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Which pairs of cardinals can be Hartogs and Lindenbaum numbers of a set?

by

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Abstract. Given any $\lambda \leqslant \kappa$, we construct a symmetric extension in which there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$. Consequently, we show that ZF + "for all pairs of infinite cardinals $\lambda \leqslant \kappa$ there is a set X such that $\aleph(X) = \lambda \leqslant \kappa = \aleph^*(X)$" is consistent.

1. Introduction. The Axiom of Choice is one of the most successful axioms in modern mathematics, generating many applications as well as several "paradoxes" (or rather counterintuitive surprises). One of its famous equivalents is Zermelo's theorem stating that every set can be well-ordered. Therefore, if the Axiom of Choice fails in a universe of set theory, some sets cannot be well-ordered. Nevertheless, we can still consider two ways in which a set X is "large", by asking how large are well-orderable subsets of X and how large are well-orderable partitions of X.

For example, if a set X can be mapped onto ω_{13}, then at the very least there is a sense in which it is large compared to ω. Moreover, if we extend the universe so that X can be well-ordered, then (at least) one of two scenarios must hold: (1) X will have the cardinality of at least $\aleph_{13}$, or (2) ω_{13} will be collapsed.

Definition 1.1. Let X be a set. The *Hartogs number* of X is

$$\aleph(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \mid \text{there is no injection } f\colon \alpha \to X\}.$$

The *Lindenbaum number* of X is

$$\aleph^*(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \setminus \{0\} \mid \text{there is no surjection } f\colon X \to \alpha\}.$$

2020 *Mathematics Subject Classification*: Primary 03E25; Secondary 03E35.
Key words and phrases: symmetric extensions, iterated forcing, iterated symmetries, symmetric iteration, Axiom of Choice, Hartogs number, Lindenbaum number.
Received 6 October 2023; revised 14 August 2024.
Published online 7 November 2024.

DOI: 10.4064/fm231006-14-8

The existence of $\aleph(X)$ is guaranteed under ZF by Hartogs's lemma, from [2] ([1]). Furthermore, $\aleph(X)$ must also be a cardinal number, and when X is well-orderable, $\aleph(X) = |X|^+$. The existence of $\aleph^*(X)$ is guaranteed under ZF by a lemma first used in the proof of Lindenbaum's theorem ([2]). Again, $\aleph^*(X)$ must also be a cardinal number, and when X is well-orderable, $\aleph^*(X) = |X|^+$.

It is also not difficult to see that for any set X, $\aleph(X) \leqslant \aleph^*(X)$. However, it need not be the case that $\aleph = \aleph^*$. Indeed, in ZF the statement $(\forall X)(\aleph(X) = \aleph^*(X))$ is equivalent to the Axiom of Choice for well-ordered families of sets, established in [8] (this axiom is weaker than the Axiom of Choice in its general form). So, if the Axiom of Choice for well-ordered families fails, there is some X such that $\aleph(X) < \aleph^*(X)$. We are concerned with a maximal possible violation of this principle.

Main Theorem. ZF *is equiconsistent with* ZF + *"for all pairs of infinite cardinals* $\lambda \leqslant \kappa$ *there is a set* X *such that* $\aleph(X) = \lambda \leqslant \kappa = \aleph^*(X)$*".*

1.1. Structure of the paper. Section 2 establishes preliminaries for the paper, in particular our conventions for handling cardinalities, forcing, and symmetric extensions. Some time is also given to permutation groups. In Section 3 we show the consistency of the existence of a single set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ for arbitrary infinite $\lambda \leqslant \kappa$. In Section 4 we use the machinery established in Section 3 to construct a class-sized product of notions of forcing to prove the main theorem.

2. Preliminaries.

Throughout this paper we work in ZFC. Our treatment of forcing will be standard. By a *notion of forcing* we mean a preordered set $\mathbb{P}$ with maximum element denoted $\mathbb{1}_{\mathbb{P}}$, or with the subscript omitted when clear from context. We write $q \leqslant p$ to mean that q *extends* p. Two conditions p, p' are said to be *compatible*, written $p \parallel p'$, if they have a common extension. We follow Goldstern's alphabet convention so p is never a stronger condition than q, etc.

When given a collection of $\mathbb{P}$-names, $\{\dot{x}_i \mid i \in I\}$, we will denote by $\{\dot{x}_i \mid i \in I\}^\bullet$ the canonical name this class generates: $\{\langle \mathbb{1}, \dot{x}_i\rangle \mid i \in I\}$. The notation extends naturally to ordered pairs and functions with domains in the ground model. An immediate application of this is a simplified definition of check names; given x, the *check name* for x is defined inductively as $\check{x} = \{\check{y} \mid y \in x\}^\bullet$.

Given a set X, we denote by $|X|$ its cardinal number. If X can be well-ordered, then $|X|$ is simply the least ordinal α such that a bijection between

([1]) One can also go to [1, Theorem 8.18] for a proof in English.

([2]) The theorem was first stated without proof in [7, Théorème 82.A_6]. The first published proof is in [10], or can be found in English in [11, Ch. XVI, Section 3, Theorem 1].

α and X exists. Otherwise, we use the Scott cardinal of X, that is, $|X| = \{Y \in V_\alpha \mid \text{there is a bijection } f\colon X \to Y\}$ with α taken minimal such that the set is non-empty. Greek letters, when used as cardinals, will always refer to well-ordered cardinals. We call an ordinal α a cardinal if $|\alpha| = \alpha$.

We write $|X| \leqslant |Y|$ to mean that there is an injection from X to Y, and $|X| \leqslant^* |Y|$ to mean that there is a surjection from Y to X or that X is empty. These notations extend to $|X| < |Y|$ (and $|X| <^* |Y|$) to mean that $|X| \leqslant |Y|$ (respectively $|X| \leqslant^* |Y|$) and there is no injection from Y to X (respectively no surjection from X to Y). Finally, $|X| = |Y|$ means that there is a bijection between X and Y.

Using this notation, one may redefine the Hartogs and Lindenbaum numbers as

$$\aleph(X) := \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\},$$
$$\aleph^*(X) := \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}.$$

2.1. Symmetric extensions. It is a fundamental property of forcing that if $V \vDash \mathsf{ZFC}$, and G is V-generic for some notion of forcing $\mathbb{P} \in V$, then $V[G] \vDash \mathsf{ZFC}$. However, this demands additional techniques for trying to establish results that are inconsistent with AC. Symmetric extensions expand the technique of forcing in this very way by constructing an intermediate model between V and $V[G]$ that is a model of ZF.

Given a notion of forcing $\mathbb{P}$, we shall denote by $\mathrm{Aut}(\mathbb{P})$ the collection of automorphisms of $\mathbb{P}$. Let $\mathbb{P}$ be a notion of forcing and $\pi \in \mathrm{Aut}(\mathbb{P})$. Then π extends naturally to act on $\mathbb{P}$-names by recursion: $\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}$.

Due to the construction of the forcing relation from the notion of forcing, we end up with the following lemma [3, Lemma 14.37].

LEMMA 2.1 (The Symmetry Lemma). *Let $\mathbb{P}$ be a notion of forcing, $\pi \in \mathrm{Aut}(\mathbb{P})$, and $\dot{x}$ a $\mathbb{P}$-name. Then $p \Vdash \varphi(\dot{x})$ if and only if $\pi p \Vdash \varphi(\pi\dot{x})$.* ■

Note in particular that for all $\pi \in \mathrm{Aut}(\mathbb{P})$ we have $\pi\mathbb{1} = \mathbb{1}$. Therefore, $\pi\check{x} = \check{x}$ for all ground model sets x, and $\pi\{\dot{x}_i \mid i \in I\}^\bullet = \{\pi\dot{x}_i \mid i \in I\}^\bullet$, similarly extending to tuples, functions, etc.

Given a group $\mathscr{G}$, a *filter of subgroups* of $\mathscr{G}$ is a set $\mathscr{F}$ of subgroups of $\mathscr{G}$ that is closed under supergroups and finite intersections. We say that $\mathscr{F}$ is *normal* if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, then $\pi H \pi^{-1} \in \mathscr{F}$.

A *symmetric system* is a triple $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ such that $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$. Given such a symmetric system, we say that a $\mathbb{P}$-name $\dot{x}$ is *$\mathscr{F}$-symmetric* if $\mathrm{sym}_{\mathscr{G}}(\dot{x}) := \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$, and that $\dot{x}$ is *hereditarily $\mathscr{F}$-symmetric* if this notion holds for every $\mathbb{P}$-name hereditarily appearing in $\dot{x}$. We denote by $\mathsf{HS}_{\mathscr{F}}$ the class of hereditarily $\mathscr{F}$-symmetric names. When clear from context, we will omit subscripts and simply write $\mathrm{sym}(\dot{x})$ or HS.

The following theorem, [3, Lemma 15.51], is foundational to the study of symmetric extensions.

THEOREM 2.2. *Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a symmetric system, $G \subseteq \mathbb{P}$ a V-generic filter, and let $\mathcal{M}$ denote the class $\mathsf{HS}_{\mathscr{F}}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}}\}$. Then $\mathcal{M}$ is a transitive model of ZF such that $V \subseteq \mathcal{M} \subseteq V[G]$.* ■

Finally, we have a forcing relation for symmetric extensions $\Vdash^{\mathsf{HS}}$ defined by relativising the forcing relation $\Vdash$ to the class HS. This relation has the same properties and behaviour of the standard forcing relation $\Vdash$. Moreover, when $\pi \in \mathscr{G}$, the Symmetry Lemma holds for $\Vdash^{\mathsf{HS}}$.

2.2. Wreath products. Frequently within this paper we will exhibit groups of automorphisms $\mathscr{G}$ as permutation groups with an action on the notion of forcing. By a *permutation group* (of a set X) we mean a subgroup of S_X, the group of bijections $X \to X$. If $\pi \in S_X$, then by the *support* of π, written $\operatorname{supp}(\pi)$, we mean the set $\{x \in X \mid \pi(x) \neq x\}$. Given an infinite cardinal λ we denote by $S_X^{<\lambda}$ the subgroup of S_X of permutations π such that $|\operatorname{supp}(\pi)| < \lambda$.

DEFINITION 2.3 (Wreath product). Given two permutation groups $G \leqslant S_X$ and $H \leqslant S_Y$, the *wreath product* of G and H, denoted $G \wr H$, is the subgroup of permutations $\pi \in S_{X \times Y}$ which have the following property:

- there is $\pi^* \in G$ and a sequence $\langle \pi_x \mid x \in X \rangle \in H^X$ such that for all $\langle x, y \rangle \in X \times Y$, $\pi(x, y) = \langle \pi^*(x), \pi_x(y) \rangle$.

That is, π first permutes each column $\{x\} \times Y$ according to some $\pi_x \in H$, and then acts on the X co-ordinate of $X \times Y$, permuting its columns via some $\pi^* \in G$.

Given $\pi \in G \wr H$, we will use the notation π^* and π_x to mean the elements of G and H respectively from the definition. Note that if $\pi, \sigma \in G \wr H$, then $(\pi\sigma)^* = \pi^*\sigma^*$.

Note also that $\{\mathrm{id}\} \wr S_Y \leqslant S_{X \times Y}$ is the group of all $\pi \in S_{X \times Y}$ such that for all $\langle x, y \rangle \in X \times Y$, $\pi(x, y) \in \{x\} \times Y$.

3. Realising a single pair as Hartogs and Lindenbaum numbers of a set. Let us spend some time establishing the consistency and construction of a single set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$. The construction used here will then be iterated in Section 4 to prove our main theorem.

THEOREM 3.1. *Let $\lambda \leqslant \kappa$ be infinite cardinals. There is a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ and a $\mathbb{P}$-name $\dot{X} \in \mathsf{HS}_{\mathscr{F}}$ such that*

$$\mathbb{1}_{\mathbb{P}} \Vdash^{\mathsf{HS}} \text{“}\aleph(\dot{X}) = \check{\lambda} \textit{ and } \aleph^*(\dot{X}) = \check{\kappa}\text{”}.$$

Proof. Let μ be a regular cardinal such that $\mu \geqslant \lambda$, and let

$$\mathbb{P} = \mathrm{Add}(\mu, \kappa \times \lambda \times \mu).$$

That is, the conditions of $\mathbb{P}$ are partial functions $p\colon \kappa \times \lambda \times \mu \times \mu \to 2$ such that $|\mathrm{dom}(p)| < \mu$, with $q \leqslant p$ if $q \supseteq p$.

For $p \in \mathbb{P}$ and $A \subseteq \kappa \times \lambda \times \mu$, we will write $p \restriction A$ to mean $p \restriction A \times \mu$, and for $B \subseteq \kappa \times \lambda$ we will write $p \restriction B$ to mean $p \restriction B \times \mu \times \mu$. Furthermore, we shall write $\mathrm{supp}(p)$ to mean the projection of the domain of p to its first three co-ordinates, so $\mathrm{supp}(p) \subseteq \kappa \times \lambda \times \mu$.

We define the following $\mathbb{P}$-names:

- $\dot{y}_{\alpha,\beta,\gamma} \coloneqq \{\langle p, \check{\delta}\rangle \mid p \in \mathbb{P},\, \delta < \mu,\, p(\alpha,\beta,\gamma,\delta) = 1\}$;
- $\dot{x}_{\alpha,\beta} \coloneqq \{\dot{y}_{\alpha,\beta,\gamma} \mid \gamma \in \mu\}^{\bullet}$;
- $\dot{X} \coloneqq \{\dot{x}_{\alpha,\beta} \mid \langle\alpha,\beta\rangle \in \kappa \times \lambda\}^{\bullet}$.

In the extension, $\dot{X}$ will be the name for the set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

Let $\mathscr{G} = S^{<\lambda}_{\kappa\times\lambda} \wr S_\mu$. That is, $\mathscr{G}$ is the group of permutations π in the wreath product $S_{\kappa\times\lambda} \wr S_\mu$ such that $\pi^* \in S_{\kappa\times\lambda}$ fixes all but fewer than λ many elements of $\kappa\times\lambda$. The group $\mathscr{G}$ acts on $\mathbb{P}$ via $\pi p(\pi(\alpha,\beta,\gamma),\delta) = p(\alpha,\beta,\gamma,\delta)$. Note that, for $\pi \in \mathscr{G}$,

$$\begin{aligned}\pi\dot{y}_{\alpha,\beta,\gamma} &= \{\langle \pi p, \pi\check{\delta}\rangle \mid p \in \mathbb{P},\, \delta < \mu,\, p(\alpha,\beta,\gamma,\delta) = 1\}\\ &= \{\langle \pi p, \check{\delta}\rangle \mid p \in \mathbb{P},\, \delta < \mu,\, \pi p(\pi(\alpha,\beta,\gamma),\delta) = 1\}\\ &= \{\langle p, \check{\delta}\rangle \mid p \in \mathbb{P},\, \delta < \mu,\, p(\pi(\alpha,\beta,\gamma),\delta) = 1\}\\ &= \dot{y}_{\pi(\alpha,\beta,\gamma)}.\end{aligned}$$

Similar verification shows that $\pi\dot{x}_{\alpha,\beta} = \dot{x}_{\pi^*(\alpha,\beta)}$ and $\pi\dot{X} = \dot{X}$. When we have defined the filter of subgroups $\mathscr{F}$ (which we shall do upon the conclusion of this sentence), it will be clear from these calculations that these names are hereditarily $\mathscr{F}$-symmetric.

For $I \in [\kappa]^{<\kappa}$, $J \in [I \times \lambda]^{<\lambda}$, and $K \in [J \times \mu]^{<\lambda}$, let $H_{I,J,K}$ be the subgroup of $\mathscr{G}$ given by those π such that

- $\pi^* \restriction I \times \lambda \in \{\mathrm{id}\} \wr S_\lambda$;
- $\pi^* \restriction J = \mathrm{id}$;
- $\pi \restriction K = \mathrm{id}$.

That is, we are taking those $\pi \in S^{<\lambda}_{\kappa\times\lambda} \wr S_\mu$ such that π^* fixes the columns taken from the set I of cardinality less than κ, and fixes pointwise the set J of cardinality less than λ. We then further require that π fixes pointwise the set K of cardinality less than λ.

Let $\mathscr{F}$ be the filter of subgroups of $\mathscr{G}$ generated by groups of the form $H_{I,J,K}$ for $I \in [\kappa]^{<\kappa}$, $J \in [I \times \lambda]^{<\lambda}$, and $K \in [J \times \mu]^{<\lambda}$ (3). We shall refer to triples I, J, K as being "appropriate" to mean that they satisfy these conditions.

By previous calculations, $\pi\dot{y}_{\alpha,\beta,\gamma} = \dot{y}_{\alpha,\beta,\gamma}$ whenever $\pi(\alpha,\beta,\gamma) = \langle\alpha,\beta,\gamma\rangle$, so that $\operatorname{sym}(\dot{y}_{\alpha,\beta,\gamma}) \geqslant H_{\{\alpha\},\{\langle\alpha,\beta\rangle\},\{\langle\alpha,\beta,\gamma\rangle\}} \in \mathscr{F}$. Similarly $\operatorname{sym}(\dot{x}_{\alpha,\beta}) \geqslant H_{\{\alpha\},\{\alpha,\beta\},\emptyset} \in \mathscr{F}$ and of course $\operatorname{sym}(\dot{X}) = \mathscr{G} \in \mathscr{F}$.

Claim 3.1.1. *$\mathscr{F}$ is normal. Hence $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$ is a symmetric system.*

Proof of claim. Note that for appropriate I, J, K and I', J', K',

$$H_{I,J,K} \cap H_{I',J',K'} = H_{I\cup I',J\cup J',K\cup K'}$$

and $I \cup I', J \cup J', K \cup K'$ is appropriate. Therefore, for all $H \leqslant \mathscr{G}$, $H \in \mathscr{F}$ if and only if there is appropriate I, J, K such that $H \geqslant H_{I,J,K}$. Hence, to show that $\mathscr{F}$ is normal, it is sufficient to show that for all appropriate I, J, K and all $\pi \in \mathscr{G}$, there is appropriate I', J', K' such that $\pi H_{I,J,K}\pi^{-1} \geqslant H_{I',J',K'}$, or equivalently that $H_{I,J,K} \geqslant \pi^{-1}H_{I',J',K'}\pi$. Given such I, J, K and π, we define

$$\begin{aligned} K' &= \pi``K,\\ J' &= \operatorname{Proj}(K') \cup \operatorname{supp}(\pi^*) \cup J \cup \pi^*``J,\\ I' &= \operatorname{Proj}(J') \cup I. \end{aligned}$$

We must first show that I', J', K' is appropriate. Firstly, note that $|K'| = |K| < \lambda$, $|J'| \leqslant 2 \cdot |J| + |\operatorname{supp}(\pi^*)| + |K'| < \lambda$, and $|I'| \leqslant |I| + |J'| < \kappa$ as required. Secondly, the inclusion of the projections in the definitions of I', J', K' guarantees that this triple is appropriate. We claim that $H_{I',J',K'}$ is the required group. Let $\sigma \in H_{I',J',K'}$; then we must show that $\pi^{-1}\sigma\pi \in H_{I,J,K}$.

Firstly, for all $\langle\alpha,\beta,\gamma\rangle \in K$, $\pi(\alpha,\beta,\gamma) \in K'$, so $\sigma(\pi(\alpha,\beta,\gamma)) = \pi(\alpha,\beta,\gamma)$ and hence $\pi^{-1}\sigma\pi(\alpha,\beta,\gamma) = \langle\alpha,\beta,\gamma\rangle$ as required.

Secondly, we claim that $(\pi^{-1}\sigma\pi)^* = \sigma^*$ and that this is sufficient. Indeed, if this is the case then, since $J \subseteq J'$ and $I \subseteq I'$, we get $\pi^{-1}\sigma\pi \in H_{I,J,K}$ as desired. Note also that since π^* is a bijection, $\langle\alpha,\beta\rangle \in \operatorname{supp}(\pi^*)$ if and only if $\pi^*(\alpha,\beta) \in \operatorname{supp}(\pi^*)$.

If $\pi^*(\alpha,\beta) \neq \langle\alpha,\beta\rangle$ then both $\langle\alpha,\beta\rangle$ and $\pi^*(\alpha,\beta)$ are in $\operatorname{supp}(\pi^*) \subseteq J'$, and thus $(\sigma\pi)^*(\alpha,\beta) = \pi^*(\alpha,\beta)$ and $\sigma^*(\alpha,\beta) = \langle\alpha,\beta\rangle$. Hence $(\pi^{-1}\sigma\pi)^*(\alpha,\beta) = \langle\alpha,\beta\rangle$ and $\langle\alpha,\beta\rangle = \sigma^*(\alpha,\beta)$ as desired.

Finally, we deal with the case $\pi^*(\alpha,\beta) = \langle\alpha,\beta\rangle$. If $(\pi\sigma)^*(\alpha,\beta) \neq \sigma^*(\alpha,\beta)$, then $\sigma^*(\alpha,\beta) \in \operatorname{supp}(\pi^*) \subseteq J'$, so $(\sigma\sigma)^*(\alpha,\beta) = \sigma^*(\alpha,\beta)$ and thus $\sigma^*(\alpha,\beta) = \langle\alpha,\beta\rangle$. Hence, $\pi^*(\alpha,\beta) \neq \langle\alpha,\beta\rangle$, and as before $(\pi^{-1}\sigma\pi)^*(\alpha,\beta) = \sigma^*(\alpha,\beta)$. Therefore, if $\pi^*(\alpha,\beta) = \langle\alpha,\beta\rangle$ then we must have $(\pi\sigma)^*(\alpha,\beta) = \sigma^*(\alpha,\beta)$.

(3) Since $\mathbb{P}$ is λ-closed and $\mathscr{F}$ is λ-complete, $\mathsf{DC}_{<\lambda}$ holds in the symmetric extension. A proof can be found in [6, Lemma 1].

Now, $\pi^*(\alpha,\beta) = \langle\alpha,\beta\rangle$, so $(\pi\sigma)^*(\alpha,\beta) = (\sigma\pi)^*(\alpha,\beta)$ and $(\pi^{-1}\sigma\pi)^*(\alpha,\beta) = \sigma^*(\alpha,\beta)$ as desired. ⊣

Claim 3.1.2. *Let $q \in \mathbb{P}$, $H = H_{I,J,K} \in \mathscr{F}$, and $\langle\alpha,\beta\rangle, \langle\alpha',\beta'\rangle \in \kappa\times\lambda$. The following are equivalent:*

(1) *There is $\pi \in H$ such that π^* is the transposition $(\langle\alpha,\beta\rangle\ \langle\alpha',\beta'\rangle)$ and $\pi q \parallel q$.*
(2) $\{\alpha,\alpha'\}\cap I \neq \emptyset \implies \alpha = \alpha'$, *and*

$$\{\langle\alpha,\beta\rangle,\langle\alpha',\beta'\rangle\}\cap J \neq \emptyset \implies \langle\alpha,\beta\rangle = \langle\alpha',\beta'\rangle.$$

Proof of claim. (1)⇒(2). By the definition of H, if there is a $\pi\in H$ as in (1) then (2) must be satisfied.

(2)⇒(1). If $\alpha,\alpha',\beta,\beta'$ satisfy (2), then any $\pi\in\mathscr{G}$ such that π^* equals $(\langle\alpha,\beta\rangle\ \langle\alpha',\beta'\rangle)$ is a candidate for an element of H (as $|\mathrm{supp}(\pi^*)| = 2 < \lambda$). Firstly, if $\langle\alpha,\beta\rangle = \langle\alpha',\beta'\rangle$ then we may take $\pi = \mathrm{id}$, so assume otherwise. Let $A = \{\gamma\in\mu \mid \langle\alpha,\beta,\gamma\rangle \in \mathrm{supp}(q)\}$, and $B = \{\gamma'\in\mu \mid \langle\alpha',\beta',\gamma'\rangle\in\mathrm{supp}(q)\}$. Since $|A|,|B| < \mu$, there is a permutation σ of μ such that $\sigma``A\cap B = \emptyset$ and $A\cap\sigma``B = \emptyset$. Therefore, setting $\pi_{\alpha,\beta} = \pi_{\alpha',\beta'} = \sigma$ we will have

$$\mathrm{supp}(q\restriction\langle\alpha,\beta\rangle)\cap\mathrm{supp}(\pi q\restriction\langle\alpha,\beta\rangle) = \sigma``A\cap B = \emptyset,$$
$$\mathrm{supp}(q\restriction\langle\alpha',\beta'\rangle)\cap\mathrm{supp}(\pi q\restriction\langle\alpha',\beta'\rangle) = A\cap\sigma``B = \emptyset.$$

Hence $q\restriction\{\langle\alpha,\beta\rangle,\langle\alpha',\beta'\rangle\} \parallel \pi q\restriction\{\langle\alpha,\beta\rangle,\langle\alpha',\beta'\rangle\}$, and for all other $\langle\alpha'',\beta''\rangle$ we have $q\restriction\langle\alpha'',\beta''\rangle = \pi q\restriction\langle\alpha'',\beta''\rangle$. ⊣

The remainder of the proof will be spent showing that the name $\dot X$ will give us the object that we are searching for, that is, $\mathbb{1}\Vdash^{\mathsf{HS}} \aleph(\dot X) = \check\lambda$ and $\mathbb{1}\Vdash^{\mathsf{HS}} \aleph^*(\dot X) = \check\kappa$. We shall first prove the inequalities $\mathbb{1}\Vdash^{\mathsf{HS}}\aleph(\dot X)\geqslant\check\lambda$ and $\mathbb{1}\Vdash^{\mathsf{HS}}\aleph^*(\dot X)\geqslant\check\kappa$, and then prove that they can be sharpened to equalities.

Towards the inequalities, for any $\alpha,\eta<\kappa$ let

$$\iota_{\alpha,\eta} := \begin{cases}\alpha & \text{if } \alpha<\eta,\\ 0 & \text{otherwise.}\end{cases}$$

Then consider the name $\dot e_\eta := \{\langle\dot x_{\alpha,\beta},\check\iota_{\alpha,\eta}\rangle^\bullet \mid \langle\alpha,\beta\rangle\in\kappa\times\lambda\}^\bullet$. Routine verification shows $\mathrm{sym}(\dot e_\eta)\geqslant H_{\eta,\emptyset,\emptyset}$, so $\dot e_\eta\in\mathsf{HS}$. Furthermore, $\mathbb{1}\Vdash$ "$\dot e_\eta\colon\dot X\to\check\eta$ is surjective", and thus $\mathbb{1}\Vdash^{\mathsf{HS}}\aleph^*(\dot X)\geqslant\check\kappa$.

Similarly, for any $\eta<\lambda$ and any $\alpha\in\kappa$, take $\dot m_\eta := \{\langle\check\beta,\dot x_{\alpha,\beta}\rangle^\bullet \mid \beta<\eta\}^\bullet$. Routine verification shows that $\mathrm{sym}(\dot m_\eta)\geqslant H_{\{\alpha\},\{\alpha\}\times\eta,\emptyset}$, so $\dot m_\eta\in\mathsf{HS}$ as well. Furthermore, $\mathbb{1}\Vdash$ "$\dot m_\eta\colon\check\eta\to\dot X$ is injective", and thus $\mathbb{1}\Vdash^{\mathsf{HS}}\aleph(\dot X)\geqslant\check\lambda$.

It remains to show that these inequalities are, in fact, equalities, starting with $\aleph^*$. Suppose that $\dot f\in\mathsf{HS}$ and $p\Vdash\dot f\colon\dot X\to\check\kappa$. Let $H = H_{I,J,K}\leqslant\mathrm{sym}(\dot f)$. Then suppose that for some $q\leqslant p$ and $\langle\alpha,\beta\rangle\in\kappa\times\lambda$ there is η such that $q\Vdash\dot f(\dot x_{\alpha,\beta}) = \check\eta$.

By Claim 3.1.2, if $\alpha \notin I$ then for any $\alpha' \notin I$ and any $\beta' \in \lambda$ there is $\pi \in H$ such that $\pi^*(\alpha, \beta) = \langle \alpha', \beta' \rangle$ and $\pi q \parallel q$. Then $\pi q \Vdash \dot{f}(\dot{x}_{\alpha',\beta'}) = \check{\eta}$, so $q \cup \pi q \leqslant q$ forces that $\dot{f}(\dot{x}_{\alpha,\beta}) = \dot{f}(\dot{x}_{\alpha',\beta'})$. Hence p forces that $\dot{f}$ is constant outside of $I \times \lambda$.

If instead $\alpha \in I$ but $\langle \alpha, \beta \rangle \notin J$ then, again by Claim 3.1.2, for any $\beta' \in \lambda$ such that $\langle \alpha, \beta' \rangle \notin J$ there is $\pi \in H$ such that $\pi^*(\alpha, \beta) = \langle \alpha, \beta' \rangle$ and $\pi q \parallel q$. Once again $\pi q \Vdash \dot{f}(\dot{x}_{\alpha',\beta'}) = \check{\eta}$, and so p forces that in $(I \times \lambda) \setminus J$ the value of $\dot{f}(\dot{x}_{\alpha,\beta})$ depends only on α. This means that $\dot{f}$ can take only at most $|J| + |I| + 1 < \kappa$ many values, so cannot be a surjection, and thus $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) = \check{\kappa}$.

Finally, if $\dot{f} \in \mathsf{HS}$ and $p \Vdash \dot{f} \colon \check{\lambda} \to \dot{X}$, then let $H = H_{I,J,K} \leqslant \operatorname{sym}(\dot{f})$. We shall show that $p \Vdash \dot{f}``\check{\lambda} \subseteq \{\dot{x}_{\alpha,\beta} \mid \langle \alpha, \beta \rangle \in J\}^\bullet$, and hence $\dot{f}$ cannot be injective.

Suppose otherwise, that for some $q \leqslant p$, $\langle \alpha, \beta \rangle \notin J$, and $\eta < \lambda$ we have $q \Vdash \dot{f}(\check{\eta}) = \dot{x}_{\alpha,\beta}$. Since $\langle \alpha, \beta \rangle \notin J$, for any $\beta' \in \lambda$ such that $\langle \alpha, \beta' \rangle \notin J$ there is $\pi \in H$ such that $\pi^*(\alpha, \beta) = \langle \alpha, \beta' \rangle$ and $\pi q \parallel q$. Since $|J| < \lambda$, we may take $\beta' \neq \beta$, and so $\pi q \Vdash \dot{f}(\check{\eta}) = \pi \dot{x}_{\alpha,\beta} = \dot{x}_{\alpha,\beta'}$. Therefore, $\pi q \cup q \Vdash \dot{x}_{\alpha,\beta} = \dot{f}(\check{\eta}) = \dot{x}_{\alpha,\beta'}$, contradicting our assumption that $\beta' \neq \beta$. Thus our assertion is proved and $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) = \check{\lambda}$. ∎

4. Realising all pairs at once. We have now constructed enough technology to prove our main theorem.

MAIN THEOREM. ZF *is equiconsistent with* ZF + "*for all pairs of infinite cardinals* $\lambda \leqslant \kappa$ *there is a set* X *such that* $\aleph(X) = \lambda$ *and* $\aleph^*(X) = \kappa$".

The structure shall be similar to the treatment of class products of symmetric extensions found in, for example, [4].

We begin in a model V of $\mathsf{ZFC} + \mathsf{GCH}$. We shall define inductively a symmetric system $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \rangle$ for each $\alpha \in \mathrm{Ord}$. Each such system will be precisely of the form described in Theorem 3.1, and so to fully define each system we need only define the parameters λ_α, κ_α, and μ_α. First, let $\{\langle \lambda_\alpha, \kappa_\alpha \rangle \mid \alpha \in \mathrm{Ord}\}$ be an enumeration of each pair $\langle \lambda, \kappa \rangle$ with $\aleph_0 \leqslant \lambda \leqslant \kappa$, using (for example) the Gödel pairing function. Then we shall define μ_α to be the least cardinal satisfying the following conditions:

- μ_α is regular;
- for all $\beta < \alpha$, $\mu_\beta < \mu_\alpha$;
- for all $\beta \leqslant \alpha$, $\kappa_\beta < \mu_\alpha$;
- setting $\mathbb{Q}$ to be the finite-support product $\prod_{\beta<\alpha} \mathbb{P}_\beta$, we require $|\mathbb{Q}| < \mu_\alpha$;
- for all $\beta < \alpha$, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} |\dot{V}_{\mu_\beta^+}| < \check{\mu}_\alpha$;
- $\aleph_\alpha < \mu_\alpha$.

Let $\mathbb{P}$ be the finite-support product of all $\mathbb{P}_\alpha$, $\mathscr{G}$ the finite-support product of all $\mathscr{G}_\alpha$, and $\mathscr{F}$ the finite-support product of all $\mathscr{F}_\alpha$. For $E \subseteq \mathrm{Ord}$, we denote by $\mathbb{P}\restriction E$ (respectively $\mathscr{G}\restriction E, \mathscr{F}\restriction E$) the restriction of $\mathbb{P}$ (respectively $\mathscr{G}, \mathscr{F}$) to the coordinates found in E. Since any $\mathbb{P}$-name $\dot{x}$ is a set, it is a $\mathbb{P}\restriction\alpha$-name for some α, and so $\dot{x}$ is hereditarily $\mathscr{F}$-symmetric if and only if it is hereditarily $\mathscr{F}\restriction\alpha$-symmetric for some α. Therefore, setting $\mathsf{HS} = \mathsf{HS}_{\mathscr{F}}$, $\mathsf{HS}_\alpha = \mathsf{HS}_{\mathscr{F}\restriction\alpha}$, and letting G be V-generic for $\mathbb{P}$, we get

$$\bigcup_{\alpha\in\mathrm{Ord}} \mathsf{HS}_\alpha^{G\restriction\alpha} = \bigcup_{\alpha\in\mathrm{Ord}} \mathsf{HS}_\alpha^G = \Big(\bigcup_{\alpha\in\mathrm{Ord}} \mathsf{HS}_\alpha\Big)^G = \mathsf{HS}^G.$$

Let $\mathcal{M} = \mathsf{HS}^G$ and $\mathcal{M}_\alpha = \mathsf{HS}_\alpha^{G\restriction\alpha}$. Then $\mathcal{M} = \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{M}_\alpha$. We wish to prove that $\mathcal{M} \vDash \mathsf{ZF}$, and shall use the following theorem [5, Theorem 9.2].

Theorem 4.1. *Let $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ be a finite-support product of symmetric extensions of homogeneous systems. Suppose that for each η there is α^* such that for all $\alpha \geqslant \alpha^*$, the αth symmetric extension does not add new sets of rank at most η. Then no sets of rank at most η are added by limit steps either. In particular, the end model satisfies* ZF. ∎

The conditions of the theorem are also desirable for our construction. We shall show that for all α there is a hereditarily symmetric name $\dot{X}_\alpha$ such that $\dot{X}_\alpha \in \mathsf{HS}_{\alpha+1}$ and $\mathcal{M}_{\alpha+1} \vDash$ "$\aleph(\dot{X}_\alpha^G) = \lambda_\alpha$ and $\aleph^*(\dot{X}_\alpha^G) = \kappa_\alpha$". In this case, if we can preserve a large enough initial segment of $\mathcal{M}_{\alpha+1}$ for the rest of the iteration, then $\dot{X}_\alpha^G$ will still have this property in $\mathcal{M}$.

We shall require the following fact [4, Lemma 2.3].

Lemma 4.2. *Let κ be a regular cardinal, $\mathbb{P}$ a κ-c.c. forcing, and $\mathbb{Q}$ a κ-distributive forcing. If $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c.", then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\mathbb{Q}}$ is $\check{\kappa}$-distributive".* ∎

Proposition 4.3. *Let $\delta < \beta < \alpha$. Then $\mathcal{M}_\beta$ and $\mathcal{M}_\alpha$ agree on sets of rank less than μ_δ^+.*

Proof. It is sufficient to prove that for all $\beta \in \mathrm{Ord}$, $\mathcal{M}_\beta$ and $\mathcal{M}_{\beta+1}$ agree on sets of rank less than μ_δ; this is the successor stage for an induction on α of the statement of the proposition, and Theorem 4.1 provides the induction at the limit stage. Towards this end, let $\delta < \beta \in \mathrm{Ord}$ and $\mathcal{N} = V[G\restriction\beta]$. We shall show that $\mathbb{P}_\beta$ adds no sets of rank less than μ_δ^+ to $\mathcal{N}$, and since $\mathcal{M}_\beta \subseteq \mathcal{N}$ and $\mathcal{M}_{\beta+1} \subseteq \mathcal{N}[G(\beta)]$, the claim will be proved.

Let $\kappa = |V_{\mu_\delta^+}^{\mathcal{N}}|$. Then, by the definition of μ_β, $\kappa < \mu_\beta$, and so it is sufficient to prove that $\mathbb{P}_\beta$ is μ_β-distributive. Since $\mathbb{P}_\beta = \mathrm{Add}(\mu_\beta, \kappa_\beta \times \lambda_\beta \times \mu_\beta)^V$, certainly in V it is μ_β^+-distributive. Furthermore, by definition, $|\mathbb{P}\restriction\beta| < \mu_\beta$, so $\mathbb{P}\restriction\beta$ is μ_β-c.c., and indeed $\mathbb{P}_\beta \Vdash$ "$\check{\mathbb{P}}\restriction\beta$ is $\check{\mu}_\beta$-c.c.". Hence, by Lemma 4.2, $\mathbb{P}\restriction\beta \Vdash$ "$\check{\mathbb{P}}_\beta$ is μ_β-distributive", as required. ∎

Proposition 4.4. *For all* $\alpha \in \mathrm{Ord}$,

$$\mathcal{M}_{\alpha+1} \vDash (\exists X_\alpha)(\aleph(X_\alpha) = \lambda_\alpha \leqslant \kappa_\alpha = \aleph^*(X_\alpha)).$$

Proof. Let $\mathcal{N}$ be the symmetric extension of $V[G{\restriction}\alpha]$ given by the symmetric system $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha\rangle$ and the generic filter $G(\alpha)$. By the usual arguments concerning product forcing, $G(\alpha)$ is $V[G{\restriction}\alpha]$-generic for $\mathbb{P}_\alpha$ whenever $G{\restriction}\alpha+1$ is V-generic for $\mathbb{P}{\restriction}\alpha+1$. Therefore, $\mathcal{M}_{\alpha+1} \subseteq \mathcal{N} \subseteq V[G{\restriction}\alpha+1]$ is a chain of transitive subclasses, and $V[G{\restriction}\alpha]$ is a transitive subclass of $\mathcal{N}$ as well.

Let $X_\alpha \in \mathcal{M}_{\alpha+1}$ be the realisation of the name $\dot{X}$ exhibited in Theorem 3.1. Since $\dot{X}$ is a $\mathbb{P}_\alpha$-name, it is equivalent to its realisation under $G{\restriction}\alpha+1$. Similarly, the realisations of $\dot{e}_\eta$ and $\dot{m}_\eta$ for appropriate values of η are still the desired functions, and so $\mathcal{M}_{\alpha+1} \vDash$ "$\aleph(X_\alpha) \geqslant \lambda_\alpha$ and $\aleph^*(X_\alpha) \geqslant \kappa_\alpha$".

Theorem 3.1 required no assumptions about the ground model other than ZFC, and so $\mathcal{N}$, in its role as a symmetric extension of $V[G{\restriction}\alpha]$, must satisfy $\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$. Therefore, since any function $f \in \mathcal{M}_{\alpha+1}$ is also in $\mathcal{N}$, we deduce that f cannot be an injection $\lambda_\alpha \to X_\alpha$ or a surjection $X_\alpha \to \kappa_\alpha$. Hence, $\mathcal{M}_{\alpha+1}$ satisfies $\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$ as well. ∎

With some care over the construction of X_α, we may now prove our main theorem.

Proof of the Main Theorem. Firstly, by Proposition 4.3 and Theorem 4.1, $\mathcal{M} \vDash$ ZF. Next, by Proposition 4.4, for all $\alpha \in \mathrm{Ord}$, $\mathcal{M}_{\alpha+1} \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". Note that, for all $\alpha \in \mathrm{Ord}$, the set X_α is constructed as an element of $\mathscr{P}^3(\mu_\alpha)$. Since $\mu_\alpha > \kappa_\alpha \geqslant \lambda_\alpha$, it must be the case that any function $\lambda_\alpha \to X_\alpha$ or $X_\alpha \to \kappa_\alpha$ must have rank less than μ_α^+. Hence, by Proposition 4.3, we see that for all $\alpha < \beta$, $\mathcal{M}_\beta \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". This, combined with Theorem 4.1, shows that for all $\alpha \in \mathrm{Ord}$, $\mathcal{M} \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". ∎

5. Open questions

Question 5.1. *What are the limitations of the spectrum of Hartogs–Lindenbaum pairs in arbitrary models of* ZF*?*

In other words, considering $\{\langle\lambda,\kappa\rangle \mid (\exists X)(\alpha(X) = \lambda, \aleph^*(X) = \kappa)\}$ in a fixed model of ZF, what are the limitations on this class? Clearly, one requirement is that $\lambda \leqslant \kappa$. We can also deduce that if this spectrum is exactly $\lambda = \kappa$, then $\mathsf{AC}_{\mathsf{WO}}$ holds. Is there anything better that we can say about the spectrum or deduce from its properties?

In [9], the spectrum of models of SVC are studied and classified into various well-behaved subclasses. For example, if $\mathcal{M} \vDash$ SVC then there is a cardinal κ such that for all X, if $\aleph^*(X) \geqslant \kappa$ then $\aleph^*(X)$ is a successor cardinal.

QUESTION 5.2. *What type of forcing notions preserve all Hartogs–Lindenbaum values from the ground model?*

Assuming ZFC holds, the above question has a simple answer: any cardinal-preserving forcing will suffice. More generally, in the case of Hartogs's number, any $<\mathrm{Ord}$-distributive forcing will not add any injections from an ordinal into a ground model set. So, for example, any σ-distributive forcing must preserve Dedekind-finiteness. But what about preservation of Lindenbaum numbers as well?

Acknowledgements. The authors would like to thank Carla Simons for her help and encouragement, as well as Andrew Brooke-Taylor for his comments on an early version of this manuscript. The authors would also like to thank the reviewer for their work in checking this manuscript.

The first author was supported by a UKRI Future Leaders Fellowship [MR/T021705/2]. The second author's work was financially supported by EPSRC via the Mathematical Sciences Doctoral Training Partnership [EP/W523860/1].

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