Scaling and universality for percolation in random networks: a unified view

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Percolation processes on random networks have been the subject of intense research activity over the last decades: the overall phenomenology of standard percolation on uncorrelated and unclustered topologies is well known. Still some critical properties of the transition, in particular for heterogeneous substrates, have not been fully elucidated and contradictory results appear in the literature. In this paper we present, by means of a generating functions approach, a thorough and complete investigation of percolation critical properties in uncorrelated locally tree-like random networks. We determine all critical exponents, the associated critical amplitude ratios and the form of the cluster size distribution for networks of any level of heterogeneity. We uncover, in particular for highly heterogeneous networks, subtle crossover phenomena, nontrivial scaling forms and violations of hyperscaling. In this way we clarify the origin of inconsistencies in the previous literature.

I. INTRODUCTION

Percolation is one of the simplest and most-studied processes in statistical physics. It consists in damaging a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $N = |\mathcal{V}|$ nodes and $E = |\mathcal{E}|$ edges, by removing some of the edges in $\mathcal{E}$. This damaging is typically pursued either by deactivating nodes and removing from $\mathcal{E}$ all the edges belonging to deactivated nodes – this is called *site percolation* – or by directly removing the edges from $\mathcal{E}$ – this is called *bond percolation*. Percolation theory studies the properties of the graph's connected components, or clusters – maximal subsets of $\mathcal{V}$ in which there is at least one path joining each pair of nodes – after the damage. Owing to its generality, percolation is used to model a wide variety of phenomena, e.g., forest fires [1], porous media [2, 3], ionic transport in composite materials [4], epidemic spreading [5–7], network robustness [8–10], and it constitutes a paradigmatic example of systems undergoing phase transitions.

If all nodes (edges) are removed, the graph does not contain any connected component (contains only components of size 1). On the other hand, if the fraction of removed nodes or edges is small, in a very large system an extensive connected component of size $\mathcal{S} \sim N$ is usually present, except for specific graphs, e.g. one-dimensional lattices. In the thermodynamic limit $N \to \infty$, this cluster is called giant component (GC), while all other clusters are small, i.e., nonextensive. The shift from a phase without a giant component (non-percolating phase) to a phase in which a giant component exists (percolating phase) is the percolation transition, and it typically involves the same kind of singularities of systems undergoing continuous phase transitions [11, 12]: the critical behavior observed in different models can be described with a set of universal critical exponents, shared by models having different microscopic details but the same symmetries.

Percolation on "finite-dimensional" topologies, such as D-dimensional regular lattices, is supported by a robust theoretical background, whose roots can be traced back to renormalization group theory [1, 13]. The exact values of the critical exponents are known for $D = 2$ [14–17], and for $D \geq 6$ the system is described by the mean-field critical exponents [18]. Even if exact solutions are in general not available for arbitrary D, scaling relations among the critical exponents and perturbative approaches provide a full understanding and precise numerical evaluations [18].

For percolation in complex networks, the situation is slightly different. Starting from the seminal works on random graphs by P. Erdős and A. Rényi [19, 20], several exact results have been obtained for percolation on various complex topologies, from uncorrelated random graphs [21–25] to random graphs with degree-degree correlations [26], to some network models with many short loops [27]. Other works [28–30] have provided a general heuristic picture of the percolation phase-transition in complex networks, that is generally believed to be complete and coherent. However, a close scrutiny reveals that several results in the literature are contradictory, while other aspects of the phenomenology have not been fully clarified. An example of this confusion concerns the value of the Fisher exponent τ (see below for a precise definition) for strongly heterogeneous networks (with degree distribution $p_k \sim k^{-\gamma_d}$ and $2 < \gamma_d < 3$). In this range of γ_d values τ was claimed to be equal to 3 [29], to $2 + 1/(\gamma_d - 2)$ [28], to γ_d [31] or even to be not defined [32]. Similarly, contradictory values are claimed for the exponent $\bar{\nu}$ (sometimes misleadingly indicated as ν [30, 32]) governing finite size scaling: $2/(3 - \gamma_d)$ [29], $(\gamma_d - 1)/(3 - \gamma_d)$ [28, 30]. In addition, the scaling property of the cluster size distribution n_s has been postulated but the explicit form of the scaling function in all the ranges of γ_d values has not been determined, although exact results for the component size distribution of uncorrelated random graphs have recently been published [33, 34]. Finally, to the best of our knowledge, universal amplitude ratios for percolation in networks have not been computed so far. In this manuscript our goal is to provide a complete and coherent summary of all critical properties of

the percolation phase-transition in uncorrelated locally tree-like random networks, that can be used for reference. To achieve this objective we fully exploit the power of the generating functions approach, which was only partly used before, to determine all critical exponents and scaling functions. In this way we point out and clarify all the inconsistencies present in the literature, derive all the missing results and independently test the validity of scaling relations.

The rest of the paper is organized as follows. Sec. II sets the stage, by means of a very general presentation of the percolation problem on networks: we define the model, the observables of interest and their critical properties. We then present the scaling ansatz for the cluster size distribution, the scaling relations among critical exponents, the definition of the correlation length and of the finite-size scaling properties. This content is not original but a uniform and complete presentation is crucial to make the more specific treatment of the other sections easy to follow. In Section III we consider an ensemble of uncorrelated locally tree-like random networks and determine closed expressions for all relevant critical properties in terms of generating functions. In Section IV we summarize the main results of the manuscript, i.e. a quantitative description of the percolation critical properties for homogeneous and for power-law degree-distributed random networks. For the sake of readability the explicit determination of these results, obtained by carefully applying the theory of Section III, is deferred to an Appendix. Some of the analytical results are tested via numerical simulations in Section V. A summary of the main findings and a discussion of their implications concludes the manuscript in Section VI.

II. PERCOLATION ON NETWORKS, SCALING AND CRITICAL PROPERTIES

In this Section, we define the main quantities of interest in percolation and we outline the differences between site and bond percolation. We then introduce a general scaling ansatz for critical properties and discuss its consequences.

A. Site and bond percolation

Let us consider first the case of uniform site percolation, in which every node is active with probability ϕ – i.e., inactive with probability $1 - \phi$. The activation probability ϕ plays the role of control parameter. The order parameter of percolation is usually defined as the relative size of the GC, averaged over the randomness of the percolation process, $P^\infty(\phi) = \lim_{N \to \infty} \langle \mathcal{S} \rangle / N$. $P^\infty(\phi)$ is also the probability that a randomly chosen node belongs to the GC. Then the percolation transition separates a phase in which $P^\infty = 0$, for $\phi \leq \phi_c$, from a phase with $P^\infty > 0$, for $\phi > \phi_c$; ϕ_c is the percolation threshold.

The basic quantity in percolation theory is the cluster size distribution, defined as the number of small clusters of size s per active node, i.e., for site percolation,

$$n_s(\phi) \equiv \frac{\mathcal{N}_s(\phi)}{\phi N}, \tag{1}$$

where $\mathcal{N}_s(\phi)$ is the number of clusters of size s. Note that the largest cluster is excluded from the enumeration of clusters in $n_s(\phi)$. A cluster size distribution including also the largest cluster of size $\mathcal{S}$ can be defined,

$$\widetilde{n}_s(\phi) = n_s(\phi) + \frac{\delta_{s,\mathcal{S}}}{\phi N}, \tag{2}$$

where $\mathcal{S}$ is a random variable distributed according to $\mathcal{P}(s = \mathcal{S}, \phi)$. Away from criticality, $\mathcal{P}(s = \mathcal{S}, \phi)$ is a Gaussian centered in $\langle \mathcal{S} \rangle$ with fluctuations of order $\sqrt{N}$. In the limit of diverging N we can thus write

$$\frac{\delta_{s,\mathcal{S}}}{\phi N} \to \frac{\delta_{s,\langle \mathcal{S} \rangle}}{\phi N} = \frac{\delta_{s,\langle \mathcal{S} \rangle}}{\phi \langle \mathcal{S} \rangle} P^\infty(\phi). \tag{3}$$

In this same limit, summing Eq. (2) over s yields $\sum_s \widetilde{n}_s(\phi) = \sum_s n_s(\phi)$, because $\langle \mathcal{S} \rangle$ in the denominator diverges. The two distributions are not normalized.

The normalized distribution is instead $s\widetilde{n}_s(\phi)$, which is the probability that a randomly chosen active node belongs to a cluster of any size s. Its normalization simply reflects the fact that any active node must be part of a cluster. Multiplying Eq. (2) by s, using Eq. (3) and summing over s we obtain

$$1 = \sum_{s \geq 1} s n_s(\phi) + \frac{P^\infty(\phi)}{\phi}. \tag{4}$$

The distribution $\pi_s(\phi) \equiv s n_s(\phi)$ is normalized only when there is no giant component, $P^\infty = 0$. Otherwise

$$P^\infty(\phi) = \phi \left[1 - \sum_{s \geq 1} \pi_s(\phi) \right]. \tag{5}$$

This equation connects the singular behavior observed in the order parameter at $\phi = \phi_c$ with $\pi_s(\phi_c) = s n_s(\phi_c)$.

The same argument developed so far holds also for bond percolation, in which edges are kept with probability φ, and all nodes are considered to be active, which implies division by N (instead of ϕN) in Eqs. (1) and (3). The theory in this paper is formally developed for site percolation, however, the same theory applies to bond percolation as well, provided Eq. (5) is replaced with

$$P^\infty_{(b)}(\varphi) = 1 - \sum_{s \geq 1} \pi_s^{(b)}(\varphi). \tag{6}$$

It is clear that $P^\infty(\phi) = \phi P^\infty_{(b)}(\phi)$ if and only if $\pi_s(\phi) = \pi_s^{(b)}(\phi)$, that is if activating a fraction ϕ of the nodes or a fraction ϕ of the edges produces exactly the same effect on the cluster size distribution. Throughout the paper we will comment on relevant differences between the behavior of site and bond percolation.

B. Observables and critical properties

The behavior of the system is characterized by considering the following observables.

$$F(\phi) \equiv \sum_{s=1}^{\infty} n_s(\phi) \qquad (7)$$

is the mean number (per active node) of finite clusters in the system. When the control parameter ϕ approaches the critical threshold, the singular part of this quantity scales as

$$F(t) \simeq A_{\pm}|\phi - \phi_c|^{2-\alpha} = A_{\pm}|t|^{2-\alpha}, \qquad (8)$$

where α is a universal critical exponent and $t = \phi - \phi_c$[1]

The relative size of the giant component, or percolation strength, is defined in Eq. (5) (for site percolation) and in Eq. (6) (for bond percolation). This quantity is zero in the nonpercolating phase, while it is finite above the threshold. Close to criticality it behaves as

$$P^{\infty}(t) \simeq Bt^{\beta} \qquad (9)$$

where β is another critical exponent. This exponent differs between site and bond percolation when $\phi_c = 0$, because of the multiplicative factor ϕ present in Eq. (5): $\beta_{site} = 1 + \beta_{bond}$ [32]. Apart from this case the two exponents always coincide and to avoid ambiguity we define as β the exponent associated to the singular behavior of the factor $1 - \sum_s \pi_s$ present in both definitions, Eq. (5) and Eq. (6).

Another observable is the mean size of the cluster to which a randomly chosen active node (not in the giant component) belongs, which is

$$\langle s \rangle = \frac{\sum_s s^2 n_s(\phi)}{\sum_s s n_s(\phi)}. \qquad (10)$$

At criticality this quantity has a singular behavior, with an associated critical exponent γ

$$\langle s \rangle - 1 \simeq C_{\pm}|t|^{-\gamma}. \qquad (11)$$

The goal of a theory for percolation is the determination of the universal critical exponents and of the universal critical amplitude ratios. Following Ref. [18], we introduce the so called "ghost field" h and generalize Eq. (7) defining

$$\Omega(\phi, h) \equiv \sum_{s=1}^{\infty} n_s(\phi)e^{-sh}. \qquad (12)$$

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[1] Here and in the rest of the paper we use, with a slight abuse of notation, the same symbols to denote functions of ϕ and functions of t.

The observables defined above can be expressed in terms of Ω and of the derivatives of Ω with respect to h. Clearly $F(\phi) = \Omega(\phi, 0)$. We further define

$$\Psi(\phi, h) \equiv -\frac{\partial \Omega}{\partial h} = \sum_{s=1}^{\infty} s n_s(\phi)e^{-sh}, \qquad (13)$$

which is related to the site percolation order parameter

$$P^{\infty}(\phi) = \phi\big[1 - \Psi(\phi, 0)\big] \qquad (14)$$

and similarly for bond percolation, without the global factor ϕ, see Eq. (6). Finally, the quantity

$$\chi(\phi, h) \equiv \frac{\partial^2 \Omega}{\partial h^2} = \sum_{s=1}^{\infty} s^2 n_s(\phi)e^{-sh}, \qquad (15)$$

is related to $\langle s \rangle$ since,

$$\langle s \rangle = \frac{\sum_s s^2 n_s(\phi)}{\sum_s s n_s(\phi)} = \frac{\chi(\phi, 0)}{\Psi(\phi, 0)}. \qquad (16)$$

Note that, using $n_s(\phi) = \pi_s(\phi)/s$ and $s^{-1} = \int_0^{\infty} dh e^{-hs}$, $F(\phi)$ can be written as

$$F(\phi) = \int_0^{\infty} dh \sum_{s \geq 1} \pi_s(\phi)e^{-sh} = \int_0^{\infty} dh \Psi(\phi, h). \qquad (17)$$

In the critical region near ϕ_c, setting $t = \phi - \phi_c$, we can write

$$\Omega(t, h) = \sum_{n=0}^{\infty} a_n(t)h^n + \{\Omega(t, h)\}_{\text{sing}}, \qquad (18)$$

where $\{\cdot\}_{\text{sing}}$ stands for "singular part". Note that Eq. (18) is completely general. Indeed, we can write $n_s(t) = n_s^{\text{small}}(t) + n_s^{\text{tail}}(t)$, where n_s^{small} decays fast enough for $s \gg 1$, and n_s^{tail} is the large s tail of the distribution. Using Eq. (12), expanding e^{-sh} in power series and changing the order of summation, the contribution from n_s^{small} gives

$$\sum_{s=1}^{\infty} n_s^{\text{small}}(t)e^{-sh} = \sum_{n=0}^{\infty} \left[(-1)^n \frac{\sum_{s=1}^{\infty} s^n n_s^{\text{small}}(t)}{n!}\right] h^n,$$

which is the first, regular, term on the r.h.s. of Eq. (18). This procedure is not mathematically justified a priori for n_s^{tail} because the series in the square brackets may diverge, if n_s^{tail} decays slowly. Hence the large s tail of the cluster size distribution is the origin of singular terms in Eq. (18). The critical behavior is due to the singular part of $\Omega(t, h)$ when the point $(t = 0, h = 0)$ is approached. In addition to the critical exponents defined above another exponent can be defined as

$$1 - \Psi(\phi_c, h) \simeq Eh^{1/\delta}, \qquad (19)$$

where in the last equation logarithmic corrections may appear for integer $1/\delta$.

In summary, the critical behavior in terms of $\{\Omega(t,h)\}_{\text{sing}}$ and its derivatives is, for $|t| \ll 1$ and $h \ll 1$,

$$\Omega(t,0) \simeq A_{\pm}|t|^{2-\alpha}, \qquad (20)$$

$$1 - \Psi(t,0) \simeq Bt^{\beta}, \qquad (21)$$

$$\frac{\chi(t,0)}{\Psi(t,0)} - 1 \simeq C_{\pm}|t|^{-\gamma}, \qquad (22)$$

$$1 - \Psi(0,h) \simeq Eh^{1/\delta}. \qquad (23)$$

C. The scaling ansatz for the cluster size distribution

At its very core, the scaling hypothesis for percolation can be reduced to the following ansatz for the tail of the cluster size distribution [35, 36]

$$n_s(t) \simeq q_0 s^{-\tau} f_{\pm}(q_1 ts^{\sigma}), \qquad (24)$$

for $s \gg 1$ and $|t| \ll 1$, where q_0, q_1 are nonuniversal positive constants, $f_{\pm}(x)$ are universal scaling functions, while τ and σ are universal critical exponents. The scaling functions decay to zero quickly for large values of their argument: $f_{\pm}(x) \ll 1$ for $|x| \gg 1$. This naturally introduces the quantity

$$s_{\xi} \simeq \Sigma|t|^{-1/\sigma}, \qquad (25)$$

which plays the role of a correlation size, where $\Sigma = q_1^{-1/\sigma}$. For $s/s_{\xi} \ll 1$, the cluster size distribution scales as $n_s(t) \sim s^{-\tau}$. This implies, if $\sigma > 0$ so that $s_{\xi} \to \infty$ as $t \to 0$, that n_s exhibits a pure power-law tail for large s with exponent τ.

Note that in the literature the scaling form Eq. (24) is often assumed [28, 37, 45] to be $s^{-\tau}\exp(-s/s_{\xi})$, which corresponds to a scaling function of the form $f_{\pm}(x) = e^{-x^{1/\sigma}}$. As it will be shown below, it is instead crucial to allow for more general functional forms.

For some particular classes of random networks, it is possible to explicitly calculate the cluster size distribution $n_s(t)$, finding an agreement with the general form, Eq. (24). Some examples are worked out in Appendix A.

The scaling ansatz implies the existence of some scaling relations among the critical exponents. From the scaling ansatz, Eq. (24), by writing $\{\Omega(t,h)\}_{\text{sing}} \simeq q_0 \int_1^{\infty} ds\, s^{-\tau} f_{\pm}(q_1 ts^{\sigma}) e^{-sh}$ and using the change of variable $x = q_1|t|s^{\sigma}$, we have

$$\{\Omega(t,h)\}_{\text{sing}} \sim |t|^{\frac{\tau-1}{\sigma}} \int_{q_1 t}^{\infty} dx\, x^{\frac{1-\tau}{\sigma}-1} f_{\pm}(x) e^{-\left[x^{\frac{1}{\sigma}} h/(q_1|t|)^{1/\sigma}\right]}.$$

The integral depends on t because of the lower extremum, and through the ratio $h/|t|^{1/\sigma}$. The integral is divergent for $t \to 0$. However, because of the multiplicative factor $|t|^{\frac{\tau-1}{\sigma}}$ this divergence is removed. It follows that $\{\Omega(t,h)\}_{\text{sing}}$ takes the form

$$\{\Omega(t,h)\}_{\text{sing}} = |t|^{\frac{\tau-1}{\sigma}} \mathcal{F}_{\pm}(h/|t|^{1/\sigma}), \qquad (26)$$

where $\mathcal{F}_{\pm}(x)$ are scaling functions such that $\mathcal{F}_{\pm}(x) \sim 1$ for $|x| \ll 1$. This directly follows from simple assumptions on the behavior of $f_{\pm}(x)$ for small $|x|$.[2]

Comparing Eq. (26) with Eq. (20) yields $2 - \alpha = (\tau - 1)/\sigma$. Taking derivatives of Eq. (26) with respect to h, evaluated at $h = 0$ and comparing with Eqs. (21)-(22) gives two other scaling relations among critical exponents: $\beta = (\tau-2)/\sigma$ and $\gamma = (3-\tau)/\sigma$. Hence all the exponents introduced so far can be expressed as functions of only two of them, for instance

$$\alpha = 2 - \frac{\tau - 1}{\sigma}, \qquad (27)$$

$$\gamma = \frac{3 - \tau}{\sigma}, \qquad (28)$$

$$\beta = \frac{\tau - 2}{\sigma}, \qquad (29)$$

$$\delta = \frac{1}{\tau - 2}. \qquad (30)$$

The last of the scaling relations in Eq. (30) is derived by means of the so-called Tauberian theorems [38]. In particular, for a function $w(h) = \sum_{s \geq 0} a_s e^{-sh}$, we have[3]

$$a_s \simeq E's^{-1/\delta-1} \iff w(h) \simeq 1 - Eh^{1/\delta}. \qquad (31)$$

This implies that, since at criticality $\Psi(\phi_c, h) = \sum_{s \geq 0} s^{1-\tau} e^{-sh}$, then $\delta = 1/(\tau - 2)$. Note that this relation holds only when $\Psi(\phi_c, h)$ is nonanalytical in h. As shown below, there are cases in which $\Psi(\phi_c, h)$ is analytical in $h = 0$; in these cases the scaling relation between δ and τ breaks down.

The framework developed here is fully general and it is applicable provided the scaling ansatz Eq. (24) is valid.

D. The correlation length

In a finite D-dimensional lattice, the probability that one node j in $\vec{y}$, at euclidean distance r_{ij} from an active node i in $\vec{x}$, is in the same connected component as i is called the pair correlation function $g(i,j)$. In particular, we can write $g(i,j) = \langle c_{ij} \rangle$, where $c_{ij} = 1$ if there is a path connecting i and j, and it is zero otherwise, and the average is taken over the randomness of the percolation process (and over the network ensemble, in the case of random networks). An important sum rule for the correlation function is

$$\frac{1}{\phi N} \sum_{i,j \in \mathcal{C}} g(i,j) = \chi(\phi, 0), \qquad (32)$$

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[2] It is sufficient to assume that $f(x) \sim x^a$ for $x \ll 1$, $a \geq 0$, including the standard case in which $f(x \ll 1)$ is a constant, i.e., $a = 0$.

[3] There are logarithmic corrections for integer $1/\delta$.

where $\mathcal{C}$ denotes the set of nodes belonging to small components. This can be seen from

$$\frac{1}{\phi N} \sum_{i,j \in \mathcal{C}} \langle c_{ij} \rangle = \frac{1}{\phi N} \sum_{\mathcal{C}} \langle s_{\mathcal{C}}^2 \rangle = \sum_{s \geq 1} \frac{\mathcal{N}_s(\phi)}{\phi N} s^2, \quad (33)$$

and Eq. (32) follows since $n_s(\phi) = \mathcal{N}_s(\phi)/(\phi N)$ by definition.

Using $g(i,j)$, one defines a (euclidean) correlation length ξ via

$$\xi^2 = \frac{\sum_{i,j \in \mathcal{C}} r_{ij}^2 g(i,j)}{\sum_{i,j \in \mathcal{C}} g(i,j)}. \quad (34)$$

Note that if $g(i,j)$ depends only on r_{ij}, summing over distances instead of summing over pairs implies

$$\xi^2 = \frac{\sum_r r^2 \widetilde{\mu}(r)}{\sum_r \widetilde{\mu}(r)},$$

where $\widetilde{\mu}(r)$ denotes the average number of nodes that are at a distance r from a given node in the same finite component, see [39].

In the case of percolation on networks, we can embed a generic finite tree-like component in a infinite-dimensional lattice by placing one node in the origin and placing each subsequent neighboring node along a new orthogonal direction. Hence, the length l of the shortest path connecting two nodes, allows us to define the euclidean distance between these two nodes as $r = \sqrt{l}$ [40]. Then a correlation length for complex networks can be computed via

$$\xi^2 = \frac{\sum_l l \mu(l)}{\sum_l \mu(l)} = \frac{\sum_l l \mu(l)}{\chi(\phi, 0)}, \quad (35)$$

where $\mu(l)$ is the average number of active nodes at distance l from a given active node in the same finite component and Eq. (32) has been used.

In all cases, close to the critical point,

$$\xi \simeq \Xi_\pm |t|^{-\nu}. \quad (36)$$

This defines a new critical exponent ν. To understand how ν is related to the other critical exponents finite-size scaling must be considered, since the role of the correlation length, and in particular of its divergence, is strongly related with another important quantity: the size N of the network.

E. Finite-size scaling

The percolation phase transition is strictly defined only in infinite-size systems.

The existence, in the thermodynamic limit, of a continuous phase transition is manifested in finite systems by fluctuations of the order parameter showing, as a function of ϕ, a maximum for a size-dependent value $\phi_c(N)$. The amplitude of this maximum increases with the system size

N. At $\phi_c(N)$ the correlation length is bounded by the maximum size that can be reached in the system, that is, in D dimensions, $\xi \sim N^{1/D}$. Observing a finite system at $\phi_c(N)$ is equivalent to observing an infinite system off-criticality at $\phi = \phi_c(N)$: in both cases the correlation length is finite and $\xi \sim N^{1/D}$. This defines $\phi_c(N)$ as an effective threshold which tends to ϕ_c in the large-N limit. From Eq. (36) then, using $t = \phi_c(N) - \phi_c$,

$$N^{\frac{1}{D}} \sim |\phi_c(N) - \phi_c|^{-\nu} \implies |\phi_c(N) - \phi_c| \sim N^{-\frac{1}{D\nu}}.$$

Percolation critical properties are described by an effective field theory, see Appendix B. For such a theory the critical behavior in a space of dimensionality $D \geq D_{\mathrm{UC}}$ larger than the *upper critical dimension* is the same (mean-field) behavior observed at $D = D_{\mathrm{UC}}$ [4]. For standard percolation on lattices the upper critical dimension is $D_{\mathrm{UC}} = 6$. Hence the scaling of the effective threshold is in general

$$|\phi_c(N) - \phi_c| \sim N^{-1/\bar{\nu}}, \quad (37)$$

where $\bar{\nu} = \nu D$ for $D < D_{\mathrm{UC}}$ while $\bar{\nu} = \nu D_{\mathrm{UC}}$ for $D \geq D_{\mathrm{UC}}$, see [41, 42].

This argument is also valid for random locally tree-like networks, where the dimension D of the space in which a finite N network can be embedded grows with N. For this reason, in the large N limit, $D(N) > D_{\mathrm{UC}}$, and the standard mean-field picture is expected to correctly capture the critical properties of the system.

At the effective threshold a GC starts to form

$$\frac{\langle \mathcal{S} \rangle \big|_{\phi_c(N)}}{\phi_c(N)} \sim N^\theta. \quad (38)$$

Close to criticality $[N \to \infty, \phi_c(N) \to \phi_c]$,

$$P^\infty = \frac{\langle \mathcal{S} \rangle \big|_{\phi_c(N)}}{\phi_c(N)N} \sim N^{\theta-1} \quad (39)$$

This quantity must also scale as $P^\infty \sim t^\beta$; since $t \sim N^{-1/\bar{\nu}}$ this implies

$$\theta = 1 - \frac{\beta}{\bar{\nu}}. \quad (40)$$

The exponent θ can be seen also as the ratio between the fractal dimension D_f of the GC and the embedding dimension D.

Fluctuations instead scale as

$$\frac{\langle (\delta \mathcal{S})^2 \rangle \big|_{\phi_c(N)}}{\phi_c(N)N} \sim t^{-\gamma'} \sim N^{\gamma'/\bar{\nu}}. \quad (41)$$

For standard percolation the mean square fluctuations of the order parameter have the same critical singularity

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[4]Actually there are logarithmic corrections exactly at $D = D_{\mathrm{UC}}$.

as the mean finite cluster size, i.e., $\gamma' = \gamma$, see, e.g., Ref. [41]. (In general this need not be the case, and in more complex percolation problems—in particular, ones involving hybrid phase transitions—the two exponents may be different [43, 44].)

Another scaling relation can be found as follows. Close to criticality the largest clusters of size s_ξ are fractals, $s_\xi \sim \xi^{D_f} \sim t^{-D_f\nu}$. Since by definition $s_\xi \sim t^{-1/\sigma}$ it follows that $D_f = \frac{1}{\sigma\nu}$ implying

$$\theta = \frac{1}{\sigma\bar{\nu}}. \tag{42}$$

Equating the two expressions for θ above implies the hyperscaling relation

$$2\beta + \gamma = \bar{\nu}. \tag{43}$$

These relations allow us to determine all critical exponents from numerical simulations. In particular, $\bar{\nu}$ and $\gamma/\bar{\nu}$ can be obtained by measuring the position and the height of the peak of $\langle(\delta\mathcal{S})^2\rangle$, while θ can be obtained from the scaling of $\langle\mathcal{S}\rangle$ at $\phi_c(N)$. Once θ, $\bar{\nu}$ and γ are known, β and σ can be obtained from the scaling relations, and all the other critical exponents from them.

The scenario just described does not strictly apply for percolation on highly hetereogeneous power-law distributed networks. In such a case, as it will be shown below, the hyperscaling relation is violated.

F. Critical amplitude ratios

Apart from critical exponents, also amplitudes of critical behaviors obey universal properties. It is possible to define [18] several amplitude ratios whose values mark the universality class of the transition. In particular we will be interested in the quantities C_+/C_-, Ξ_+/Ξ_- and $R_\chi = C_+ E^{-\delta} B^{\delta-1}$.

G. Summary of the definitions

We summarize in Table I definitions and physical meaning of all the critical exponents introduced in this section.

TABLE I. Summary of the definitions of the various critical exponents. In the first seven lines the thermodynamic critical exponents; all the scalings hold for $t \ll 1$, $h \ll 1$. In the last two lines, critical exponents related to finite-size scaling, valid for large N.

Exponent	Definition	Physical meaning
α	$F \simeq A|t|^{2-\alpha}$	Mean number of finite clusters per active node, see Eq. (7).
β	$P^\infty/\phi \simeq Bt^\beta$	Relative size (per active node) of the GC, see Eq. (9).
γ	$\langle s \rangle - 1 \simeq C_\pm|t|^{-\gamma}$	Mean size of small components to which a randomly chosen active node belongs, see Eq. (11).
δ	$1 - \Psi(\phi_c, h) \simeq Eh^{1/\delta}$	Singular behaviour of the h-derivative of $\Omega(t, h)$ at criticality, see Eq. (19).
τ	$n_s(t) \simeq q_0 s^{-\tau} f(q_1 s^\sigma t)$	Scaling of the large s tail of the cluster size distribution, see Eq. (24)
σ	$s_\xi \simeq \Sigma|t|^{-1/\sigma}$	Correlation size for finite components, see Eq. (25)
ν	$\xi \simeq \Xi|t|^{-\nu}$	Average distance between nodes in the same finite component, see Eq. (36).
$\bar{\nu}$	$|\phi_c - \phi_c(N)| \sim N^{-1/\bar{\nu}}$	Effective finite-size threshold $\phi_c(N)$, see Eq. (38).
θ	$\frac{\langle\mathcal{S}\rangle|_{\phi_c(N)}}{\phi_c(N)} \sim N^\theta$	Average largest cluster at the effective threshold $\phi_c(N)$, see Eq. (37)

III. GENERATING FUNCTION APPROACH FOR UNCORRELATED RANDOM GRAPHS

In this Section we calculate all the critical exponents for percolation on uncorrelated random graphs, by analyzing the singular behavior of generating functions. This strategy was already developed in part in Refs. [28, 37]. The treatment here is for an ensemble of uncorrelated locally tree-like networks with generic degree distribution p_k (excess degree distribution $q_k = (k + 1)p_{k+1}/\langle k \rangle$) and completely random in all other respects. We denote

by $g_0(z) = \sum_k p_k z^k$ and $g_1(z) = \sum_r q_r z^r$ the generating functions of the network's degree and excess degree distributions, respectively.

The application of the formulas derived in this Section to the specific case of homogeneous or heterogeneous random networks is presented in Appendix C and the results are summarized in Sec. IV.

The generating function of the distribution $\pi_s(\phi)$ is

$$H_0(z) \equiv \sum_{s=1}^{\infty} \pi_s(\phi)z^s \tag{44}$$

For uncorrelated locally tree-like random graphs, it is well known [9, 33] [5] that $H_0(z)$ obeys

$$H_0(z) = zg_0\big(\phi H_1(z) + 1 - \phi\big) \equiv zG_0\big(H_1(z)\big), \quad (45)$$

where we defined $G_0(x) \equiv g_0(\phi x + 1 - \phi)$, and

$$H_1(z) \equiv \sum_{s=1}^{\infty} \rho_s(\phi) z^s \quad (46)$$

is the generating function of $\rho_s(\phi)$, the distribution of the total number of nodes reachable via a randomly chosen edge that leads to an active node. Note that the giant component, if there is one, is excluded from $H_1(z)$, i.e., $u \equiv H_1(1)$ is the probability that a randomly chosen edge, ending in an active node, leads to a finite component of any size. $H_1(z)$ is determined by the implicit equation

$$H_1(z) = zg_1\big(\phi H_1(z) + 1 - \phi\big) \equiv zG_1\big(H_1(z)\big), \quad (47)$$

where we defined $G_1(x) \equiv g_1(\phi x + 1 - \phi)$.

The percolation threshold is determined by the condition $1 = \phi_c g_1'(1)$ [9], implying $\phi_c = b^{-1} = \langle k \rangle/(\langle k^2 \rangle - \langle k \rangle)$ [21, 22]. This holds if the network has a finite branching factor b. If the branching factor diverges in the thermodynamic limit, then only the percolating phase exists for any value of ϕ, i.e., $\phi_c = 0$.

By setting $z = e^{-h}$, and $m = 1 - H_1(e^{-h})$ Eqs. (45) and (47) can be rewritten as

$$\Psi(\phi, h) = e^{-h} g_0\big(1 - \phi m\big), \quad (48)$$
$$m = 1 - e^{-h} g_1\big(1 - \phi m\big). \quad (49)$$

A. The exponents β and δ

Solving Eq. (49) for $m(\phi, h)$ and inserting the solution into Eq. (48) allows to determine the exponents β and δ. Everything depends on the behavior of the generating functions $g_0(z)$ and $g_1(z)$ close to $z = 1$. For $x \ll 1$ it is always possible to write

$$g_0(1-x) \simeq 1 - \langle k \rangle x + \frac{1}{2}\langle k \rangle b x^2 + \{g_0(1-x)\}_{\text{sing}}, \quad (50)$$

$$g_1(1-x) \simeq 1 - bx + \frac{1}{2}dx^2 + \{g_1(1-x)\}_{\text{sing}}, \quad (51)$$

where $\langle k \rangle$ is the average degree, always assumed to be finite, and b, d are constants. Their values are $b = \langle k(k-1)\rangle/\langle k\rangle$, $d = \langle k(k-1)(k-2)\rangle/\langle k\rangle$ if the degree distribution has finite second and third moment, respectively.

Otherwise b and d are numerical values depending on the degree distribution, and the divergence of the moments shows up in the singular part of the generating function.

We will always consider networks with a finite average degree $\langle k \rangle$. Hence the singular contribution in $g_0(1-x)$ can always be neglected because the finiteness of $\langle k \rangle$ implies that $\langle k \rangle x \gg \{g_0(1-x)\}_{\text{sing}}$ for $x \to 0$ [in other words, $g_0(z)$ is continuous and differentiable in $z = 1$, and any singularity can appear only in the second derivative or higher]. We can then always consider only the constant and the linear terms in the r.h.s. of Eq. (48) so that

$$\Psi(\phi, h) \simeq 1 - \langle k \rangle \phi m(\phi, h) - h. \quad (52)$$

Expanding Eq. (49) for $|t| \ll 1$ and $h \ll 1$, we can obtain an approximate solution for $m(\phi, h)$. Plugging such a solution into Eq. (52) then allows us to evaluate the exponents β and δ and the amplitudes B and E.

B. The exponent α

From Eq. (17), using the expression for $\Psi(\phi, h)$ in Eq. (48), yields, after integrating by parts [45],

$$F(\phi) = \Psi(\phi, 0) - \phi \int_0^\infty dh e^{-h} g_0'(1 - \phi m)\frac{\partial m}{\partial h}. \quad (53)$$

Then, $g_0'(x) = \langle k \rangle g_1(x)$, and Eq. (49)

$$F(\phi) = \Psi(\phi, 0) - \phi \int_0^\infty dh(1 - m)\frac{\partial m}{\partial h}. \quad (54)$$

After the change of variable $x = m(\phi, h)$ at fixed ϕ, and the simple integration, we get

$$F(\phi) = \Psi(\phi, 0) - \frac{\phi\langle k \rangle}{2}[1 - m(\phi, 0)]^2, \quad (55)$$

where $m(\phi, 0)$ is the solution of Eq. (49) for $h = 0$ [6] Once the behavior of $m(t, 0)$ for small t is known, expanding Eq. (55) for small t, allows to determine the exponent α.

C. The exponent γ

We are interested in the average size of the small cluster to which a randomly chosen active node belongs, given by Eq. (16). The numerator of this quantity, $\chi(\phi, 0)$, may diverge at criticality because of the power-law tail in $n_s(\phi_c)$ and determine the exponent γ, while $\Psi(\phi, h)$ is always finite. From Eq. (48) and from the relation $\chi(\phi, h) = -\partial\Psi(\phi, h)/\partial h$ we find

$$\chi(\phi, h) = \Psi(\phi, h) + e^{-h}g_0'(1 - \phi m)\phi\frac{\partial m}{\partial h}, \quad (56)$$

---

[5] Eq. (45) and (47) correspond to the equations in [9] via the mapping $H_0^{[9]}(z) = 1 - \phi + \phi H_0(z)$, $H_1^{[9]}(z) = 1 - \phi + \phi H_1(z)$. This is because we conditioned $H_0(z)$ and $H_1(z)$ on picking active nodes (see Eq. (1)), while the authors in [9] did not: hence the factor ϕ and the additional term $\pi_0 = \rho_0 = 1 - \phi$ expressing the probability of picking an inactive node.

[6] This quantity, $m(\phi, 0)$, is the same used to evaluate $P^\infty(\phi)$ and β: it is the probability of reaching the GC following a randomly chosen edge ending in an active node.

where m is the solution of Eq. (49), from which,

$$\frac{\partial m}{\partial h} = e^{-h} g_1(1 - \phi m) + e^{-h} g_1'(1 - \phi m)\phi \frac{\partial m}{\partial h},$$

whose solution is

$$\frac{\partial m}{\partial h} = \frac{e^{-h} g_1(1 - \phi m)}{1 - e^{-h}\phi g_1'(1 - \phi m)} = \frac{1 - m}{1 - e^{-h}\phi g_1'(1 - \phi m)}.$$

Using $g_0'(x) = \langle k \rangle g_1(x)$, and again Eq. (49), we finally obtain, from Eq. (16),

$$\langle s \rangle - 1 = \frac{\phi}{\Psi(\phi, 0)} \frac{\langle k \rangle (1 - m)^2}{1 - \phi g_1'(1 - \phi m)}. \tag{57}$$

Inserting the expression of $m(\phi, 0)$ into Eq. (57) and expanding for small $t = \phi - \phi_c$ provides the singular behavior as criticality is approached.

D. The exponent ν

Using Eq. (35), the problem of deriving the correlation length ξ is reduced to the computation of $\mu(l)$. If there is no GC, all nodes are in finite components and since the network is uncorrelated and locally tree-like, we have

$$\mu(l) = \langle k \rangle \phi^l b^{l-1}. \tag{58}$$

Hence, using the formula $\sum_{l \geq 0} l z^{l-1} = (1 - z)^{-2}$, we get

$$\xi^2 = \frac{\phi \langle k \rangle [1 - \phi b]^{-2}}{\chi(\phi, 0)}. \tag{59}$$

Here we generalize this argument to the case in which a giant component is present, so that in the computation of $\mu(l)$ one must explicitly impose that the GC is not reached. Let us define $P_l(n)$ as the probability that in the shell at distance l from a given active node, there are exactly n nodes, in the same finite component to which the original active node belongs. Then $\mu(l) = \sum_n n P_l(n) = L_l'(1)$, where $L_l(z) = \sum_n P_l(n) z^n$ are the generating functions of the distributions $P_l(n)$. Writing down an explicit expression for $P_l(n)$ is rather involved. However, it is possible to write, proceding step by step, a recursive expression for the generating functions $L_l(z)$. First of all, at level 0 $P_0(n) = \delta_{1,n}\Psi(\phi, 0)$, since an active node at distance 0 from a given active node, (i.e., itself) is in a finite component with probability $\Psi(\phi, 0)$. At distance 1 it is easy to recognize that

$$P_1(n) = \sum_k p_k \binom{k}{n} (\phi u)^n (1 - \phi)^{k-n}, \tag{60}$$

where we remind the reader that u is the probability that following a randomly chosen edge we reach a finite component. Multiplying Eq. (60) by z^n and summing over n we obtain

$$L_1(z) = g_0(\phi u z + 1 - \phi) = G_0(uz). \tag{61}$$

Let us consider now the nodes at distance 2. It is possible to condition on the number of nodes which were reachable at the previous step by writing

$$P_2(n) = \sum_m P_2(n|m) P_1(m). \tag{62}$$

The expression for $P_2(n|m)$ is simply given by

$$P_2(n|m) = \frac{1}{u^m} \sum_{n_1,\ldots,n_m} \delta_{n,n_1+\cdots+n_m} \sum_{r_1,\ldots,r_k} q_{r_1} \cdots q_{r_m} \binom{r_1}{n_1} \cdots \binom{r_m}{n_m} (\phi u)^{n_1} \cdots (\phi u)^{n_m} (1 - \phi)^{r_1 - n_1 + \cdots + r_m - n_m}. \tag{63}$$

Note the factor u^m in the denominator. It serves to correctly count the number of u's in the expression. In $P_1(m)$ there is a factor u for each of the m branches, and at the second level these factors u are replaced with the probabilities coming from the branches encountered in the further exploration of the tree. Multiplying Eq. (62) by z^n and using the expression for $P_2(n)$, a straightforward computation gives yields

$$L_2(z) = L_1(G_1(uz)/u) = g_0\big(\phi g_1(\phi u z + 1 - \phi) + 1 - \phi\big). \tag{64}$$

This argument can be used at any shell l, by conditioning on the previous shell $l-1$ to have m active nodes in finite components. We arrive at the recursive equations

$$L_l(z) = L_{l-1}\big(G_1(uz)/u\big), \tag{65}$$

$$L_1(z) = G_0(uz). \tag{66}$$

Essentially, as we proceed, we replace all the factors uz at level $l-1$ with $G_1(uz)$. Taking the derivative of these recursive equations, and evaluating them at $z = 1$, leads to, recalling that $u = G_1(u)$,

$$\mu(l) = \mu(l-1)\phi g_1'(\phi u + 1 - \phi), \tag{67}$$

$$\mu(1) = \phi \langle k \rangle u^2, \tag{68}$$

whose solution is simply given by

$$\mu(l) = \phi \langle k \rangle u^2 [\phi g_1'(\phi u + 1 - \phi)]^{l-1}. \tag{69}$$

Note that this expression reduces to the one shown before if $t < 0$, that is for $u = 1$, i.e., in the absence of a GC. Eq.(69) is formally analogous to Eq. (58), but with an "effective" branching factor which is reduced by the presence of the GC. Inserting Eq. (69) into $\chi(\phi, 0) =$

$\sum_{l\geq 0}\mu(l)$,

$$\chi(\phi,0) = \Psi(\phi,0) + \phi\langle k\rangle u^2 \sum_{l\geq 1}[\phi g_1'(\phi u + 1 - \phi)]^{l-1}$$

$$= \Psi(\phi,0) + \frac{\phi\langle k\rangle u^2}{1 - \phi g_1'(\phi u + 1 - \phi)}, \qquad (70)$$

which coincides with Eq. (56) evaluated at $h = 0$. For ξ^2 we have, summing again the derivative of the geometric series,

$$\xi^2 = \frac{\phi\langle k\rangle u^2[1 - \phi g_1'(\phi u + 1 - \phi)]^{-2}}{\Psi(\phi,0) + \phi\langle k\rangle u^2[1 - \phi g_1'(\phi u + 1 - \phi)]^{-1}}. \qquad (71)$$

Expanding the numerator and the denominator for t close to 0, both below ($u = 1$) and above ($u < 1$) the transition, allows us to compute the critical exponent ν.

E. The exponents σ and τ

From the singular behavior of $H_0(z)$ and $H_1(z)$, also the scaling form of $\pi_s(\phi)$ and hence the exponents τ and σ can be worked out explicitly. Indeed, if the generating function $H_0(z)$ has a convergence radius ρ_{H_0},

$$H_0(z) = A_0 - B_0(\rho_{H_0} - z)^{a_0}, \qquad (72)$$

inversion (Tauberian) theorems [38] guarantee that, for $s \gg 1$, π_s scales as

$$\pi_s(\phi) \simeq \frac{B_0}{\Gamma(-a_0)} s^{-a_0-1} e^{-s/s_\xi}, \qquad (73)$$

in agreement with Eq. (24), with $\tau = 2 + a_0$, and

$$s_\xi = \frac{1}{\log(\rho_{H_0})}. \qquad (74)$$

Hence in order to find σ and τ it is necessary to determine the singular behavior of $H_0(z)$ which, from Eq. (45), is related to the singular behavior of $H_1(z)$, that we now analyze.

By setting $u = H_1(z)$, Eq. (47) can be rewritten as

$$z = \psi(H_1(z)), \qquad (75)$$

where $\psi(u) = u/G_1(u)$ is the inverse function of $H_1(z)$. Assuming $G_1(0) \neq 0$ (as it is always the case for $0 < \phi \leq 1$) and $\lim_{u\to R^-} uG_1'(u)/G_1(u) > 1$ [where R is the convergence radius of $G_1(u)$] there exists (see Ref. [38], Proposition IV.5, page 278) a unique solution $u^* \in (0, R)$ of the *characteristic equation* $\psi'(u^*) = 0$, that is, of the equation

$$G_1(u^*) - u^* G_1'(u^*) = 0. \qquad (76)$$

Then, the convergence radius of the power series $H_1(z)$ is

$$\rho_{H_1} = \psi(u^*) = \frac{u^*}{G_1(u^*)}. \qquad (77)$$

In other words, the inversion of the function $\psi(u)$ can be performed only up to u^*, where its first derivative vanishes. As a consequence, the function $H_1(z)$ has a singularity at $z = \rho_{H_1} = \psi(u^*)$, and this is the closest singularity to the origin.

The singular behavior of $H_1(z)$ close to ρ_{H_1} is

$$H_1(z) \simeq u^* - A_1(\rho_{H_1} - z)^{a_1}, \quad z \to \rho_{H_1}^-, \qquad (78)$$

with A_1 a positive constant[7]. Note that $H_1(z = \rho_{H_1}) = H_1(\psi(u^*)) = u^*$.

If $G_0(H_1)$ is analytic at $H_1 = u^*$, insert Eq. (78) into Eq. (45) and expanding $G_0(H_1(z))$ leads to

$$H_0(z) \simeq \rho_{H_1}G_0(u^*) - \rho_{H_1}G_0'(u^*)A_1(\rho_{H_1} - z)^{a_1}, \quad (79)$$

that is, Eq. (72) with $\rho_{H_0} = \rho_{H_1}$, $A_0 = \rho_{H_1}G_0(u^*)$, $B_0 = \rho_{H_1}G_0'(u^*)A_1$ and $a_0 = a_1$.

The case in which $G_0(H_1)$ is nonanalytic at $H_1 = u^*$ may occur when $u^* = 1 = \rho_{H_1}$. In such a case we can insert Eq. (78) into Eq. (45) and use now the singular expansion for $G_0(H_1)$ around $u^* = 1$, see Eq. (50). Since the singularity in $G_0(H_1)$ has an exponent larger than 1, the leading singularity in $H_0(z)$ is still given by Eq. (79).

From now on we denote $\rho_{H_0} = \rho_{H_1} = \rho$. $H_1(z)$ is the generating function of a probability distribution, hence its convergence radius must be $\rho \geq 1$ (the coefficients cannot diverge exponentially).

Recalling that $u = H_1(z)$, the singular behavior of $H_1(z)$ [Eq. (78)] is found by determining how $H_1(z) - u^* = u - u^*$ depends on $\rho - z$. Since $\rho = \psi(u^*)$ and $z = \psi(u)$ this implies that we have to write

$$\rho - z = \psi(u^*) - \psi(u), \qquad (80)$$

expand the r.h.s. for small $\epsilon = u^* - u$ and invert to obtain how ϵ is singular as a function of $\rho - z$: $\epsilon \sim (\rho - z)^{a_1}$. The exponent $a_1 = a_0$ is the one appearing in Eq. (78) and hence it determines τ. The type of singularity obtained by this inversion depends on the expansion of the function $\psi(u)$, which in turn strongly depends on the degree distribution and whether supercritical ($t > 0$), critical ($t = 0$) or subcritical ($t < 0$) properties are considered, as shown in detail in Appendix D.

F. The exponent $\bar{\nu}$

Consider a network with finite maximum degree k_c and large N. Regardless of the degree-distribution, such a system is effectively homogeneous. In the infinite-size limit, this system has a critical point $\phi_c(k_c) = b(k_c)^{-1}$ which depends on the maximum degree k_c. In the denominator of Eq. (71), for ϕ close to $\phi_c(k_c)$, $\Psi(\phi,0)$ can be neglected, leading to (see Appendix C) $\xi^2 \simeq |1 - \phi b(k_c)|$. Hence

$$\xi \simeq b(k_c)^{-1/2}|\phi - b(k_c)^{-1}|^{-1/2}. \qquad (81)$$

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[7]There are logarithmic corrections to Eq. (78) if a_1 is an integer.

According to our finite-size scaling argument, the effective critical point is determined by this expression, evaluated at $\phi = \phi_c(N)$, with the left hand side proportional to $N^{1/6}$, since the system is homogeneous and $D_{\mathrm{UC}} = 6$. Neglecting for simplicity the absolute value, this yields

$$\phi_c(N, k_c) \simeq b(k_c)^{-1} \left(1 + C_1 N^{-1/3} \right), \qquad (82)$$

where C_1 is a constant. If k_c does not depend on N, or the dependence is subalgebraic, Eq. (82) leads to $\bar{\nu} = 3$. However, if $k_c \sim N^\omega$, as for heterogeneous networks, a deviation from the standard mean-field exponent may occur.

$(p_k \sim k^{-\gamma_d})$ is detailed in Appendix C. For what concerns thermodynamic critical exponents and amplitude ratios, results are summarized in Table II. Note that all exponents are derived independently, scaling relations are not used (yet they are verified by our results except for the relation $\delta = 1/(\tau - 2)$ for $2 < \gamma_d < 3$). Some of the values in Table II (for exponents τ and γ in the case $2 < \gamma_d < 3$) disagree with some values reported previously in the literature. In the concluding section we discuss in detail the origin of these discrepancies and what led to the previous incorrect claims.

IV. RESULTS

The application of this general approach to the specific cases of homogeneous and power-law degree distributions

TABLE II. Values of the thermodynamic critical exponents and critical amplitude ratios for uncorrelated power-law distributed networks in the various γ_d ranges, computed from the exact solution with the generating functions. Values indicated by [a] were first calculated in Ref. [28]. Values indicated by [b] were first calculated in Ref. [39]. Values indicated by [c] were first calculated in Ref. [37]. See Appendix C 5 b for a discussion of values indicated by [*].

Network type	β	δ	α	γ	ν	σ	τ	C_+/C_-	Ξ_+/Ξ_-	R_χ
Homogeneous (and $\gamma_d > 4$)	1^a	2	-1	1^a	$\frac{1}{2}^b$	$\frac{1}{2}^a$	$\frac{5}{2}^c$	1	1	1
$3 < \gamma_d < 4$	$\frac{1}{(\gamma_d - 3)}^a$	$\gamma_d - 2$	$-\frac{5 - \gamma_d}{\gamma_d - 3}$	1^a	$\frac{1}{2}^b$	$\frac{\gamma_d - 3}{\gamma_d - 2}^a$	$2 + \frac{1}{\gamma_d - 2}^a$	$\frac{1}{\gamma_d - 3}$	1	$\frac{1}{\gamma_d - 3}$
$2 < \gamma_d < 3$	$\frac{1}{(3 - \gamma_d)}^a$	1^*	$-\frac{3(\gamma_d - 7/3)}{3 - \gamma_d}$	-1^a	$-\frac{1}{2}$	$\frac{3 - \gamma_d}{\gamma_d - 2}^a$	$2 + \frac{1}{\gamma_d - 2}^a$	N/A	N/A	$\frac{1}{3 - \gamma_d}^*$

Beyond the value of the exponents, the analysis yields a very rich picture for the cluster size distribution $n_s(t)$, with nontrivial preasymptotic behaviors and violations of the scaling ansatz, as summarized here.

On homogeneous networks percolation is perfectly described by the scaling ansatz (24) [see Fig. 1(a)]: the cluster size distribution is a power-law with exponent $\tau = 5/2$ cut off exponentially at the correlation size s_ξ. Both for positive and negative t, s_ξ diverges as $|t|^{-1/\sigma}$ as criticality is approached, with the mean-field exponent $\sigma = 1/2$.

The same scenario applies for power-law networks with $\gamma_d > 4$ and $t > 0$, but for $t < 0$ the picture is different [see Fig. 1(b)]: after an exponential cutoff, occurring for $s_0 \sim |t|^{-1/\sigma}$, the cluster size distribution decays indefinitely as $s^{-\gamma_d}$, a legacy of the degree distribution of the original network [33]. This violation of the scaling ansatz Eq. (24) is a direct effect of the heterogeneity of the substrate but it is not related to different critical properties of the percolation process. Indeed, for $t \to 0^-$ the asymptotic tail with exponent $\gamma_d > \tau$ is not responsible for the divergence of the fluctuations, which are determined in-

stead by the preasymptotic scaling with exponent τ. The scale $s_0 \sim |t|^{-1/\sigma}$ plays a role similar to the correlation size s_ξ for $t > 0$ and scales in the same way.

For more heterogeneous networks, $3 < \gamma_d < 4$, both σ and τ become γ_d-dependent, but the most relevant change is that for $t > 0$ the asymptotic decay of $n_s(t)$ is of the form $s^{-5/2} e^{-s/s_\xi}$, i.e., it is different from the decay $s^{-\tau}$ occurring exactly at criticality ($t = 0$). The way these two apparently incompatible behaviors are reconciled is depicted in Fig. 1(c): a crossover scale s_*, smaller than s_ξ but diverging with t in the same way, separates the preasymptotic decay $s^{-\tau}$ from the regime $s^{-5/2} e^{-s/s_\xi}$. As criticality is approached, both s_* and s_ξ diverge and the regime $s^{-\tau}$ extends to all scales. For $t < 0$ instead the phenomenology is analogous to the case for $\gamma_d > 4$ only with a different value of τ.

Finally, for $2 < \gamma_d < 3$ only the percolating phase $t > 0$ exists, as $\phi_c = 0$. In contrast to what happens for the other types of networks discussed above, when $\phi \to 0$ the system exhibits no critical behavior: $\langle s \rangle$ does not diverge ($\gamma = -1$), the cluster size distribution tends to be delta distributed ($n_s(0) = \delta_{s,1}$), and the correlation length ξ
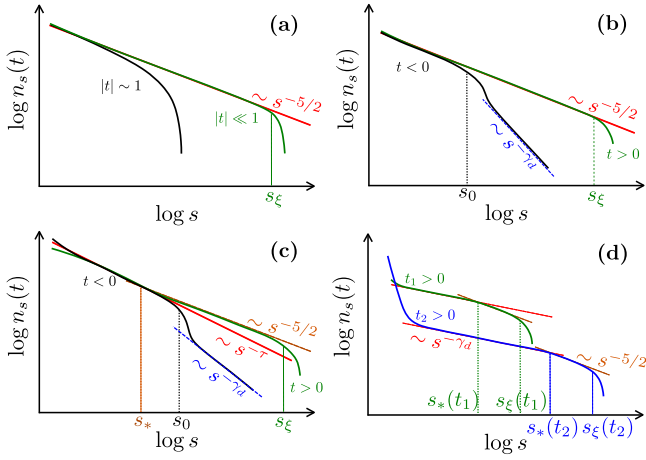
FIG. 1. Figure with schematic log-log plots of $n_s(t)$ in the various regimes. (a) Homogeneous networks. The red line corresponds to the critical scaling $s^{-\tau}$ with $\tau = 5/2$. For $|t| \ll 1$, both positive and negative, an exponential cutoff occurs for for $s = s_\xi$ with the correlation scale s_ξ diverging as criticality is approached. (b) $\gamma_d > 4$. The red line corresponds to the critical scaling $s^{-\tau}$ with $\tau = 5/2$. For positive $t \ll 1$ (green line), the same picture as in (a) holds. However, for $|t| \ll 1$ but $t < 0$ (black line), the asymptotic scaling is given by $s^{-\gamma_d}$ (blue dashed line), and the exponent τ is visible only up to the crossover scale s_0. (c) $3 < \gamma_d < 4$. The red line corresponds to the critical scaling $s^{-\tau}$ with $\tau = 2 + 1/(\gamma_d - 2)$. For small positive t (green line), the asymptotic scaling is with an exponent $5/2$ (orange dashed line) followed by an exponential cutoff occurring at s_ξ, while in the preasymptotic regime $1 \ll s \ll s_*$ the scaling with exponent τ is observed. For small $t < 0$ (black line) the same picture as in (b) holds. (d) $2 < \gamma_d < 3$. Only the case $t > 0$ is possible. The two cases $t_2 < t_1 \ll 1$ exhibit the same scaling behavior: a preasymptotic scaling with exponent γ_d, followed (after the scale s_*) by the exponent $5/2$ and then an exponential cutoff at s_ξ. However, as t is decreased, the amplitude of these decays decreases and vanishes asymptotically, leaving only $n_s(t=0) = \delta_{s,1}$.

goes to zero ($\nu = -1/2$). However, for small t the cluster size distribution exhibits a nontrivial behavior, displaying some features of a critical transition, see Fig. 1(d). For fixed small t, similarly to the case $3 < \gamma_d < 4$, n_s exhibits two power-law decays, separated by a crossover scale s_*. For $1 \ll s \ll s_*$, it decays as $s^{-\gamma_d}$, while for $s \gg s_*$ we find $n_s \sim s^{-5/2} e^{-s/s_\xi}$, where $s_\xi \sim t^{-1/\sigma}$, with $\sigma = (3 - \gamma_d)/(\gamma_d - 2)$. Also the crossover scale s_* diverges as $t^{-1/\sigma}$. Hence one could be tempted to conclude that $\tau = \gamma_d$, in agreement with some theoretical results in the literature [31], and in contrast with others [28, 30]. Note that with this value the scaling relations (30) would not be satisfied. But crucially the large-s tail of n_s is multiplied by a t-dependent factor, vanishing as $t \to 0$, in agreement with the fact that $n_s(\phi) \to \delta_{s,1}$. Despite the fact that $\phi_c = 0$ is not a veritable critical point it is still possible to write this behavior of $n_s(t)$ in a scaling form (24), where now $\tau = 2 + 1/(\gamma_d - 2)$ (in agreement with [28] and with the scaling relations) and the scaling function $f_+(x)$

does not go to a constant for $x \to 0$, vanishing instead as $x^{\gamma_d - 1}$. This nontrivial form of the scaling function $f_+(x)$ has the consequence that for fixed t the cluster size distribution decays with an exponent γ_d different from τ and a t-dependent prefactor vanishing as $t \to 0$.

All these results about $n_s(t)$ are in agreement with what one can deduce from the theory for the network component size distribution formulated by Kryven [34], adapting the results therein by assuming that nodes are diluted with probability ϕ (see Appendix E).

TABLE III. Values of size-related critical exponents for uncorrelated power-law distributed networks in the various γ_d ranges, where the hard structural cutoff of the network is taken as $k_{\max} \sim N^{1/\omega}$, with ω a positive parameter obeying the constraint $\omega \geq 1$ ($\omega \geq 2$ for $2 < \gamma_d < 3$). The homogeneous case corresponds to a sub-algebraically growing $k_c(N)$. The hyperscaling relation, Eq. (43), is satisfied when the values in the last two columns coincide (see Appendix C 7). Values indicated by ^{a} were first calculated in Ref. [28].

Network type		$\bar{\nu}$	θ	$\frac{1}{\sigma\bar{\nu}}$
Homogeneous		3^a	$\frac{2}{3}$	$\frac{2}{3}^a$
$4 < \gamma_d$	$\omega \leq 3(\gamma_d - 3)$	3	$\frac{2}{3}$	$\frac{2}{3}$
	$\omega \geq 3(\gamma_d - 3)$	$\frac{\omega}{\gamma_d - 3}$	$1 - \frac{\gamma_d - 3}{\omega}$	$\frac{2(\gamma_d - 3)}{\omega}$
$3 < \gamma_d < 4$	$\omega \leq \gamma_d - 1$	$\frac{\gamma_d - 1}{\gamma_d - 3}$	$\frac{\gamma_d - 2}{\gamma_d - 1}$	$\frac{\gamma_d - 2}{\gamma_d - 1}$
	$\omega \geq \gamma_d - 1$	$\frac{\omega}{\gamma_d - 3}$	$1 - \frac{1}{\omega}$	$\frac{\gamma_d - 2}{\omega}$
$2 < \gamma_d < 3$		$\frac{\omega}{3 - \gamma_d}$	$1 - \frac{1}{\omega}$	$\frac{\gamma_d - 2}{\omega}$

Also concerning finite-size scaling the picture is surprisingly rich (see Table III for a summary of results and Appendix C 7 for a detailed discussion). Contrary to naive expectation and to what is reported in the literature [28–30], the exponent $\bar{\nu}$, governing how the effective threshold approaches the asymptotic limit, does not necessarily assume the mean-field value $\bar{\nu} = 3$ for power-law degree-distributed networks with $\gamma_d > 4$. Indeed, $\bar{\nu}$ assumes its mean-field value 3 only for small enough ω, the exponent governing how the network hard structural cutoff $k_{\max}$ grows with the system size N, $k_{\max} \sim N^{1/\omega}$. Specifically, for $\gamma_d > 4$ we have $\bar{\nu} = 3$ only for $\omega \leq 3(\gamma_d - 3)$. For $\omega \geq 3(\gamma_d - 3)$ the exponents $\bar{\nu}$ and θ depend also on ω and hyperscaling relations do not hold. The crossover value for $3 < \gamma_d < 4$, above which hyperscaling relations are violated, is $\omega = \gamma_d - 1$. For $2 < \gamma_d < 3$ the transition occurring at $\phi = 0$ is not really critical and the whole finite size scaling framework must be interpreted differently (see Appendix C 7).

V. NUMERICAL SIMULATIONS

We performed numerical simulations to test some of the analytical results, in particular with regard to finite size scaling. We considered networks built according to the uncorrelated configuration model [46], with $k_{\max} =$
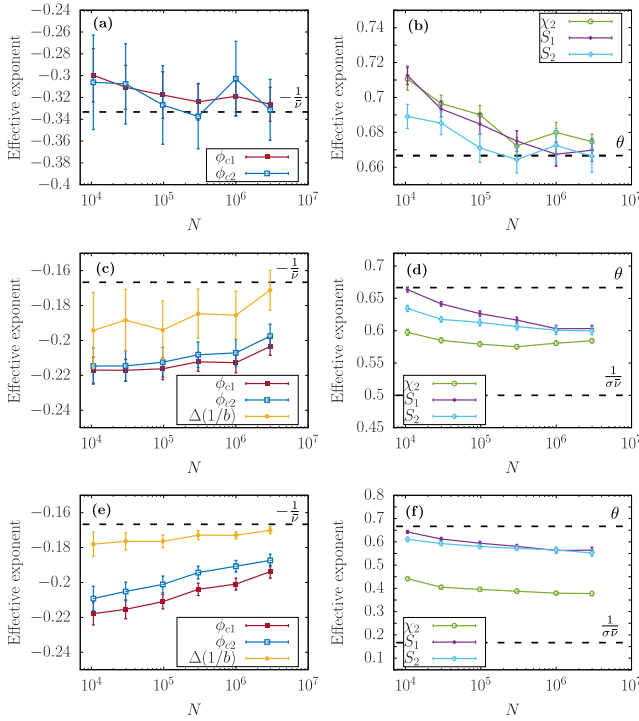
FIG. 2. Effective exponents of the observables ϕ_{c1}, ϕ_{c2}, χ_2, S_1, and S_2 measured via Eq. (83) in numerical simulations for power-law networks, with (a),(b) $\gamma_d = 5$, (c),(d) $\gamma_d = 3.5$, and (e),(f) $\gamma_d = 2.5$. Dashed lines correspond to the theoretical predictions of the exponent as in Table III. Note that in (b), where the hyperscaling is expected to hold, $\theta = 1/(\sigma\bar{\nu})$. In (c) and (e), we also report the scaling of the branching factor measured over the network samples, showing the presence of preasymptotic effects.

$k_{\min}N^{1/\omega}$, $\omega = 3$, $k_{\min} = 3$ and various γ_d. Site percolation was simulated by means of the efficient Newman-Ziff algorithm [47] where for each network $M = 1000$ realizations were run. To identify the critical point we considered two different susceptibilities [48]: ϕ_{c1} is the position of the peak of $\chi_1 = (\langle \mathcal{S}^2 \rangle - \langle \mathcal{S} \rangle^2)/(\phi N)$, while ϕ_{c2} is the position of the peak of $\chi_2 = (\langle \mathcal{S}^2 \rangle - \langle \mathcal{S} \rangle^2)/\langle \mathcal{S} \rangle$. We evaluate the scaling of the size of the largest cluster at criticality for both determinations of the critical point, $S_1(N) = \langle \mathcal{S} \rangle |_{\phi_{c1}}(N)/\phi_{c1}(N) \sim N^\theta$ and analogously for $S_2(N)$. The height of the peak of χ_1 scales as $N^{\gamma/\bar{\nu}}$ [see Eq. (41)], while the peak of χ_2 diverges with an exponent $(\gamma + \beta)/\bar{\nu} = 1/(\sigma\bar{\nu})$. This latter exponent coincides with θ if hyperscaling holds. We then average all the measured quantities over 100 to 1000 different realizations of the network substrate. We finally evaluate the effective exponent λ of the observable $O \sim N^\lambda$ as a function of size by performing simulations for several sizes N_i and calculating

$$\lambda(N_i) = \frac{\log(O(N_{i+1})/O(N_i))}{\log(N_{i+1}/N_i)}. \qquad (83)$$

For $\gamma_d = 5$, finite size scaling is the same valid for homogeneous networks: $\bar{\nu} = 3$, $\theta = 2/3$, and hyperscal-

ing holds. Fig. 2(a) confirms that this picture holds for networks with size of the order of $N = 10^6$ or larger. For $\gamma_d = 2.5$ instead, we do not expect hyperscaling to hold and the predicted value of $\bar{\nu}$ is 6. While the violation of hyperscaling is clear in Fig. 2(f), it is evident that the effective exponents are far from the predicted asymptotic values. Effective exponents differ from those predicted analytically also for $\gamma_d = 3.5$ [Fig. 2(d)]. Also in this case strong preasymptotic effects dominate, as shown by Fig. 2(c): while the scaling of $\Delta(1/b) = 1/b(N) - 1/b_\infty$ approaches the expected behavior $N^{-1/6}$ for the largest sizes considered, the effective exponents derived from the scaling of $\phi_{c1}(N)$ and $\phi_{c2}(N)$ are close to -0.2. This indicates that the term proportional to $N^{-1/3}$ in Eq. (C33) cannot be safely neglected, even for the largest values of N considered. The effective exponent close to -0.2 can be interpreted, since $1/\bar{\nu} = (\gamma_d - 3)/\omega$, as the consequence of an effective value of $\omega \approx 2.5$ which, inserted into $\theta = 1 - 1/\omega$, accounts for the effective exponent of S_1 and S_2 close to 0.6. The true asymptotic behavior could be observed only for system sizes such that the blue and red curves in Fig. 2(c),(e) reach the dashed horizontal line. Networks of several orders of magnitude larger than those we can consider would be needed. For the same reason, it is impossible to verify numerically the breakdown of hyperscaling and the validity of the prediction $\bar{\nu} = \omega/(\gamma_d - 3)$ for $\gamma_d > 4$ and $\omega \geq 3(\gamma_d - 3)$.

VI. CONCLUSIONS

In this paper we have provided a complete and coherent analysis of the critical behavior of standard site and bond percolation models on uncorrelated locally tree-like random networks with generic degree distribution p_k. Even if recent exact results about anomalous exponents on finite-N scale-free networks [49–51] suggest that a full mathematical theory is still lacking, percolation on random graphs has been the subject of intense research activity for decades and a general understanding of many nontrivial features had already been reached a long time ago. In particular the vanishing threshold for strongly heterogeneous networks and the dependence of some critical exponents on the exponent γ_d have been recognized over 20 years ago [8, 28]. However, results about critical properties are scattered in a large body of literature and often interspersed with imprecise or incorrect statements.

Moreover a complete theory for all exponents and the scaling functions was lacking. In this work we have filled these gaps. Exploiting the locally tree-like nature of the networks considered, by means of the generating functions approach we have derived all critical properties for both infinite and finite systems.

In this way we have clarified the true value of some exponents for scale-free networks, over which confusion existed in the literature. In particular: the Fisher exponent τ, that was claimed to be equal to 3 [29], to $2+1/(\gamma_d-2)$ [28], to γ_d [31] or even to be not defined [32] is found to be

$2 + 1/(\gamma_d - 2)$. The exponent γ, claimed to be not defined [32] turns out to be equal to -1. The exponent $\bar{\nu}$, governing finite size scaling, has an anomalous behavior (see Table III) at odds with the previously claimed values of $2/(3 - \gamma_d)$ [32] and $(\gamma_d - 1)/(3 - \gamma_d)$ [28]. We have derived the value of other exponents (α, δ) not explicity calculated before, and we have determined for the first time critical amplitude ratios.

A crucial finding is the detailed understanding of how the cluster size distribution $n_s(t)$ behaves in the various ranges of γ_d values. It turns out that the usual scaling assumption $n_s(t) \sim s^{-\tau} f(s/s_\xi(t))$ with $f(x) = \exp(-x^{1/\sigma})$ is never fully correct for power-law distributed networks. For $\gamma_d > 4$ in the subcritical case the exponential cutoff is followed by an asymptotic decay $s^{-\gamma_d}$. For $3 < \gamma_d < 4$ this subcritical feature is accompanied, above the threshold, by a crossover such that the asymptotic decay occurs with an exponent that differs from the critical value τ. For strongly heterogeneous networks $(2 < \gamma_d < 3)$ the vanishing threshold implies that $\phi_c = 0$ is not a true critical point. It is still possible to write $n_s(t)$ in a scaling form but $f(x)$ vanishes for $x \to 0$. One of the consequences of this nontrivial scaling is that the decay of $n_s(t)$ for fixed t is governed by an exponent γ_d differing from the Fisher exponent $\tau = 2 + 1/(\gamma_d - 2)$ and is multiplied by a factor which vanishes as $t \to 0$. The claims in the literature that τ is equal to γ_d or not even well defined reflect a partial understanding of the scaling properties, that we have fully elucidated here.

We have also presented a consistent theory for finite-size scaling properties, showing that the exponent $\bar{\nu}$ may depend on how the maximum degree k_c diverges with the system size. At odds with all previous literature, we find that this may happen even for $\gamma_d > 4$, (when all thermodynamic exponents are equal to those for homogeneous systems) provided k_c diverges sufficiently slowly. This dependence on the network maximum degree implies that hyperscaling relations, usually assumed to be valid, are violated in this case.

Our work may stimulate further in depth investigation (or reinvestigation) of critical properties for other types of percolation processes on networks. What happens when nodes are removed in a non fully random way, for example targeting first highly [52] or poorly [53, 54] connected nodes? Or if connected components are defined allowing for gaps in paths connecting nodes (extended-range percolation [55, 56])? What can be said about critical exponents and scaling relations when the system undergoes a discontinuous or hybrid transition?

Our results may have implications for epidemic models and spreading processes in general, due to well-established links between such models and percolation theory [57]. Our generating functions approach may also be useful in the analysis of critical properties in other models of interacting systems on networks exhibiting continuous phase transitions, e.g., spin systems, models of synchronization and opinions dynamics [29].

Finally, an interesting question is to understand how

our results, obtained within the assumption of locally tree-like networks, are affected by the presence of clustering and short loops.

The exploration and understanding of universal behaviors in models defined on complex networks – often qualitatively different from those observed on regular topologies – offer a pathway to deeper theoretical insights into the physics of complex systems, remaining an intriguing challenge for future research.

Appendix A: Cluster size distribution of some graph models close to criticality

It is interesting to show the correctness of the scaling ansatz Eq. (24) in cases for which the form of $n_s(t)$ can be exactly derived.

1. 1-d chain

For the $1d$ chain $n_s(\phi) = \phi^2(1 - \phi)$ [13]. In this case $\phi_c = 1$, so that $\phi = 1 + t$, hence

$$n_s(t) = (-t)^2(1 + t)^s = (-t)^2 \exp(s \ln(1 + t))$$
$$= s^{-\tau} f(s^\sigma t), \tag{A1}$$

where $s_\xi = -1/\ln(1 + t) \simeq -t^{-1}$ and $\tau = 2$, $\sigma = 1$. Note that in this case $f(x) = x^2 e^{-x}$, hence $f(x) \sim x^2$ for $|x| \ll 1$.

2. Erdős-Rényi graphs

Unfortunately, there are no other finite dimensional systems for which an exact expression for $n_s(t)$ is available. However, in Ref. [33], Newman brilliantly derived a general formula for the cluster size distribution in random graphs, where he showed that

$$\pi_s(\phi) = sn_s(\phi) = \frac{\phi^{s-1}\langle k \rangle}{(s - 1)!} \left[\frac{d^{s-2}}{dz^{s-2}} [g_1(z)]^s \right]_{z=1-\phi}. \tag{A2}$$

For ER graphs with $p_k = e^{-c}c^k/k!$ this implies

$$\pi_s(\phi) = \frac{e^{-cs\phi}(cs\phi)^{s-1}}{s!}, \tag{A3}$$

from which it follows, using Stirling's formula $s! \simeq s^s e^{-s} \sqrt{2\pi s}$ for $s \gg 1$, working close to $\phi_c = 1/c$, hence

$$c\phi = ct - 1,$$

$$n_s(t) \simeq \frac{s^{-5/2}}{\sqrt{2\pi}} \exp(-s[ct - \ln(1+ct)])$$

$$\simeq \frac{s^{-5/2}}{\sqrt{2\pi}} \exp(-(cs^{1/2}t)^2/2]) = s^{-\tau} f(cs^\sigma t), \quad \text{(A4)}$$

which is exactly in the form of Eq. (24), with $f(x)$ Gaussian, and $\tau = 5/2$, $\sigma = 1/2$.

3. Random regular graphs

For random regular networks with degree $c > 2$[8] (c-RRN), the generating functions are $g_0(z) = z^c$ and $g_1(z) = z^{c-1}$, hence from Eq. (A2)

$$\pi_s(\phi) = \phi^{s-1}\langle k \rangle \frac{\Gamma[s(c-1)+1]}{(s-1)!\Gamma[s(c-2)+3]}(1-\phi)^{s(c-2)+2}.$$

Using $\Gamma(x+3) \simeq x^2\Gamma(x+1)$, and again Stirling's formula for $s \gg 1$, we find, after some tedious but straightforward computations,

$$\frac{\Gamma[s(c-1)+1]}{(s-1)!\Gamma[s(c-2)+3]} = s\frac{\Gamma[s(c-1)+1]}{s!\Gamma[s(c-2)+3]} \simeq s[s(c-2)]^{-2}\frac{[s(c-1)]^{s(c-1)}e^{-s(c-1)}\sqrt{2\pi s(c-1)}}{s^s e^{-s}\sqrt{2\pi s}[s(c-2)]^{s(c-2)}e^{-s(c-2)}\sqrt{2\pi s(c-2)}}$$

$$= s^{-3/2}\sqrt{\frac{c-1}{2\pi(c-2)}}(c-2)^{-2}\exp\left[s(c-1)\ln(c-1) - s(c-2)\ln(c-2)\right]$$

from which it follows that

$$n_s(\phi) \simeq \langle k \rangle \frac{s^{-5/2}}{\phi}\sqrt{\frac{c-1}{2\pi(c-2)}}\left(\frac{1-\phi}{c-2}\right)^2 \exp\left[-s/s_\xi\right],$$

where

$$s_\xi^{-1} = -(c-2)\ln\left[\frac{(c-1)(1-\phi)}{(c-2)}\right] - \ln[\phi(c-1)].$$

Since $\phi_c = 1/(c-1)$, we can write $c - 2 = (1-\phi_c)/\phi_c$, $(c-1)/(c-2) = 1/(1-\phi_c)$, and $(c-1)(1-\phi)/(c-2) = 1 - t/(1-\phi_c)$. Therefore, expanding $\ln(1+x)$ for small x and keeping the lowest orders in t

$$s_\xi^{-1} \simeq \left(\frac{1}{2\phi_c(1-\phi_c)} + \frac{1}{2\phi_c^2}\right)t^2 = \frac{1}{2\phi_c^2(1-\phi_c)}t^2,$$

from which $\sigma = 1/2$. For $t \ll 1$, we then have

$$n_s(\phi) \simeq \sqrt{\frac{(1+\phi_c)^2}{2\pi(1-\phi_c)}}s^{-5/2}e^{-s/s_\xi}, \quad \text{(A5)}$$

Hence the scaling ansatz Eq. (24) holds, with a Gaussian scaling function, $\tau = 5/2$ and $\sigma = 1/2$.

a. Explicit solution for $c = 3$

For $c = 3$ it is possible to solve explicitly for $H_0(z)$, since the generating functions $G_0(z)$ and $G_1(z)$ are simple monomials of third and second order, respectively. Indeed, the equation for H_1,

$$H_1 = z(\phi H_1 + 1 - \phi)^2,$$

can be explicitly solved

$$H_1(z) = \frac{1 - 2z\phi(1-\phi) - \sqrt{1 - 4z\phi(1-\phi)}}{2z\phi^2},$$

which substituted in the equation

$$H_0(z) = z\left(\phi H_1(z) + 1 - \phi\right)^3,$$

gives

$$H_0(z) = \frac{\left[1 - \sqrt{1 - 4z\phi(1-\phi)}\right]^3}{8z^2\phi^3}.$$

The convergence radius of $H_0(z)$ is $z^* = 1/(4\phi(1-\phi))$. Note that its singular behavior close to z^* is determined by the singular behavior of $H_1(z)$. The singularity is always outsite the unitary circle apart from the critical point $\phi = \phi_c = 1/2$. Expanding $H_0(z)$ close to z^*, and z^* for ϕ close to ϕ_c, yields $\tau = 5/2$, $\sigma = 1/2$.

Appendix B: Mean-field theory for percolation and upper critical dimension

A standard procedure in statistical physics to study the critical behavior in finite-dimensional systems is to find

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[8]The requirement $c > 2$ is needed to avoid the case in which the network is originally made up entirely of loops. In this case, Newman's results, which are exact only for tree-like networks, fail. However, for any $\phi < 1$ almost all loops break up into trees, and Eq. (A2) can be used, even though it is not well defined in the limit $\phi \to 1$.

an equivalent, coarse-grained, description of the system of interest in terms of a field theory, i.e. a partition function

$$Z = \int \mathcal{D}\varphi e^{-\mathcal{S}[\varphi]} \tag{B1}$$

where $\mathcal{S}[\varphi]$ is

$$\mathcal{S}[\varphi] = \int d^D x \left[\frac{1}{2} \varphi \triangle \varphi + V(\varphi) \right], \tag{B2}$$

$\triangle$ is the Laplacian operator, and $V(\varphi)$ is in general a function of the ordering field φ, whose average defines the order parameter. The potential V can be determined on the basis of symmetry considerations, and it strongly depends on the values of the control parameters. If one finds a proper functional form for $\mathcal{S}[\varphi]$, then the study of the critical behavior of the effective field-theory described by Eq. (B1) is equivalent to the study of the critical behavior of the finite-dimensional system of interest. For standard percolation, the effective potential contains a cubic term, [18, 58, 59]

$$V(\varphi) = \frac{1}{2} c_2 \varphi^2 + \frac{1}{3!} c_3 \varphi^3 \tag{B3}$$

where $c_2 \sim \phi - \phi_c$ and c_3 is constant. Since in general it is not possible to solve the integral in Eq. (B1), many approximations, perturbative and non-perturbative methods have been developed in the last decades. The starting point remains, however, the so-called Landau approximation. In this zero-th order approximation, one simply neglects spatial fluctuations. This can be physically interpreted as a coarse-graining procedure performed over length scales larger than the correlation length ξ. Writing $\varphi(x) = \varphi_0 + \delta\varphi(x)$, in practice one solves the integral in Eq. (B1) ignoring the contributions in $\delta\varphi(x)$. The spatial integral over $d^D x$ gives a volume factor, hence one can compute Z using the saddle-point method, which is to find the minimum of $V(\varphi)$, φ_0 such that

$$V'(\varphi_0) = 0. \tag{B4}$$

Within this approximation, we can study the critical behavior and compute the critical exponents. However, we must always question whether or not the approximation is justified. We can find the answer in the Ginzburg criterion [11][9], which tells us that for a system whose Landau theory is described by a set of critical exponents $\beta, \gamma, \nu, \ldots$, the validity of the description provided by the Landau approximation is conditioned on

$$|t|^{-\gamma - 2\beta + D\nu} \ll 1. \tag{B5}$$

Given D, this criterion provides an argument to understand whether or not we are "close but not too close" to

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[9]See, for instance, Ref. [60] for a generalization of the Ginzburg criterion to Ising model in spatial complex networks.

$t = 0$ in order to see the mean-field Landau exponents rather than the D-dependent ones. On the other hand, this criterion tells us what is the minimum dimension D of the space above which we are safe in describing our system using the Landau approximation. This dimension is called upper critical dimension and is given by

$$D_{\text{UC}} = \frac{\gamma + 2\beta}{\nu}. \tag{B6}$$

In particular, the Landau theory for standard percolation using Eq. (B3) gives $\beta = 1$, $\gamma = 1$, $\nu = 1/2$, hence we recover the well-known result $D_{\text{UC}} = 6$ [58]. Since Landau theory correctly describes the critical behavior for $D \geq D_{\text{UC}}$, we can conclude that the mean-field Landau theory for percolation, from Eq. (B3), is equivalent to percolation on (infinite-dimensional) homogeneous networks. Hence to introduce the idea of space in the infinite-dimensional world of networks we can use standard finite-dimensional results replacing D with D_{UC}.

Homogeneous networks are effectively infinite-dimensional systems and mean-field theory describes well the critical properties of percolation on top of them. However, the effective field theory associated with standard percolation fails to describe the critical behavior of systems with strong heterogeneity in the connectivities and a different theory is necessary [61]. In other words, while it is true that percolation on homogeneous networks is equivalent to the mean-field regime of percolation on finite-dimensional systems with $D > 6$, percolation on heterogeneous networks obeys the mean-field theory of a different system, in which a singular γ_d-dependent term appears in the effective Hamiltonian [61]. In particular, it has been shown [61] that the correct mean-field theory associated to percolation on heterogeneous networks with power-law degree distribution $p_k \sim k^{-\gamma_d}$ for large k is given by

$$V(\varphi) = \frac{1}{2} c_2 \varphi^2 + \frac{1}{3!} c_3 \varphi^3 + c_{\gamma_d} \varphi^{\gamma_d - 1}. \tag{B7}$$

We can interpret Eq. (B7) as the effective potential of a finite D-dimensional system with heterogeneous connectivity of each site. Therefore we can insert the critical exponents of the Landau theory associated to Eq. (B7), i.e., the exponents computed in this paper for heterogeneous networks, see Table II, into the Ginzburg criterion, (B6), to determine the upper critical dimension D_{UC}. Using the values of β, γ and ν in Table II, we find

$$D_{\text{UC}} = \begin{cases} 6, & \gamma_d \geq 4, \\ \frac{2(\gamma_d - 1)}{(\gamma_d - 3)}, & 3 < \gamma_d < 4. \end{cases} \tag{B8}$$

These values are in agreement with previous results in the literature [39] obtained via different arguments. For $2 < \gamma_d < 3$ Eq. (B8) would predict $D_{\text{UC}} < 0$. This reflects the fact that in no finite-dimensional space, of any arbitrary large dimension D, is the critical behavior described by the Landau theory of Eq. (B1) and Eq. (B7) equivalent to percolation on heterogeneous networks.

Appendix C: Calculations of the critical exponents for random graphs

In this section we develop in detail the computation of all critical exponents for random graphs with homogeneous or power-law degree distributions, following the general recipe described in Sec.III.

For homogeneous degree distributions, such as Erdős-Rényi (ER) or random regular networks, $g_0(z)$ and $g_1(z)$ are analytic functions, hence they don't have any singular part. In this case $b = \langle k^2 \rangle / \langle k \rangle - 1$ is the branching factor and $d = \langle k(k-1)(k-2) \rangle / \langle k \rangle$. The case of heterogeneous networks is provided by random graphs with a power-law degree distribution $p_k \sim k^{-\gamma_d}$, asymptotically for large k. In this case it is always possible to write[10]

$$g_0(1-x) \simeq 1 - \langle k \rangle x + \frac{1}{2}\langle k \rangle b x^2 + C(\gamma_d - 1)x^{\gamma_d - 1}, \tag{C1}$$

$$g_1(1-x) \simeq 1 - bx + \frac{1}{2}dx^2 + C(\gamma_d - 2)x^{\gamma_d - 2}, \tag{C2}$$

where b, d and $C(\cdot)$ depend on the value of γ_d. For the explicit values of these non-universal coefficients for graphs with $p(k) = (\gamma_d - 1)k_{\min}^{\gamma_d - 1}k^{-\gamma_d}$, within the continuous degree approximation, see Appendix G in [55].

1. The exponents β and δ

a. Homogeneous degree distributions

From Eqs. (49) and (51) it follows, using $1 - b\phi_c = 0$, that

$$-btm + \frac{1}{2}d\phi^2 m^2 \simeq h, \tag{C3}$$

which can be solved for m. In particular, at criticality the linear term vanishes and $m \sim h^{1/2}$, hence $\delta = 2$ and $E = \phi_c \langle k \rangle [2/(\phi_c^2 d)]^{1/2}$. At $h = 0$ instead, $m = 0$ for $t \leq 0$ and

$$m \simeq \left(\frac{2}{d\phi_c^3} \right) t, \tag{C4}$$

for $t > 0$, hence $1 - \Psi(t, 0) = 2\langle k \rangle t/(d\phi_c^2)$, i.e., $\beta = 1$ and $B = 2\langle k \rangle/(d\phi_c^2)$.

b. PL degree distributions

For $\gamma_d > 4$, $d = \langle k(k-1)(k-2) \rangle$ is finite and the singular part in $g_1(1-x)$ does not modify Eq. (C3), which is still valid. Hence $\beta = 1$, $\delta = 2$. In this case, the

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[10]There are logarithmic corrections for integer γ_d.

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degree heterogeneity is not strong enough to produce a scenario different from the homogeneous case.

For $3 < \gamma_d < 4$ instead, the singularity $x^{\gamma_d - 2}$ in $g_1(1-x)$ (Eq. (C2)) is dominant with respect to the term x^2 for small x. Eq. (49) becomes, using again $1 - b\phi_c = 0$,

$$C(\gamma_d - 2)\phi^{\gamma_d - 2}m^{\gamma_d - 2} - btm = h, \tag{C5}$$

from which it follows that $m \sim h^{1/(\gamma_d - 2)}$ at criticality, hence $\delta = \gamma_d - 2$, while $E = \langle k \rangle / [\phi_c C(\gamma_d - 2)]^{1/(\gamma_d - 2)}$. For $h = 0$ instead, $m = 0$ for $t \leq 0$, while

$$m \simeq \left[\frac{1}{C(\gamma_d - 2)\phi_c^{\gamma_d - 1}} \right]^{1/(\gamma_d - 3)} t^{1/(\gamma_d - 3)}, \tag{C6}$$

for $t > 0$, hence $\beta = 1/(\gamma_d - 3)$ and $B = \langle k \rangle [C(\gamma_d - 2)\phi_c^2]^{1/(3-\gamma_d)}$.

The case $2 < \gamma_d < 3$ requires more care. From Eq. (C2), the equation for m is, ($t = \phi$ since $\phi_c = 0$),

$$m(1 + |b|t) + C(\gamma_d - 2)t^{\gamma_d - 2}m^{\gamma_d - 2} = h, \tag{C7}$$

where now b is a negative constant and it is no longer the branching factor of the network (which is infinite). Setting $t = 0$, we get $m = h$, and from Eq. (52) $\delta = 1$. Note that in this case $\Psi(0, h) = e^{-h} \simeq 1 - h$ is analytical and no information about the coefficients of its power series for large s can be extracted from its behavior around the origin. Hence formally we have $\delta = 1$, $E = 1$, but these values are not associated to a singular behaviour, and it is no longer true that $\delta = 1/(\tau - 2)$, as it will be shown below when τ is evaluated.

For $h = 0$ instead

$$m \simeq [-C(\gamma_d - 2)]^{\frac{1}{(3-\gamma_d)}} t^{(\gamma_d - 2)/(3-\gamma_d)}, \tag{C8}$$

from which, using Eq. (52) with $\phi = t$, we get $\beta = 1 + (\gamma_d - 2)/(3 - \gamma_d) = 1/(3 - \gamma_d)$ and $B = \langle k \rangle [-C(\gamma_d - 2)]^{1/(3-\gamma_d)}$. For $2 < \gamma_d < 3$ this exponent describes the critical properties of the percolation strength for bond percolation: $\beta_{bond} = \beta$. As explained in the main text, for site percolation the presence of an additional multiplicative factor ϕ in the equation for P^∞, Eq. (5), implies instead [32] $\beta_{\text{site}} = 1 + \beta_{\text{bond}} = 1 + 1/(3 - \gamma_d) = (4 - \gamma_d)/(3 - \gamma_d)$.

2. The exponent α

a. Homogeneous degree distributions

From Eq. (55), expanding $\Psi(\phi, 0)$ for small t, using $m = m(t, 0)$ as in Eq. (C4) and $b\phi - 1 = bt$ we get

$$F(t) \simeq 1 - \frac{\langle k \rangle \phi_c}{2} - \frac{\langle k \rangle}{2}t + \frac{\langle k \rangle}{2}tm^2 - \frac{1}{3!}\langle k \rangle d\phi_c^3 m^3. \tag{C9}$$

Eq. (C9) shows that $F(t)$ approaches a constant value linearly in t as $t \to 0$. However, the exponent α accounts for the behavior of the singular part of $F(t)$, which is given by the terms involving m. For homogeneous distributions,

since $m \sim t$ for $t > 0$ [see Eq. (C4)] we get $F(t) \sim t^3$, hence $\alpha = -1$, in agreement with the scaling relations. In this case, the exponent governing the behavior of the singular part is, by chance, an integer. As shown below, this is not the case for PL degree distributions.

Note also that, for any $t < 0$, m is identically zero. This means that there is no singular contribution in $F(t)$ for $t < 0$. Hence the exponent α and the associated amplitude A_- are not defined in the nonpercolating phase. For this reason, we did not consider in our analysis two additional amplitude ratios containing A_- defined in [18].

b. PL degree distributions

From Eq. (55), expanding $\Psi(\phi, 0)$ for small t and using $m = m(t, 0)$ we find, neglecting the non-singular term $1 - \phi \langle k \rangle / 2$,

$$F(t) \simeq \frac{\phi \langle k \rangle}{2}(b\phi - 1)m^2 - \frac{\langle k \rangle d}{3!}\phi^3 m^3 + C(\gamma_d - 1)\phi^{\gamma_d - 1}m^{\gamma_d - 1}. \tag{C10}$$

For $\gamma_d > 4$, the singular term proportional to $m^{\gamma_d - 1}$ can be neglected. Since $m \sim t$ and $b\phi - 1 = bt$, as for homogeneous degree distributions we get $\alpha = -1$.

For $3 < \gamma_d < 4$, the only difference is that $m \sim t^{1/(\gamma_d - 3)}$ [see Eq. (C6)], hence at leading singular order

$$F(t) \sim t^{(\gamma_d - 1)/(\gamma_d - 3)}, \tag{C11}$$

from which $\alpha = (\gamma_d - 5)/(\gamma_d - 3)$.

For $2 < \gamma_d < 3$, $\phi = t$, and $m \sim t^{(\gamma_d - 2)/(3 - \gamma_d)}$ [see Eq. (C8)]. Now the leading singular term is the one proportional to $t^{\gamma_d - 1}m^{\gamma_d - 1}$ (since $\gamma_d - 1 < 2$), so that

$$F(t) \sim t^{(\gamma_d - 1)/(3 - \gamma_d)}, \tag{C12}$$

from which $\alpha = (7 - 3\gamma_d)/(3 - \gamma_d)$.

Again the exponent α and the associated amplitude A_- are not defined in the nonpercolating phase.

3. The exponent γ

a. Homogeneous degree distributions

For $t < 0$, at $h = 0$ we have $m = 0$. Hence from Eq. (57)

$$\langle s \rangle - 1 \simeq \frac{\phi_c \langle k \rangle}{-bt} = \phi_c^2 \langle k \rangle |t|^{-1},$$

giving $\gamma = 1$ and $C_- = \phi_c^2 \langle k \rangle$. As expected, C_- is a nonuniversal constant, since it depends on the details of the model. For $t > 0$ instead, m is given by Eq. (C4), and

$$\langle s \rangle - 1 \simeq \frac{\phi_c \langle k \rangle}{-tb + 2bt} = \phi_c^2 \langle k \rangle t^{-1},$$

from which again $\gamma = 1$ and the (universal) amplitude ratio is $C_+/C_- = 1$.

b. PL degree distributions

For $\gamma_d > 4$, the results obtained for homogeneous distributions hold.

For $3 < \gamma_d < 4$, still $m = 0$ for $t < 0$, from which we get again the same behavior approaching the threshold from below. For $t > 0$ instead, inserting Eq. (C6) into Eq. (57) we have at leading order

$$\langle s \rangle - 1 \simeq \frac{\phi_c \langle k \rangle}{1 - \phi[b - (\gamma_d - 2)C(\gamma_d - 2)\phi_c^{\gamma_d - 3}m^{\gamma_d - 3}]}$$

$$= \frac{\phi_c \langle k \rangle}{-bt + (\gamma_d - 2)bt} = \frac{\phi_c^2 \langle k \rangle}{(\gamma_d - 3)}t^{-1}.$$

In this case the exponent is $\gamma = 1$ and the critical amplitude ratio is $C_+/C_- = 1/(\gamma_d - 3)$.

For $2 < \gamma_d < 3$, inserting Eq. (C8) into Eq. (57) at leading order we get

$$\langle s \rangle - 1 \simeq \frac{t \langle k \rangle}{1 - t[-|b| - (\gamma_d - 2)C(\gamma_d - 2)t^{\gamma_d - 3}m^{\gamma_d - 3}]}$$

$$\simeq \frac{\langle k \rangle}{3 - \gamma_d}t.$$

This implies $\gamma = -1$ and $C_+ = \langle k \rangle/(3 - \gamma_d)$.

4. The exponent ν

a. Homogeneous degree distributions

Close to criticality, taking $\phi < \phi_c$, since $u = 1$ and $g_1'(1) = b$, from Eq. (71) it follows $\xi \simeq (1 - \phi b)^{-1/2} = b^{-1/2}(-t)^{-1/2}$. If $t > 0$, we have $u = 1 - m$ and $g_1'(1 - \phi m)$ can be expanded using Eq. (C4) obtaining $\xi \simeq (\phi b - 1)^{-1/2} = b^{-1/2}t^{-1/2}$. In both cases the prefactor is the same so that the amplitude ratio $\Xi_+/\Xi_- = 1$.

b. PL degree distributions

For any $\gamma_d > 3$, since b is finite an analogous argument also applies yielding $\nu = 1/2$ and $\Xi_+/\Xi_- = 1$. When $2 < \gamma_d < 3$, instead, since $\phi = t$, the denominator in Eq. (71) goes to a constant, while the numerator is of order t. As a consequence, $\xi \sim t^{1/2}$ and hence $\nu = -1/2$.

5. The amplitude ratio R_χ

a. Homogeneous degree distributions

Inserting in the definition

$$R_\chi = C_+ E^{-\delta} B^{\delta - 1} \tag{C13}$$

the value $\delta = 2$ and the expressions for C_+, E and B derived above for homogeneous distributions yields

$$R_\chi = 1, \qquad (C14)$$

the same universal value obtained on lattices above the upper critical dimension [18].

b.   PL degree distributions

For $\gamma_d > 4$ the amplitudes are the same of the homogeneous case so that $R_\chi = 1$. For $3 < \gamma_d < 4$ instead, $\delta = \gamma_d - 2$ and the expressions of the amplitudes C_+, B and E are different (see Sec. C 1 b). Inserting them into the definition yields $R_\chi = 1/(\gamma_d - 3)$, which has the same level of universality of the exponents, i.e., it depends only on γ_d but not on details of the degree distribution. For $2 < \gamma_d < 3$, since $\delta = 1$ and $E = 1$ we find $R_\chi = C_+ = \langle k \rangle/(3 - \gamma_d)$. Note that this value is not universal (it depends on $\langle k \rangle$), as it should be. However, we already pointed out that the point $t = 0$ for $2 < \gamma_d < 3$ is not a true critical point, and the exponent $\delta = 1$, together with the amplitude $E = 1$, coming from the analytic part of Ψ are not associated to a critical behavior. However, a nonrigorous argument can be developed to restore criticality and define a universal R_χ even in this case. The argument is as follows. The singular part of $\Psi(t, h)$ can be written, from Eq. (26), as $\{\Psi(t, h)\}_{\text{sing}} = |t|^{\frac{\tau - 2}{\sigma}} \mathcal{F}'_\pm(x)$, where $\mathcal{F}'_\pm$ is a scaling function of the variable $x = h/|t|^{1/\sigma}$. The critical behavior – the exponents β, δ and the associated critical amplitudes – can be recovered from the cases $x \ll 1$, i.e. $h \to 0$ and t small, and $x \gg 1$, i.e. $t \to 0$ and h small. For $\gamma_d > 3$ as well as for homogeneous degree distributions, $\mathcal{F}'_\pm(x) \sim 1$ for $x \ll 1$ and $\mathcal{F}'_\pm(x) \sim x^{\tau - 2}$ for $x \gg 1$. However, for $2 < \gamma_d < 3$, as no singular part of Ψ remains when $t \to 0$, the scaling behavior of $\mathcal{F}'_\pm(x)$ for $x \gg 1$ must be different, and must correspond to $\{\Psi(t \to 0, h)\}_{\text{sing}} \sim 0$. Working instead at $x = \mathcal{O}(1)$, i.e., small but finite h and t, from Eq. (C7) it follows, using the value of σ in Table II, at lowest order $m \simeq [-C(\gamma_d - 2)]^{1/(3 - \gamma_d)} x^{-1} h$. Plugged into Eq. (52), with $\phi = t = x^{-\sigma} h^\sigma$, we finally get

$$\Psi(\phi, h) \simeq 1 - \langle k \rangle [-C(\gamma_d - 2)]^{\frac{1}{(3 - \gamma_d)}} x^{\frac{1}{2 - \gamma_d}} h^{1/(\gamma_d - 2)},$$

from which $\delta' = 1/(\gamma_d - 2)$ and $E' = \langle k \rangle [-C(\gamma_d - 2)]^{\frac{1}{(3 - \gamma_d)}} x^{\frac{1}{2 - \gamma_d}}$. Using these values, from Eq. (C13), we get

$$R_\chi = \frac{1}{3 - \gamma_d} [-C(\gamma_d - 2)]^{\frac{1}{\gamma_d - 3}} x.$$

Fixing $x = [-C(\gamma_d - 2)]^{\frac{1}{3 - \gamma_d}}$ we then find the universal value $R_\chi = 1/(3 - \gamma_d)$. Thus for $2 < \gamma_d < 3$ we can formally study the singular behavior of Ψ in the limit $t \to 0$, $h \to 0$ but keeping x fixed as above. Note that δ' is also in agreement with the scaling relation $\tau = 2 + 1/\delta'$. The physical motivation for the choice of x remains to be understood.

6.   The exponents σ and τ

a.   Homogeneous degree distributions

Using Eq. (D3) and (D4) in Appendix D, the solution of the characteristic equation is $\eta = t/(d\phi_c^3)$. Hence, since $u^* = 1 - \eta$

$$\rho = \psi(u^*) \simeq 1 + \frac{1}{d\phi_c^4} t^2, \qquad (C15)$$

from which, using the definition of s_ξ in Eq. (74), $\sigma = 1/2$. For $t > 0$, ρ is far away from the unitary disk: the presence of the GC strongly hinders the formation of other large clusters, introducing an exponential cutoff [33]. At criticality $\phi = \phi_c$, $t = 0$, the singularity ρ touches the unitary disk $\rho = 1$, and s_ξ diverges. For $t < 0$, in the phase without a GC, the singularity ρ is again pushed away from the unitary circle since, also in this case, only finite clusters can form.

In order to derive τ we must invert Eq. (80). From the expansion of $\psi(u)$ around u^*, setting $\epsilon = u^* - u$, see Eq. (D1), we obtain

$$\rho - z = \psi(u^*) - \psi(u) \simeq \frac{1}{2} |\psi''(u^*)| \epsilon^2,$$

which can be inverted[11] to give $\epsilon \sim (\rho - z)^{1/2}$, yielding

$$H_1(z) \simeq u^* - A_1 (\rho - z)^{1/2}, \qquad (C16)$$

with A_1 a positive constant. This result implies

$$\pi_s(\phi) \simeq q_0 s^{-3/2} e^{-s/s_\xi} = q_0 s^{-3/2} f(q_1 s^{1/2} t), \qquad (C17)$$

from which we find that $\tau = 5/2$ and the universal scaling function is Gaussian $f(x) = e^{-x^2}$. As noted in [34], this behavior is a consequence of the central limit theorem. Note that Eq. (C17) holds for any $|t| \ll 1$ (see Fig. 1(a)).

b.   PL degree distributions

For power-law degree-distributed networks it is necessary to analyze separately what happens above and below the threshold and consider various ranges of γ values.

a. $\gamma_d > 4$

$t > 0$: For positive $t \ll 1$, the expansions of $\psi(u^*)$ and $\psi'(u^*)$ in Eqs. (D5), (D6) are equivalent to Eqs. (D3),(D4), hence we recover again Eq. (C15), and $\sigma = 1/2$. Furthermore, from Eq. (D1) we get for $s \gg 1$

$$n_s(t) \sim s^{-5/2} e^{-s/s_\xi}. \qquad (C18)$$

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[11]The inversion produces an ambiguity in the determination of the sign. The sign is taken in order to have an increasing $H_1(z)$ for $z \to \rho^-$.

$t = 0$: At criticality $\psi(u)$ is singular for $u = 1$ and its expansion is now given by Eq. (D7). However, since $\epsilon^{\gamma_d-2} \ll \epsilon^2$, we get the same asymptotic behavior as in Eq. (C18), even though γ_d-dependent subleading corrections are present. We can conclude that $\tau = 5/2$.

$t < 0$: In this case the convergence radius of $H_1(z)$ remains $\rho = 1$, since the characteristic equation $\psi'(u^*) = 0$ admits a solution only for $t > 0$. Hence $s_\xi = \infty$. In the expansion of $\psi(u)$ around 1 (see Eq. (D8)) if the condition

$$-t \ll \epsilon \qquad (C19)$$

holds, then the term proportional to ϵ can be neglected compared to the term ϵ^2. In such a case the inversion of Eq. (D9) yields

$$\epsilon \simeq (1-z)^{1/2}. \qquad (C20)$$

Inserting this into Eq. (C19) consistency is found as long as $1 - z \gg (-t)^2 = (-t)^{1/\sigma}$. Hence n_s decays with exponent $\tau = 5/2$ for $s \ll s_0 \sim (-t)^{-1/\sigma}$. When this condition is not fulfilled, then at leading order from Eq. (D9) $\epsilon \sim (1-z)$ which, replaced back into Eq. (D9), gives

$$1 - H_1(z) = \epsilon = c_1(1-z) + \text{other reg. terms} + c_2(1-z)^{\gamma_d-2}. \qquad (C21)$$

Since the first term in the r.h.s. is regular, this expression implies, via Eq. (73), that

$$n_s(t) \sim s^{-\gamma_d}. \qquad (C22)$$

In summary, we find the critical decay $n_s \sim s^{-5/2}$ for $1 \ll s \ll s_0$, with $s_0 \sim |t|^{-1/\sigma}$; for $s \sim s_0$, $n_s(t)$ exhibits an exponential cutoff and then, for $s \gg s_0$, a tail $s^{-\gamma_d}$ (see Fig. 1(b)).

Hence, even though the correlation size s_ξ is formally infinite for $t < 0$, there is another characteristic size s_0 such that the cluster size distribution can be written as

$$n_s(t) \simeq q_0 s^{-\tau} f_-(q_1 s^\sigma t) + q_2 s^{-\gamma_d}, \qquad (C23)$$

with $f_-(x) \ll 1$ for $|x| \ll 1$, showing that the scaling ansatz Eq. (24) must be modified.

Two distinct mechanisms contribute to the cluster size distribution tail in Eq. (C23). The first is the collective phenomenon of percolation, where finite clusters merge into larger and larger clusters as $t \to 0^-$. This mechanism is responsible for the critical properties such as the divergence of $\langle s \rangle$. The second is the presence for any finite $\phi > 0$ of a small but finite fraction of nodes with arbitrarily high degree, surrounded by active neighbors, as already noticed in [33]. The contribution of the first mechanism is larger than the second, all the more so when criticality is approached, because $\gamma_d > \tau$, so that only the first tail remains as s_0 diverges. Therefore, even if Eq. (24) is not strictly obeyed, the correction in Eq. (C23) can be neglected if we are interested in critical properties.

b. $3 < \gamma_d < 4$

$t > 0$: For positive $t \ll 1$, in Eq. (D6) the subleading term proportional to η can be neglected. Inserting the solution of the characteristic equation into Eq. (D5) we obtain

$$\rho \simeq 1 + \frac{\phi_c^{-(\gamma_d-2)/(\gamma_d-3)}}{[(\gamma_d - 2)C(\gamma_d - 2)]^{1/(\gamma_d-3)}} t^{(\gamma_d-2)/(\gamma_d-3)}, \qquad (C24)$$

from which $\sigma = (\gamma_d - 2)/(\gamma_d - 3)$. Since $t > 0$, Eq. (D1) can be used again, from which we find again the scaling in Eq. (C18). However, now Eq. (D1) is valid conditioned on $\epsilon \ll 1 - u^* = \eta$, because of the presence of the singularity of $G_1(u)$ in $u = 1$[12]. If we consider the inverse function $H_1(z)$ (see Fig. 5) the inversion works only far from the singularity in $z = 1$, which implies $\rho - z \ll \rho - 1 = \psi(u^*) - 1 \ll 1$. Hence the asymptotic behavior obtained from Eq. (D1) holds provided $s \gg s_*$, where

$$s_* = \frac{1}{\psi(u^*) - 1}. \qquad (C25)$$

In the opposite regime $\psi(u^*) - 1 \ll \rho - z \ll 1$, that is for $1 \ll s \ll s_*$, it is not possible to distinguish u^* from 1, hence ρ from 1, and the validity of the Taylor expansion breaks down. We can use instead the asymptotic expansion close to 1 – see Eq. (D7) – rather than the Taylor expansion close to u^*. In this regime, we observe the critical behavior, see the case $t = 0$ below. Note that since $s_\xi = 1/\log(\rho) = 1/\log(1 + \psi(u^*) - 1)$, and $\log(1 + x) \le x$, we have $s_* \le s_\xi$. In particular, $s_* \sim s_\xi$ for $t \ll 1$.

$t = 0$: In this case Eq. (D7) must be used, where now $\epsilon^2 \ll \epsilon^{\gamma_d-2}$. Inverting Eq. (80) implies $\epsilon \sim (\rho - z)^{1/(\gamma_d-2)}$ which leads to

$$n_s(0) \sim s^{-[2+1/(\gamma_d-2)]}, \qquad (C26)$$

i.e., $\tau = 2 + 1/(\gamma_d - 2)$.

There is an apparent contradiction: the asymptotic scaling with exponent $5/2$ for any $t > 0$ is different from what occurs for $t = 0$, where $\tau = 2 + 1/(\gamma_d-2) > 5/2$. This is at odds with what happens for homogeneous systems (and for $\gamma_d > 4$) where the value of the exponent is the same and only the cutoff scale s_ξ changes as criticality is approached. The conundrum is solved by the presence of the crossover scale s_* (discussed above for $t > 0$), which separates a decay as $s^{-\tau}$ for $1 \ll s \ll s_*$ from the asymptotic decay $s^{-5/2} e^{-s/s_\xi}$. When $t \to 0$ the crossover

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[12] Note that such a singularity is present also for $\gamma_d > 4$, but in that case the leading term produced by the inversion is still given by a square root singularity, as already noted above. The singular behaviour close to and at criticality are the same and no crossover is observed.

size diverges, $s_* \sim t^{-1/\sigma}$, while still remaining smaller than the correlation size $s_* \leq s_\xi$ This does not imply a violation of the scaling ansatz. Eq. (24) is still valid but, analogously to what happens in the case $2 < \gamma_d < 3$ (see below), the scaling function $f_+(x)$ has a nonstandard form: it is constant for $x \ll x^* = s_*/s_\xi$; it vanishes exponentially for $x \gg 1$; it behaves as $x^{(4-\gamma_d)/[2(\gamma_d-3)]}$ in the interval between x^* and 1.

$t < 0$: The phenomenology is perfectly analogous to the case $\gamma_d > 4$: $n_s(t)$ decays as $s^{-\tau}$ for $1 \ll s \ll s_0$, where s_0 diverges as $|t|^{-1/\sigma}$ as criticality is approached. For $s \gg s_0$ instead $n_s(t) \sim s^{-\gamma_d}$. The only difference with respect to the case $\gamma_d > 4$ is that now $\tau = 2 + 1/(\gamma_d - 2)$, because the term competing with the linear term in Eq. (D8) is ϵ^{γ_d-2} instead of ϵ^2.

c. $2 < \gamma_d < 3$

In this range of γ_d values, only the phase $t > 0$ exists and $\phi = t$. We can solve again the characteristic equation using the expansion Eq. (D6) for small η, and from Eq. (D5) obtain

$$\rho = 1 - C(\gamma_d-2)[-(\gamma_d-2)C(\gamma_d-2)]^{1/(3-\gamma_d)}t^{(\gamma_d-2)/(3-\gamma_d)}, \tag{C27}$$

from which $\sigma = (3-\gamma_d)/(\gamma_d-2)$. This result implies the existence of a diverging correlation size $s_\xi \sim t^{-1/\sigma}$ even if for $t = 0$ no critical behavior occurs.

The asymptotic decay of $n_s(t)$ is again determined by the singularity produced by the inversion of $\psi(u)$ close to u^*

$$H_1(z) \simeq u^* - c_1|\psi''(u^*)|^{-1/2}(\rho - z)^{1/2}. \tag{C28}$$

Hence n_s exhibits a tail $s^{-5/2}$ with an exponential cutoff at s_ξ. As in the case $3 < \gamma_d < 4$, this expression holds provided that u is close enough to u^*, hence the asymptotic behavior extracted from it holds only for $s \gg s_* \sim t^{-1/\sigma}$. For $1 \ll s \ll s_*$, we can use the expansion in Eq. (D2), in perfect analogy to the cases of PL networks for $t < 0$ considered above, which gives

$$H_1(z) \simeq 1 - c_2\phi^{\gamma_d-2}(1-z)^{\gamma_d-2}, \tag{C29}$$

corresponding to a power law scaling of n_s with exponent γ_d. Following the analogy with the case $3 < \gamma_d < 4$, we may be tempted to identify $\tau = \gamma_d$: as $t \to 0$, only the decay with exponent γ_d survives. However it has to be noted that there are multiplicative factors depending on ϕ. While they do not play any essential role for $\gamma_d > 3$, as for $t \ll 1$ they are simply $\phi_c > 0$ at leading order, the situation now requires a more careful analysis. From Eq. (D2), and the expression of $\eta = 1 - u^*$ in terms of t, we see that $|\psi''(u^*)| \sim t^{(\gamma_d-3)/(\gamma_d-2)} = t^{-1/\sigma}$. The concavity of $\psi(u)$ in u^* diverges as $t \to 0$. This means that the peak around u^* gets not only smaller, but also narrower, as $t \to 0$. This reflects the fact that $\psi(u) = u$ is analytic

for $\phi = 0$ and there is no peak at all. Hence a prefactor proportional to $t^{1/\sigma}$ multiplies the decay of n_s for large s. Similarly, the preasymptotic scaling for $1 \ll s \ll s_*$ contains a prefactor t^{γ_d-2}. Finally, an additional factor t in $n_s(t)$ comes from the mapping between the singular behavior of H_1 and the singular behavior of H_0, Eq. (72), since $B_0 = \rho G_0'(u^*)A_1 \sim t$. Putting all these results together

$$n_s(t) \simeq \begin{cases} c_a t^{\tilde{a}} s^{-\gamma_d}, & 1 \ll s \ll s_*, \\ c_b t^{\tilde{b}} s^{-\frac{5}{2}} e^{-s/s_\xi}, & s \gg s_*, \end{cases} \tag{C30}$$

where $s_\xi \sim s_* \sim t^{-1/\sigma}$, $\sigma = (3-\gamma_d)/(\gamma_d-2)$, $\tilde{a} = \gamma_d - 1$, $\tilde{b} = 1/(2\sigma) + 1$, and c_a, c_b are constants independent of t. The expression in Eq. (C30) can be cast in a scaling form like Eq. (24), in which the dependence on t is absorbed in the scaling function. Hence

$$n_s(t) \simeq s^{-\tau} f_+(q_1 s^\sigma t), \tag{C31}$$

where $\tau = 2 + 1/(\gamma_d - 2)$ and the scaling function $f_+(x)$ behaves now as

$$f_+(x) \simeq \begin{cases} (c_a/q_1^{\tilde{a}})x^{\tilde{a}}, & x \ll x_*, \\ (c_b/q_1^{\tilde{b}})x^{\tilde{b}} e^{-x^{1/\sigma}}, & x \gg x_*, \end{cases} \tag{C32}$$

with $x_* = q_1\tilde{q}$. Note that if $x_* > 1$, only the first scaling, followed by the exponential cutoff for $x \sim 1$, is observable. This form of the scaling function with $f_+(x) \to 0$ as $x \to 0$ is unusual, but note that a similar behavior of the scaling function appears also for the linear chain (see Appendix A), where n_s can be put in a scaling form like Eq. (24) with $\tau = 2$ and $f_-(x) \sim x^2 \to 0$ for $x \to 0$.

In conclusion, the analysis developed here shows that for $2 < \gamma_d < 3$ the cluster size distribution obeys the scaling form in Eq. (24) with $\sigma = (3-\gamma_d)/(\gamma_d-2)$, $\tau = 2 + 1/(\gamma_d - 2)$ but with a scaling function $f_+(x) \sim x^{\gamma_d-1}$ for $x \to 0$. This implies that if the decay of $n_s(t)$ with s is studied at fixed t (as it is usually the case) the exponent measured is not τ but γ_d with a prefactor decreasing as $t \to 0$.

7.   The exponent $\bar{\nu}$

For power-law degree-distributed networks with $\gamma_d > 3$, $b(k_c) \simeq b_\infty - C_2 k_c^{-(\gamma_d-3)}$, where $b_\infty = b(k_c = \infty)$ and C_2 is a positive constant. Inserting this expression into Eq. (82) we obtain, at lowest order,

$$\phi_c(N, k_c) \simeq b_\infty^{-1} + \frac{C_2}{b_\infty^2} k_c^{-(\gamma_d-3)} + b_\infty^{-1} C_1 N^{-1/3}. \tag{C33}$$

If degrees are sampled from a distribution with a hard (structural) cutoff $k_{\max}(N) \sim N^{1/\omega}$ the actual cutoff in a realization of the network is the minimum between the structural cutoff and the natural one, $N^{1/(\gamma_d-1)}$. This

implies that in full generality $k_c(N) \sim N^{1/\tilde{\omega}}$ with $\tilde{\omega} = \max[\omega, \gamma_d - 1]$ [13].

Then, considering Eq. (C33), the scaling of the effective threshold depends on whether $(\gamma_d - 3)/\tilde{\omega}$ is larger or smaller than $1/3$.

For $\tilde{\omega} \leq 3(\gamma_d - 3)$ the second to last term in Eq. (C33) decays faster than the last one; as a consequence $\bar{\nu} = 3$ as in the fully homogeneous case: the scaling of the effective threshold $\phi_c(N)$ is governed by bona fide critical fluctuations. This occurs for $\gamma_d \geq 4$ and $\omega \leq 3(\gamma_d - 3)$.

If instead $\tilde{\omega} \geq 3(\gamma_d - 3)$ the role of the two contributions is reversed: in this case, for asymptotically large N the second to last contribution in Eq. (C33) dictates the scaling of $\phi_c(N)$, yielding $\bar{\nu} = \tilde{\omega}/(\gamma_d - 3)$. For $3 < \gamma_d < 4$ and $\omega \leq \gamma_d - 1$ (so that $\tilde{\omega} = \gamma_d - 1$) this implies $\bar{\nu} = (\gamma_d - 1)/(\gamma_d - 3)$. Otherwise $\tilde{\omega} = \omega$ and $\bar{\nu} = \omega/(\gamma_d - 3)$, see Fig. 3. The fact that $\bar{\nu}$ depends on the parameter ω, which governs how the thermodynamic limit is reached, immediately implies that the hyperscaling relation (43) is violated as the left hand side cannot depend on ω. The violation of (43) can be traced back to the breakdown of Eq. (42). In its turn this is interpreted as follows: the slow growth of the maximum degree k_c with N introduces an additional cutoff smaller than s_ξ in the cluster size distribution.

For $2 < \gamma_d < 3$ the picture is completely different. The argument based on the mapping to an appropriate field-theory puzzlingly yields a negative upper critical dimension. It is more useful to observe that the negative value of the exponent $\nu = -1/2$ indicates that the correlation length does not diverge for $\phi \to \phi_c = 0$. The threshold $\phi_c = 0$ is not a true critical point. If one insists on determining numerically the behavior of the correlation length ξ as a function of ϕ, one finds a peak in a position $\phi_c(N) \approx b(k_c(N))^{-1}$ but with an amplitude that *decreases* with the system size ($\gamma = -1$). In principle it is possible to define in this way an exponent $\bar{\nu} = \omega/(3 - \gamma_d)$, but with this value hyperscaling relation (43) is violated.

The rich finite size scaling phenomenology just described is summarized in Fig. 3.

The exponent θ is readily obtained from Eq. (40). In the region where hyperscaling holds $\theta = 2/3$ for $\gamma_d > 4$ and $\theta = (\gamma_d - 2)/(\gamma_d - 1)$ for $3 < \gamma_d < 4$. These values coincide with those obtained using Eq. (42). Instead, when hyperscaling is violated, we find: $\theta = 1 - (\gamma_d - 3)/\omega$ for $\gamma_d > 4$; $\theta = 1 - 1/\omega$ for $\gamma_d < 4$. These values do not coincide with those obtained from Eq. (42): $1/(\sigma\bar{\nu}) = 2(\gamma_d - 3)/\omega$ for $\gamma_d > 4$; $1/(\sigma\bar{\nu}) = (\gamma_d - 2)/\omega$ for $\gamma_d < 4$. Table III presents all these results as a function of ω for the different ranges of γ_d.
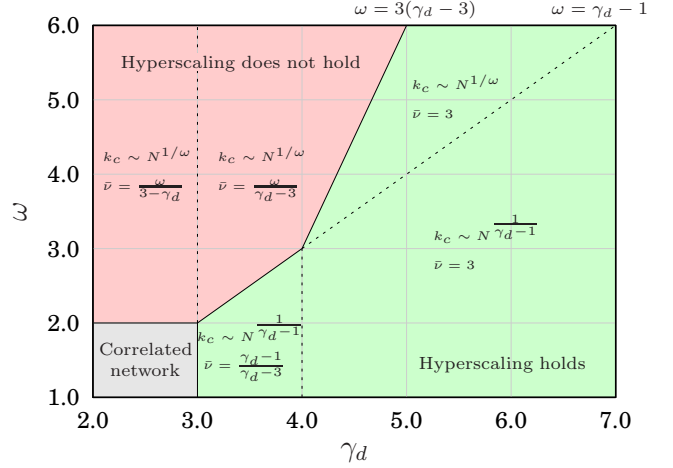
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[13] The exponent $\tilde{\omega}$ must be larger than or equal to 2, otherwise the network is necessarily correlated [46]

FIG. 3. Finite size scaling behavior as a function of γ_d and ω.

Appendix D: Asymptotic expansions of $\psi(u)$ and $\psi'(u)$

In this Appendix we write the asymptotic expansions of $\psi(u^*)$ and $\psi'(u^*)$ [see Eq.(75)] for u^* close to 1 and also the expansion of $\psi'(u)$ for u close to u^*. We remind that $\psi(u) = u/G_1(u)$, where $G_1(u) = g_1(\phi u + 1 - \phi)$ and that u^* is the solution of the characteristic equation $\psi'(u^*) = 0$. These expansions are needed in Appendix C to determine the form of the cluster size distribution $n_s(\phi)$. Indeed, in order to determine σ we need s_ξ for $|t| \ll 1$, which can be evaluated, via Eq. (74), using the expression of $\rho = \psi(u^*)$. The condition $|t| \ll 1$ implies $u^* = 1 - \eta(t)$ with $\eta \ll 1$. Moreover, to evaluate τ we must invert Eq. (80) to find $\epsilon = u^* - u$ as a function of $\rho - z$ and hence the exponent $a_1 = a_0$, to be inserted in Eq. (73). In some cases this inversion process is far from trivial and is characterized by crossovers between competing behaviors, giving rise to preasymptotic effects in the shape of $n_s(\phi)$, see Fig. 1. The main formulas we are going to use are the Taylor expansion of $\psi(u)$ close to u^*, setting $\epsilon = u^* - u$,

$$\psi(u) \simeq \psi(u^*) - \frac{1}{2}|\psi''(u^*)|^2\epsilon^2, \quad (D1)$$

where we assume $|\psi''(u^*)| \neq 0$, and the asymptotic expansion for u close to 1, setting $\varepsilon = 1 - u$,

$$\psi(u) \simeq 1 - (1 - \phi b)\varepsilon - \frac{1}{2}d\phi^2\varepsilon^2 - C(\gamma_d - 2)\phi^{\gamma_d - 2}\varepsilon^{\gamma_d - 2}. \quad (D2)$$

1. Homogeneous degree distributions

For homogeneous distributions $G_1(z)$ is analytic and both $\psi(u^*)$ and $\psi'(u^*)$ can be Taylor expanded [see Fig. 4]. In practice this amounts to use Eq. (D2) without the singular term on the r.h.s.. Evaluating it in $u^* = 1 - \eta(t)$, that is $\varepsilon = \eta(t)$, we have for $|t| \ll 1$, $\eta \ll 1$, keeping only
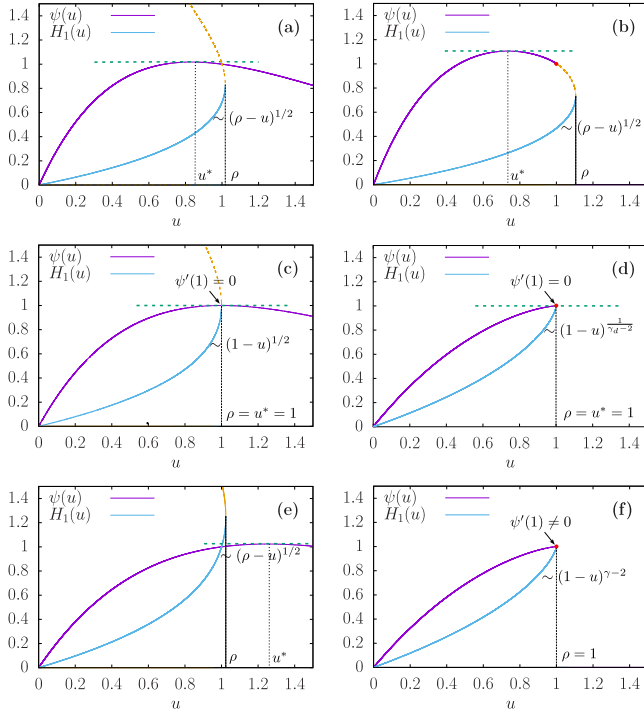
FIG. 4. Various plots of the functions $\psi(u)$ (purple) and its inversion $H_1(u)$ (blue). The yellow dashed line is the non-physical part produced by the inversion of $\psi(u)$. In the different rows the cases $t > 0$, $t = 0$ and $t < 0$, respectively, are considered. In the left column results are for homogeneous networks, in particular ER graphs, in the right column for power-law network with $\gamma = 3.5$. The asymptotic scaling of H_1 for $\rho - u \ll 1$ is reported. Dashed lines serve as guide to the eye to indicate when the derivative of $\psi(u)$ vanishes (green dashed), in u^* (black dotted), and the corresponding convergence radius ρ of H_1 (black dashed). The red dot signals the presence of a singularity at $u = 1$ in $\psi(u)$, but not necessarily in H_1, see panel (b). While for homogeneous networks the inversion always produces a square-root singularity, i.e. for $t > 0$ (a), $t = 0$ (c), $t < 0$ (e), for power-law networks the inversion produces three different exponents in (b), (d), and (f) corresponding to different scaling asymptotes for $n_s(t)$. Note that the exponents in (d) and (f) look similar, but they are different, as explained in detail in Fig. 5.

the lowest orders in η

$$\psi(u^*) \simeq 1 + bt\eta, \tag{D3}$$

$$\psi'(u^*) \simeq -bt + d\phi_c^2\eta. \tag{D4}$$

The behavior of $\psi(u)$ for u close to u^*, setting $\epsilon = u^* - u$, is instead given by Eq. (D1). Note that these expansions are valid for any t, both positive and negative.

2. Power-law degree distributions

For power-law degree distributions more care must be taken. The main problem is that $\psi(u)$ is no longer analytic in $u = 1$, because of the singularity of $G_1(u)$ in $u = 1$.
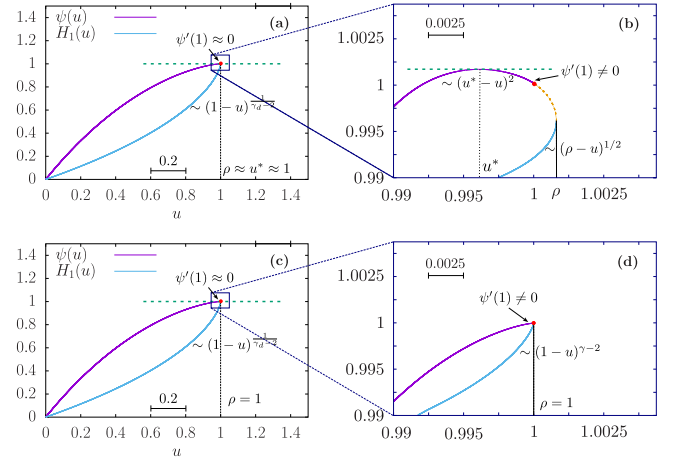

FIG. 5. The origin of the crossovers in the cluster size distributions for power-law networks. In this case, $\gamma_d = 3.5$ is considered. In (a), the functions $\psi(u)$ and its inverse $H_1(u)$ are plotted for $t > 0$ but $t \ll 1$. In particular, on the scale in (a) it is impossible to detect the difference between ρ, u^* and 1, and the system seems to be at criticality $t = 0$ and $\rho = 1$, with a singular behaviour associated with the critical exponent τ. On a closer look (b), however – hence on a smaller scale – we can distinguish u^* from 1 and a square-root singularity determines the behaviour close to the singularity at $\rho > 1$. The asymptotic scaling of n_s for large $s \gg s_*$ is then determined by the inversion in (b), but for $1 \ll s \ll s_*$ the picture in (a) is valid and determines the preasymptotic scaling with exponent τ. Similarly, in (c) the situation $t < 0$ but $|t| \ll 1$ is considered. At first sight, the system seems to be at criticality $t = 0$ in (c), where $\psi'(1) = 0$. However, on a closer look (d) we realize that $t < 0$ since $\psi'(1) \neq 0$. Again, the picture in (c) determines the scaling of n_s for $1 \ll s \ll s_0$, while asymptotically we must look closer (d) to find the scaling $n_s \sim s^{-\gamma_d}$ for $s \gg s_0$.

Let us discuss separately the three cases $t > 0$, $t = 0$, and $t < 0$. In the first case, $u^* < 1$ and $\psi(u)$ can still be Taylor expanded around u^*, using Eq. (D1). However, as it will be shown, the validity of such expansion breaks down as t gets small. For $t \leq 0$ instead, the singular behavior of $G_1(u)$ around 1 must be considered and hence Eq. (D2) must be used.

a. Expansions for $t > 0$

In this case, setting $u^* = 1 - \eta(t)$ we can write, at lowest order in η, using Eq. (D2)

$$\psi(u^*) \simeq 1 - (1 - \phi b)\eta - C(\gamma_d - 2)\phi^{\gamma_d - 2}\eta^{\gamma_d - 2}, \tag{D5}$$

$$\psi'(u^*) \simeq (1 - \phi b) + d\phi^2\eta + (\gamma_d - 2)C(\gamma_d - 2)\phi^{\gamma_d - 2}\eta^{\gamma_d - 3}. \tag{D6}$$

Note that these expansions are defined only for $\eta > 0$, hence for $t > 0$, in contrast to the case of homogeneous degree distributions, where $\eta < 0$ for $t < 0$.

Expanding $\psi(u)$ for u close to u^* we get again Eq. (D1)

since, provided that $u^* < 1$, we can Taylor expand $G_1(u)$ in a neighborhood of u^*.

However, Eq. (D1) is valid conditioned on $\epsilon \ll 1 - u^* = \eta$, because of the presence of the singularity of $G_1(u)$ in $u = 1$ (see Fig. 4(b)). As shown in Appendix C, this feature leads to preasymptotic effects in the behavior of $n_s(t)$.

b. Expansions for $t = 0$

At criticality $u^* = 1$ and the local behavior of $\psi(u)$ for u close to u^* now strongly depends on the value of γ_d. We can no longer Taylor expand $\psi(u)$ (see Fig. 4(d)) and the asymptotic expansion in Eq. (D2) must be used instead, obtaining

$$\psi(u) \simeq 1 - \frac{1}{2}d\phi^2\epsilon^2 - C(\gamma_d - 2)\phi^{\gamma_d - 2}\epsilon^{\gamma_d - 2}, \quad \text{(D7)}$$

where $d = \langle k(k-1)(k-2)\rangle/\langle k\rangle > 0$ for $\gamma_d > 4$, while d is a negative constant for $\gamma_d < 4$ when $\langle k^3\rangle = \infty$. As explicitly discussed in Appendix C, the value of the exponent τ depends on which of the two terms depending on ϵ in Eq. (D7) is the leading order.

c. Expansion for $t < 0$

In this case, the maximum of $\psi(u)$ is now reached in $u^* = 1$, not because the first derivative vanishes but because $\psi(u)$ is defined only for $u \leq 1$ (see Fig. 4(f)). Hence, setting $1 - b\phi = -bt$,

$$\psi(u) \simeq 1 + bt\epsilon - \frac{1}{2}d\phi^2\epsilon^2 - C(\gamma_d - 2)\phi^{\gamma_d - 2}\epsilon^{\gamma_d - 2}, \quad \text{(D8)}$$

that can be rewritten as

$$1 - z \simeq -bt\epsilon + \frac{1}{2}d\phi^2\epsilon^2 + C(\gamma_d - 2)\phi^{\gamma_d - 2}\epsilon^{\gamma_d - 2}. \quad \text{(D9)}$$

The effect of the competition between the different contributions in these expressions on the behavior of $n_s(t)$ is nontrivial, as reported in Appendix C

Appendix E: Connection with Kryven's results

In Ref. [34] Kryven presented a general theory to determine the size distribution of connected components in an infinite network built according to the configuration model and specified by an arbitrary degree distribution. The expressions presented there can be used to determine asymptotic properties of the cluster size distribution for a percolation process on the same network by considering that, under dilution, the tail of the degree distribution remains a power-law with the same exponent. In particular, if $q_k = Ak^{-(\gamma_d - 1)}$ is the excess degree distribution of the network with $\phi = 1$, after the dilution $q_k(\phi) \sim A_\phi k^{-(\gamma_d - 1)}$ for large k, where $A_\phi = A\phi^{\gamma_d - 2}$. In other words, dilution affects the small k part of the degree distribution, while the tail remains with the same exponent, only with a different prefactor. All the nontrivial behaviors derived by means of the generating function approach and in particular in Fig. 1, can be worked out from the expressions in Table II in [34]. In particular, it is interesting to see this connection for the case $2 < \gamma_d < 3$. From Table II in [34] and the preasymptotic scaling of the cluster size distribution (see the unnumbered equation below Table II) we find, using the crucial observations that $s_{\text{Kryven}} \sim t^{\gamma_d - 2}$ and $\mu_{1,\text{Kryven}} = t\langle k\rangle$,

$$\pi_s(t) \sim \begin{cases} q_a t^{\tilde{a}} s^{-(\gamma_d - 1)}, & 1 \ll s \ll s_*, \\ q_b t^{\tilde{b}} s^{-\frac{3}{2}} e^{-s/s_\xi}, & s \gg s_*, \end{cases} \quad \text{(E1)}$$

where $s_\xi \sim t^{-1/\sigma}$, $\sigma = (3 - \gamma_d)/(\gamma_d - 2)$, $\tilde{a} = \gamma_d - 1$, $\tilde{b} = 1/(2\sigma) + 1$, and q_a, q_b are constants independent of t. This is in perfect agreement with Eq. (C30). Similar arguments can be developed for $\gamma_d > 3$, recovering the rich picture of crossover phenomena described here.

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