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Holonomic modules and 1-generation in the Jacobian Conjecture

Modules holonomes et 1-génération dans la conjecture jacobienne

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Abstract. Let P_n be a polynomial algebra in n indeterminates over a field K of characteristic zero. An endomorphism $\sigma \in \mathrm{End}_K(P_n)$ is called a *Jacobian map* if its Jacobian is a nonzero scalar. Each Jacobian map σ is extended to an endomorphism σ of the Weyl algebra A_n.

The *Jacobian Conjecture* (JC) says that every Jacobian map is an automorphism. Clearly, the Jacobian Conjecture is true iff the twisted (by σ) P_n*-module* ${}^{\sigma}P_n$ is cyclic for all Jacobian maps σ. It is shown that the A_n*-module* ${}^{\sigma}P_n$ is cyclic for all Jacobian maps σ. Furthermore, the A_n-module ${}^{\sigma}P_n$ is holonomic and as a result has finite length. An explicit upper bound is found for the length of the A_n-module ${}^{\sigma}P_n$ in terms of the degree $\deg(\sigma)$ of the Jacobian map σ. Analogous results are given for the Conjecture of Dixmier and the Poisson Conjecture. These results show that the Jacobian Conjecture, the Conjecture of Dixmier and the Poisson Conjecture are questions about holonomic modules for the Weyl algebra A_n and the images of the Jacobian maps, of the endomorphisms of the Weyl algebra A_n and of the Poisson endomorphisms are large in the sense that further strengthening of the results on largeness would be either to prove the conjectures or produce counter examples.

A short direct algebraic (without reduction to prime characteristic) proof is given of the equivalence of the Jacobian and the Poisson Conjectures (this gives a new short proof of the equivalence of the Jacobian, Poisson and Dixmier Conjectures).

Résumé. Soit P_n un polynôme à n indéterminées sur un corps K de caractéristique zéro. Un endomorphisme $\sigma \in \mathrm{End}_K(P_n)$ est appelé une *application jacobienne* si son jacobien est un scalaire non nul. Chaque application jacobienne σ est étendue en un endomorphisme σ de l'algèbre de Weyl A_n.

La *Conjecture Jacobienne* (CJ) affirme que chaque application jacobienne est un automorphisme. Clairement, la Conjecture Jacobienne est vraie si le module tordu (par σ) ${}^{\sigma}P_n$ est cyclique pour toutes les applications jacobienne σ. Il est démontré que le module A_n ${}^{\sigma}P_n$ est cyclique pour toutes les applications jacobienne σ. De plus, le module A_n- ${}^{\sigma}P_n$ est holonomique et, par conséquent, de longueur finie. Une borne supérieure explicite est trouvée pour la longueur du module A_n- ${}^{\sigma}P_n$ en fonction du degré $\deg(\sigma)$ de l'application jacobienne σ. Des résultats analogues sont donnés pour la Conjecture de Dixmier et la Conjecture Poisson. Ces résultats montrent que la Conjecture Jacobienne, la Conjecture de Dixmier et la Conjecture Poisson sont des questions sur les modules holonomiques pour l'algèbre de Weyl A_n et que les images des applications jacobienne, des endomorphismes de l'algèbre de Weyl A_n et des endomorphismes de Poisson sont importantes au sens où des renforcements supplémentaires des résultats sur l'importance consisteraient soit à prouver les conjectures, soit à produire des contre-exemples.

Une démonstration directe et brève (sans réduction à la caractéristique première) est donnée de l'équivalence des Conjectures Jacobienne et Poisson (ceci donne une nouvelle démonstration brève de l'équivalence des Conjectures Jacobienne, Poisson et Dixmier).

Keywords. The Jacobian Conjecture, the Conjecture of Dixmier, the Weyl algebra, the holonomic module, the endomorphism algebra, the length, the multiplicity.

Mots-clés. La conjecture jacobienne, la conjecture de Dixmier, l'algèbre de Weyl, le module holonomique, l'algèbre des endomorphismes, la longueur, la multiplicité.

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In this paper, K is a field of characteristic zero and $K^\times := K\backslash\{0\}$, $P_n = K[x_1,\dots,x_n]$ is a polynomial algebra in n the variables, $\mathrm{Der}_K(P_n)$ is the set of all K-derivations of the polynomial algebra P_n. For a K-algebra A, the set $\mathrm{End}_K(A)$ is the monoid of K-algebra endomorphisms of A and $\mathrm{Aut}_K(A)$ is the automorphism group of A.

1. The Conjecture of Dixmier, holonomic A_n-modules and finite length.

For an endomorphism $\sigma \in \mathrm{End}_K(A)$ and an A-module M we denote by ${}^\sigma M$ the A-module M *twisted by* σ: ${}^\sigma M = M$ (as a vector space) and

$$a \cdot m := \sigma(a)m \quad \text{for all } a \in A, \ m \in M.$$

The ring of differential operators $A_n := \mathscr{D}(P_n)$ on the polynomial algebra P_n is called the *Weyl algebra*. The Weyl algebra A_n is generated by the elements $x_1,\dots,x_n,\partial_1,\dots,\partial_n$ subject the defining relations: $[x_i,x_j] = 0$, $[\partial_i,\partial_j] =$ and $[\partial_i,x_j] = \delta_{ij}$ for all $i,j = 1,\dots,n$ where $\partial_i := \frac{\partial}{\partial x_i}$, $[a,b] := ab - ba$, and δ_{ij} is the Kronecker delta. The Weyl algebra A_n is a simple Noetherian domain of Gelfand–Kirillov dimension $\mathrm{GK}(A_n) = 2n$.

Inequality of Bernstein. *For all nonzero finitely generated A_n-modules M,*

$$\mathrm{GK}(M) \geq n.$$

A finitely generated A_n-module M is called *holonomic* if $\mathrm{GK}(M) = n$. Each holonomic module has finite length and is a cyclic A_n-module, i.e. 1-generated. Each nonzero sub- or factor module of a holonomic module is holonomic.

Conjecture of Dixmier DC$_n$ ([9]). $\mathrm{End}_K(A_n) = \mathrm{Aut}_K(A_n)$.

The Weyl algebra A_n is isomorphic to its opposite algebra A_n^{op} via $A_n \to A_n^{op}$, $x_i \mapsto x_i$, $\partial_i \mapsto \partial_i$ for $i = 1,\dots,n$. So, the algebra $A_n \otimes A_n^{op} \simeq A_{2n}$ is isomorphic to the Weyl algebra A_{2n}. In particular, every A_n-*bimodule* N is a *left* A_{2n}-module (${}_{A_n}N_{A_n} = {}_{A_n\otimes A_n^{op}}N \simeq {}_{A_{2n}}N$). When we say that an A_n-*bimodule* N is *holonomic* we mean that the corresponding left A_{2n}-module N is holonomic. The Weyl algebra A_n is a simple holonomic A_n-bimodule (since $\mathrm{GK}(A_n) = 2n$ and $\mathrm{GK}(A_{2n}) = 4n$).

Theorem 1 ([3, Theorem 1.3]). *If M is a holonomic A_n-module and $\sigma \in \mathrm{End}_K(A_n)$ then the A_n-module ${}^\sigma M$ is also a holonomic A_n-module (and as a result has finite length and is 1-generated over A_n).*

The Weyl algebra A_n is a simple algebra. So, for each $\sigma \in \mathrm{End}_K(A_n)$, the image $\sigma(A_n)$ is isomorphic to the Weyl algebra A_n.

- *The Conjecture of Dixmier is true if for every endomorphism $\sigma \in \mathrm{End}(A_n)$, the $\sigma(A_n)$-bimodule A_n is simple.*

Corollary 2 ([3, Corollary 3.4]). *For each algebra endomorphism $\sigma : A_n \to A_n$, the Weyl algebra A_n is a holonomic $\sigma(A_n)$-bimodule, hence, of finite length and 1-generated.*

Each nonzero element $a\in A_n$ is a unique sum $a=\sum_{\alpha,\beta\in\mathbb{N}^n}\lambda_{\alpha\beta}x^\alpha\partial^\beta$ for some scalars $\lambda_{\alpha\beta}\in K$ where $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\partial^\beta=\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$. The natural number

$$\deg(a):=\max\{|\alpha|+|\beta|\,|\,\lambda_{\alpha\beta}\neq 0\}$$

is called the *degree* of the element a. Then $\{A_{n,i}\}_{i\geq 0}$ is a finite dimensional filtration of the Weyl algebra A_n where $A_{n,i}:=\{a\in A_n\,|\,\deg(a)\leq i\}$ and $\deg(0):=-\infty$ ($A_n=\bigcup_{i\geq 0}A_{n,i}$ and $A_{n,i}A_{n,j}\subseteq A_{n,i+j}$ for all $i,j\geq 0$).

Each endomorphism $\sigma\in\mathrm{End}_K(A_n)$ is uniquely determined by the elements

$$x_1':=\sigma(x_1),\dots,x_n':=\sigma(x_n),\partial_i':=\sigma(\partial_1),\dots,\partial_n':=\sigma(\partial_n).$$

The natural number $\deg(\sigma):=\max\{\deg(x_i'),\deg(\partial_i')\ \ i=1,\dots,n\}$ is called the *degree* of σ.

Theorem 3. *Let $\sigma\in\mathrm{End}(A_n)$ and $d:=\deg(\sigma)$. Then $L_{\sigma(A_n)}(A_n)\leq d^{2n}$ where $L_{\sigma(A_n)}(M)$ is the length of a $\sigma(A_n)$-bimodule M.*

Proof. Since $\deg(x_i')\leq d$ and $\deg(\partial_i')\leq d$,

$$x_i'A_{n,ds}\subseteq A_{n,d(s+1)},\ A_{n,ds}x_i'\subseteq A_{n,d(s+1)},\ \partial_i'A_{n,ds}\subseteq A_{n,d(s+1)}\quad\text{and}\quad A_{n,ds}\partial_i'\subseteq A_{n,d(s+1)}$$

for all $i=1,\dots,n$ and $s\geq 0$. Therefore, $\{A_{n,ds}\}_{s\geq 0}$ is a finite dimensional filtration of the $\sigma(A_n)$-bimodule A_n such that

$$\dim_K(A_{n,ds})=\binom{ds+2n}{2n}=\frac{1}{(2n)!}(ds+2n)(ds+2n-1)\cdots(ds+1)=\frac{d^{2n}}{(2n)!}s^{2n}+\cdots$$

where three dots denote smaller terms. By [10, Lemma 8.5.9], $L_{\sigma(A_n)}(A_n)\leq d^{2n}$. □

2. The Jacobian Conjecture, holonomic A_n-modules and 1-generation

Each endomorphism $\sigma\in\mathrm{End}_K(P_n)$ is uniquely determined by the polynomials

$$x_1':=\sigma(x_1),\dots,x_n':=\sigma(x_n).$$

The matrix of partial derivatives,

$$\mathscr{J}(\sigma):=\frac{\partial x'}{\partial x}:=\left(\frac{\partial x_i'}{\partial x_j}\right),\quad\text{where}\quad \mathscr{J}(\sigma)_{ij}:=\frac{\partial x_i'}{\partial x_j},$$

is called the *Jacobian matrix* of σ. An endomorphism $\sigma\in\mathrm{End}(P_n)$ with $\det(\mathscr{J}(\sigma))\in K^\times$ is called a *Jacobian map*.

Jacobian Conjecture JC$_n$. *Every Jacobian map is an automorphism.*

Theorem 4 ([2, Theorem 2.1]). *A Jacobian map $\sigma\in\mathrm{End}(P_n)$ is an automorphism of P_n if the P_n-module ${}^\sigma P_n$ is finitely generated.*

- *The Jacobian Conjecture is true iff the P_n-module ${}^\sigma P_n$ is 1-generated for all Jacobian maps σ*

Each Jacobian map σ is extended to a (necessarily) monomorphism of the Weyl algebra A_n:

$$\sigma: A_n\to A_n,\ \partial_i\mapsto\partial_i',\ i=1,\dots,n, \tag{1}$$

where ∂_i' is a K-derivation of the polynomial algebra P_n which is given by the rule:

$$\partial_i'(p):=\frac{1}{\det\mathscr{J}(\sigma)}\mathscr{J}(\sigma(x_1),\dots,\sigma(x_{i-1}),p,\sigma(x_{i+1}),\dots,\sigma(x_n))\quad\text{for all}\quad p\in P_n. \tag{2}$$

For an algebra A and a non-empty subset S of A, we define the *centralizer* of S in A by $C_A(S):=\{a\in A\,|\,as=sa \text{ for all } s\in S\}$. Let $\widehat{P}_n:=K[\![x_1,\dots,x_n]\!]$ and $\widehat{A}_n:=\bigoplus_{\alpha\in\mathbb{N}^n}\widehat{P}_n\partial^\alpha$. Proposition 5 is a description of all extensions of a Jacobian map of P_n to an endomorphism of the Weyl algebra A_n.

Proposition 5. *Let σ be a Jacobian map of P_n, σ be its extension to an endomorphism of the Weyl algebra A_n as in* (1), $x'_1 = \sigma(x_1), \dots, x'_n = \sigma(x_n)$ *and* $\partial'_1 = \sigma(\partial_1), \dots, \partial'_n = \sigma(\partial_n)$, *see* (2).

(1) *If σ' is another extension of the Jacobian map σ then $\sigma'(\partial_i) = \partial'_i + \partial'_i(p)$, $i = 1, \dots, n$ where $p \in P_n$, and vice versa.*
(2) *An extension of the Jacobian map σ of P_n is unique if the images of the elements $\partial_1, \dots, \partial_n$ are derivations of P_n. So, the extension σ in* (2) *is such a unique extension, and* $\mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial'_i$.
(3) $C_{A_n}(x'_1, \dots, x'_n) = P_n$.
(4) *Suppose that $x'_i = x_i + \cdots$ for $i = 1, \dots, n$ where the three dots denote higher terms. Then $\sigma \in \mathrm{Aut}_K(\widehat{P}_n)$ and every extension σ' of the Jacobian map σ to an endomorphism of the Weyl algebra A_n belongs to* $\mathrm{Aut}_K(\widehat{A}_n)$.

Proof.

(1)–(3). Up to an affine change of variables in the polynomial algebra P_n, we can assume that $\sigma(x_i) = x_i + \cdots$ for $i = 1, \dots, n$ where the three dots denote *higher* terms. Since $\det(\mathscr{J}(\sigma)) \in K^\times$, we have that $\mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial'_i$ (as $\partial_i = \sum_{j=1}^n \frac{\partial x'_j}{\partial x_i} \partial'_j$ for all $i = 1, \dots, n$), and so

$$\widehat{P}_n = K[\![x'_1, \dots, x'_n]\!] \quad \text{and} \quad \sigma \in \mathrm{Aut}_K(\widehat{P}_n).$$

If the elements $\sigma'(\partial_i)$ are derivations of the polynomial algebra P_n then

$$\sigma'(\partial_i)(x'_j) = [\sigma'(\partial_i), x'_j] = [\sigma'(\partial_i), \sigma'(x_j)] = \sigma'([\partial_i, x_j]) = \sigma'(\delta_{ij}) = \delta_{ij} \quad \text{for} \quad i, j = 1, \dots, n.$$

Hence, $\sigma'(\partial_i) = \frac{\partial}{\partial x'_i}$ for $i = 1, \dots, n$, and so $\sigma'(\partial_i) = \partial'_i$, see (2).

In the general case,

$$[\sigma'(\partial_i) - \partial'_i, x'_j] = [\sigma'(\partial_i), \sigma'(x_j)] - [\partial'_i, x'_j] = \delta_{ij} - \delta_{ij} = 0,$$

and so $d_i := \sigma'(\partial_i) - \partial'_i \in C_{A_n}(x'_1, \dots, x'_n)$. Clearly,

$$P_n \subseteq C_{A_n}(x'_1, \dots, x'_n) \subseteq C_{\widehat{A}_n}(x'_1, \dots, x'_n) = \widehat{P}_n,$$

and so $C_{A_n}(x'_1, \dots, x'_n) = P_n$. Therefore, $d_i \in P_n \subseteq \widehat{P}_n = K[\![x'_1, \dots, x'_n]\!]$ for all $i = 1, \dots, n$. For all $i, j = 1, \dots, n$,

$$0 = \sigma'([\partial_i, \partial_j]) = [\sigma'(\partial_i), \sigma'(\partial_j)] = [\partial'_i + d_i, \partial'_j + d_j] = \partial'_i(d_j) - \partial'_j(d_i).$$

Therefore, there is an element $p \in K[\![x'_1, \dots, x'_n]\!]$ such that $d_i = \partial'_i(p)$ for $i = 1, \dots, n$, by the Poincaré Lemma. Since all $d_j \in P_n$, we must have

$$\partial_i(p) = \sum_{j=1}^n \frac{\partial x'_j}{\partial x_i} \partial'_j(p) = \sum_{j=1}^n \frac{\partial x'_j}{\partial x_i} d_j \in P_n.$$

Hence, $p \in P_n$ since $K[\![x'_1, \dots, x'_n]\!] = \widehat{P}_n$.

(4). It follows from statement (1). □

Theorem 6. *Let $\sigma \in \mathrm{End}(P_n)$ be Jacobian map and $d := \deg(\sigma)$. Then*

(1) *The A_n-module ${}^{\sigma}P_n$ is holonomic, hence of finite length and 1-generated as an A_n-module.*
(2) $l_{A_n}({}^{\sigma}P_n) \le m^n$ *where* $m := \max\{d, (d-1)^{n-1} - 1\}$ *where $l_{A_n}(M)$ is the length of an A_n-module M.*

Proof.

(1). The A_n-module P_n is holonomic. By Theorem 1, the A_n-module ${}^{\sigma}P_n$ is holonomic, hence of finite length and 1-generated as an A_n-module.

(2). Since $\deg(x_i') \le d$,

$$\partial_i' = \sum_{j=1}^{n} \frac{\partial x_j}{\partial x_i'}\partial_j = \sum_{j=1}^{n}\left(\mathscr{J}(\sigma)^{-1}\right)_{ij}\partial_j, \quad \text{and} \quad \deg\left(\mathscr{J}(\sigma)^{-1}\right)_{ij} \le (d-1)^{n-1},$$

we have that

$$x_i' P_{n,ms} \subseteq P_{n,m(s+1)}, \quad \text{and} \quad \partial_i' P_{n,ms} \subseteq P_{n,m(s+1)} \quad \text{for all} \quad i = 1, \dots, n \quad \text{and} \quad s \ge 0.$$

Therefore, $\{P_{n,ms}\}_{s\ge 0}$ is a finite dimensional filtration of the A_n-module ${}^{\sigma}P_n$ such that

$$\dim_K(P_{n,ms}) = \binom{ms+n}{n} = \frac{1}{n!}(ms+n)(ms+n-1)\cdots(ms+1) = \frac{m^n}{n!}s^n + \cdots$$

where three dots denote smaller terms. By [10, Lemma 8.5.9], $l_{A_n}({}^{\sigma}P_n) \le m^n$. □

The Dixmier Conjecture implies the *Jacobian Conjecture*, [2, p. 297]), and the inverse implication is also true, as shown by Tsuchimoto [11] and Belov–Kanel and Kontsevich [8] (a short proof is given in [4]).

3. Equivalence of the Jacobian and the Poisson Conjectures

The Weyl algebra $A_n = \mathscr{D}(P_n) = \bigcup_{i\ge 0}\mathscr{D}(P_n)_i$ is a ring of differential operators on P_n and hence admits the *degree filtration* $\{\mathscr{D}(P_n)_i\}_{i\ge 0}$ where $\mathscr{D}(P_n)_i = \bigoplus_{\{\alpha\in\mathbb{N}^n \mid |\alpha|\le i\}} P_n\partial^{\alpha}$. The *associated graded algebra*

$$\mathrm{gr}(A_n) := \bigoplus_{i\ge 0}\mathrm{gr}(A_n)_i,$$

where $\mathrm{gr}(A_n)_i := \mathscr{D}(P_n)_i/\mathscr{D}(P_n)_{i-1}$ and $\mathscr{D}(P_n)_{-1} := 0$, is a polynomial algebra P_{2n} in $2n$ variables $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+n}$ (where $x_{n+i} := \partial_i + P_n$) that admits the canonical Poisson structure given by the rule:

$$\{\cdot,\cdot\} : \mathrm{gr}(A_n)_i \otimes_K \mathrm{gr}(A_n)_j \to \mathrm{gr}(A_n)_{i+j-1}, \; (\bar{a},\bar{b}) \mapsto \{\bar{a},\bar{b}\} := [a,b] + \mathscr{D}(P_n)_{i+j-2} \tag{3}$$

where $\bar{a} := a + \mathscr{D}(P_n)_{i-1}$ and $\bar{b} := b + \mathscr{D}(P_n)_{j-1}$ (since $[\mathscr{D}(P_n)_i, \mathscr{D}(P_n)_j] \subseteq \mathscr{D}(P_n)_{i+j-1}$ for all $i, j \ge 0$). Equivalently,

$$\{x_i, x_j\} = 0, \; \{x_{n+i}, x_{n+j}\} = 0 \quad \text{and} \quad \{x_{n+i}, x_j\} = \delta_{ij} \quad \text{for all} \quad i, j = 1, \dots, n. \tag{4}$$

Poisson Conjecture $\mathbf{PC_{2n}}$. $\mathrm{End}_{\mathrm{Pois}}(P_{2n}) = \mathrm{Aut}_{\mathrm{Pois}}(P_{2n})$.

The Poisson Conjecture and the Conjecture of Dixmier are equivalent (Adjamagbo and van den Essen [1]).

Theorem 7.

(1) $\mathrm{JC}_{2n} \Rightarrow \mathrm{PC}_{2n}$.
(2) $\mathrm{DC}_{2n} \Rightarrow \mathrm{PC}_{2n}$.
(3) $\mathrm{PC}_{2n} \Rightarrow \mathrm{JC}_n$.
(4) *The Jacobian Conjecture and the Poisson Conjecture are equivalent.*
(5) *The Jacobian Conjecture, the Conjecture of Dixmier and the Poisson Conjecture are equivalent.*

Proof.

(1). Given $\sigma \in \mathrm{End}_{\mathrm{Pois}}(P_{2n})$. Then $\det(\mathscr{J}(\sigma)) \in \{\pm 1\}$ (see the proof of Step 6 of [4, Theorem 3] of the fact that $\mathrm{JC}_{2n} \Rightarrow \mathrm{DC}_n$): Notice that $\det(\{x_i, x_j\}) \in \{\pm 1\}$ where $1 \le i, j \le 2n$, and so

$$\begin{aligned}\{\pm 1\} \ni \det(\{x_i, x_j\}) &= \sigma(\det(\{x_i, x_j\})) = \det(\sigma(\{x_i, x_j\}))\\ &= \det(\{\sigma(x_i), \sigma(x_j)\}) = \det\big(\mathscr{J}^t(\sigma)\cdot(\{x_i, x_j\})\cdot \mathscr{J}(\sigma)\big)\\ &= \det(\mathscr{J}(\sigma))^2 \det(\{x_i, x_j\}),\end{aligned}$$

and so $\det(\mathscr{J}(\sigma)) \in \{\pm 1\}$. By JC_{2n}, $\sigma \in \mathrm{Aut}_K(P_{2n})$, and statement 1 follows.

(2). Given $\sigma \in \mathrm{End}_{\mathrm{Pois}}(P_{2n})$. The map

$$\sigma : A_{2n} \to A_{2n},\ x_i \mapsto x_i' := \sigma(x_i),\ \partial_i \mapsto \partial_i'(\cdot) := \begin{cases}\{x'_{n+i}, \cdot\} & \text{if } i = 1, \dots, n,\\ \{-x'_{n-i}, \cdot\} & \text{if } i = n+1, \dots, 2n,\end{cases} \tag{5}$$

is an algebra endomorphism of the Weyl algebra A_{2n} where $\partial_i' \in \mathrm{Der}_K(P_{2n})$. By DC_{2n}, $\sigma \in \mathrm{Aut}_K(A_{2n})$, and so

$$A_{2n} = \bigoplus_{\alpha,\beta\in\mathbb{N}^{2n}} K x'^{\alpha}\partial'^{\beta}.$$

It follows that $\mathscr{D}(P_{2n})_i = \bigoplus_{\{\alpha,\beta\in\mathbb{N}^{2n}\,|\,|\beta|\le i\}} K x'^{\alpha}\partial'^{\beta}$ (use the defining relations in the new variables of A_{2n}). By the very definition, the automorphism σ respects the degree filtration on A_{2n}. In particular, $\sigma(P_{2n}) = P_{2n}$ since $\mathscr{D}(P_{2n})_0 = P_{2n}$, i.e. $\sigma \in \mathrm{Aut}_{\mathrm{Pois}}(P_{2n})$.

(3). Given a Jacobian map $\sigma \in \mathrm{End}_K(P_n)$. Let $\sigma \in \mathrm{End}_K(A_n)$ be its extension given by (1). By (2),

$$\sigma(\mathscr{D}(P_n)_i) \subseteq \mathscr{D}(P_n)_i \quad \text{for all} \quad i \ge 0,$$

i.e. the endomorphism σ of the Weyl algebra A_n respects the degree filtration and so the associated graded map

$$\mathrm{gr}(\sigma) : \mathrm{gr}(A_n) \to \mathrm{gr}(A_n),\ \ a + \mathscr{D}(P_n)_{i-1} \mapsto \sigma(a) + \mathscr{D}(P_n)_{i-1} \tag{6}$$

respects the Poisson structure, i.e. $\mathrm{gr}(\sigma) \in \mathrm{End}_{\mathrm{Pois}}(\mathrm{gr}(A_n))$. By PC_{2n}, $\mathrm{gr}(\sigma) \in \mathrm{Aut}_{\mathrm{Pois}}(\mathrm{gr}(A_n))$, hence $\sigma \in \mathrm{Aut}_K(P_n)$ since the automorphism $\mathrm{gr}(\sigma) \in \mathrm{Aut}_{\mathrm{Pois}}(\mathrm{gr}(A_n))$ is a *graded* automorphism and $\mathscr{D}(P_n)_0 = P_n$.

(4). It follows from statements (1) and (3).

(5). It follows from statement (4) and the equivalence $\mathrm{JC}_{2n} \Longleftrightarrow \mathrm{DC}_n$. □

By (5),

- PC_{2n} *is true iff the* A_{2n}*-module* ${}^{\sigma}P_{2n}$ *is simple for all* $\sigma \in \mathrm{End}_{\mathrm{Pois}}(P_{2n})$.

Theorem 8. *Let* $\sigma \in \mathrm{End}_{\mathrm{Pois}}(P_{2n})$ *and* $d := \deg(\sigma)$. *Then*

1. *The* A_{2n}*-module* ${}^{\sigma}P_{2n}$ *is holonomic, hence of finite length and 1-generated as an* A_{2n}*-module.*
2. $l_{A_{2n}}({}^{\sigma}P_{2n}) \le d^{2n}$.

Proof.

(1). The A_{2n}-module P_{2n} is holonomic. By Theorem 1, the A_{2n}-module ${}^{\sigma}P_{2n}$ is holonomic, hence of finite length and 1-generated as an A_{2n}-module.

(2). Since $\deg(x_i') \le d$ and $\deg(\partial_i') \le d$ (see (5)),

$$x_i' P_{2n,ds} \subseteq P_{2n,d(s+1)}, \quad \text{and} \quad \partial_i' P_{2n,ds} \subseteq P_{2n,d(s+1)} \quad \text{for all} \quad i = 1,\dots,2n \quad \text{and} \quad s \ge 0.$$

Therefore, $\{P_{2n,ds}\}_{s\ge 0}$ is a finite dimensional filtration of the A_{2n}-module ${}^{\sigma}P_{2n}$ such that

$$\dim_K(P_{2n,ds}) = \binom{ds+2n}{2n} = \frac{1}{(2n)!}(ds+2n)(ds+2n-1)\cdots(ds+1) = \frac{d^{2n}}{(2n)!}s^{2n} + \cdots$$

where three dots denote smaller terms. By [10, Lemma 8.5.9], $l_{A_{2n}}({}^{\sigma}P_{2n}) \le d^{2n}$. □

4. An analogue of the Conjecture of Dixmier for the algebras $\mathbb{I}_n$ of integro-differential operators

Let $\mathbb{I}_n := K\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n,\int_1,\dots,\int_n\rangle$ be the algebra of polynomial integro-differential operators where $\int_i : P_n \to P_n$, $p \mapsto \int p\,\mathrm{d}x_i$, i.e. $\int_i x^\alpha = (\alpha_i+1)^{-1}x_i x^\alpha$ for all $\alpha \in \mathbb{N}^n$, [6].

Conjecture ([7]). $\mathrm{End}_K(\mathbb{I}_n) = \mathrm{Aut}_K(\mathbb{I}_n)$.

Theorem 9 ([7, Theorem 1.1]). $\mathrm{End}_K(\mathbb{I}_1) = \mathrm{Aut}_K(\mathbb{I}_1)$.

5. An analogue of the Jacobian Conjecture and the Conjecture of Dixmier for the algebras $A_{n,m} := A_n \otimes P_m$

The centre of the algebra $A_{n,m}$ is P_m. Hence, for all $\sigma \in \mathrm{Aut}_K(A_{n,m})$, $\sigma(P_m) = P_m$.

Conjecture $\mathbf{JD}_{n,m}$ ([5]). *Every endomorphism $\sigma : A_n \otimes P_m \to A_n \otimes P_m$ such that $\sigma(P_m) \subseteq P_m$ and $\det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right) \in K^*$ is an automorphism.*

Theorem 10 ([5, Theorem 5.8, Proposition 5.9]). $\mathrm{JD}_{n,m} \Longleftrightarrow \mathrm{JC}_m + \mathrm{DC}_n$.

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Declaration of interests

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