The Parameterized Complexity of the Survivable Network Design Problem*

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Abstract

For the well-known Survivable Network Design Problem (SNDP) we are given an undirected graph G with edge costs, a set R of terminal vertices, and an integer demand $d_{s,t}$ for every terminal pair $s,t \in R$. The task is to compute a subgraph H of G of minimum cost, such that there are at least $d_{s,t}$ disjoint paths between s and t in H. Depending on the type of disjointness we obtain several variants of SNDP that have been widely studied in the literature: if the paths are required to be edge-disjoint we obtain the edge-connectivity variant (EC-SNDP), while internally vertex-disjoint paths result in the vertex-connectivity variant (VC-SNDP). Another important case is the element-connectivity variant (LC-SNDP), where the paths are disjoint on edges and non-terminals, i.e., they may only share terminals.

In this work we shed light on the parameterized complexity of the above problems. We consider several natural parameters, which include the solution size ℓ , the sum of demands D, the number of terminals k, and the maximum demand d_{max} . Using simple, elegant arguments, we prove the following results.

- We give a complete picture of the parameterized tractability of the three variants w.r.t. parameter ℓ : both EC-SNDP and LC-SNDP are FPT, while VC-SNDP is W[1]-hard (even in the uniform single-source case with k=3).
- We identify some special cases of VC-SNDP that are FPT:
 - when $d_{\text{max}} \leq 3$ for parameter ℓ ,
 - on locally bounded treewidth graphs (e.g., planar graphs) for parameter ℓ , and
 - on graphs of treewidth tw for parameter tw + D, which is in contrast to a result by Bateni et al. [JACM 2011] who show NP-hardness for tw = 3 (even if $d_{\text{max}} = 1$, i.e., the STEINER FOREST problem).
- The well-known DIRECTED STEINER TREE (DST) problem can be seen as single-source EC-SNDP with $d_{\text{max}} = 1$ on directed graphs, and is FPT parameterized by k [Dreyfus & Wagner 1971]. We show that in contrast, the 2-DST problem, where $d_{\text{max}} = 2$, is W[1]-hard, even when parameterized by ℓ (which is always larger than k).

1 Introduction

Network design is an algorithmic research area that investigates connecting a set of nodes in a network in the cheapest possible way. A well-known, classical example is the STEINER TREE problem, where we are given an undirected graph G = (V, E) with non-negative edge costs and a terminal set $R \subseteq V$. The aim is to find the cheapest tree in G containing all terminals of R. This is one of the first NP-hard problems given by Karp [46] in 1975. The STEINER TREE problem finds many applications (see surveys [19, 59, 61]), and can for instance model a scenario in which several nodes in a telecommunication network (e.g., the Internet) should form a subnetwork while minimizing the cost of paying for the involved connections (cf. [61]).

Many variations of STEINER TREE have been studied in this active research area (see surveys [29, 40, 44]). One well-known generalization is obtained by introducing a redundancy requirement so that terminals are connected not only with one path but several, in order to guarantee connectivity even if some parts of the network should fail. This leads to the so-called SURVIVABLE NETWORK DESIGN PROBLEM (SNDP), which is the main focus

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of this paper and is central to the area of network design (see surveys [47, 49, 58]). Formally, we are given an undirected graph G = (V, E) together with a non-negative edge cost function denoted by cost : $E \to \mathbb{R}^+$, a terminal set $R \subseteq V$, and a non-negative integer demand $d_{s,t} \in \mathbb{N}_0$ for each terminal pair $s, t \in R$. The aim is to find a subgraph $H \subseteq G$ containing R so that every terminal pair $s, t \in R$ is connected by at least $d_{s,t}$ disjoint paths in H, while minimizing $\cot(H) = \sum_{e \in E(H)} \cot(e)$. Depending on the type of path-disjointness we obtain several variants of this problem. In particular, the edge-connectivity $\lambda_H(s,t)$ between two terminals s and t in H is the maximum number of edge-disjoint paths between them, while the vertex-connectivity $\kappa_H(s,t)$ is the maximum number of internally vertex-disjoint paths. If the aim is to compute a solution H such that $\lambda_H(s,t) \geq d_{s,t}$ for every terminal pair $s, t \in R$, we obtain the edge-connectivity variant called EC-SNDP, and if the constraint is replaced by $\kappa_H(s,t) \geq d_{s,t}$ we have the vertex-connectivity variant denoted by VC-SNDP. Note that if $d_{s,t} = 1$ for all $s,t \in R$, each of these problems becomes the STEINER TREE problem, while if $d_{s,t} \in \{0,1\}$ we obtain the so-called STEINER FOREST problem (since now an optimum solution will be a forest).

Because the STEINER TREE problem is NP-hard [46] we do not expect any polynomial-time algorithms to solve SNDP. One approach to get around this issue that has been thoroughly studied for SNDP, is to find polynomial-time approximation algorithms [60, 62]: for some $\alpha > 1$, such an algorithm computes an α -approximate solution, which costs at most α times more than the optimum. Jain [45] showed that EC-SNDP has a polynomial-time 2-approximation algorithm. In this seminal work he introduced the iterative rounding technique for linear programs, which is considered a milestone in the development of approximation algorithms and has since then been applied to many different problems [50]. In contrast to EC-SNDP, it is known that VC-SNDP is much harder to approximate: given a graph with n vertices, when $d_{\max} = \max\{d_{s,t} \mid s,t \in R\}$ is polynomial in n, there is no $(2^{\log^{1-\varepsilon}n})$ -approximation [48] for any $\varepsilon > 0$, unless NP admits quasi-polynomial-time algorithms. For constant values of d_{\max} , no d_{\max}^{ε} -approximation algorithm exists [14] given that $d_{\max} \geq d_0$ for some universal constants d_0 and ε , unless P=NP. On the positive side, if k = |R|, Chuzhoy and Khanna [23] show that an $O(d_{\max}^3 \log k)$ -approximation can be computed in polynomial time. These results reflect a typical behaviour of network design (and other) problems, namely that the "vertex-version" of a problem (in this case VC-SNDP) is usually computationally harder than its "edge-version" (in this case EC-SNDP).

The algorithm by Chuzhoy and Khanna [23] for VC-SNDP exploits known results for another important variant of SNDP, for which the paths connecting terminals are supposed to be element-disjoint, where the "elements" are the non-terminals (also called Steiner vertices) and edges. That is, the element-connectivity between s and t in H, denoted by $\kappa'_H(s,t)$, is the maximum number of paths from s to t that may only share terminals, and for the element-connectivity variant called LC-SNDP the aim is to compute a minimum cost solution H for which $\kappa'_H(s,t) \geq d_{s,t}$ for all $s,t \in R$. While at first glance the element-connectivity seems more akin to the vertex-connectivity, surprisingly the iterative rounding 2-approximation algorithm for EC-SNDP can be generalized [37] to LC-SNDP, making this problem computationally similar to EC-SNDP instead of VC-SNDP.

In this paper we shed new light on these problems from the point-of-view of parameterized complexity [24], which is a different popular approach to obtain efficient algorithms for NP-hard problems: we are given a parameter $p \in \mathbb{N}_0$ and the aim is to compute an optimum solution in $f(p) \cdot n^{O(1)}$ time for some computable function f. If such an algorithm exists the problem is called fixed-parameter tractable (FPT) for parameter p, and the algorithm is correspondingly called an FPT algorithm. While for NP-hard problems we expect this function f to be super-polynomial (e.g., exponential), the rationale is that for applications in which the parameter is small such an algorithm solves the problem efficiently.

Some special cases of SNDP have been considered from the parameterized perspective before. Most prominently, the classic Dreyfus and Wagner [28] algorithm can be used to solve the STEINER TREE problem in $3^k \cdot n^{O(1)}$ time. This runtime was later improved [38] to $(2+\varepsilon)^k \cdot n^{O(\sqrt{1/\varepsilon}\log(1/\varepsilon))}$ for any $\varepsilon > 0$, and to $2^k \cdot n^{O(1)}$ if the edge weights are polynomially bounded [9, 55]. Moreover, in the special case of EC-SNDP when V = R and all vertex pairs $s, t \in V$ have uniform demand $d_{s,t} = d$ for some given d, we obtain the d-EDGE CONNECTED SUBGRAPH problem. Bang-Jensen et al. [7] show that this problem is FPT for the combined parameter of d and the size of a deletion set, i.e., the number of edges to be removed from the input graph in order to obtain a minimum cost solution. For the same parametrization, Gutin et al. [41] provide a (non-uniform) FPT algorithm for the vertex-connectivity variant called d-Vertex Connected Subgraph on unweighted graphs. The authors of [7] also note that requiring a spanning solution H (i.e., when $d_{s,t} \geq 1$ for all $s,t \in V$) makes SNDP trivially FPT when parameterizing by the solution size (i.e., the number of edges of the solution H), since any spanning

subgraph has at least n-1 edges. On the other hand, for non-spanning solutions Bang-Jensen et al. [7] prove that the parameterization by the deletion set already renders the STEINER FOREST problem W[1]-hard.

In this paper we consider the above three variants of SNDP in their full generality, and we obtain a complete classification for the natural parameterization by the solution size. As summarized by the following two theorems, our results reflect the previous findings for approximation algorithms that EC-SNDP and LC-SNDP have similar complexities and are computationally easier than VC-SNDP.

THEOREM 1.1. Both EC-SNDP and LC-SNDP can be solved in $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

In contrast, the following result shows that VC-SNDP is not FPT for parameter ℓ , unless FPT=W[1]. This hardness result is true even in the *single-source* case where we have a fixed terminal $r \in R$ called the *root* and any terminal only needs to be connected to the root, i.e., a demand $d_{s,t}$ is positive only if $r \in \{s,t\}$, and otherwise $d_{s,t} = 0$. It is known that single-source VC-SNDP has a polynomial-time $O(d_{\max}^2)$ -approximation algorithm [56, 57], improving over the known approximation for the general case. An even better $O(d \log d)$ -approximation can be computed [56, 57] for *uniform* single-source VC-SNDP, meaning that $d_{r,t} = d$ for a given d and all terminals $t \in R$. Furthermore, note that if there are only two terminals then a simple mincost flow computation will solve VC-SNDP in polynomial-time. We show that already increasing the number of terminals by one makes the problem hard.

THEOREM 1.2. Uniform single-source VC-SNDP is W[1]-hard parameterized by the number ℓ of edges of the solution, even if the number k of terminals is 3.

We remark that the previously mentioned approximation lower bound of [48] uses a reduction that together with the results of [27, 53] implies that VC-SNDP parameterized by the number k of terminals has no $f(k) \cdot n^{O(1)}$ time algorithm for any function f that computes a $k^{1/4-o(1)}$ -approximation under Gap-ETH, or computes a $k^{1/2-o(1)}$ -approximation under the Strongish Planted Clique Hypothesis. Theorem 1.2 nicely complements these hardness results, since the maximum demand d_{max} is unbounded in the reduction given in [48], and thus does not provide hardness parameterized by the solution size ℓ , as $\ell \geq d_{\text{max}}$.

In light of Theorem 1.2 we identify several special cases of VC-SNDP that are FPT. When $d_{\text{max}} \leq 2$, Fleischer [36] showed that the iterative rounding 2-approximation algorithms for EC-SNDP and LC-SNDP can be generalized to VC-SNDP. On the other hand, she also showed that this approach cannot be generalized to $d_{\text{max}} = 3$. Interestingly, we prove that our techniques to obtain the FPT algorithms of Theorem 1.1 can be generalized to $d_{\text{max}} \leq 3$ (but no further; see Section 1.1).

THEOREM 1.3. VC-SNDP can be solved in $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time if $d_{\max} \leq 3$, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

An important graph class that has been thoroughly studied in network design (and elsewhere) are planar graphs. For instance Borradaile et al. [12] show that on such graphs a polynomial-time approximation scheme (PTAS) exists for the STEINER TREE problem, and also for the STEINER FOREST problem as shown by Bateni et al. [8]. For SNDP on the other hand, to the best of our knowledge it is not known whether better approximations can be computed for planar graphs than in general. We prove that VC-SNDP is FPT for parameter ℓ on planar input graphs. In fact our result holds for the more general class of locally bounded treewidth graphs, for which any subgraph has treewidth bounded as a function of its diameter (see Section 3 for formal definitions). This also means that our result extends to apex-minor-free graphs.

THEOREM 1.4. VC-SNDP on graphs of locally bounded treewidth (e.g., planar and apex-minor-free graphs) can be solved in $f(\ell) \cdot n$ time for some function f, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

To obtain the algorithm of Theorem 1.4 we rely on an FPT algorithm parameterized by the treewidth (i.e., not locally bounded). While the STEINER TREE problem is solvable [10] in $2^{O(tw)} \cdot n$ time on graphs of treewidth tw, we cannot hope for an FPT algorithm for SNDP when using only the treewidth as a parameter, since it was shown by Bateni et al. [8] that the special case of STEINER FOREST is NP-hard for graphs of treewidth 3. Instead

we combine the parameter with the sum of demands $D = \sum_{s,t \in R} d_{s,t}$ to obtain the following result. Note that D can be linearly upper bounded by ℓ and thus D is the stronger parameter, which means that Theorem 1.2 excludes an FPT algorithm using only D.

THEOREM 1.5. VC-SNDP can be solved in $f(tw + D) \cdot n$ time for some function f, where tw is the treewidth and n the number of vertices of the input graph, and D is the sum of demands.

When considering directed graphs as inputs, several well-studied problems can be seen as special cases of SNDP, and are typically much harder than their undirected counterparts. For instance, already the special case when $d_{\rm max}=1$, i.e., the Directed Steiner Network (DSN) problem, is very hard as shown by Dinur and Manurangsi [27]: no $k^{1/4-o(1)}$ -approximation algorithm with runtime $f(k) \cdot n^{O(1)}$ exists for any computable function f, under Gap-ETH. More recently, under the Strongish Planted Clique Hypothesis, Manurangsi et al. [53] showed that no $k^{1/2-o(1)}$ -approximation can be computed with this runtime. This result is also asymptotically tight, since an $O(k^{1/2+\varepsilon})$ -approximation can be computed [17, 32] in polynomial time for any $\varepsilon > 0$. Feldmann and Marx [33] gave a complete characterization of which special cases of DSN are FPT and which are W[1]-hard parameterized by k, depending on the structure of the non-zero demands. For instance, the single-source case with $d_{max} = 1$, which is called the DIRECTED STEINER TREE (DST) problem (since an optimum solution is an arborescence), is FPT [28, 33]. However, in polynomial time no $O(\log^{2-\varepsilon} k)$ -approximation can be computed for DST [42], unless NP admits randomized quasi-polynomial-time algorithms. The currently best polynomial-time approximation algorithm [16] achieves a ratio of $O(k^{\varepsilon})$ for any constant $\varepsilon > 0$. The generalization known as d-DST is the directed edge-connectivity variant with uniform single-source demands d. For quasibipartite input graphs, where Steiner vertices form an independent set, an $O(\log k \log d)$ -approximation algorithm was obtained for d-DST by Chan et al. [15]. For 2-DST on general input graphs, Grandoni and Laekhanukit [39] give an $O(k^{\varepsilon} \log n)$ -approximation algorithm for any $\varepsilon > 0$.

While DST is FPT [28, 33] parameterized by k, we show that increasing the demand by only one compared to DST (i.e., the 2-DST problem) already yields W[1]-hardness, even when using the much weaker parameter ℓ instead of k. This is in stark contrast to undirected graphs, as witnessed by Theorems 1.1 and 1.3.

THEOREM 1.6. The 2-DST problem is W[1]-hard parameterized by the number ℓ of edges of the solution.

Our techniques. We use simple, elegant techniques to prove our theorems. The main ingredient for the FPT algorithms for EC-SNDP and LC-SNDP of Theorem 1.1 is the "reduction lemma" of Chekuri and Korula [18]. They prove that it is possible to either delete or contract any edge between two Steiner vertices without reducing the element-connectivity between any terminal pair. As we prove in Section 2.1, this implies that an optimum solution to LC-SNDP has no cycles on the Steiner vertices and can thus be decomposed into internally vertex-disjoint trees for which the leaves are the terminals. We then use the well-known colour coding technique [3] to colour all vertices of the input graph in such a way that every tree of the decomposition of the optimum solution is coloured by a unique colour. Finally, we invoke known FPT algorithms [28] for the STEINER TREE problem in order to compute each tree of the decomposition of the LC-SNDP solution independently. To solve EC-SNDP we show that our structural insights on solutions to LC-SNDP implies that optimum solutions to EC-SNDP can be decomposed into edge-disjoint trees, via a simple reduction. We can then again use colour coding, where now we colour terminals and edges, to obtain a very similar algorithm to before.

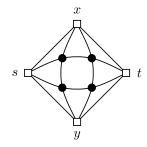


Figure 1: If the demand between every terminal pair (white boxes) is 4, then the only VC-SNDP solution in the shown graph is the whole graph, which has a cycle on the Steiner vertices (black dots).

Also our FPT algorithm of Theorem 1.3 for VC-SNDP with $d_{\text{max}} \leq 3$ uses the same approach. However, for this we need to extend the reduction lemma of Chekuri and Korula [18] to vertex-connectivity. We do this by adding some observations to their proof in Section 2.2, to show that when deleting or contracting an edge the vertex-connectivity between any terminal pair can only decrease if their connectivity is at least 4. As a consequence, any optimum solution to VC-SNDP can be decomposed into trees if the demands are

These hardness results also imply the above mentioned approximation hardness for VC-SNDP, via a reduction from DSN.

at most 3. In fact this is best possible, since there is an example with $d_{\text{max}} = 4$ for which the optimum solution cannot be decomposed into trees in this way, since there exists a cycle among the Steiner vertices (cf. Fig. 1).

The algorithm of Theorem 1.5 for the parameterization by the treewidth and sum of demands can be obtained via a straight-forward dynamic program on the tree decomposition of the graph. In the interest of simplicity, however, in this paper we do not present this dynamic program directly, as it would take a lengthy formal argument that would not add much insight into the problem. Instead, in Section 3 we express the problem using an MSOL formula and then invoke a very general Courcelle-type theorem (which at its heart also involves a dynamic program). This means that the runtime we obtain is far from optimal compared to a direct formulation of the needed dynamic program, but the proof that there is an FPT algorithm for parameter tw+D is very simple and short. Given this algorithm it is then easy to obtain the FPT algorithm of Theorem 1.4 for graphs of locally bounded treewidth parameterized by the solution size, as also detailed in Section 3.

The hardness result of Theorem 1.6 for the 2-DST problem is a rather straightforward reduction from the W[1]-hard MULTICOLOURED CLIQUE problem and can be found in Section 5. To prove hardness of VC-SNDP in Theorem 1.2 however, a standard reduction from MULTICOLOURED CLIQUE seems hard to obtain due to the undirectedness of the input graphs: it is not clear how to prevent paths from passing through the wrong gadgets. To overcome this obstacle, in Section 4 we reduce from the W[1]-hard GRID TILING problem instead, and exploit its grid structure in order to control the routes taken by the paths.

Finally, in Section 6 we list some open problems.

1.2 Additional related results. As can be seen from above, a vast literature exists on SNDP and related problems. We add just a few more closely related results here.

In addition to undirected graphs, Bang-Jensen et al. [7] also show that in directed input graphs both d-EDGE and d-Vertex Connected Subgraph are FPT parameterized by d and the size of a deletion set. While a brute-force algorithm for these problems can find an optimum solution in $2^{O(dn(\log d + \log n))}$ time, for d-EDGE Connected Subgraph this was improved to a single-exponential $2^{O(dn)}$ runtime by Agrawal et al. [1] for both directed and undirected graphs.

The STEINER TREE problem was shown to admit a $(\ln(4)+\varepsilon)$ -approximation algorithm in the seminal work of Byrka et al. [13]. It is also known that this problem is APX-hard [22]. Using an FPT algorithm [28] for STEINER TREE it is not hard to prove that also STEINER FOREST is FPT for the number of terminals (even in single-exponential time; cf. [21]). For the parameterization by the number of Steiner vertices in the optimum solution, a folklore result says that STEINER TREE is W[2]-hard (cf. [24, 30]). However a parameterized approximation scheme exists for this parameter [30]. Similar results have been found for special cases of DSN [21].

A useful and well-known graph operation introduced by Lovász [51] is called *splitting-off* and entails replacing two edges uv and wv incident to a vertex v by a direct edge uw. This operation can be performed in such a way that the edge-connectivity of the graph remains unchanged, i.e., it preserves the global edge-connectivity. Mader [52] generalized this to preserve local edge-connectivity, i.e., the edge-connectivity between every vertex pair remains unchanged. In a similar spirit, Hind and Oellermann [43], and independently later also Cheriyan and Salavatipour [20], introduced the global reduction lemma, which entails contracting or deleting an edge between Steiner vertices such that the element-connectivity of a terminal set remains unchanged. Chekuri and Korula [18] then generalized this to preserve local element-connectivity, and this is the starting point for our FPT algorithm for EC-SNDP and LC-SNDP.

Finally, we note that several works consider variations of SNDP where $d_{s,t} > 1$ for all $s, t \in X$ for a constant size set X, while the demands between all other pairs of vertices in the graph are in $\{0,1\}$ or equal to 1. For example, Balakrishnan et al. [6] gave a 2-approximation algorithm when |X| = 2 and all remaining demands are in $\{0,1\}$, assuming the costs satisfy the triangle inequality. Arkin and Hassin [4] showed that when |X| = 2 and all other demands are 1, the problem can be solved in polynomial time. They also described approximation algorithms for various other cases that require a specific size of X, the demands on pairs in X to be a specified constant, and all other demands to be 1.

2 Tractability of general SNDP parameterized by solution size

In this section we prove Theorems 1.1 and 1.3. We first show how to obtain an FPT algorithm for LC-SNDP, and how this also leads to an algorithm for EC-SNDP. For VC-SNDP with $d_{\text{max}} \leq 3$ we need to generalize some of the used arguments.

2.1 Element- and edge-connectivity SNDP. Chekuri and Korula [18] proved the so-called reduction lemma that preserves element-connectivity under deletion or contraction of edges. Here, deleting an edge simply means to remove it from the graph, while contracting an edge means to identify its incident vertices and removing all resulting loops and parallel edges. In the following, for an edge $e \in E$, as usual G - e and G/e denote the graph obtained from G by deleting e and contracting e, respectively.

LEMMA 2.1. (LC REDUCTION LEMMA [18]) Let G = (V, E) be an undirected graph and $R \subseteq V$ be a terminal set. Let $e \in E$ be any edge where $e \cap R = \emptyset$, and let $G_1 = G - e$ and $G_2 = G/e$. Then at least one of the following holds:

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 \begin{array}{l} (i) \ \forall s,t \in R: \ \kappa'_{G_1}(s,t) = \kappa'_G(s,t), \ or \\ (ii) \ \forall s,t \in R: \ \kappa'_{G_2}(s,t) = \kappa'_G(s,t). \end{array}
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Chekuri and Korula [18] remark that their reduction lemma can be applied repeatedly until the Steiner vertices form an independent set. By subdividing edges between terminals, we may also assume that the terminals form an independent set. We will exploit this to prove the following structure of *minimal* solutions to LC-SNDP, which are solutions for which no edge can be removed without making the solution infeasible. In particular, any optimum solution is minimal.

LEMMA 2.2. Let H be a minimal solution to an LC-SNDP instance. Then there exist trees $T_1, \ldots, T_b \subseteq H$ such that $H = \bigcup_{i=1}^b T_i$, no two trees share a Steiner vertex, all leaves of any tree are terminals, and all internal vertices of any tree are Steiner vertices. Moreover, for any terminal pair $s, t \in R$ there exist $d_{s,t}$ element-disjoint paths between s and t in H, such that the edge set of any tree T_i , $i \in \{1, \ldots, b\}$, intersects with the edge set of at most one of these paths.

Proof. To prove the claim, we repeatedly apply Lemma 2.1 on the minimal solution H. In particular, let H_0, H_1, \ldots, H_h be any sequence of graphs we obtain from $H_0 = H$ as follows. For each $j \geq 0$ we pick some edge e of H_j where $e \cap R = \emptyset$ and either delete or contract e without decreasing the element-connectivity between any terminal pair, to obtain H_{j+1} . Note that $R \subseteq V(H_j)$ for all j. According to Lemma 2.1 we can do this for every H_j , until every edge is incident to some terminal.

We claim that in each step j we can only contract the chosen edge e when applying Lemma 2.1, as deleting e will be decreasing the element-connectivity between some terminal pair. The proof is by induction. In the first step j=0 this is true since $H_0=H$ is a minimal solution. For a step $j\geq 1$, assume that some edge e of H_j can be deleted without reducing the element-connectivity between any terminal pair. This means that there exist $\kappa'_{H_j}(s,t)$ element-disjoint paths in H_j between each terminal pair $s,t\in R$ such that none of these paths contain e. By the induction hypothesis, some edge e' was contracted in H_{j-1} to obtain H_j . The element-disjoint paths in H_j also exist in H_{j-1} , after uncontracting the edge e' along some of these paths. Also the edge e exists in H_{j-1} (or an edge corresponding to e after uncontracting e') but is not used by any of the resulting paths in H_{j-1} . But then e can be deleted in H_{j-1} without reducing the element-connectivity between any terminal pair, contradicting the induction hypothesis.

Now consider the graph H' obtained from H after exhaustively contracting edges between Steiner vertices. From above we know that by Lemma 2.1, $\kappa'_H(s,t) = \kappa'_{H'}(s,t)$ for all terminal pairs $s,t \in R$. Note that in H' the remaining Steiner vertices form an independent set. Furthermore, w.l.o.g. no two terminals are adjacent in H (otherwise we can subdivide such an edge using a Steiner vertex). Hence, we may assume that H' is a bipartite graph, with R and $V \setminus R$ forming the bipartition. Thus we can decompose H' into edge-disjoint stars S_1, \ldots, S_b (which partition the edge set of H') such that no two stars share a Steiner vertex.

Now fix $\kappa'_H(s,t)$ element-disjoint paths between each terminal pair s,t in H'. No edge set of a star S_i , $i \in \{1,\ldots,b\}$, can intersect with more than one of the $\kappa'_H(s,t)$ paths for any terminal pair $s,t \in R$, as these paths would otherwise intersect in the center vertex of the star, which is a Steiner vertex. Note that for each of these $\kappa'_H(s,t)$ paths there exist $\kappa'_H(s,t)$ corresponding element-disjoint paths in H, by uncontracting edges along the paths. Let T_i be the subgraph of H that is obtained from S_i after uncontracting all edges. Also the edge set of T_i intersects with at most one of the $\kappa'_H(s,t)$ paths between any s and t. This means that if T_i contains a cycle, then we may delete any edge of the cycle, while maintaining a path between s and t through T_i (if the path passed through the deleted edge, we may reroute it through the remaining edges of the cycle). However this would contradict the minimality of H, and thus T_i is a tree, which concludes the proof. \Box

We also need a similar structural result for minimal EC-SNDP solutions. Note that in contrast to Lemma 2.2, the trees of the following lemma are edge-disjoint instead of internally vertex-disjoint.

LEMMA 2.3. Let H be a minimal solution to an EC-SNDP instance. Then there exist trees $T_1, \ldots, T_b \subseteq H$ such that $H = \bigcup_{i=1}^b T_i$, no two trees share an edge, all leaves of any tree are terminals, and all internal vertices of any tree are Steiner vertices. Moreover, for any terminal pair $s, t \in R$ there exist $d_{s,t}$ edge-disjoint paths between s and t in H, such that the edge set of any tree T_i , $i \in \{1, \ldots, b\}$, intersects with the edge set of at most one of these paths.

Proof. We use a standard reduction² from EC-SNDP to LC-SNDP and then invoke Lemma 2.2. The reduction takes every Steiner vertex v of the input graph G of the instance I to EC-SNDP and replaces it by a clique K_v of size equal to the degree of v in G, in order to obtain a new graph G'. Two such cliques K_u and K_v are then connected by an edge in G' if uv was an edge in G, and every terminal in G' is connected to the cliques corresponding to its Steiner neighbours in G (edges between terminals are untouched). Note that it is possible to obtain these connections in such a way that every vertex of a clique K_v has exactly one neighbour outside the clique in G', since the size of the clique is equal to the degree of v in G. The new instance G' of LC-SNDP is given by G' where the edges of cliques all have cost 0 and all other edges have cost corresponding to their edge in the instance G'. It is now easy to see that there is an LC-SNDP solution G' in the new graph if and only if there is an EC-SNDP solution G' in the original graph: to convert a solution G' to a solution in G' we simply contract all edges that belong to a clique G', which means that any two element-disjoint paths of G' we can just add all edges of every clique G', which means that any two edge-disjoint paths of G' that meet in a Steiner vertex G' can be extended by using two vertex-disjoint edges of G' to make the paths element-disjoint.

By Lemma 2.2 we know that any minimal solution H' to LC-SNDP in G' can be decomposed into internally vertex-disjoint trees $T'_1, \ldots, T'_b \subseteq H'$. When converting the solution H' to a solution H in G as described above, these trees are converted into edge-disjoint trees $T_1, \ldots, T_b \subseteq H$ with the required properties. \square

Using colour coding and known FPT algorithms for STEINER TREE, we exploit Lemmas 2.2 and 2.3 to obtain the following result.

Theorem 1.1. Both EC-SNDP and LC-SNDP can be solved in $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

Proof. We first consider the LC-SNDP problem. Recall that by Lemma 2.2, an optimum solution H can be partitioned into b internally vertex-disjoint trees T_1, \ldots, T_b for some b. Note that b is bounded by ℓ , and recall that the trees only overlap on the terminals. For a terminal $t \in R$, let c_t denote the number of trees incident on t. Then we observe that $f := \sum_{t \in R} c_t \leq 2\ell$, because any terminal is incident on a tree T_i via a unique edge of H, unless this edge goes between two terminals.

We now describe the algorithm. Given an instance to LC-SNDP, we first guess the number b of trees into which the optimum solution H can be partitioned according to Lemma 2.2. Then we guess the total number f of edges of the trees incident on the terminals of G. We now use colour coding [3]. First, we randomly colour the Steiner vertices of the input graph G = (V, E) with b colours. Then we randomly colour the terminals of G by creating a bin with k balls of each of the b colours (so kb balls in total), randomly taking f balls from the bin, and if the j-th ball of colour $i \in [b]$ was taken, assigning the j-th terminal colour i. Note that terminals are thus assigned a set of colours. That is, we pick a function φ that maps $V \setminus R$ to [b] and R to $2^{[b]}$, such that $\sum_{t \in R} |\varphi(t)| = f$. We then condition on the event that every Steiner vertex $v \in V(H) \setminus R$ of $H \subseteq G$ has colour $\varphi(v) = i$ if it belongs to tree T_i of the decomposition, and that every terminal $t \in R$ is coloured with a subset $\varphi(t) = C$ of [b] such that $i \in C$ if and only if the terminal belongs to tree T_i . For every colour $i \in [b]$ we then define the graph G_i induced by all vertices of colour i in G, and compute an optimum Steiner tree for the terminal set $R_i = \{t \in R \mid i \in \varphi(t)\}$ of G_i . Such a tree can be computed using an FPT algorithm for STEINER

 $^{^2}$ In the conference version of this paper [34] it was claimed that the parameter ℓ only grows by a factor of 2 in this reduction, which if true would mean that it could be used in combination with the FPT algorithm for LC-SNDP to obtain an algorithm for EC-SNDP with the same asymptotic running time. However, the parameter growth might in fact be quadratic, leading to a worse runtime than claimed in Theorem 1.1 for EC-SNDP using this direct approach.

Tree parameterized by the number of terminals (e.g., using the Dreyfus and Wagner [28] algorithm). The union of these Steiner trees is the computed LC-SNDP solution.

We now argue that the above algorithm outputs an optimum LC-SNDP solution, conditioned on the correct colouring. By Lemma 2.2 there exist $d_{s,t}$ element-disjoint paths between any terminal pair $s,t \in R$ in any optimum solution H, such that each tree T_i of the decomposition contains at most one such path. If $x, y \in R$ are terminals of T_i used by such a path P between s and t in H, then the Steiner tree computed for G_i also contains a path between x and y. Hence, we can find a path corresponding to P in the computed solution that contains edges of a Steiner tree computed for G_i if and only if P contains edges of T_i . The colour coding ensures that the computed Steiner trees do not share Steiner vertices. This means that we can find $d_{s,t}$ element-disjoint paths between s and t in the union of the computed Steiner trees. Furthermore, the computed solution must have minimum cost, since otherwise some tree $T_i \subseteq H$ could be replaced by an optimum Steiner tree of G_i , yielding a feasible solution of smaller cost.

The success probability of the above algorithm is the probability with which the algorithm picks a colouring φ such that every Steiner vertex v of H has colour $\varphi(v) = i$ if it belongs to tree T_i , and every terminal $t \in R$ has set of colours $\varphi(t) = C$ such that t belongs to tree T_i if and only if $i \in C$. Note that we may permute the b colours arbitrarily without affecting the success probability. Hence, there are $b! \cdot b^{n-|V(H)|}$ correct ways to colour the input graph G, since there are n-|V(H)| vertices in G that are not part of H, and all of these are Steiner vertices that each receive one colour from [b], while the vertices in H are assigned a unique (set of) colours (determined by the permutation). The total number of ways to colour the input graph is $\binom{kb}{f}b^{n-k}$, since there are $\binom{kb}{f}$ possible ways to assign sets of colours to the terminals, and b ways to colour each of the n-k Steiner vertices. Thus the success probability is at least

$$\frac{b! \cdot b^{n-|V(H)|}}{(kb)^f b^{n-k}} \geq \frac{b!}{k^{2\ell}} b^{k-4\ell} \geq 2^{-O(\ell \log \ell)},$$

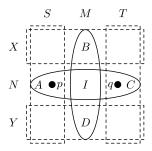
since $b, k \in \{1, \dots, \ell\}$, and $|V(H)|, f \leq 2\ell$. Using a standard argument, we may run this algorithm $2^{O(\ell \log \ell)} \cdot n$ times in order to compute the optimum solution with high probability. Since the Dreyfus and Wagner [28] algorithm has a single-exponential runtime in the number of terminals, which is upper bounded by the solution size, each optimum Steiner tree can be computed in $2^{O(\ell)} \cdot n^{O(1)}$ time. Furthermore, the number of trees to be computed is also at most the solution size, and thus this randomized algorithm takes $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time. Alternatively, we may derandomize [3] the colour coding algorithm to compute the optimum deterministically in $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time.

For EC-SNDP we use essentially the same technique, but instead of colouring Steiner vertices we colour edges. That is, based on the existence of b edge-disjoint trees T_1, \ldots, T_b as given by Lemma 2.3, we pick a function φ that maps E to [b] and R to $2^{[b]}$, such that $\sum_{t\in R} |\varphi(t)| = f$ for the corresponding value f. We then compute an optimum Steiner tree in each subgraph G_i spanned by edges of colour i for terminal set $R_i = \{t \in R \mid i \in \varphi(t)\}$. An analogous argument to above then shows that due to the properties of Lemma 2.3 the union of these Steiner trees is an optimum EC-SNDP solution, conditioned on the event that $\varphi(e) = i$ for each edge e of tree T_i and that every terminal $t \in R$ is coloured by a subset $\varphi(t) = C$ of [b] such that $i \in C$ if and only if t belongs to T_i . To calculate the success probability, note that, if m is the number of edges of the input graph, there are now $b! \cdot b^{m-\ell}$ correct ways to colour the graph, while the total number of ways to colour the input graph is $\binom{kb}{f}b^m$. Thus the success probability is again at least $2^{-O(\ell \log \ell)}$, leading to the same asymptotic running time as above for the FPT algorithm.

Vertex-connectivity SNDP. In order to prove Theorem 1.3 for VC-SNDP with $d_{\text{max}} \leq 3$, we extend the reduction lemma of Chekuri and Korula [18] by showing that when contracting an edge of G, the vertex connectivity $\kappa(s,t)$ between any terminals s, t never drops below 3 if its connectivity was larger in G.

LEMMA 2.4. (VC REDUCTION LEMMA) Let G = (V, E) be an undirected graph and $R \subseteq V$ be a terminal set. Let $e \in E$ be any edge where $e \cap R = \emptyset$, and let $G_1 = G - e$ and $G_2 = G/e$. Then at least one of the following

- (i) $\forall s, t \in R : \kappa_{G_1}(s, t) = \kappa_G(s, t)$, or
- (ii) $\forall s, t \in R$:
 - if $\kappa_G(s,t) \ge 4$, then $\kappa_{G_2}(s,t) \in {\kappa_G(s,t) 1, \kappa_G(s,t)}$, and if $\kappa_G(s,t) \le 3$, then $\kappa_{G_2}(s,t) = \kappa_G(s,t)$.



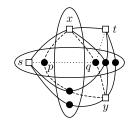


Figure 2: The structure of the vertex-tri-partitions of Proposition 2.1 on the left. A possible setup of the terminals (white boxes) on the right, where the dotted path does not exist in G_1 but in G. The dashed paths between x and y imply $|B|, |D| \ge 1$, while $s, p \in A$ implies $|A| \ge 2$, and $t \in T \cap X$ implies $|C| \ge |D| + 1$. In particular, as $|N| \ge |A| + |C|$, in this example we have $\kappa_G(x, y) \ge 5$.

Proof. We extend the proof of Lemma 2.1 by Chekuri and Korula [18]. In particular, we use the same setup, which we summarize below. Here a vertex-tri-partition (A, B, C) of a graph is a partition of its vertex set into non-empty parts A, B, and C, such that B separates A and C, i.e., every path from A to C contains a vertex of B. Also, for vertices u and v, a (u, v)-separator is a vertex set B such that there exists a vertex-tri-partition (A, B, C) with $u \in A$ and $v \in C$ (in particular, u and v are not contained in the separator). We will repeatedly use the well-known fact that the maximum number of internally vertex-disjoint paths between u and v is equal to the size of a minimum (u, v)-separator, i.e., Menger's theorem [26].

Note that in Proposition 2.1 below, the vertex-tri-partitions and separators are of the graph $G_1 = G - e$. We remark that all of the following properties are true regardless of whether we consider vertex- or element-connectivity. In contrast to element-connectivity though, as studied by Chekuri and Korula [18], for vertex-connectivity it may happen that a minimum (s, t)-separator for a terminal pair $s, t \in R$ contains other terminals (see Fig. 2).

PROPOSITION 2.1. ([18]) Let $e = \{p, q\}$ be the edge that is deleted or contracted to obtain G_1 and G_2 . Assuming there are terminal pairs $s, t \in R$ and $x, y \in R$ such that $\kappa_{G_1}(s, t) = \kappa_G(s, t) - 1$ and $\kappa_{G_2}(x, y) = \kappa_G(x, y) - 1$, the following holds:

- 1. every minimum (s,t)-separator of G_1 is also a (not necessarily minimum) (p,q)-separator of G_1 ,
- 2. there is a vertex-tri-partition (S, M, T) of G_1 such that $s \in S$, $t \in T$, and M is a minimum (s, t)-separator in G_1 with $|M| = \kappa_{G_1}(s, t) = \kappa_{G}(s, t) 1$ where, w.l.o.g., $p \in S$ and $q \in T$,
- 3. there is a vertex-tri-partition (X, N, Y) of G_1 such that $x \in X$, $y \in Y$, and N is a minimum (x, y)-separator in G_1 with $|N| = \kappa_{G_1}(x, y) = \kappa_{G}(x, y)$ where $p, q \in N$, and
- 4. if A, B, C, D, and I respectively denote $S \cap N$, $X \cap M$, $T \cap N$, $Y \cap M$, and $N \cap M$, we have $p \in A$ and $q \in C$, and
 - if $s \in S \cap X$ then $|A| \ge |D| + 1$,
 - if $s \in S \cap Y$ then $|A| \ge |B| + 1$,
 - if $t \in T \cap X$ then $|C| \ge |D| + 1$,
 - if $t \in T \cap Y$ then |C| > |B| + 1.

Proof. For completeness we provide a short proof of this proposition as can also be found in [18]. We proceed point by point:

- 1. holds since otherwise it would be an (s,t)-separator of size $\kappa_{G_1}(s,t)$ in G so that $\kappa_G(s,t) \leq \kappa_{G_1}(s,t)$,
- 2. holds since M is (p,q)-separator due to point 1,
- 3. holds since in G_2 every minimum (x,y)-separator N' of size $\kappa_{G_2}(x,y) = \kappa_G(x,y) 1$ must contain the vertex into which e is contracted, and so any (x,y)-separator of G_1 has size at least $\kappa_G(x,y)$; the one in G_1 corresponding to N' contains p and q, has size $\kappa_G(x,y)$, and is thus minimum,
- 4. $p \in A$ and $q \in C$ holds due to the previous points, and
 - holds since then $A \cup I \cup B$ is an (s,t)-separator, but not a minimum one due to point 1 as $p \in A$, in contrast to $M = B \cup I \cup D$, so that $|B \cup I \cup D| < |A \cup I \cup B| 1$,
 - ditto, since then $A \cup I \cup D$ is a non-minimal (s,t)-separator as $p \in A$,

- ditto, since then $B \cup I \cup C$ is a non-minimal (s,t)-separator as $q \in C$,
- ditto, since then $C \cup I \cup D$ is a non-minimal (s, t)-separator as $q \in C$.

Now assume that the statement of the lemma does not hold. This means that there exists a terminal pair $s, t \in R$ for which $\kappa_{G_1}(s, t) \neq \kappa_{G}(s, t)$ and a terminal pair $x, y \in R$ for which either

- $\kappa_G(x,y) \ge 4$ and $\kappa_{G_2}(x,y) \notin {\kappa_G(x,y) 1, \kappa_G(x,y)}$, or
- $\kappa_G(x,y) \leq 3$ and $\kappa_{G_2}(x,y) \neq \kappa_G(x,y)$.

Observe that deleting or contracting e can only decrease the connectivity between any terminal pair by 1, and thus $\kappa_{G_1}(s,t)=\kappa_G(s,t)-1$ and $\kappa_{G_2}(x,y)\in\{\kappa_G(x,y)-1,\kappa_G(x,y)\}$. Hence, only the latter of the two cases can apply to x and y, i.e., we have $\kappa_G(x,y)\leq 3$ and $\kappa_{G_2}(x,y)=\kappa_G(x,y)-1$. Furthermore, we obtain the properties of Proposition 2.1. We will show that both B and D contain at least one vertex each, and as a consequence A and C contain at least two vertices each. The latter implies that $|N|\geq |A|+|C|\geq 4$ and thus $\kappa_G(x,y)=\kappa_{G_1}(x,y)\geq 4$ — a contradiction.

To prove that $B, D \neq \emptyset$, fix $\kappa_{G_1}(x, y)$ internally vertex-disjoint paths between x and y in G_1 . By point 3 of Proposition 2.1, there are two such paths P, Q such that P contains $p \in N$ and Q contains $q \in N$ (cf. Fig. 2). Since each of these paths can only contain one vertex of the minimum (x, y)-separator N (due to Menger's theorem), the union of these two paths forms a cycle containing two paths from p to q, such that the internal vertices of one path are only from X (including x) and the internal vertices of the other are only from Y (including y). Since M separates p and q by point 1 of Proposition 2.1, the first path contains a vertex of $B = M \cap X$, while the second one contains a vertex of $D = M \cap Y$, i.e., both B and D are non-empty.

This implies that A and C contain two vertices each, as follows. If $t \in C$ then $|C| \ge 2$, since $q \in C$ by point 4 of Proposition 2.1, but $q \ne t$ as q is not a terminal in contrast to t. Otherwise, $t \in T \cap X$ or $t \in T \cap Y$, which by Proposition 2.1 means that $|C| \ge |D| + 1$ or $|C| \ge |B| + 1$. From above we know that $|B| \ge 1$ and $|D| \ge 1$ and so $|C| \ge 2$ in either case. Analogously for A, if $s \in A$ we are done as $p \in A$ and $p \ne s$, and if $s \notin A$ we get $|A| \ge |B| + 1$ or $|A| \ge |D| + 1$ by Proposition 2.1. Hence, we also have $|A| \ge 2$, which concludes the proof.

As a consequence, we obtain the same structure of minimal solutions for VC-SNDP with $d_{\text{max}} \leq 3$ as for LC-SNDP, by simply replacing the application of Lemma 2.1 by Lemma 2.4 in the proof of Lemma 2.2 to show the following.

LEMMA 2.5. Let H be a minimal solution to a VC-SNDP instance with maximum demand at most 3. Then there exist trees $T_1, \ldots, T_b \subseteq H$ in H such that $H = \bigcup_{i=1}^b T_i$, no two trees share a Steiner vertex, all leaves of any tree are terminals, and all internal vertices of any tree are Steiner vertices. Moreover, for any terminal pair $s, t \in R$ there exist $d_{s,t}$ vertex-disjoint paths between s and t in H, such that the edge set of any tree T_i , $i \in \{1, \ldots, b\}$, intersects with the edge set of at most one of these paths.

Lemma 2.5 implies the same FPT algorithm parameterized by the solution size as for LC-SNDP as found in the proof of Theorem 1.1, and thus we obtain the following theorem.

THEOREM 1.3. VC-SNDP can be solved in $2^{O(\ell \log \ell)} \cdot n^{O(1)}$ time if $d_{\max} \leq 3$, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

3 Tractability of VC-SNDP for treewidth and planar graphs

We first prove that VC-SNDP is FPT parameterized by the treewidth tw plus the sum of demands D. To this end, we provide a simple formulation of the problem in MSOL and apply a result by Arnborg et al. [5]. We then argue that this implies that VC-SNDP is FPT on planar graphs parameterized by the solution size ℓ .

For sake of completeness, we define treewidth and describe the result of Arnborg et al. [5] that we rely on.

DEFINITION 3.1. A tree decomposition of a graph G = (V, E) is a tree T and a family B of subsets of vertices (called bags), one bag B(t) per vertex t of the tree, such that the following holds:

- 1. $\bigcup_{t \in V(T)} B(t) = V;$
- 2. for each edge $uv \in E$, there is a $t \in V(T)$ such that $u, v \in B(t)$;
- 3. for each vertex $v \in V$, the vertices $t \in V(T)$ for which $v \in B(t)$ induce a subtree of T.

The width of a tree decomposition is $\max_{t \in V(T)} |B(t)| - 1$. The treewidth of a graph is the minimum width over any of its tree decompositions.

The result of Arnborg et al. [5, Theorem 5.6] applies Monadic Second Order Logic (MSOL) to graphs. An MSOL formula is a logical formula, allowing universal and existential quantifiers over variables that are single vertices or edges, variables that are sets of vertices or edges, basic logical formulas $(\lor, \land, \neg, \rightarrow)$, basic binary relations $(\in, =)$, and the special binary relation $\operatorname{inc}(v, e)$ that is true if and only if edge e is incident on vertex v. Note that using the primitives of MSOL, it is easy to define other common primitives, such as \notin, \neq, \subseteq , on the way to creating more complex formulas.

THEOREM 3.1. ([5, THEOREM 5.6]) Let G = (V, E) be an n-vertex graph of treewidth tw with a constant number of special sets S_1, \ldots, S_q of vertices or edges. Let ϕ be an MSOL formula that uses S_1, \ldots, S_q and with p free set variables X_1, \ldots, X_p , let f_1, \ldots, f_p be p functions with the same domain as X_1, \ldots, X_p respectively that assign integer values, and let F be a linear function on p variables. Then there is an algorithm that finds in $f(tw, |\phi|) \cdot n$ time the minimum or maximum value of $F(\sum_{x \in X_1} f_1(x), \ldots, \sum_{x \in X_p} f_p(x))$ for sets X_1, \ldots, X_p such that $\phi(X_1, \ldots, X_p)$ is satisfied, for some function f.

We now formulate VC-SNDP as an MSOL formula to obtain the following result.

THEOREM 1.5. VC-SNDP can be solved in $f(tw + D) \cdot n$ time for some function f, where tw is the treewidth and n the number of vertices of the input graph, and D is the sum of demands.

Proof. In this proof, we only consider terminal pairs $s,t \in R$ for which $d_{s,t} > 0$ (often implicitly). Let z be the number of terminal pairs $s,t \in R$ for which $d_{s,t} > 0$, so $z \leq {k \choose 2}$. Order the pairs of distinct terminals of R arbitrarily. Use S_1, \ldots, S_z to denote the sets of terminal pairs, that is, $S_i = \{s_i, t_i\}$ where s_i, t_i is the i-th terminal pair (with $d_{s_i,t_i} > 0$). We define the following primitive to describe that, given a set of vertices V_P and a set of edges X, is true if and only if there exists a path between the terminal pair s_i, t_i on the vertices of V_P that uses a subset of X.

$$\operatorname{path}_{i}(V_{P}, X) = S_{i} \subseteq V_{P} \land \forall Z \subseteq V_{P} \Big(\big(\exists u \in S_{i} : u \in Z \land \exists v \in S_{i} : v \notin Z \big) \to \Big(\exists e \in X \ \exists u, v \in V_{P} : \operatorname{inc}(u, e) \land \operatorname{inc}(v, e) \land u \in Z \land v \notin Z \Big) \Big).$$

Note that this formula ensures that $s, t \in V_P$ and that for any partition of V_P that separates s_i and t_i , there is an edge of X crossing the partition. Hence, the subgraph $(V_P, X \cap (V_P \times V_P))$ contains an s_i - t_i path as required (by Menger's theorem [26]).

We also define a primitive disjoint_i $(V_1, ..., V_j)$, which is true if and only if the input sets $V_1, ..., V_j$ are pairwise disjoint except that all sets contain S_i as a subset. We omit the straightforward definition.

We can then use these primitives for the following function:

$$\phi(X) = X \subseteq E \land \forall i \in \{1, \dots, z\} \ \exists V_1, \dots, V_{d_{s_i, t_i}} \subseteq V :$$

$$\left(\text{disjoint}_i(V_1, \dots, V_{d_{s_i, t_i}}) \land \forall j \in \{1, \dots, d_{s_i, t_i}\} : \text{path}_i(V_j, X) \right).$$

Here we are using $\forall i \in \{1, ..., z\}$: ϕ_i as a shortcut for $\phi_1 \wedge \cdots \wedge \phi_z$. Each ϕ_i must be explicitly written out, because the demands d_{s_i,t_i} are not an input to the formula or the graph structure. This is also the reason why we need the sets S_i . Therefore, the length of $\phi(X)$ depends only on $D = \sum_{i=1}^{z} d_{s_i,t_i}$.

Assume now that a tree decomposition is known; otherwise, compute one using Bodlaender's algorithm [11]. Then, apply the minimization version of Theorem 3.1 on ϕ with f_1 being the edge cost function, F the identity function, and S_1, \ldots, S_z being the special sets. The theorem follows.

We now show that our result extends to graphs of bounded local treewidth for parameter solution size. A graph has bounded local treewidth if there is a function g such that the treewidth of any subgraph induced by the vertices within (shortest path) distance r of any vertex is at most g(r). Alber et al. [2] showed that planar graphs have bounded local treewidth for a linear function g. Eppstein [31] proved that the graphs of bounded local treewidth are exactly the apex-minor-free graphs (graphs that exclude a fixed apex graph as a minor) and Demaine and Hajiaghayi [25] showed that the function g is linear here as well.

THEOREM 1.4. VC-SNDP on graphs of locally bounded treewidth (e.g., planar and apex-minor-free graphs) can be solved in $f(\ell) \cdot n$ time for some function f, where ℓ is the number of edges of the solution and n is the number of vertices of the input graph.

Proof. Observe that all edges of any solution must be within (shortest path) distance at most ℓ of some terminal. Hence, we may restrict the graph to all vertices and edges within distance ℓ of any terminal and remove all others. The resulting graph has diameter $\ell \cdot k \leq \ell^2$ and thus has treewidth bounded as a function of ℓ . The result then follows from Theorem 1.5, as $D \leq 2\ell$.

4 Hardness of VC-SNDP

In this section we prove Theorem 1.2, i.e., that uniform single-source VC-SNDP with k=3 terminals is W[1]-hard parameterized by the solution size ℓ . We give a reduction from the GRID TILING problem, where the input consists of integers K, n and a collection of K^2 non-empty sets $S_{i,j} \subseteq [n] \times [n]$ of integer pairs for $i, j \in [K]$. We think of such an instance as a $K \times K$ grid, where each grid cell contains a set $S_{i,j}$ of integer pairs. The goal is to find one ordered pair $s_{i,j} \in S_{i,j}$ for every $i, j \in [K]$, such that if (i,j) and (i',j') are adjacent in the first or the second coordinate, then $s_{i,j}$ and $s_{i',j'}$ agree in the first or the second coordinate, respectively. More formally,

- if $s_{i,j} = (a,b)$ and $s_{i+1,j} = (a',b')$ then a = a', and
- if $s_{i,j} = (a,b)$ and $s_{i,j+1} = (a',b')$ then b = b'.

If such pairs exist, the instance is a YES-instance, and otherwise it is a NO-instance (i.e., this is a decision problem). GRID TILING is known to be W[1]-hard when parameterized by K [54].

In the following we first give a polynomial time construction of a VC-SNDP instance given a GRID TILING instance, followed by the proof of correctness of our reduction.

4.1 Construction. Given an instance for GRID TILING we first create a graph G = (V, E) with $V = \bigcup_{i,j \in [K]} V_{i,j}$ and $E = \bigcup_{i,j \in [K]} E_{i,j}$ as follows. For each pair $(a,b) \in S_{i,j}$, add two new vertices $v_{(a,b)}, v_{(a,b)}^*$ to $V_{i,j}$ and put an edge between them, i.e., $\{v_{(a,b)}, v_{(a,b)}^*\} \in E_{i,j}$. For $i,j \in [K]$, we call the graph $(V_{i,j}, E_{i,j})$ a cell $c_{i,j}$ of G, and we use the notation $V_{i,j}^* = \bigcup_{(a,b) \in S_{i,j}} \{v_{(a,b)}^*\}$. Note that each cell $c_{i,j}$ is simply a perfect matching between $V_{i,j}^*$ and $V_{i,j} \setminus V_{i,j}^*$. We also put edges across these cells if they are adjacent, i.e., we add an edge between $v_{(a,b)}^* \in V_{i,j}^*$ and $v_{(a',b')} \in V_{i',j'}$ whenever |i'-i|+|j'-j|=1, $i' \geq i$, $j' \geq j$, and if i=i' then a=a', or if j=j' then b=b' (cf. Fig. 3).

Next we do the following set of modifications to the graph G:

- 1. Create 4 new sets of vertices $V_{(l)}, V_{(r)}, V_{(t)}, V_{(b)}$ each of size K where v_{xi} belongs to $V_{(x)}$ for $x \in \{l, t, r, b\}$ and $i \in [K]$.
- 2. For $i, j \in [K]$, add all possible edges between v_{li} and $V_{i,1} \setminus V_{i,1}^*$, and all possible edges between v_{tj} and $V_{1,j} \setminus V_{1,j}^*$. Similarly, add all possible edges between v_{ri} and $V_{i,K}^*$, and also between v_{bj} and $V_{K,j}^*$.
- 3. Add 3 new vertices $R = \{s, t_1, t_2\}$, which form the terminal set.
- 4. Connect t_1 to all the vertices in $V_{(t)} \cup V_{(r)}$ and connect t_2 to $V_{(l)} \cup V_{(b)}$.
- 5. Connect s to all the vertices in $V_{(t)}$ and $V_{(l)}$.
- 6. Add an edge each between s and t_1 , between s and t_2 , and also between t_1 and t_2 .

Let G'=(V',E') be the resulting graph with $V'=V\cup R\cup V_{(l)}\cup V_{(r)}\cup V_{(t)}\cup V_{(b)}$. The target VC-SNDP instance contains G', where the terminals are $R=\{s,t_1,t_2\}$ and the demands between the source s and the sinks t_1,t_2 are $d_{s,t_1}=d_{s,t_2}=2K+2$. Note that this is a uniform single-source instance with root s. All the edges $e\in E'$ are undirected and of unit cost, i.e., $\cos(e)=1$. This completes the description of our construction. It is easy to see that it can be done in polynomial time.

4.2 Correctness. To prove the correctness of our reduction we show that the VC-SNDP instance has a solution $H \subseteq G'$ of cost at most $3K^2 + 8K + 3$ if and only if the given GRID TILING instance is a YES-instance.

Before moving on to establish the correctness, we first define the following notions. Call the vertices $V_{(x)}$ for $x \in \{l, t, r, b\}$ as boundary vertices. Any path connecting a vertex from $V_{(l)}$ with a vertex from $V_{(r)}$ is called a horizontal path, and a path connecting $V_{(t)}$ with $V_{(b)}$ is a vertical path. Call the subgraph induced by vertices in $\bigcup_{i \in [K]} V_{i,j}$ the horizontal layer i and the subgraph induced by vertices in $\bigcup_{i \in [K]} V_{i,j}$ the vertical layer j. Note that horizontal paths do not necessarily lie in a horizontal layer, and also vertical paths are not bound to any vertical layer. We call the edges corresponding to pairs in $S_{i,j}$ (i.e., edges of all $E_{i,j}$) the cell edges and edges

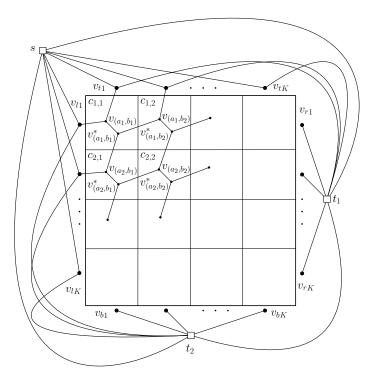


Figure 3: Graph G' = (V', E') with source s and sinks t_1, t_2 . The annotated cells depict some edge connections inside and across the cells.

going across cells between adjacent layers the connector edges. Note that the horizontal layer i and the vertical layer j only intersect in the cell edges of $c_{i,j}$. Any connector edge is only contained in either a horizontal or a vertical layer, and we correspondingly refer to the former connector edges as horizontal and the latter as vertical. We are now ready to prove the easy part of the reduction, which is captured in the following lemma.

LEMMA 4.1. If the VC-SNDP instance is constructed for a YES-instances of GRID TILING, then there is solution $H \subseteq G'$ with $cost(H) = 3K^2 + 8K + 3$.

Proof. Since the given GRID TILING instance is a YES-instance, for each $i, j \in [K]$, we have a pair $s_{i,j} \in S_{i,j}$ such that if (i,j) and (i',j') are adjacent in the first or the second coordinate, then $s_{i,j} = (a,b)$ and $s_{i',j'} = (a',b')$ agree in the first or the second coordinate, respectively. In our VC-SNDP instance each pair $s_{i,j}$ corresponds to a unique cell edge $\{v_{(a,b)}, v_{(a,b)}^*\}$ in the cell $c_{i,j}$. Furthermore, by construction, in each horizontal (resp., vertical) layer these cell edges of two adjacent cells are joined via a connector edge, due to the agreement of the corresponding pairs on the first (resp., second) coordinate.

We construct a solution H for the VC-SNDP instance as follows. For each $i \in [K]$, we have a horizontal path connecting v_{li} to v_{ri} strictly through the horizontal layer i, alternately using cell edges and horizontal connector edges. Similarly, for each $j \in [K]$, we have a path connecting v_{tj} to v_{bj} through the vertical layer j, alternately using cell edges and vertical connector edges. Note that we can find such paths so that a single cell edge is used in each cell $c_{i,j}$, which is shared by the i-th horizontal and the j-th vertical path. Altogether these amount to K^2 cell edges, 2K(K-1) connector edges joining them inside the grid, and an additional 4K edges connecting the boundary vertices to the outermost cell edges of the grid. To complete the solution we further add the 3 direct edges between the terminals, and also all the 6K edges between the terminals and the boundary vertices. As all edges have unit cost, the cost of this solution is exactly $K^2 + 2K(K-1) + 4K + 3 + 6K = 3K^2 + 8K + 3$.

For correctness, we have K horizontal paths, one in each horizontal layer i (and hence, pairwise vertex-disjoint), which must exist due to the agreement of some K pairs $s_{i,j}$ (one corresponding to each cell $c_{i,j}$) on their first coordinate in each row in the GRID TILING instance. Similarly, there are K vertex-disjoint vertical paths, one in each vertical layer j, due to the agreement on the second coordinate in each column in the GRID TILING instance. This means that there are K vertex-disjoint paths from s to t_1 via $V_{(l)}$ and $V_{(r)}$ in H, and

also K vertex-disjoint paths from s to t_2 via $V_{(t)}$ and $V_{(b)}$. In addition, there are K vertex-disjoint paths from s to t_1 , each of length two, via $V_{(t)}$ that are vertex-disjoint from the former paths, and also K vertex-disjoint paths from s to t_2 , each of length two, via $V_{(t)}$ disjoint from the former. Hence, there are a total of $d_{s,t_1}=2K+2$ vertex-disjoint paths between s and t_1 , after adding the path of length two via t_2 and the path of length one from s to t_1 . Similarly there are also $d_{s,t_2}=2K+2$ vertex-disjoint paths between s and t_2 .

In the rest of this section, we prove the other direction of the reduction, as follows. First in Lemma 4.2 we identify some basic structure of any solution of cost $3K^2 + 8K + 3$ to the VC-SNDP instance. Next, we argue that the proposed solution in Lemma 4.1 is the only way to achieve the demands with this cost. In particular, in Lemma 4.3 we prove that any other way of connecting the terminals to meet the demands must incur a total cost of more than $3K^2 + 8K + 3$. Note that the solution size is $\ell = \cos(H)$ for any solution H, since all edges have unit cost. Hence, the parameter ℓ is bounded by $O(K^2)$ and after establishing the above mentioned lemmas we obtain the following.

THEOREM 1.2. Uniform single-source VC-SNDP is W[1]-hard parameterized by the number ℓ of edges of the solution, even if the number k of terminals is 3.

We start with two easy claims about the structural properties of the instance and the solution. In the following, we consider the sets $V_{(l)}$, $V_{(t)}$ as vertical and horizontal layer 0, respectively, and the sets $V_{(r)}$, $V_{(b)}$ as vertical and horizontal layer K+1, respectively. This also means that the edges between layers 0 and 1, and between layers K+1, are considered connector edges.

CLAIM 4.1. Inside any cell $c_{i,j}$ where $i, j \in [K]$, vertices adjacent to horizontal layer i-1 and vertical layer j-1 are disjoint from the set of vertices adjacent to horizontal layer i+1 and vertical layer j+1. The former set of vertices is exactly $V_{i,j} \setminus V_{i,j}^*$, while the latter is exactly $V_{i,j}^*$, and they are only connected via the cell edges of $c_{i,j}$.

Proof. Follows directly from the construction of the VC-SNDP instance in Section 4.1.

CLAIM 4.2. Any solution to the VC-SNDP instance contains all edges incident to the three terminals s, t_1, t_2 . Moreover, in every solution there are K vertex-disjoint horizontal paths P_1, \ldots, P_K that do not go via t_2 or $V_{(t)}$, and K vertex-disjoint vertical paths Q_1, \ldots, Q_K that do not go via t_1 or $V_{(t)}$.

Proof. Note that the degrees of the terminals are all equal to the demand 2K+2. Hence we need to add all edges incident to the terminals to any solution. Considering the terminal pair s, t_1 , the edges incident to the terminals already include K+2 paths: the K paths of length two via the vertices in $V_{(t)}$, the path of length two via t_2 , and the edge between s and t_1 . The only remaining paths between s and t_1 must use the boundary vertices $V_{(l)}$ and $V_{(r)}$ and are thus the vertex-disjoint horizontal paths P_1, \ldots, P_K . As these are vertex-disjoint from the K+2 paths implied by edges incident to the terminals, they cannot use t_2 or $V_{(t)}$. The argument for the vertex-disjoint vertical paths Q_1, \ldots, Q_K is analogous. \square

In the following lemmas we heavily exploit the grid structure of the graph G' in order to control the routes of the horizontal and vertical paths given by Claim 4.2. This is crucial in our reduction for undirected graphs, as alluded to in Section 1.1. In the following, let P_1, \ldots, P_K and Q_1, \ldots, Q_K denote the paths given by Claim 4.2 for any solution.

LEMMA 4.2. Any solution to the VC-SNDP instance has cost at least $3K^2 + 8K + 3$. If the solution has cost exactly $3K^2 + 8K + 3$, then (1) every vertical (horizontal) layer $j \in \{1, ..., K\}$ contains exactly K edges, each being part of a horizontal path $P_1, ..., P_K$ (vertical path $Q_1, ..., Q_K$), and (2) between adjacent vertical (horizontal) layers $j \in \{0, ..., K\}$ and j+1 there are exactly K edges, each being part of a horizontal path $P_1, ..., P_K$ (vertical path $Q_1, ..., Q_K$).

Proof. By Claim 4.2, no horizontal path P_1, \ldots, P_K contains t_2 or any vertex from $V_{(t)}$. Consider the graph G'' obtained by removing all terminals and $V_{(t)}$ from G', and note that any vertex $v_{bj} \in V_{(b)}$ is only adjacent to $V_{K,j}^*$ in this graph. Now, due to Claim 4.1, in G'' each of the two sets $\bigcup_{i=1}^K V_{i,j}^*$ and $\bigcup_{i=1}^K V_{i,j} \setminus V_{i,j}^*$ of a vertical layer j forms a vertex cut between $V_{(l)}$ and $V_{(r)}$. Thus the vertex disjoint paths P_1, \ldots, P_K connecting $V_{(l)}$ and $V_{(r)}$

in G'' must use at least K vertices from each such vertex cut. As a consequence these paths must also use at least K edges from each vertical layer and at least K edges between adjacent vertical layers. An analogous argument shows that the K vertical paths must use at least K edges from each horizontal layer and at least K edges between adjacent horizontal layers.

Counting the number of edges used by a solution, between any adjacent vertical and horizontal layers (including layers 0 and K+1) there are K edges of the solution giving 2(K+1)K edges, while each of the K vertical layers also contains K edges giving K^2 more edges (the horizontal layers intersect the vertical ones in cells and thus may not contribute more edges). Additionally, by Claim 4.2 any solution uses all 6K+3 edges incident to the terminals. This amounts to a total of at least $2(K+1)K+K^2+6K+3=3K^2+8K+3$ edges, which is also the cost of the solution as each edge has cost 1. Given the above structural considerations of the horizontal and vertical paths, this proves the first part of the lemma, but also proves points (1) and (2), since in a solution of exactly this cost we cannot afford any more edges than those that we argued are always present.

Finally, we show that in a solution of cost $3K^2 + 8K + 3$, the K horizontal (resp., vertical) paths through the grid must join the corresponding pairs from $V_{(l)}$ and $V_{(r)}$ (resp., $V_{(t)}$ and $V_{(b)}$) through distinct layers: although it might seem possible to route these paths in other ways, e.g., a horizontal path connecting v_{li} and $v_{ri'}$ for $i \neq i'$, we show that such a solution must use some extra cell edges or connector edges and have a total cost of more than $3K^2 + 8K + 3$.

LEMMA 4.3. For a solution of cost $3K^2 + 8K + 3$ to the VC-SNDP instance and any $i \in [K]$, w.l.o.g., the horizontal path P_i (vertical path Q_i) connects v_{li} and v_{ri} (v_{ti} and v_{bi}) using only vertices of the horizontal layer i (vertical layer i).

Proof. Since the horizontal paths P_1, \ldots, P_K are vertex disjoint, for any $i \in [K]$, w.l.o.g., each path P_i starts in vertex v_{li} and ends in a distinct vertex of $V_{(r)}$ (potentially not v_{ri}). In particular, a path P_i does not contain any other vertices from $V_{(l)}$ and $V_{(r)}$. By Claim 4.2, P_i also does not contain any vertex from $V_{(t)}$, nor the terminal t_2 . This means that the internal vertices of P_i are all inside the grid or from $V_{(b)}$. Now assume that in some vertical layer j the path P_i moves downwards from some horizontal layer $i' \in [K]$ to horizontal layer i' + 1. If $i' \leq K - 1$ so that horizontal layer i' + 1 is not the set $V_{(b)}$, then due to Claim 4.1, the path P_i can only arrive at a vertex from $V_{i'+1,j} \setminus V_{i'+1,j}^*$ of cell $c_{i'+1,j}$ from a vertex of $V_{i',j}^*$ in cell $c_{i',j}$, and can then only use a cell edge of $c_{i'+1,j}$, a horizontal connector edge to reach vertical layer j - 1, or a vertical connector edge to go back up to $c_{i',j}$. In case i' = K, the path reaches vertex $v_{bj} \in V_{(b)}$ from a vertex of $V_{i',j}^*$ of cell $c_{i',j}$, and since the path does not contain t_2 it can only use a vertical connector edge to go back up to another vertex of $V_{i',j}^*$ of $c_{i',j}$.

Note that by Claim 4.1, to reach a vertex of $V_{i',j}^*$ from v_{il} before moving downwards, in some horizontal layer the path P_i needs to use a connector edge between vertical layers j-1 and j, and it needs to use a cell edge of vertical layer j. Thus in the first case when $i' \leq K-1$ and P_i uses a cell edge of $c_{i'+1,j}$, this path uses two cell edges of vertical layer j. However, this contradicts property (1) of Lemma 4.2. In the second case, when P_i uses a horizontal connector edge to reach vertical layer j-1 after moving downwards, it uses two connector edges between vertical layers j-1 and j, which however contradicts property (2) of Lemma 4.2. The remaining case is when $i' \leq K$ and P_i uses a vertical connector edge to go back up to $c_{i',j}$. This implies that between horizontal layers i' and i'+1 the solution contains two vertical connector edges, which are incident to the same vertex of horizontal layer i'+1. From property (2) of Lemma 4.2, these two edges must be used by two distinct vertical paths of Q_1, \ldots, Q_K , which however contradicts the fact that these paths are vertex disjoint.

Hence, P_i cannot move downwards into a different horizontal layer. Using this observation, note that the horizontal paths induce a permutation p on $\{1, 2, ..., K\}$, where for $i, j \in [K]$ we have p(i) = j if a path starting at the vertex v_{li} on the left ends up at the vertex v_{rj} on the right. If p is not the identity function, then there must be indices i < j such that p(i) = j. But this also means that the corresponding horizontal path moves down at some point to reach the layer j, which is a contradiction. Now that we know that every horizontal path starts and ends in the same layers, suppose there is a path that does not stick to its layer in between and takes a detour through some other layers. However, the path has to go downwards from a layer (not necessarily the one it started in) at some point, which again is a contradiction.

Through an analogous argument we can first argue that a vertical path from Q_1, \ldots, Q_K cannot move rightwards, and similarly if any path goes out of the corresponding vertical layer at any point we arrive at a contradiction as before. \Box

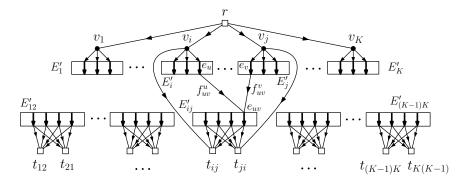


Figure 4: Graph G' with some edge connections shown for the root r and the terminals t_{ij} and t_{ji} . The thicker edges denote the edges of cost 1 in the graph and the rest of the edges are of cost 0.

In conclusion, any solution of cost $3K^2 + 8K + 3$ to the VC-SNDP instance implies a solution to the GRID TILING instance: by Lemmas 4.2 and 4.3, each cell contains exactly one cell edge of the solution, which encodes an integer pair (a, b) of the GRID TILING instance. Since the solution connects these cell edges by connector edges, the coordinates of the integer pairs agree accordingly. Thus together with Lemma 4.1 and the W[1]-hardness of the GRID TILING problem we obtain Theorem 1.2.

5 Hardness of 2-DST

In this section we consider arguably one of the simplest versions of SNDP in directed graphs: the 2-Connected Directed Steiner Tree (2-DST) problem. We are given a directed graph G = (V, E) with edge-costs cost(e) for $e \in E$, and a set of k terminals $R \subseteq V$ with root $r \in R$. The goal is to compute a subgraph H of G with minimum cost, that has at least 2 edge-disjoint paths from r to each $t \in R \setminus \{r\}$. The 2-DST problem is a natural generalization of the classical Directed Steiner Tree (DST) problem, where only one $r \to t$ path is required to exist for each $t \in R \setminus \{r\}$. While DST is know to be FPT [28] for parameter k, for the 2-DST problem we show the following:

THEOREM 1.6. The 2-DST problem is W[1]-hard parameterized by the number ℓ of edges of the solution.

We show a reduction from the MULTICOLOURED CLIQUE problem to the 2-DST problem. The input of MULTICOLOURED CLIQUE consists of an undirected graph G, an integer K, and a partition (V_1, \ldots, V_K) of the vertices of G. The goal is to decide if there is a K-clique containing exactly one vertex from each set V_i . If such a clique exists the instance is a YES-instance and otherwise a NO-instance (i.e., it is a decision problem). MULTICOLOURED CLIQUE is W[1]-hard when parameterized by K [35].

- **5.1 Construction.** Let $(G, K, (V_1, ..., V_K))$ be an *n*-vertex instance of MULTICOLOURED CLIQUE. For $i, j \in [K]$ where i < j, let E_{ij} be the set of (undirected) edges between V_i and V_j . For any directed edge (u, v) we call u the tail and v the head. We construct an instance G' of 2-DST from it as follows (cf. Fig. 4):
 - 1. For each $i \in [K]$, create a directed edge set E'_i with a distinct edge e_v for each $v \in V_i$.
 - 2. For $i, j \in [K]$ where i < j, create a directed edge set E'_{ij} with an edge e_{uv} for each (undirected) edge $\{u, v\} \in E_{ij}$.
 - 3. For $i, j \in [K]$ where i < j and each edge $\{u, v\} \in E_{i,j}$, add two directed edges f_{uv}^u and f_{uv}^v from the heads of $e_u \in E_i'$ and $e_v \in E_j'$, respectively, to the tail of $e_{uv} \in E_{ij}'$.
 - 4. For $i, j \in [K]$ where i < j, create two vertices t_{ij} and t_{ji} , and put directed edges from all tails of edges in E'_{ij} to both t_{ij} and t_{ji} .
 - 5. Add K new vertices v_1, \ldots, v_K and for each $i \in [K]$, add a directed edge from v_i to all tails of edges in E'_i and also add an edge to t_{ij} for each $j \in [K]$.
 - 6. Add a vertex r and put edges from r to the vertices v_1, \ldots, v_K .
 - 7. The edges in E'_i and E'_{ij} are of cost 1, and all other edges are of cost 0.

This completes the construction of the target 2-DST instance G' where r is the root vertex and the terminal set is $R = \{r, t_{ij} \mid i \neq j\}$. It is easy to see that the above construction can be done in polynomial time.

5.2 Correctness. To show the correctness of the reduction, we argue that the 2-DST instance has a solution H with $cost(H) \leq K + {K \choose 2}$ if and only if the given MULTICOLOURED CLIQUE instance is a YES-instance. In particular, first in Lemma 5.1 we show how to construct a solution of $cost\ K + {K \choose 2}$ given a YES-instance of MULTICOLOURED CLIQUE. Next in Lemma 5.2 we prove that any solution to the 2-DST instance has a cost at least $K + {K \choose 2}$. Finally, in Lemma 5.3 we show that every $cost\ K + {K \choose 2}$ solution to the 2-DST instance must correspond to a solution to the MULTICOLOURED CLIQUE instance. Since Lemma 5.1 also bounds the solution size, Theorem 1.6 follows.

LEMMA 5.1. Given a YES-instance of MULTICOLOURED CLIQUE, the constructed 2-DST instance has a solution of cost $K + {K \choose 2}$. Furthermore, its solution size is $\ell = O(K^2)$.

Proof. Since the given MULTICOLOURED CLIQUE instance is a YES-instance, for each $1 \le i < j \le K$, there is an edge $\{u,v\} \in E_{ij}$ for a unique vertex $u \in V_i$ and a unique vertex $v \in V_j$, that are part of a K-clique, say C. In the 2-DST instance we construct a solution from C as follows: for each $\{u,v\} \in E_{ij}$ in C add $e_u \in E'_i$, $e_v \in E'_j$, and $e_{uv} \in E'_{ij}$ to the solution and also add the two edges f^u_{uv} and f^v_{uv} connecting e_u and e_v to e_{uv} . Further, add the edges connecting each v_i with the corresponding edge e_v where $v \in V_i$, the two edges connecting e_{uv} to t_{ij}, t_{ji} , and also the two edges (v_i, t_{ij}) and (v_j, t_{ji}) . Finally, to complete the solution add the K edges (r, v_i) for each $i \in [K]$.

This indeed gives a solution to the 2-DST instance, since for each terminal t_{ij} we have two edge-disjoint paths from r: a length-two path via v_i and a path from v_j to some $e_{uv} \in E'_{ij}$ via $e_v \in E'_j$, which must exist as the given MULTICOLOURED CLIQUE instance is a YES-instance. Since there is a unique vertex u in each V_i associated with a K-clique C, we add to our solution in the 2-DST instance exactly the K edges associated with those vertices, one e_u in each E'_i . Also, we add to the solution the $\binom{K}{2}$ edges, corresponding to the edges in the K-clique C, one e_{uv} in each E'_{ij} . Hence, the cost of our solution is exactly $K + \binom{K}{2}$ as all other edges added are of cost 0.

From our construction, the total number of remaining edges in the solution is bounded by $2K + K(K-1) + 2K(K-1) = O(K^2)$. Here the first term corresponds to the K length-two paths connecting r to each E'_i via v_i , for $i \in [K]$. The second term is due the edges from v_i to t_{ij} , one for each of the K(K-1) terminals. For each $e_{uv} \in E'_{ij}$ in the solution, the final term correspond to the two edges from E'_i and E'_j to e_{uv} and the two edges from e_{uv} to t_{ij} and t_{ji} . Hence, we obtain $\ell = O(K^2)$, as required.

We remark that the following lemma crucially uses the directedness of G' in order to control the routes of the paths. For undirected graphs such a (rather straightforward) reduction seems hard to achieve, as mentioned in Section 1.1.

LEMMA 5.2. Any solution to the 2-DST instance has a cost of at least $K + {K \choose 2}$. Moreover, in any solution of $cost K + {K \choose 2}$ there is exactly one edge picked from each E'_i and each E'_{ij} , and it contains some edge f^u_{uv} (resp., f^v_{uv}) connecting the selected edge $e_u \in E'_i$ (resp., $e_v \in E'_j$) to the selected edge $e_{uv} \in E'_{ij}$.

Proof. To construct a solution to the 2-DST instance, for each terminal t_{ij} we must pick two edge-disjoint paths from r to t_{ij} . Notice that all the paths from r to t_{ij} go via v_i or v_j , since t_{ij} can only be reached from v_i , E'_{ij} , E'_{ij} , and E'_{j} , in addition to v_j and r. As there is only one edge from r to each v_i , due to the edge-disjointness condition we must pick one path through v_i and another through v_j to reach t_{ij} . Though it is possible to use the (v_i, t_{ij}) edge, which is of cost 0, any path connecting v_j to t_{ij} must use an edge from E'_{j} and E'_{ij} , which have cost 1 each. Similarly for t_{ji} the paths must go via both v_i, v_j and one of the paths goes via E'_{i} and E'_{ij} . Thus together the terminals t_{ij}, t_{ji} ensure that any solution uses at least one edge from each of E'_{i} , E'_{j} , and E'_{ij} , together with some edges f^u_{uv} and f^v_{uv} connecting these. Hence any solution must incur a cost of at least $K + {K \choose 2}$. Moreover, any solution of cost $K + {K \choose 2}$ can only afford one edge from each of the edges sets E'_{i} and E'_{ij} .

LEMMA 5.3. Any solution of cost $K + {K \choose 2}$ for the 2-DST instance implies a solution to the MULTICOLOURED CLIQUE instance.

Proof. From Lemma 5.2 we know that in any solution to the 2-DST instance of cost at most $K + {K \choose 2}$, for any indices i < j there is a unique edge e_u used in E'_i , a unique edge e_v used in E'_j , and a unique edge e_{uv} used in E'_{ij} .

This means that the solution must also contain edges f_{uv}^u and f_{uv}^v connecting E_i' and E_j' to E_{ij}' . We can now find a clique in the MULTICOLOURED CLIQUE instance by selecting the vertices $u \in V_i$ and $v \in V_j$ for each $i, j \in [K]$ where i < j. The existence of the connecting edges f_{uv}^u and f_{uv}^v imply the existence of an edge $\{u, v\}$ between any pair of selected vertices, i.e., the vertices induce a clique.

6 Open problems

While we settle the parameterized complexity of quite a few cases of SNDP in this paper, we leave some open questions as well:

- Given that VC-SNDP is W[1]-hard parameterized by the solution size, an obvious question becomes: is it possible to approximate this problem in FPT time within a factor better than the known polynomial-time approximation algorithms, possibly even beating the known approximation lower bounds? The same question can be posed for d-DST.
- Our reduction for VC-SNDP excludes FPT algorithms parameterized by the maximum demand d_{max} , even for a constant number k of terminals. At the same time, the reduction in [48] together with the results of [27, 53] show hardness for the parameterization by k, but with unbounded d_{max} . Note that these results do not exclude an algorithm with runtime $f(k) \cdot n^{g(d_{\text{max}})}$ for some functions f and g. Such an algorithm would nicely bridge the gap to the FPT algorithm [28] for STEINER FOREST, where $d_{\text{max}} = 1$ and k is the parameter. In particular, can VC-SNDP be solved in $2^{O(k)} \cdot n^{O(d_{\text{max}})}$ time?
- We showed that EC-SNDP and LC-SNDP are FPT parameterized by the solution size ℓ and gave algorithms with runtime $2^{O(\ell)} \cdot n^{O(1)}$. What is the parameterized complexity of these problems for stronger parameters, such as the sum of demands D or even the number k of terminals?
- When considering input graphs of bounded treewidth, we gave an FPT algorithm for VC-SNDP parameterized by the treewidth tw and the sum of demands D. However, what is the parameterized complexity of this problem when combining the treewidth tw with the stronger parameter given by the number k of terminals?
- Bateni et al. [8] give an approximation scheme for STEINER FOREST with XP runtime parameterized by the treewidth alone. They leave open whether this can be improved to FPT runtime. Considering SNDP, is there an approximation scheme parameterized by only the treewidth tw with either XP or FPT runtime?

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