ADELIC C^*-CORRESPONDENCES AND PARABOLIC INDUCTION

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Abstract. In analogy with the construction of representations of adelic groups as restricted products of representations of local groups, we study restricted tensor products of Hilbert C^*-modules and of C^*-correspondences. The construction produces global C^*-correspondences from compatible collections of local C^*-correspondences. When applied to the collection of C^*-correspondences capturing local parabolic induction, the construction produces a global C^*-correspondence that captures adelic parabolic induction.

Contents

1. Introduction

Historically, C^*-algebras and representation theory share a common ancestry [15, 18, 23]. In the 1960's, representation theory of reductive groups over rings of adeles of number fields has started to acquire a central position in number theory and automorphic forms theory, and this motivated some C^*-algebraists to consider restricted tensor products of C^*-algebras [17, 5]. Recent years have seen several striking applications of the notion of C^*-correspondences from the theory of C^*-algebras to the theory of representations of reductive groups over local fields [6, 7, 8, 21]. Motivated by these developments, we explore notion of restricted tensor product in the setting of C^*-correspondences with applications to representation theory of adelic reductive groups.

The main results of this paper are of a technical nature showing how to form the restricted tensor product of C^*-correspondences, followed by an example placing parabolic induction for adelic reductive groups in the context of C^*-correspondences. In more detail, the technical motivation for the paper is the following question. Consider two collections of C^*-algebras $(A_v)_{v\in I}$ and $(A'_v)_{v\in I}$ and a collection of (A_v, A'_v)-correspondences $(X_v)_{v\in I}$.

How do we patch these collections together in an infinite restricted tensor product? We answer this question in this paper by enhancing the C^*-algebras with appropriate projections $p_v \in A_v$, $p'_v \in A'_v$ and the modules distinguished vectors $x_v \in X_v$, satisfying certain compatibility conditions. The restricted tensor product $\bigotimes'_{v\in I}(X_v, x_v)$ can then be constructed as an invariant of the collection $(X_v, x_v)_{v\in I}$, and becomes a C^*-correspondence over for the pair of restricted tensor product C^*-algebras $(\bigotimes'_{v\in I}(A'_v, p'_v), \bigotimes'_{v\in I}(A_v, p_v))$. Without the extra data of projections and distinguished vectors, a restricted tensor product would not make sense.

To illustrate the utility of our construction, we turn to representation theory which was our main motivation as mentioned earlier. Suppose that G is a reductive algebraic group defined over the rationals, and $P \subseteq G$ a rational parabolic subgroup with Langlands decomposition $P = LN$. In [6], Clare constructs a $(C^*(G(\mathbb{R})), C^*(L(\mathbb{R})))$-correspondence which implements the foundational procedure of parabolic induction in the representation theory of real reductive groups. First, we apply the method of [6] to the case non-archimedean setting and obtain $(C^*(G(\mathbb{Q}_p)), C^*(L(\mathbb{Q}_p)))$-correspondences capturing parabolic induction in the p-adic reductive group case. Then we show in Theorem 6.10 (see also Proposition 6.11 and Remark 6.12) our main construction applies to this collection of C^*-correspondences giving us a $(C^*(G(\mathbb{A})), C^*(L(\mathbb{A})))$-correspondence which implements the adelic parabolic induction for the group $G(\mathbb{A})$. In the forthcoming work [16], we apply the main construction of this paper to global theta correspondence.

The paper is organized as follows. In Section 2 we recall the relevant background on restricted tensor products of vector spaces, Hilbert spaces and C^*-algebras. Further details for adelic groups are discussed in Section 3, with a particular focus on their representation theory and C^*-algebras. The main construction can be found in Section 4, where we construct the restricted tensor product of a collection of Hilbert C^*-modules with a distinguished vector chosen. In the cases of interest in representation theory, one considers restricted tensor products of a collection not just of Hilbert C^*-modules but of C^*-correspondences and considerations thereof can be found in Section 5. We discuss applications to parabolic induction for adelic groups in Section 6.

2. Background

The key technical tool in constructing the adeles as well as groups thereover comes from restricted products or tensor products. We here recall the incarnation of restricted (tensor) products in various classes of objects. The history of infinite tensor products can be traced back to von Neumann [24]. A good sources for the materials below is [17]. See also [5].

2.1. **Vector spaces.** Let $(W_v)_{v\in I}$ be a family of vector spaces indexed by a countable index set I. For all indices v, fix a vector $x_v \in W_v$, in such a way that $x_v = 0$ for at most finitely many v. For a finite subset $F \subseteq I$, let $W_F := \bigotimes_{v\in F} W_v$. Then for $F' = F \sqcup \{v'\}$, we will consider the linear map

$$W_F \to W_{F'} = W_F \otimes W_{v'}, \quad w \mapsto w \otimes x_{v'}, \tag{2.1}$$

which is an embedding for all but finitely many v. Using these maps, we turn the collection W_F into a directed system and form

$$\bigotimes_{v\in I}{}' (W_v, x_v) := \varinjlim_F W_F = \bigcup_S W_S.$$

The latter union is over all finite sets S that contain all v with $x_v = 0$, so that the W_S form an ascending chain.

Note that $\bigotimes'_{v\in I}(W_v, x_v)$ is spanned by elements $w = \otimes_v w_v$ such that for all but finitely many v, we have w_v is the distinguished vector x_v.

If $B_v : W_v \to W_v$ are linear maps such that $B_v(x_v) = x_v$ for all but finitely many v, we have a linear operator

$$\bigotimes_{v\in I}{}' B_v : \bigotimes_{v\in I}{}' (W_v, x_v) \longrightarrow \bigotimes_{v\in I}{}' (W_v, x_v), \qquad \otimes_v w_v \mapsto \otimes_v B_v(w_v).$$

For notational simplicity, we also include the case that x_v can be zero infinitely often. In this case, we define $\bigotimes'_{v\in I}(W_v, x_v) := 0$.

2.2. **Hilbert spaces.** When the vector spaces in the preceding section are Hilbert spaces $\mathcal{H}_v = W_v$, then the restricted tensor product can be given a Hilbert space structure provided that the distinguished vectors $h_v = x_v$ are in the unit ball, and $\|h_v\| = 1$ for all but finitely many v. Then the maps (2.1) are isometries for all but finitely many indices and the direct limit $\varinjlim_F \mathcal{H}_F = \bigcup_S \mathcal{H}_S$ carries a canonical pre-Hilbert space structure. We denote its completion as

$$\bigotimes_{v\in I}{}' (\mathcal{H}_v, h_v).$$

Note that for simple tensors $e = \otimes_v e_v, f = \otimes_v f_v \in \bigcup_S \mathcal{H}_S$, we have

$$\langle e, f\rangle = \prod_v \langle e_v, f_v\rangle_v$$

which is well defined because for all but finitely many places, we have $e_v = f_v = h_v$ so that

$$\langle e_v, f_v\rangle_v = \langle h_v, h_v\rangle_v = 1$$

by our assumption that h_v are unit vectors.

If $B_v : \mathcal{H}_v \to \mathcal{H}'_v$ are bounded linear maps between two collections of Hilbert spaces, with distinguished unit vectors (h_v) and (h'_v), such that

(1) $B_v(h_v) = h'_v$ for all but finitely many v,
(2) $\|B_v\| \le 1$ for all but finitely many v,

then the linear map

$$\bigotimes_{v\in I}{}' B_v : \bigotimes_{v\in I}{}' (\mathcal{H}_v, h_v) \longrightarrow \bigotimes_{v\in I}{}' (\mathcal{H}'_v, h'_v), \qquad \otimes_v w_v \mapsto \otimes_v B_v(w_v), \tag{2.2}$$

is bounded as well. Note that the two assumptions above in fact imply that $\|B_v\| = 1$ for all but finitely many v, and so

$$\left\| \bigotimes_{v\in I}{}' B_v \right\| = \prod_{v\in I} \|B_v\|, \tag{2.3}$$

where the factors in the product are 1 for all but finitely many v. It follows from Proposition 4.7 below that all compact operators $\bigotimes'(\mathcal{H}_v, h_v) \longrightarrow \bigotimes'(\mathcal{H}'_v, h'_v)$ can be approximated in norm by linear combinations of operators of the form (2.2) (for B_v compact). The analogous statement fails for bounded operators.

2.3. **C^*-algebras.** Let $(A_v)_{v\in I}$ be a collection of C^*-algebras indexed by a countable set I. The C^*-algebras A_v's will in our applications be type I and nuclear, but in this section we state explicitly when such assumptions are used. We shall use $\otimes$ to denote the spatial tensor product of C^*-algebras. Assume that we have a distinguished projection $p_v \in A_v$ that we fix, and that $p_v \neq 0$ for all but finitely many v. We will denote such a family simply by

$$(A_v, p_v)_{v\in I}.$$

Given a finite subset F of I, we define the C^*-algebra

$$A_F := \bigotimes_{j\in F} A_v.$$

Then for $F' = F \sqcup \{k\}$, we form the $*$-homomorphism

$$A_F \hookrightarrow A_{F'} = A_F \otimes A_k, \quad a \mapsto a \otimes p_k, \tag{2.4}$$

which is injective, hence isometric, for all but finitely many v. These turn the collection $(A_F)_F$, for F ranging over finite subsets of I into a directed system of C^*-algebras. We define the restricted product of the collection $(A_v, p_v)_{v\in I}$ as the direct limit of this directed system

$$\bigotimes_{v\in I}{}'(A_v, p_v) := \varinjlim_F A_F.$$

The direct limit is taken in the category of C^*-algebras and we can describe the C^*-algebra structure explicitly. The direct limit $\varinjlim_F A_F$ is defined as the closure of $\bigcup_S A_S$ in the C^*-norm defined from each A_S, where S runs over all finite subsets containing all v for which $p_v = 0$. This is well defined since for such S, S' the map in (2.4) is isometric.

2.4. **Representations.** Let $(A_v, p_v)_{v\in I}$ be as in Section 2.3. Given v, let us fix a representation $(\pi_v, \mathcal{H}_v)$ of A_v, that is, $*$-homomorphisms

$$\pi_v : A_v \to \mathbb{B}(\mathcal{H}_v)$$

for Hilbert spaces $\mathcal{H}_v$.

If for all but finitely many indices v, we fix unit vectors $h_v \in \mathcal{H}_v$ such that $\pi_v(p_v)(h_v) = h_v$ then we can, using the norm computation (2.3), form a representation

$$\pi := \bigotimes_{v\in I}{}' \pi_v \ : \bigotimes_{v\in I}{}'(A_v, p_v) \longrightarrow \mathbb{B}\left(\bigotimes_{v\in I}{}'(\mathcal{H}_v, h_v)\right), \tag{2.5}$$

as follows: given $a = \otimes_v a_v \in \bigotimes'(A_v, p_v)$ and $w = \otimes_v w_v \in \bigotimes'(\mathcal{H}_v, h_v)$, we define $\pi(a)(w)$ via the rule

$$\pi(a)(w) := \otimes_v \pi_v(a_v)(w_v).$$

At all but finitely many indices v, we have $a_v = p_v$ and $w_v = h_v$, and thus $\pi_v(a_v)(w_v) = w_v$. It follows that $\pi(a)(w)$ belongs to $\bigotimes'(\mathcal{H}_v, h_v)$, and from the discussion in Section 2.2 we deduce that $\pi(a)$ extends by linearity and continuity to a bounded linear operator. This construction extends by linearity and continuity to all of $\bigotimes'_{v\in I}(A_v, p_v)$ as claimed in (2.5).

The representation π is irreducible if all the π_v are irreducible. Modifying the fixed vectors h_v at finitely many indices v does not change the isomorphism class of π. The same is true if we replace h_v's with a scalar multiple. Thus, if the projections $\pi_v(p_v)$ all have rank one for all but finitely many v, then π does not depend on the choice of h_v.

Definition 2.1 (Definition 12 in [17])**.** A projection p in a C^*-algebra A has *rank at most one* if for every irreducible representation $(\pi, \mathcal{H})$ of A, the projection $\pi(p)$ has rank at most one.

The next result is due to Guichardet (see [17, Section 13]).

Theorem 2.2. *Let $(A_v, p_v)_{v\in I}$ be a collection of Type I C^*-algebras with all but finitely many of the distinguished projections p_v having rank at most one. Every irreducible representation of $\bigotimes'(A_v, p_v)$ is equivalent to a restricted tensor product as in* (2.5) *of irreducible representations π_v of A_v, $v \in I$.*

The reader should note that it is crucial that we use the spatial tensor product in Theorem 2.2. For the maximal tensor product there are non-factorizable irreducible representations.

3. Adelic groups

We now turn to discuss how restricted products of groups produce adelic groups and describe their representation theory. Some useful sources are [11], [13, Section 3] and [22, Sections 5,6].

3.1. Restricted products of groups.

Let $(G_v)_{v\in I}$ be a countable collection of locally compact, second countable, Hausdorff groups. We assume that each G_v is a Type I group, that is, its C^*-algebra $C^*(G_v)$ is of Type I. Since each group G_v is type I, each of the C^*-algebras $C^*(G_v)$ are nuclear by [25]. Assume that for all $v \in I$ we fix a subgroup $K_v \subset G_v$.

Definition 3.1. A countable collection $(G_v, K_v)_{v\in I}$ consisting of type I groups G_v with subgroups $K_v \subset G_v$ is called an *admissible family* if there exists a possibly empty finite set E of indices such that for all $v \notin E$, K_v is a *compact open* subgroup of G_v.

For convenience, we will denote such an admissible family as $(G_v, K_v)_{v\in I}$. The *restricted product* of the admissible family $(G_v, K_v)_{v\in I}$ is defined as

$$\begin{aligned}\prod_{v\in I}{}^{'}(G_v : K_v) \;&:=\; \Big\{(g_v)_v \in \prod\nolimits_{v\in I} G_v \mid g_v \in K_v \text{ for all but finitely many } v\Big\}\\ &=\; \bigcup_S \prod_{v\in S} G_v \times \prod_{v\notin S} K_v\end{aligned}$$

where S varies over finite subsets of I containing E. Equipping with the direct limit topology, we obtain a locally compact, Hausdorff topological group.

In the rest of the paper, we will put

$$G_S := \prod_{v\in S} G_v \times \prod_{v\notin S} K_v$$

so that

$$\prod_{v\in I}{}^{'}(G_v : K_v) = \bigcup_S G_S.$$

Note that each G_S is an open subgroup of $\prod{}^{'}(G_v : K_v)$.

3.2. **Representations of restricted products of groups.** Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups as in Definition 3.1. Let unitary representations

$$\pi_v : G_v \to \mathcal{U}(\mathcal{H}_v)$$

be given. Assume that for all but finitely many v, there exist a nonzero vector $h_v \in \mathcal{H}_v$ such that

$$h_v \text{ is fixed by } K_v.$$

Using the previous constructions, we can put the π_v's together

$$\pi := \bigotimes{}^{'} \pi_v.$$

This is an irreducible unitary representation of $\prod{}^{'}(G_v : K_v)$ on the Hilbert space $\bigotimes{}^{'} \mathcal{H}_v$.

In order to describe the dependence of π on the distinguished vectors, we recall that notion of a *Gelfand pair*. Let G be a topological group, $K \subset G$ a compact subgroup and $\mathcal{H}(G, K)$ the Hecke algebra , that is, the convolution algebra of continuous, compactly supported bi-K-invariant complex valued functions on G. Then (G, K) is called a Gelfand pair if $\mathcal{H}(G, K)$ is commutative.

Two collections $\{h_v\}$, $\{h'_v\}$ of distinguished vectors give rise to isomorphic representations if h_v is a scalar multiple of h'_v all but finitely many v. Thus the following proposition (see e.g. [10, Prop. 6.3.1]) is relevant.

Proposition 3.2. *Let (G, K) be a Gelfand pair. For any irreducible unitary G-module V, the subspace V^K is at most one-dimensional.*

Thus if all but finitely many (G_v, K_v) form a Gelfand pair, then the restricted product representation π will be independent of the chosen collection of distinguished vectors.

In the converse direction, we have the following factorization result.

Theorem 3.3. *Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups such that (G_v, K_v) is a Gelfand pair for all but finitely many v. Every irreducible unitary representation π of the restricted product group $\prod'(G_v : K_v)$ is factorizable into local unitary irreducible representations:*

$$\pi \simeq \bigotimes{}' \pi_v.$$

The isomorphism classes of the unitary representations $(\pi_v, \mathcal{H}_v)$ are determined by that of π. For all but finitely many v, the dimension of $\mathcal{H}_v^{K_v}$ is one.

We note that Theorem 3.3 follows from Guichardet's Theorem 2.2. There is an analogous factorization theorem of Flath [11] (see also [14, Section 5.7]) for irreducible "admissible" representations of a reductive group over the adeles. Such representations form a larger class that contain the unitary ones and accordingly one has to apply the strategy of the proof of Guichardet's Theorem to the Hecke algebra of the reductive group in order to achieve the factorization.

3.3. **Adeles and groups over adeles.** Let F be a number field. If v is a place of F, let F_v denote the completion of F at v. All F_v are locally compact. Recall that if v is a finite place, then F_v has a unique maximal compact open subring $\mathcal{O}_v$. The restricted product

$$\mathbb{A} = \mathbb{A}_F := \prod{}'(F_v : \mathcal{O}_v),$$

can be made into a locally compact ring called the *ring of adeles* of F. We embed F diagonally into $\mathbb{A}$. Then F sits discretely inside $\mathbb{A}$ and the quotient $F\backslash\mathbb{A}$ is compact.

Let G be a linear[1] algebraic group defined over a number field F. For convenience, let us put

$$G_v := G(F_v)$$

for a place v of F. For a finite place v, let us put

$$K_v := G(\mathcal{O}_v) := G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$$

Note that K_v is a *compact open* subgroup of $G(F_v)$.

The *adelic group* $G(\mathbb{A})$ obtained by taking the adelic points of G is isomorphic as a topological group to the restricted product of G_v with respect to K_v:

$$G(\mathbb{A}) \simeq \prod{}'(G_v : K_v).$$

The group $G(F)$ embeds diagonally into $G(\mathbb{A})$. The image is *discrete* but note that $G(F)\backslash G(\mathbb{A})$ is not necessarily compact.

Proposition 3.4. *If G is reductive, then for all but finitely many of the finite places v, (G_v, K_v) is a Gelfand pair. Thus, irreducible unitary representation of $G(\mathbb{A})$ are factorizable in the sense of Theorem 3.3.*

[1] That is, G embeds into GL_n for some n.

See [27] for a proof. We mention in passing that when G is reductive, $G(\mathbb{A})$ is a Type I group. This is not necessarily true if G is not reductive (see [22]).

3.4. **C^*-algebras of restricted products of groups.** Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups with $K_v \subset G_v$ compact open for all $v \in I \setminus E$, with E a possibly empty finite set of exceptional indices. For a finite subset $S \subset I$ containing E, for convenience, recall that

$$G_S := \prod_{v\in S} G_v \times K_S, \quad \text{where} \quad K_S := \prod_{v\notin S} K_v$$

so that

$$\prod{}' (G_v : K_v) = \bigcup_S G_S. \tag{3.1}$$

For a group G, we write ϵ_G for the trivial G-representation and if G is compact we tacitly identify the trivial representation with its support projection in $C^*(G)$. We note that for a compact group K the support projection ϵ_K is represented by the constant function $1_K \in C(K) \subseteq C^*(K)$ (assuming the volume is normalized to one).

Proposition 3.5. *Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups and $S \subseteq I$ a finite subset containing E. We have that*

$$C^*(G_S) \simeq \bigotimes_{v\in S} C^*(G_v) \otimes C^*(K_S),$$

and

$$C^*_r(G_S) \simeq \bigotimes_{v\in S} C^*_r(G_v) \otimes C^*(K_S),$$

and the C^*-algebra of K_S is the restricted tensor product

$$C^*(K_S) \simeq C^*_r(K_S) \simeq \bigotimes_{v\notin S}{}' (C^*(K_v), \epsilon_{K_v}).$$

Proof. The first two isomorphisms follow from the corresponding product decomposition at the level of groups. A priori, the product decomposition for the first isomorphism gives that $C^*(G_S)$ is the maximal tensor product $\bigotimes^{\max}_{v\in S} C^*(G_v) \otimes C^*(K_S)$ but each G_v is type I so $C^*(G_v)$ is nuclear by [25] and K_S is compact so the full and reduced C^*-algebras of K_S coincide and are nuclear

We define $A_0 \subseteq C(K_S)$ as the subalgebra of continuous functions that are constant except in finitely many places, i.e. functions $f \in C(K_S)$ of the form $f((k_v)_{v\notin S}) = f_0((k_v)_{v\in S'\setminus S})$ for a finite super set $S' \supseteq S$. Since the support projection of the trivial representation of a compact group is represented by the constant function (assuming the volume is normalized to one), it is clear that A_0 naturally embeds as a dense $*$-subalgebra of both $C^*(K_S)$ and $\bigotimes_{v\notin S}(C^*(K_v), \epsilon_{K_v})$ so the isomorphism exists by the universal property of group C^*-algebras. ☐

Proposition 3.5 can also be found in the literature as [5, Example (3), page 316].

Lemma 3.6. *(Baum-Millington-Plymen) Let G be a group that is the union of ascending chain of open subgroups G_n. Then the inclusions*

$$\bigcup_n C^*(G_n) \subseteq C^*(G) \quad \text{and} \quad \bigcup_n C^*_r(G_n) \subseteq C^*_r(G)$$

are dense, or in other words $C^*(G) = \varinjlim C^*(G_n)$ and $C^*_r(G) = \varinjlim C^*_r(G_n)$ in the category of C^*-algebras.

In fact, this lemma is proven for the reduced group C^*-algebra in [2] but the proof works for the maximal group C^*-algebra as well.

Lemma 3.7. *For an admissible family $(G_v, K_v)_{v\in I}$, we have*

$$C^*\left(\prod\nolimits'(G_v : K_v)\right) \simeq \bigotimes_v{}'(C^*(G_v), p_{K_v}),$$

and

$$C^*_r\left(\prod\nolimits'(G_v : K_v)\right) \simeq \bigotimes_v{}'(C^*_r(G_v), p_{K_v}),$$

where p_{K_v} is the projection in $C^*(G_v)$, or $C^*_r(G_v)$, given by the characteristic function of the compact open subgroup K_v whose volume is normalized to be 1.

For compact open subgroups K_v, we have that $C^*(K_v)$ is a subalgebra of $C^*(G_v)$. Recall that when viewed as an element of the subalgebra $C^*(K_v)$, the projection p_{K_v} is simply the constant function 1, that is, ϵ_{K_v} that we used earlier.

Proof. The admissible families $(G_v, K_v)_{v\in I}$ are assumed to be type I, so by [25] each $C^*(G_v)$ is nuclear. Therefore, we need only consider the spatial tensor product.

By definition $\prod'(G_v : K_v) = \bigcup_S G_S$ (cf. (3.1)) and each G_S is open. From Lemma 3.6 below, we conclude that we have a dense inclusion

$$\bigcup_S C^*(G_S) \subseteq C^*\left(\prod\nolimits'(G_v : K_v)\right).$$

The inclusion is isometric on each $C^*(G_S)$. Since

$$\bigcup_S C^*(G_S) \simeq \bigcup_S \left(\bigotimes_{v\in S} C^*(G_v) \otimes \bigotimes_{v\notin S}(C^*(K_v), p_{K_v})\right),$$

by Proposition 3.5, we conclude

$$C^*\left(\prod\nolimits'(G_v : K_v)\right) \simeq \overline{\bigcup_S C^*(G_S)} \simeq \overline{\bigcup_S \left(\bigotimes_{v\in S} C^*(G_v) \otimes \bigotimes_{v\notin S}(C^*(K_v), p_{K_v})\right)}.$$

The subgroup K_v is compact open in G_v and therefore $p_{K_v} \in C^*(K_v) \subseteq C^*(G_v)$. It follows that

$$\bigcup_S \left(\bigotimes_{v\in S} C^*(G_v) \otimes \bigotimes_{v\notin S}(C^*(K_v), p_{K_v})\right) = \bigcup_S C^*(G_S), \tag{3.2}$$

where we identify $C^*(G_S)$ with its image in $C^*\left(\prod'(G_v : K_v)\right)$ in the right hand side. After taking closures in (3.2), the proof is complete. □

Corollary 3.8. *Let G be a reductive linear algebraic group over a number field F. Then*

$$C^*\left(G(\mathbb{A})\right) \simeq \bigotimes_v{}'(C^*(G_v), p_{K_v}), \quad \textit{and} \quad C_r^*\left(G(\mathbb{A})\right) \simeq \bigotimes_v{}'(C_r^*(G_v), p_{K_v}),$$

where p_{K_v} is the projection in $C^*(G_v)$, or $C_r^*(G_v)$, given by the characteristic function of the compact open subgroup K_v whose volume is normalized to be 1.

Remark 3.9. Corollary 3.8 is equivalent with the result of Tadic [26] that describes the the unitary dual of $G(\mathbb{A})$ as a restricted product in terms of those of G_v's.

4. Restricted tensor products of Hilbert C^*-modules

We now turn our attention to defining restricted tensor products of Hilbert C^*-modules. For Hilbert C^*-modules, restricted tensor products will be defined in a similar spirit as for Hilbert spaces (see Subsection 2.2) but more care is needed for the fixed vectors.

Recall that a Hilbert C^*-module X over a C^*-algebra A is a right A-module X equipped with an A-valued inner product $\langle\cdot,\cdot\rangle_A$ making X into a Banach space in the norm

$$\|x\|_X := \|\langle x, x\rangle_A{}^{1/2}\|_A.$$

The C^*-algebra of adjointable A-linear operators on X is denoted by $\mathrm{End}_A^*(X)$. An element $T \in \mathrm{End}_A^*(X)$ is said to be A-compact if it is a norm limit of sums of rank one module operators

$$T_{\xi,\eta} : x \mapsto \xi\langle\eta, x\rangle_X,$$

for $\xi, \eta \in X$. We write $\mathbb{K}_A(X) \subseteq \mathrm{End}_A^*(X)$ for the ideal of A-compact operators. For more details on Hilbert C^*-modules, see [19].

4.1. Compatible collections.

Let $(A_v, p_v)_{v\in I}$ be a collection of C^*-algebras equipped with distinguished projections as in Subsection 2.3. We also consider a collection $(X_v)_{v\in I}$ of Hilbert C^*-modules X_v over A_v. For notational simplicity, we write $\langle\cdot,\cdot\rangle_v$ for the inner product on X_v.

Definition 4.1. Assume that (x_v) is a collection of vectors $x_v \in X_v$. We say that $(X_v, x_v)_{v\in I}$ *is compatible with* $(A_v, p_v)_{v\in I}$ if for all but finitely many indices $v \in I$, we have that $\langle x_v, x_v\rangle_v = p_v$. In this case say that $(X_v, x_v)_{v\in I}$ is a compatible collection of right Hilbert C^*-modules over $(A_v, p_v)_{v\in I}$.

Remark 4.2. Note that compatibility implies that for all but finitely many v we have $x_v = x_v p_v$, as $\langle x_v - x_v p_v, x_v - x_v p_v\rangle_v = 0$. Also, the rank one operators $p_{x_v} := T_{x_v,x_v} \in \mathbb{K}(X_v)$ are projections.

We introduce the notations

$$A := \bigotimes_{v\in I}{}'(A_v, p_v) \quad \text{and} \quad A_0 := \bigcup_S A_S.$$

Note that $A_0 \subseteq A$ is a dense $*$-subalgebra.

We shall now show that a compatible collection $(X_v, x_v)_{v\in I}$ of Hilbert C^*-modules over $(A_v, p_v)_{v\in I}$ admits a restricted tensor product, which is a Hilbert C^*-module over the restricted tensor product C^*-algebra A of $(A_v, p_v)_{v\in I}$.

Following Subsection 2.1, for any finite subset F of I, we define the right A_F-Hilbert C^*-module X_F as the exterior tensor product

$$X_F := \bigotimes_{v\in F} X_v. \tag{4.1}$$

Indeed, X_F is a Hilbert C^*-module over A_F because it is a finite exterior tensor product. Then for $F' = F \sqcup \{v'\}$, we will consider the map

$$\iota_F^{F'} : X_F \hookrightarrow X_{F'} = X_F \otimes x_{v'}, \quad x \mapsto x \otimes x_{v'}, \tag{4.2}$$

which is isometric whenever F contains the set of exceptional indices E. Using these embeddings, we turn the collection X_S, where S runs over finite sets containing E, into an ascending chain and form

$$X_0 := \bigcup_S X_S. \tag{4.3}$$

Note that X_0 is spanned by elements $w = \otimes_v w_v$ such that for all but finitely many v, we have w_v is the distinguished vector x_v. Given such an element $w = \otimes_v w_v \in X_0$ with all but finitely many w_v equals the distinguished vectors x_v, and $a = \otimes_v a_v \in A_0$ we define $w{\cdot}a$ via the rule

$$w{\cdot}a := \otimes_v (w_v{\cdot}a_v)$$

that is, we act component-wise. It holds that $w_v{\cdot}a_v = w_v = x_v$, for all but finitely many v, because (x_v) is compatible with (p_v), so $w{\cdot}a$ lies in X_0. We can extend this operation by linearity to a right action of A_0 on X_0.

We will now describe the pre-Hilbert A_0-module structure on X_0. For elements $w = \otimes_v w_v, z = \otimes_v z_v \in X_0$, we put

$$\langle w, z\rangle := (\otimes_v \langle w_v, z_v\rangle). \tag{4.4}$$

Then for all but finitely many places we have

$$\langle w_v, z_v\rangle = \langle x_v, x_v\rangle = p_v.$$

Hence the inner product $\langle w, z\rangle$ lands in A_0. We can extend this inner product by sesquilinearity over A_0 to all of X_0, and we wish to show that for all $z \in X_0$ we have $\langle z, z\rangle \geq 0$ and that $\langle z, z\rangle = 0$ only if $z = 0$.

By definition, $X_0 := \bigcup_S X_S$ under the inclusions

$$\iota_S^X : X_S \hookrightarrow X_0 \subset X, \quad \iota_S^A : A_S \hookrightarrow A_0, \tag{4.5}$$

where ι_S^X is A_S-linear and inner-product preserving in the sense that for $w, x \in X_S$ we have

$$\langle \iota_S^X(w), \iota_S^X(z)\rangle_{X_0} = \iota_S^A\left(\langle w, z\rangle_{X_S}\right). \tag{4.6}$$

Each X_S is a finite exterior tensor product and thus a Hilbert C^*-module, and for $z \in X_S$ we have

$$\langle \iota_S^X(z), \iota_S^X(z) \rangle = \langle z, z \rangle_S \geq 0, \quad \langle \iota_S^X(z), \iota_S^X(z) \rangle = 0 \Rightarrow \iota_S^X(z) = 0,$$

so Equation (4.4) defines an positive and non-degenerate inner product on $X_0 := \bigcup_S X_S$.

We have shown that X_0 is a right pre-Hilbert C^*-module over A_0. The above discussion can be summarized into the following definition.

Definition 4.3. Let $(X_v, x_v)_{v \in I}$ by a compatible collection of Hilbert C^*-modules over $(A_v, p_v)_{v \in I}$ as in Definition 4.1. The *restricted tensor product of the compatible collection* $(X_v, x_v)_{v \in I}$ is the right Hilbert C^*-module over $A := \bigotimes'_{v \in I}(A_v, p_v)$ obtained as the completion

$$\bigotimes_{v \in I}{}' (X_v, x_v) := \overline{X_0}$$

of X_0 with respect to the norm arising from the inner product (4.4).

4.2. **Direct limit construction.** A robust construction of $\bigotimes'_{v \in I}(X_v, x_v)$ can be achieved by promoting $(X_F)_F$ to a directed system of Hilbert C^*-modules over A and taking the direct limit in this category.

Observe that the space X_F is a finite exterior tensor product that forms a right Hilbert A_F-module in the obvious way. We have the following.

Proposition 4.4. *For finite sets $F \subseteq F'$, the map*

$$X_F \otimes_{A_F} A_{F'} \to X_{F'},$$

induced from (4.2), is $A_{F'}$-linear, adjointable and for all but finitely many v, it is isometric. In particular, the system

$$\{X_F \otimes_{A_F} A\}_F,$$

is a directed system of right A-Hilbert C^*-modules.

We now show that the direct limit of $\{X_F \otimes_{A_F} A\}_F$ captures the restricted tensor product $\bigotimes'_{v \in I}(X_v, x_v)$ of Definition 4.3.

Theorem 4.5. *We have*

$$\bigotimes_{v \in I}{}' (X_v, x_v) \simeq \varinjlim_F X_F \otimes_{A_S} A$$

as right A-Hilbert C^*-modules.

Proof. The proposition follows if we can produce a bounded A_0-linear inner product preserving map

$$\Phi : \varinjlim_F{}^{\mathrm{alg}} X_F \otimes_{A_F} A_0 \to X,$$

with dense range. The superscript indicates we take the direct limit in the algebraic category of modules for the ring A_0. First of all observe that we may take the direct limit over all finite sets S containing the exceptional set E instead. On simple tensors we define

$$\Phi_S : A_S \otimes_{A_S} A_0 \to X, \quad ((\otimes_{v\in S} w_v) \otimes_{A_S} a) := wa.$$

The map Φ_S is well defined, innerproduct preserving, and compatible with the directed system $\{X_S \otimes_{A_S} A_0\}$, so the universal property of direct limits gives a map

$$\Phi : \varinjlim_{S} X_S \otimes^{\mathrm{alg}}_{A_S} A_0 \to X.$$

The range of Φ is $X_0 \cdot A_0$ which is dense in $\bigotimes'_{v\in I}(X_v, x_v)$, and the result follows. □

4.3. **Induction commutes with restricted tensor product.** The following relates the construction above to how local induction functors glue together to give a global one.

Proposition 4.6. *Let $(X_v, x_v)_{v\in I}$ be a compatible collection of Hilbert C^*-modules over $(A_v, p_v)_{v\in I}$, and $X := \bigotimes'_{v\in I}(X_v, x_v)$ their restricted tensor product. If*

$$(\pi, \mathcal{H}) \simeq \left(\bigotimes{}' \pi_v, \bigotimes_v{}' (\mathcal{H}_v, h_v) \right)$$

is a representation of A, then we have a canonical, unitary isomorphism

$$X \otimes_A \mathcal{H} \simeq \bigotimes{}' \left(X_v \otimes_{A_v} \mathcal{H}_v, x_v \otimes_{A_v} h_v\right),$$

defined on simple tensors by

$$(\otimes_v w_v) \otimes (\otimes_v y_v) \mapsto \otimes_v (w_v \otimes y_v). \tag{4.7}$$

Proof. We note that if π is a non-zero representation then $\pi(p_v)h_v = h_v$ for all but finitely many v and $x_v \otimes_{A_v} h_v \in X_v \otimes_{A_v} \mathcal{H}_v$ is a unit vector for all but finitely many v. By linearity, the map (4.7) extends to a surjection from a dense subspace of $X \otimes_A \mathcal{H}$ to a dense subspace of $\bigotimes' (X_v \otimes_{A_v} \mathcal{H}_v, x_v \otimes_{A_v} h_v)$. A short computation shows that on this dense subspace the map (4.7) is isometric. Therefore, (4.7) extends to an isometry with dense range, i.e. a unitary isomorphism. □

4.4. **Compact operators on restricted tensor products.** We now turn to describing the compact operators on a restricted tensor product of Hilbert C^*-modules. Consider a compatible collection $(X_v, x_v)_{v\in I}$ of Hilbert C^*-modules over $(A_v, p_v)_{v\in I}$, with

$$X := \bigotimes_{v\in I}{}' (X_v, x_v),$$

their restricted tensor product. Recall from Remark 4.2 that the operators

$$p_{x_v}\xi := x_v \langle x_v, \xi\rangle_v \in \mathbb{K}(X_v)$$

are projections.

Proposition 4.7. *The inner-product preserving inclusions $\iota_S^X : X_S \to X$ from Equation (4.5) induce an isomorphism of C^*-algebras*

$$\mathbb{K}_A(X) \simeq \bigotimes_{v\in I}{}' (\mathbb{K}_{A_v}(X_v), p_{x_v}).$$

Proof. Let S be a set containing the set of exceptional indices E and $S' = S \sqcup \{v\}$. The inner-product preserving inclusions $\iota_S^X : X_S \to X$ and $\iota_S^{S'} : X_S \to X_{S'}$ from Equations (4.2) and (4.5) induce isometric $*$-homomorphisms

$$\alpha_S^{S'} : \mathbb{K}(X_S) \to \mathbb{K}(X_{S'}), \quad \alpha_S^X : \mathbb{K}(X_S) \to \mathbb{K}(X),$$

defined on rank one operators by $\alpha_S^{S'}(T_{\xi,\eta}) := T_{\iota_S^{S'}(\xi),\iota_S^{S'}(\eta)}$, and similarly for ι_S^X. We have commutative diagrams

$$\begin{array}{ccc} X_S & \xrightarrow{\iota_S^X} & X \\ {\scriptstyle \iota_S^{S'}}\downarrow & \nearrow {\scriptstyle \iota_{S'}^X} & \\ X_{S'} & & \end{array} \qquad \begin{array}{ccc} \mathbb{K}_{A_S}(X_S) & \xrightarrow{\alpha_S^X} & \mathbb{K}_A(X) \\ {\scriptstyle \alpha_S^{S'}}\downarrow & \nearrow {\scriptstyle \alpha_{S'}^X} & \\ \mathbb{K}_{A_{S'}}(X_{S'}). & & \end{array}$$

A short computation shows that for $T \in \mathbb{K}(X_S)$ we have $\alpha_S^{S'}(T) = T \otimes p_{x_v}$, so the direct limit of the $\mathbb{K}(X_S)$ along the maps $\alpha_S^{S'}$ coincides with $\bigotimes'_{v\in I}(\mathbb{K}_{A_v}(X_v), p_{x_v})$, and we obtain an isometric $*$-homomorphism

$$\bigotimes_{v\in I}{}' (\mathbb{K}_{A_v}(X_v), p_{x_v}) \to \mathbb{K}(X).$$

We need to show that the image of this embedding is dense. Since

$$X_0 = \bigcup_S X_S \subset X,$$

is dense, the operators $T_{\xi,\eta}$, with $\xi, \eta \in X_S$, belong to the image of $\mathbb{K}_{A_S}(X_S) \hookrightarrow \mathbb{K}_A(X)$ and the proof is complete. □

5. Restricted tensor products of C^*-correspondences

The application of Hilbert C^*-modules we are interested is hinted at in Proposition 4.6, but what is missing is how the Hilbert C^*-module transfers a representation of A to another C^*-algebra A'. The appropriate tool for such a construction is a C^*-correspondence. Recall that an (A', A)-correspondence is an A-Hilbert C^*-module X equipped with a left action of A' defined in terms of a $*$-homomorphism

$$\alpha : A' \to \mathrm{End}_A^*(X).$$

We will in this section define restricted tensor products of C^*-correspondences.

We remain in the set-up of Section 4. That is $(A_v, p_v)_{v\in I}$ will be a collection of C^*-algebras equipped with distinguished projections, $(X_v, x_v)_{v\in I}$ is a compatible collection of Hilbert C^*-modules over $(A_v, p_v)_{v\in I}$ as in Definition 4.1. We now also introduce a second collection of C^*-algebras equipped with distinguished projections $(A'_v, p'_v)_{v\in I}$ as in Subsection 2.3.

Definition 5.1. We say that a collection $(\alpha_v)_{v\in I}$ of $*$-homomorphisms

$$\alpha_v : A'_v \longrightarrow \mathrm{End}^*_{A_v}(X_v), \tag{5.1}$$

is *compatible with* $(X_v, x_v)_{v\in I}$ if for all but finitely many places it holds that

$$\alpha_v(p'_v)\cdot x_v = x_v.$$

If this is the case, we also say that $(X_v, x_v)_{v\in I}$ is a collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences.

As before, let us put

$$A' := \bigotimes_{v\in I}{}' (A'_v, p'_v). \qquad A := \bigotimes_{v\in I}{}' (A_v, p_v).$$

Let

$$X := \bigotimes_{v\in I}{}' (X_v, x_v)$$

denote the restricted product of the family $(X_v, x_v)_{v\in I}$.

Proposition 5.2. *Let $(X_v, x_v)_{v\in I}$ be a collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences as in Definition 5.1. Then, we can define the $*$-homomorphism*

$$\alpha : A' \longrightarrow \mathrm{End}^*_A(X)$$

by declaring the action for simple tensors $w = \otimes_v w_v \in X$ and $a' = \otimes_v a'_v \in A'$ to be

$$\alpha(a')(w) = \otimes_v(\alpha_v(a'_v)(w_v)).$$

Moreover, the (A', A)-correspondence X has the property that if

$$(\pi, \mathcal{H}) \simeq \left(\bigotimes{}' \pi_v, \bigotimes_v{}' (\mathcal{H}_v, h_v) \right)$$

is a representation of A, the canonical, unitary isomorphism from Proposition 4.6 is an isomorphism of A'-representations

$$X \otimes_A \mathcal{H} \simeq \bigotimes{}' (X_v \otimes_{A_v} \mathcal{H}_v, x_v \otimes_{A_v} h_v)\,.$$

Proof. We prove the proposition by giving a functorial definition of α. There is an obvious left action $\alpha_S : A'_S \longrightarrow \mathrm{End}_{A_S}(X_S)$. The connecting maps $\iota_S^{S'}$ defined in Equation (4.2) are clearly compatible with $(\alpha_S)_{S\,finite}$, so there is an induced map $\alpha : A' \longrightarrow \mathrm{End}_A(X)$ in the limit. □

In many applications the left action of A'_v is by compact Hilbert C^*-module operators. It is of interest to know whether these compact actions bundle up to a left A'-action by A-compact operators.

Definition 5.3. Assume that $(X_v, x_v)_{v\in I}$ is a collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$ correspondences as in Definition 5.1.

- We say that $(X_v, x_v)_{v\in I}$ is a *coherent collection of correspondences* if at all but finitely many places,
$$\alpha_v(p'_v) = p_{x_v},$$
where $p_{x_v} \in \mathbb{K}_{A_v}(X_v)$ is the projection along $x_v \in X_v$ (cf. Proposition 4.7).
- A coherent collection of correspondences (X_v, x_v) is *compact* if also the range of the map α_v is contained in $\mathbb{K}_{A_v}(X_v)$ at all places v.
- A coherent collection of correspondences (X_v, x_v) is *type I* if also the range of the map α_v contains $\mathbb{K}_{A_v}(X_v)$ at all places v.

Remark 5.4. We note that $(X_v, x_v)_{v\in I}$ is a coherent collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences if and only if for all but finitely many places
$$\alpha_v(p'_v)X_v = x_v A.$$
This follows from the compatibility condition of Definition 4.1.

Proposition 5.5. *When $(X_v, x_v)_{v\in I}$ is a coherent collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences, the map α from Proposition 5.2 satisfies the range condition that*
$$\alpha(A') \subseteq {\bigotimes_{v\in I}}' (\mathrm{End}^*_{A_v}(X_v), p_{x_v}).$$
Moreover, if (X_v, x_v) is compact, the map α from Proposition 5.2 satisfies the range condition that
$$\alpha(A') \subseteq \mathbb{K}_A(X),$$
that is, A' acts by A-compact operators on X. And finally, if (X_v, x_v) is type I, the map α from Proposition 5.2 satisfies the range condition that
$$\alpha(A') \supseteq \mathbb{K}_A(X),$$
that is, the image of A' contains all A-compact operators on X.

Proof. We write $\iota_S : X_S \to X$ for the isometry defined from the connecting isometries $\iota_S^{S'} : X_S\otimes \to X_{S'}$ defined in Equation (4.2). Because we have a coherent collection of correspondences, $\alpha_v(p'_v) = p_{x_v}$ and for any finite S the following diagram commutes:

$$\begin{array}{ccc} A'_S & \xrightarrow{\alpha_S} & \mathrm{End}^*_{A_S}(X_S) \\ \downarrow \alpha & \swarrow \mathrm{Ad}(\iota_S) & \downarrow \\ \mathrm{End}^*_A(X) & \longleftarrow & {\bigotimes}'_{v\in I}(\mathrm{End}^*_{A_v}(X_v), p_{x_v}) \end{array} \tag{5.2}$$

Here we use the functorial maps
$$\mathrm{End}^*_{A_S}(X_S) = \otimes_{v\in S}\mathrm{End}^*_{A_v}(X_v) \hookrightarrow {\bigotimes_{v\in I}}'(\mathrm{End}^*_{A_v}(X_v), p_{x_v}),$$

in the vertical right arrow and the natural inclusion ${\bigotimes}'_{v\in I}(\mathrm{End}^*_{A_v}(X_v), p_{x_v}) \hookrightarrow \mathrm{End}^*_A(X)$ in the bottom horizontal arrow. By taking the limit in the commuting diagram (5.2) we conclude that $\alpha(A') \subseteq {\bigotimes}'_{v\in I}(\mathrm{End}^*_{A_v}(X_v), p_{x_v})$.

If $(X_v, x_v)_{v\in I}$ is a compact collection of $(A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences, then the algebra A'_S acts as A_S-compact operators on X_S. Since $A_S \subseteq A$ is a subalgebra, we conclude that as soon as $\alpha_v : A'_v \longrightarrow \mathbb{K}_{A_v}(X_v)$ for all places v, we have the following range condition for any finite set of places S:

$$\alpha_S : A'_S \longrightarrow \mathbb{K}_A(X_S).$$

Since ι_S is inner-product preserving, we have a $*$-homomorphism

$$\alpha_S^X : \mathbb{K}_{A_S}(X_S) \to \mathbb{K}_A(X).$$

We can then combine these two facts with Proposition 4.7, into an argument similar to above showing that if we have a compact correspondence, then $\alpha(A') \subseteq \mathbb{K}_A(X)$.

The proof that for a type I collection we have $\alpha(A') \supseteq \mathbb{K}_A(X)$ can with Proposition 4.7 be reduced to a straightforward density argument. $\square$

Remark 5.6. The condition of coherence in Definition 5.3 is necessary in compact compatibility for the conclusion of Proposition 5.5 to hold. The example $I = \mathbb{N}$, $A'_v = M_2(\mathbb{C})$, $p'_v = 1$, $A_v = \mathbb{C}$, $p_v = 1$, $X_v = \mathbb{C}^2$ and $x_v = (1,0)^T$ provides an example where the range of the map α_v is contained in $\mathbb{K}_{A_v}(X_v)$ at all places v holds but we do not have a coherent collection of correspondences and the conclusion of Proposition 5.5 fails to hold since in this case A' is unital and X is an infinite-dimensional Hilbert space.

5.1. A digression on coherent collections of correspondences.

We discuss how the condition of coherence in a collection of correspondences in Definition 5.3 holds under some natural hypotheses.

Proposition 5.7. *Let $(X_v, x_v)_{v\in I}$ be a collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences. If for all but finitely many indices v, we have that*

1. *the balanced tensor product with X_v, i.e. $(\pi, \mathcal{H}) \mapsto (\alpha_v \otimes_\pi 1, X_v \otimes_{A_v} \mathcal{H})$, takes irreducible representations of A_v to irreducible representations of A'_v,*
2. *the projections $p'_v \in A'_v$ are of rank at most one, and*
3. *if the balanced tensor product with X_v takes an irreducible representation π_v of A_v to a representation π'_v of A'_v such that $\pi'_v(p'_v) \neq 0$, then $\pi_v(p_v) \neq 0$,*

then $(X_v, x_v)_{v\in I}$ is coherent collection of correspondences.

Proof. For a representation (π, H_π) of A_v we write $Q_\pi := p'_v \otimes 1_{H_\pi}$ and $P_\pi := p_{x_v} \otimes 1_{H_\pi}$. In case π is injective, the associated representation $\mathrm{End}^*(X_v) \to \mathbb{B}(X_v \otimes_{A_v} H_\pi)$ is injective. Since the irreducible representations separate the points of A_v, in order to show that $p'_v = p_{x_v}$ it suffices to show that have $Q_\pi = P_\pi$ for all irreducible representations (π, H_π).

By Definition 5.1, the projections $p'_v, p_{x_v} \in \mathrm{End}^*_{A_v}(X_v)$ satisfy $p'_v p_{x_v} = p_{x_v}$. For an irreducible representation (π, H_π) we thus have $Q_\pi P_\pi = P_\pi$ and $\mathrm{im}\,\mathrm{P}_\pi \subset \mathrm{im}\,\mathrm{Q}_\pi$. Therefore $Q_\pi = 0$ implies $P_\pi = 0$ and in particular then $P_\pi = Q_\pi$. When $Q_\pi \neq 0$, properties (1) and (2) give that Q_π has rank one. Moreover, property (3) implies that also $P_\pi \neq 0$, and since $\mathrm{im}\,\mathrm{P}_\pi \subset \mathrm{im}\,\mathrm{Q}_\pi$, we must have $P_\pi = Q_\pi$ as well. $\square$

5.2. **Type I representations and type I collections.** We discuss the reason for the term type I collection in Definition 5.3. We say that an irreducible representation (π, H) of a C^*-algebra is *type I* if $\pi(A) \supseteq \mathbb{K}(H)$. Note that by Guichardet's theorem (see Theorem 2.2) if $A = \bigotimes'_{v\in I}(A_v, p_v)$, for $(A_v, p_v)_{v\in I}$ a collection of type I C^*-algebras equipped with distinguished projections of rank at most one as in Subsection 2.3, then any irreducible representation (π, H) factors in the form $H = \bigotimes'_{v\in I}(H_v, h_v)$ and $\pi = \bigotimes'_{v\in I}\pi_v$. Proposition 4.7 implies that π is always type I, given that each A_v is type I.

Proposition 5.8. *Let $(X_v, x_v)_{v\in I}$ be a type I collection of $((A'_v, p'_v), (A_v, p_v))_{v\in I}$-correspondences. Then the induction map from representations of A to representations of A' defined from X as in Proposition 5.2 satisfies that on factorizable A-representations*

$$(\pi, \mathcal{H}) \simeq \left(\bigotimes{}' \pi_v, \bigotimes_v{}'(\mathcal{H}_v, h_v)\right),$$

the A'-representation

$$X \otimes_A \mathcal{H} \simeq \bigotimes{}' (X_v \otimes_{A_v} \mathcal{H}_v, x_v \otimes_{A_v} h_v),$$

is type I as soon as $(\pi, \mathcal{H})$ is.

Proof. Write $(\pi', \mathcal{H}') = (\alpha \otimes_\pi 1_\mathcal{H}, X \otimes_A \mathcal{H})$ and identify $\mathbb{K}_A(X) \otimes 1_{\mathcal{H}_\pi}$ with a subalgebra of $\mathbb{B}(\mathcal{H}')$. If π is type I, Proposition 4.7 implies that we have the inclusion

$$\mathbb{K}(\mathcal{H}') \subseteq \mathbb{K}_A(X) \otimes 1_{\mathcal{H}_\pi}.$$

Since (X_v, x_v) is type I we can by Proposition 5.5 conclude that $\mathbb{K}_A(X) \subseteq \alpha(A')$. Therefore, if π is type I, we have the inclusions

$$\mathbb{K}(\mathcal{H}') \subseteq \mathbb{K}_A(X) \otimes 1_{\mathcal{H}_\pi} \subseteq \alpha(A') \otimes 1_{\mathcal{H}_\pi} = \pi'(A').$$

This completes the proof. □

6. C^*-correspondences for adelic parabolic induction

A direct application of the results of Section 5 is to the theory of parabolic induction. For real reductive groups, parabolic induction of representations was described in terms of a C^*-correspondence in the series of works [6, 7, 8]. We will now extend this construction to the reductive groups over non-archimedean local fields of characteristic 0 and glue the local C^*-correspondences together to a global parabolic induction module.

6.1. **Induced representations.** We start with a quick recall of the theory of induction of unitary representations (for details, see [12]) after which we discuss parabolic induction (see more in [14, 20, 28]). Let G be a locally compact Hausdorff group with a closed subgroup H. Given a unitary representation (π, V_π) of H, we consider the space $L^G_H(V_\pi)$ of continuous functions $f : G \to V_\pi$ satisfying the conditions

(i) the support of f has compact image under the projection $G \to G/H$,
(ii) $f(gh) = \pi(h^{-1})f(g)$ for all $h \in H$ and $g \in G$.

We fix a quasi-invariant Borel measure μ on G/H. The Hermitian the form

$$\langle f_1, f_2\rangle := \int_{G/H} \langle f_1(g), f_2(g)\rangle_{V_\pi} d\mu(g)$$

equips $L^G_H(V_\pi)$ with a pre-Hilbert space structure. We denote the Hilbert space completion of $L^G_H(V_\pi)$ by $\mathrm{Ind}^G_H(V_\pi)$. The group G acts on $L^G_H(V_\pi)$ via left translation:

$$(g\cdot f)(g') = f(g^{-1}g').$$

To obtain a unitary representation of G on $\mathrm{Ind}^G_H(V_\pi)$, we need to tweak the action of G as follows. Let δ_G, δ_H denote the modulus characters of G and H. The function $h \mapsto \delta_G(h)/\delta_H(h)$ on H admits an extension, denoted Δ, to G. One can verify that the action

$$(g\cdot f)(g') = \Delta(g)^{1/2} f(g^{-1}g')$$

defined a unitary action of G on $\mathrm{Ind}^G_H(V_\pi)$. The equivalence class of the unitary representation $\mathrm{Ind}^G_H(\pi)$ does not depend on the choice of the quasi-invariant measure μ.

The case when G is a reductive group over the integers and $H = P$ for a parabolic subgroup $P \subseteq G$ plays a particular role. For a local field F, assume that (π, V_π) is a unitary representation of the Levi factor $L(F)$ in the Langlands factorization $P = LN$, and extend π trivially to $P(F)$ by $\pi(ln) = \pi(l)$, then $\mathrm{Ind}^{G(F)}_{P(F)}(V_\pi)$ is called a parabolically induced representation.

Parabolic induction plays an important role in parametrizing all admissible representations of G, for the precise definition of an admissible representation see [14, 20, 28]. Indeed, there is a bijective correspondence between pairs (P, π), where P is a standard parabolic subgroup and π a tempered representation of the semisimple part of $L = AM$ restricting to a character of A from the positive Weyl chamber, and admissible representations of G. The bijection takes (P, π) to the unique irreducible quotient of $\mathrm{Ind}^G_P(\pi)$. For F archimedean, this is known as Langlands classification [20, Theorem 8.54 and §XIV.17] and in the non-archimedean case, this is known as the Bernstein-Zelevinsky classification [3]; these classifications and their relation are discussed in [14]. The class of admissible representations contains all the unitary representations, but it is a hard problem to characterize the subset of all unitary representations.

6.2. **Local-global compatibility.** To go to the global picture, we discuss how infinite restricted products interact with induction. Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups as in Definition 3.1 and for all $v \in I$, we fix a closed subgroup $H_v \subseteq G_v$. As in Subsection 3.1, we can form the locally compact groups

$$\mathbb{G} := \prod\nolimits' (G_v : K_v) \quad \text{and} \quad \mathbb{H} := \prod\nolimits' (H_v : K_v \cap H_v)$$

The inclusions $H_v \hookrightarrow G_v$ allow us to view $\mathbb{H}$ as a closed subgroup of $\mathbb{G}$. Recall that by Theorem 3.3, there is a one-to-one correspondence between unitary, irreducible representations π of $\mathbb{H}$ and collections $(\pi_v, V_v)_{v\in I}$ for unitary, irreducible representations (π_v, V_v) of H_v such that for all but finitely many $v \in I$ the invariant subspace $V_v^{H_v\cap K_v}$ is one-dimensional.

Lemma 6.1. *Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups and $H_v \subseteq G_v$ closed subgroups. Assume that $\pi \simeq \bigotimes' \pi_v$ is a unitary, irreducible representation of $\mathbb{H}$ and that $\mathrm{Ind}_{H_v}^{G_v}(\pi_v)$ has one-dimensional K_v-invariant subspace for all but finitely many v. We then have*

$$\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\pi) \simeq \bigotimes_{v\in I}{}' \mathrm{Ind}_{H_v}^{G_v}(\pi_v),$$

as unitary representations of $\mathbb{G}$.

The reader should note that the assumption of Lemma 6.1 – that $\mathrm{Ind}_{H_v}^{G_v}(\pi_v)$ has one-dimensional K_v-invariant subspace for all but finitely many v – is necessary for the right hand side $\bigotimes' \mathrm{Ind}_{H_v}^{G_v}(\pi_v)$ to be well defined.

Proof. The proof is structurally analogous to that of Proposition 4.6, but given the different viewpoint of using induction rather than an induction module we provide some details. Fix K_v-invariant unit vectors $\phi_v \in \mathrm{Ind}_{H_v}^{G_v}(V_v)$, something we can do for all but finitely many v by assumption on $\mathrm{Ind}_{H_v}^{G_v}(\pi_v)$. The proof follows from noting that we can define a unitary $\mathbb{G}$-isomorphism $\bigotimes'(\mathrm{Ind}_{H_v}^{G_v}(V_v), \phi_v) \to \mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\bigotimes' V_v)$ from mapping a simple tensor $\bigotimes' f_v$, with $f_v = \phi_v$ for all but finitely many v, to $f \in \mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\bigotimes' V_v)$ defined by $f((g_v)_v) := \bigotimes' f_v(g_v)$. Since $\phi_v^0 := \phi_v|_{eH_v} \in V_v$ is $K_v \cap H_v$-invariant, f is a well defined element of $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\bigotimes' V_v)$. □

The same argument provides a more general statement in the case of $H_v \subseteq G_v$ being cocompact, as is the case for parabolic induction.

Lemma 6.2. *Let $(G_v, K_v)_{v\in I}$ be an admissible family of groups and $H_v \subseteq G_v$ closed subgroups such that K_v acts transitively on G_v/H_v for all but finitely many places. Assume that $\pi \simeq \bigotimes' \pi_v$ is a unitary, irreducible representation of $\mathbb{H}$ that for all but finitely many places admits a unit vector $\phi_v^0 \in V_v$ spanning the $K_v \cap H_v$-invariant subspace of V_v. Write $\phi_v \in \mathrm{Ind}_{H_v}^{G_v}(V_v)$ for the unit vector defined by normalizing the extension of the constant function $K_v \ni k \mapsto \phi_v^0 \in V_v$ to an element of $L_{H_v}^{G_v}(V_v)$. We then have*

$$\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\pi) \simeq \bigotimes{}'(\mathrm{Ind}_{H_v}^{G_v}(\pi_v), \phi_v),$$

as unitary representations of $\mathbb{G}$.

The element ϕ_v is well defined since when K_v acts transitively on G_v/H_v, $G_v/H_v = K_v/(K_v \cap H_v)$.

Proof. The proof goes in the same way as that of Lemma 6.1, noting that the topology of $\mathbb{G}$ ensures that the unitary $\mathbb{G}$-isomorphism $\bigotimes'(\mathrm{Ind}_{H_v}^{G_v}(V_v), \phi_v) \to \mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}(\bigotimes' V_v)$ has dense range. □

Now let G be a reductive linear algebraic group over the integers and F a number field. The product $\mu := \prod' \mu_v$ gives a Haar measure on $G(\mathbb{A})$. As G is reductive, the groups $G(F_v)$ and $G(\mathbb{A})$ are unimodular. As above we fix $K_v \subseteq G(F_v)$ as maximal compact subgroups at the infinite places and maximal compact open subgroups at the finite places. We take the Haar measures μ_v on each $G(F_v)$ normalized by $\mu_v(K_v) = 1$. Combining Theorem 3.3 with Lemma 6.2, we arrive at the following corollary.

Corollary 6.3. *Consider the situation of the preceding paragraph and assume that for each place v, we have chosen a parabolic subgroup $P_v \subseteq G(F_v)$ defined over F_v. Write $P_v = L_v N_v$ for the Langlands decomposition of P_v and*

$$\mathbb{L} := \prod{}'(L_v : K_v \cap L_v) \quad \textit{as well as} \quad \mathbb{P} := \prod{}'(P_v : K_v \cap P_v).$$

Assume that $(L_v, K_v \cap L_v)$ is a Gelfand pair for all but finitely many places v. Then given a collection $(\pi_v, V_v)_v$ if irreducible, unitary representations of L_v that forms an irreducible, unitary representation $\pi = \bigotimes' \pi_v$ of $\mathbb{L}$ we have that

$$\mathrm{Ind}_{\mathbb{P}}^{G(\mathbb{A})}(\pi) \simeq \bigotimes{}'(\mathrm{Ind}_{P_v}^{G(F_v)}(\pi_v), \phi_v),$$

as unitary representations of $G(\mathbb{A})$, where ϕ_v is the K_v-invariant unit vector defined as in Lemma 6.2.

6.3. **The local C^*-correspondence.** For the rest of this section, we will fix a parabolic subgroup $P = LN$ of G that we assume is defined over the integers.

Fix a place v of F. By an abuse of notation we write $G_v = G(F_v)$, and so on. Note that the parabolic subgroups $P_v \subseteq G_v$ and their Langlands decompositions are already over F_v. The Lie algebra of G and the relevant subgroups are defined over F, so we can also form Lie algebraic objects such as $\mathfrak{g}_v := \mathfrak{g} \otimes_F F_v$, $\mathfrak{n}_v := \mathfrak{n} \otimes_F F_v$ et cetera and the same goes for $\mathfrak{p}_v$. As in [7], we consider the $G_v \times L_v$-action on $C_c(G_v/N_v)$ defined by

$$(g, l) \cdot f(x) := \delta_v(l)^{-1/2} f(g^{-1}xl),$$

where $\delta_v : L_v \to \mathbb{R}_+$ is the character

$$\delta_v(l) := |\det(\mathrm{Ad}(l) : \mathfrak{n}_v \to \mathfrak{n}_v)|_v.$$

As in [6], we write $\mathcal{E}(G_v/N_v)$ for the completion of $C_c(G_v/N_v)$ in the $C^*(L_v)$-valued inner product defined from matrix coefficients in the right L_v-action. The right $C^*(L_v)$-Hilbert C^*-module $\mathcal{E}(G_v/N_v)$ carries a left action of $C^*(G_v)$. Following [7, Proposition 4.4], we have a left action by compact operators

$$\alpha_v : C^*(G_v) \to \mathbb{K}_{C^*(L_v)}(\mathcal{E}(G_v/N_v)).$$

More precisely, the argument for compactness in [7, Proposition 4.4] is only for a real place v but since it shows how to approximate the left action of $C_c(G_v)$ by finite rank operators defined from vectors in $C_c(G_v/N_v)$ it applies both the the full and reduced completion in any place. We conclude the following result from [6, Section 3] stating that $\mathcal{E}(G_v/N_v)$ implements parabolic induction.

Proposition 6.4. *Suppose that $(\mathcal{H}_\rho, \rho)$ is a unitary representation of L_v extended to P_v by the trivial representation on N_v. Then as unitary representations of G_v,*

$$\mathrm{Ind}_{P_v}^{G_v}(\mathcal{H}_\rho, \rho) \cong \mathcal{E}(G_v/N_v) \otimes_{C^*(L_v)} \mathcal{H}_\rho.$$

In particular, the Hilbert C^*-module $\mathcal{E}(G_v/N_v)$ implements parabolic induction from P_v to G_v.

Remark 6.5. The constructions above follow the work of Clare-Crisp-Higson [7], that covered the case of the real place. In turn, [7, Remark 4.3] discusses the relation of [7] to constructions in the p-adic case by Bezrukavnikov-Kazhdan [4]. In these works, as well as the follow up work [9], the left action of $C^*(G(\mathbb{R}))$ or the Hecke algebra of G in the p-adic case plays an important role via relations to Bernstein's second adjoint theorem describing the Jacquet functor (also known as parabolic restriction).

6.4. **The global C^*-correspondence.** We now glue together the parabolic induction correspondences built in the preceding subsection. There will in the construction be a freedom to choose parabolic subgroups independently at each place, so at each place v we shall have a parabolic subgroup $P_v \subseteq G$ defined over the integers.

To construct the restricted tensor product, we need only to prescribe distinguished vectors and projections in the finite places v. For a finite place v, we can define the vectors $x_v := \chi_{G/N(\mathcal{O}_v)} \in \mathcal{E}(G_v/N_v)$. Here $G/N(\mathcal{O}_v) \subseteq G/N(F_v) = G_v/L_v$ denotes the set of points over $\mathcal{O}_v$ of G/N.

Proposition 6.6. *The vector $x_v \in \mathcal{E}(G_v/N_v)$ is fixed by the left action of $G(\mathcal{O}_v)$ and the right action of $L(\mathcal{O}_v)$.*

Proof. It follows from the definition that x_v is fixed by $G(\mathcal{O}_v)$. Moreover, δ_v restricts to the trivial character on $L(\mathcal{O}_v)$ since the latter is compact. Therefore it follows from the definition that x_v is fixed also by $L(\mathcal{O}_v)$. □

Proposition 6.7. *Let v be a finite place. The distinguished vector $x_v := \chi_{G/N(\mathcal{O}_v)} \in \mathcal{E}(G_v/N_v)$ spans the right $C^*(L_v)$-submodule $\alpha_v(\chi_{G(\mathcal{O}_v)})X_v$.*

Proof. Take a $\xi \in C_c^\infty(G_v/N_v)$ such that $\alpha_v(\chi_{G(\mathcal{O}_v)})\xi = \xi$, or in other words, for $g \in G(\mathcal{O}_v)$ and $x \in G_v/N_v$

$$\xi(gx) = \xi(x).$$

By a similar argument as in [14, Lemma 5.2.2], there is a compact open subgroup $K_v \subseteq L(\mathcal{O}_v)$ under which ξ is right invariant. In fact, a similar argument as in [14, Lemma 5.2.2] proves the existence of a finite collection $x_1, \ldots, x_N \in G_v/N_v$ such that ξ is in the linear span of $\{\chi_{Y_i} : i = 1, \ldots N\}$ where

$$Y_i = G(\mathcal{O}_v)x_i K_v L(\mathcal{O}_v).$$

In particular, ξ is in the linear span of $x_v a_i$ where $a_i := \chi_{x_i K_v} \in C^*(L)$. Since $C_c^\infty(G_v/N_v)$ is dense in $\mathcal{E}(G_v/N_v)$ the proposition follows. □

Remark 6.8. We note that in the case at hand, the assumptions of Proposition 5.7 would read as follows

(1) parabolic induction takes irreducible representations of $C^*(L_v)$ to irreducible representations of $C^*(G_v)$,
(2) the space of $G(\mathcal{O}_v)$-fixed vectors in any irreducible unitary representation of G_v is at most one-dimensional, and
(3) if $\mathcal{E}(G_v/N_v)$ takes an irreducible unitary representation ρ_v of L_v to a representation π_v of G_v such that π_v admits a non-zero $G(\mathcal{O}_v)$-fixed vector, then ρ_v admits a non-zero $L(\mathcal{O}_v)$-fixed vector.

Item (2) follows from Proposition 3.2 (in all but finitely many places). Item (3) follows from the fact that if $\xi \in \mathcal{H}_\pi = \mathcal{E}(G_v/N_v) \otimes_{C^*(L_v)} \mathcal{H}_\rho$ is fixed under $\pi_v(G(\mathcal{O}_v))$, we can identify ξ with a smooth P_v-invariant function $G_v \to \mathcal{H}_\rho$, and $\xi_0 := \xi(1_G) \in \mathcal{H}_\rho$ provides an explicit non-zero element fixed under $\rho(L(\mathcal{O}_v))$.

The reason we do not invoke Proposition 5.7 to ensure coherence is that item (1) listed above fails. Indeed, if we consider an irreducible representation of L_v for which the Langlands quotient differs from the parabolically induced representation, the latter is not irreducible.

Proposition 6.9. *The collection of vectors $x_v := \chi_{G/N(\mathcal{O}_v)} \in \mathcal{E}(G_v/N_v)$ constructed above fits into a compactly compatible collection of C^*-correspondences in the sense of Definition 5.3 for $A_v = C^*(L_v)$, $p_v = \chi_{L(\mathcal{O}_v)}$, $A'_v = C^*(G_v)$, $p'_v = \chi_{G(\mathcal{O}_v)}$.*

Proof. Compatibility in the sense of the Definitions 4.1 and 5.1 hold by Proposition 6.6. Proposition 6.7 proves that we have a coherent collection of correspondences. By the discussion above, $C^*(G_v)$ acts by compact operators in all places, so compact compatibility holds. □

Theorem 6.10. *Let G be a reductive linear algebraic group over the integers, $P_v = L_vN_v$ be a collection of parabolic subgroups of G defined over the integers and F a number field. The local C^*-correspondences $\mathcal{E}(G_v/N_v)$ defined above glue together to a global parabolic induction module*

$$\mathcal{E}_{\mathbb{A}} := \bigotimes_v{}' \left(\mathcal{E}(G_v/N_v), x_v\right),$$

which is a right $C^*(L(\mathbb{A}))$-Hilbert C^*-module with a left action by compact operators

$$C^*(G(\mathbb{A})) \to \mathbb{K}_{C^*(L(\mathbb{A}))}(\mathcal{E}_{\mathbb{A}}).$$

If $K_v \subseteq G_v$ are such that $(L_v, K_v \cap L_v)$ is a Gelfand pair for all but finitely many places v, then the Hilbert C^*-module $\mathcal{E}_{\mathbb{A}}$ implements parabolic induction of representations of $\mathbb{L}$ to representations of $G(\mathbb{A})$ in the following sense: if $(\pi, \mathcal{H}_\pi)$ is a unitary, irreducible representation of $L(\mathbb{A})$, then as unitary representations of $G(\mathbb{A})$,

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\mathcal{H}_\pi) \cong \mathcal{E}_{\mathbb{A}} \otimes_{C^*(L(\mathbb{A}))} \mathcal{H}_\pi.$$

Proof. The first part of the theorem follows from the results of Section 5, that applies due to Proposition 6.9. The second part follows from combining Proposition 4.6 and 6.4 with Corollary 6.3. □

Arguing as in Subsection 3.4, we can give the following more concrete description of the correspondence $\mathcal{E}_{\mathbb{A}}$ constructed in Theorem 6.10,

Proposition 6.11. *Under the assumption of Theorem 6.10, we can form the locally compact space*

$$X_{\mathbb{A}} := \prod{}'(G_v/N_v : K_v) = G(\mathbb{A})/\mathbb{N}, \quad \textit{where} \quad \mathbb{N} := \prod{}'(N_v : 1).$$

There is a left action of $G(\mathbb{A})$ commuting with a proper right action by $\mathbb{L} := \prod'(L_v : K_v \cap L_v)$, making the dense, functorial inclusion

$$C_c(X_{\mathbb{A}}) \hookrightarrow \mathcal{E}_{\mathbb{A}},$$

linear for the left action by $C_c(G(\mathbb{A})) \subseteq C^*(G(\mathbb{A}))$ and the right action of $C_c(\mathbb{L}) \subseteq C^*(\mathbb{L})$.

Remark 6.12. The reader should note that by Proposition 3.4, the assumption that $(L_v, K_v \cap L_v)$ is a Gelfand pair for all but finitely many places v automatically holds if $P_v = P$ is the constant family of parabolic subgroups of G. In this case, the space in Proposition 6.11 is the homogeneous space

$$X_{\mathbb{A}} = G(\mathbb{A})/N(\mathbb{A}).$$

Remark 6.13. Theorem 6.10 provides a Hilbert C^*-module approach to global parabolic induction of adelic groups. It leaves open several questions and problems:

1. The problem of obtaining results similar to Theorem 6.10, in the vein of Langlands- and Bernstein-Zelevinsky classification, for admissible representations would rely on a framework beyond C^*–algebras.
2. Jacquet functors and Bernstein's second adjoint theorem, as discussed in Remark 6.5, can be attacked starting from the search for a left inner product. For the real place [7], there is a left inner product if we restrict to the reduced group C^*-algebra that was used in [9] for constructing adjoint functors in the case of $G = SL_2(\mathbb{R})$.
3. The problem of characterizing the unitary dual of $G(\mathbb{A})$, or already the reduced unitary dual, is by Remark 3.9 equivalent in the local case and the global case. It is not clear to the authors if elegant work such as [1, 7] can be emulated and compute the reduced unitary dual directly in the global case of $G(\mathbb{A})$.

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