded discriminant. Also, there exists a $g \in C(N)$ for some bounded N such that the corresponding coordinate of $M_{k'}$ is given by $j(gz)$ and the final coordinate of $M_{k'}$ is given by $j(z)$. Hence, there is $z \in \mathbb{H}$ such that

$$j(z) = y \text{ and } j(gz) = x.$$

In particular, the discriminant of y must be bounded. Hence, we may assume that the corresponding coordinates of M_k and $M_{k'}$ are either both constant coordinates or both non-constant coordinates.

Suppose the corresponding coordinates of both M_k and $M_{k'}$ are non-constant. So there are $g \in C(N)$ and $g' \in C(M)$ for some bounded M, N such that there exist $w, z \in \mathbb{H}$ such that

$$y = j(z) = j(w)$$

and

$$x_i = j(gz) = j(g'w).$$

Since $j(z) = j(w)$, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $w = \gamma z$. Hence,

$$j(gz) = j(g'\gamma z).$$

In particular, either $g = g'$ or, by Proposition 2.6, y is a singular modulus of bounded discriminant. In the latter case we are done, so we may assume that $g = g'$.

Thus, provided $|\Delta(y)|$ is larger than some suitable constant C, there exist j-maps $f_1, \ldots, f_n$, not all constant, such that $f_1, \ldots, f_n, j$ are pairwise distinct,

$$(x_1, \ldots, x_n, y) \in \{(f_1(z), \ldots, f_n(z), j(z)) : z \in \mathbb{H}\}$$

and, for every $k \in \{1, \ldots, n\}$ and $i \in I_k$, there exist $a_{k,i} \in \mathbb{Z} \setminus \{0\}$ such that

$$\prod_{i \in I_k} (f_i(z) - j(z))^{a_{k,i}} = 1$$

for all $z \in \mathbb{H}$. Since $k \in I_k$ always, we may use Lemma 6.5 to find $b_1, \ldots, b_n \in \mathbb{Z} \setminus \{0\}$ such that

$$\prod_{i=1}^{n} (f_i(z) - j(z))^{b_i} = 1$$

for all $z \in \mathbb{H}$. Thus, the set

$$\{(f_1(z), \ldots, f_n(z), j(z)) : z \in \mathbb{H}\}$$

is a multiplicative special curve in $\mathbb{C}^{n+1}$ which contains $(x_1, \ldots, x_n, y)$. □

References

1. J. Adamus and S. Randriambololona, *Tameness of holomorphic closure dimension in a semialgebraic set*, Math. Ann. **355** (2013), no. 3, 985–1005.
2. J. Ax, *On Schanuel's conjectures*, Ann. of Math. (2) **93** (1971), 252–268.
3. W. Berwick, *Modular Invariants Expressible in Terms of Quadratic and Cubic Irrationalities*, Proc. London Math. Soc. (2) **28** (1928), no. 1, 53–69.
4. Yu. Bilu, S. Gun, and E. Tron, *Effective multiplicative independence of 3 singular moduli*, preprint, arXiv:2207.05183v2 (2022).
5. Yu. Bilu, P. Habegger, and L. Kühne, *No singular modulus is a unit*, Int. Math. Res. Not. IMRN (2020), no. 24, 10005–10041.
6. Yu. Bilu, F. Luca, and D. Masser, *Collinear CM-points*, Algebra & Number Theory **11** (2017), no. 5, 1047–1087.
7. Yu. Bilu, F. Luca, and A. Pizarro-Madariaga, *Rational products of singular moduli*, J. Number Theory **158** (2016), 397–410.
8. E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006.
9. E. Bombieri, D. Masser, and U. Zannier, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Res. Not. IMRN (2007), no. 19.
10. D. Cox, *Primes of the form $x^2 + ny^2$*, John Wiley & Sons, Inc., New York, 1989.
11. L. van den Dries and C. Miller, *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), no. 1-3, 19–56.
12. G. Fowler, *Triples of singular moduli with rational product*, Int. J. Number Theory **16** (2020), no. 10, 2149–2166.
13. G. Fowler, *Multiplicative independence of modular functions*, J. Théor. Nombres Bordeaux **33** (2021), no. 2, 459–509.
14. G. Fowler, *Equations in three singular moduli: the equal exponent case*, J. Number Theory **243** (2023), 256–297.
15. B. Gross and D. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220.
16. P. Habegger, *Singular moduli that are algebraic units*, Algebra Number Theory **9** (2015), no. 7, 1515–1524.
17. P. Habegger and J. Pila, *Some unlikely intersections beyond André–Oort*, Compos. Math. **148** (2012), no. 1, 1–27.
18. P. Habegger and J. Pila, *O-minimality and certain atypical intersections*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 813–858.

19. S. Lang, *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987, With an appendix by J. Tate.
20. K. Lauter and B. Viray, *On singular moduli for arbitrary discriminants*, Int. Math. Res. Not. IMRN (2015), no. 19, 9206–9250.
21. Y. Li, *Singular units and isogenies between CM elliptic curves*, Compos. Math. **157** (2021), no. 5, 1022–1035.
22. T. Loher and D. Masser, *Uniformly counting points of bounded height*, Acta Arith. **111** (2004), no. 3, 277–297.
23. R. Paulin, *An explicit André–Oort type result for* $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_{\mathrm{m}}(\mathbb{C})$ *based on logarithmic forms*, Publ. Math. Debrecen **88** (2016), no. 1-2, 21–33.
24. J. Pila, *O-minimality and the André–Oort conjecture for* $\mathbb{C}^n$, Ann. of Math. (2) **173** (2011), no. 3, 1779–1840.
25. J. Pila, *Point-counting and the Zilber-Pink conjecture*, Cambridge Tracts in Mathematics, vol. 228, Cambridge University Press, Cambridge, 2022.
26. J. Pila and J. Tsimerman, *Ax–Schanuel for the* j*-function*, Duke Math. J. **165** (2016), no. 13, 2587–2605.
27. J. Pila and J. Tsimerman, *Multiplicative relations among singular moduli*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 4, 1357–1382.
28. J. Pila and A. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616.
29. J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **19** (2008), no. 2, 149–162.
30. R. Pink, *A combination of the conjectures of Mordell–Lang and André–Oort*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 251–282.
31. A. Riffaut, *Equations with powers of singular moduli*, Int. J. Number Theory **15** (2019), no. 3, 445–468.
32. T. Schneider, *Arithmetische Untersuchungen elliptischer Integrale*, Math. Ann. **113** (1937), no. 1, 1–13.
33. C. Siegel, *Über die Klassenzahl quadratischer Zahlkörper*, Acta Arith. **1** (1935), 83–86.
34. U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies, vol. 181, Princeton University Press, Princeton, NJ, 2012, With appendices by D. Masser.
35. B. Zilber, *Exponential sums equations and the Schanuel conjecture*, J. London Math. Soc. (2) **65** (2002), no. 1, 27–44.

School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK
Email address: `v.aslanyan@leeds.ac.uk`

School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK
Email address: `s.eterovic@leeds.ac.uk`

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, 30167 Hannover, Germany
Email address: `fowler@math.uni-hannover.de`