Article:

Hayashi, M. orcid.org/0000-0003-3104-1000 and Ouyang, Y. orcid.org/0000-0003-1115-0074 (2024) Finding the optimal probe state for multiparameter quantum metrology using conic programming. npj Quantum Information, 10 (1). 111. ISSN 2056-6387

https://doi.org/10.1038/s41534-024-00905-x npj | quantum information

Article

Published in partnership with The University of New South Wales

https://doi.org/10.1038/s41534-024-00905-x

Finding the optimal probe state for multiparameter quantum metrology using conic programming

Check for updates

Masahito Hayashi [1,2,3] ✉ & Yingkai Ouyang [4] ✉

The ultimate precision in quantum sensing could be achieved using optimal quantum probe states. However, current quantum sensing protocols do not use probe states optimally. Indeed, the calculation of optimal probe states remains an outstanding challenge. Here, we present an algorithm that efficiently calculates a probe state for correlated and uncorrelated measurement strategies. The algorithm involves a conic program, which minimizes a linear objective function subject to conic constraints on a operator-valued variable. Our algorithm outputs a probe state that is a simple function of the optimal variable. We prove that our algorithm finds the optimal probe state for channel estimation problems, even in the multiparameter setting. For many noiseless quantum sensing problems, we prove the optimality of maximally entangled probe states. We also analyze the performance of 3D-field sensing using various probe states. Our work opens the door for a plethora of applications in quantum metrology.

Quantum sensors promise to estimate parameters with unprecedented precision using quantum resources. A canonical problem in quantum sensing is that of channel estimation[1–4], where a quantum channel that embeds physical parameters of interest acts on an initial probe state. The quantum channel describes the dynamics of the quantum system embeds the unknown physical parameters, and acts on the initial state, or probe state, of the quantum system. With access to multiple queries of the quantum channel, the objective is to estimate the embedded physical parameters with maximum precision.

There are two main families of measurement strategies, namely correlated strategies and uncorrelated strategies. In correlated strategies, one may perform measurements across multiple copies of evolved probe states. After each batch of correlated measurements, we can update our choice of probe states and correlated measurement strategies on subsequent batches of evolved probe states. In uncorrelated strategies, one may only perform measurements on individual copies of evolved probe states.

Regardless of the choice of measurement strategy, the most important question in channel estimation is: What is the best probe state to use? Without knowledge of the optimal probe state, a channel estimation protocol will invariably be suboptimal. Suboptimal channel estimation protocols in turn impedes us from realizing the maximum potential of quantum sensors.

Unfortunately, there is no systematic way to find the best probe state efficiently. Typical approaches in the optimization of the probe state in channel estimation entail a two-step optimization process. First, for a fixed probe state, one can solve a minimization problem with optimal value equal to a precision bound that corresponds to a particular type of measurement strategy. Second, we further optimize these precision bounds by changing the probe state. The main difficulty is that the ultimate precision achievable in channel estimation is a non-trivial function of the input probe state, and need not have a convex structure. Hence, the numerical optimization of such a function will be inefficient; such a function has no guarantee of fast convergence, and its optimization would require a forbiddingly enormous computational cost.

In this Article, we resolve the fundamental question of how to find the optimal probe state efficiently. Namely, we present a simple and efficient algorithm that always finds the optimal probe state for channel estimation estimation problems in the multi-parameter setting. The algorithm involves a conic program, which minimizes a linear objective function subject to conic constraints on a operator-valued variable. Our algorithm then outputs a probe state that is a simple function of the optimal variable. We prove that our algorithm finds the optimal probe state for channel estimation problems, even in the multiparameter setting. Using our framework, we unraveled numerous situations where the maximally entangled state is optimal. We furthermore study numerically 3D-field sensing in the

[1]School of Data Science, The Chinese University of Hong Kong, 518172 Shenzhen, Longgang District, China. [2]International Quantum Academy (SIQA), Futian District, 518048 Shenzhen, China. [3]Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan. [4]School of Mathematical and Physical Sciences, University of Sheffield, Sheffield, S3 7RH, UK. ✉e-mail: hmasahito@cuhk.edu.cn; y.ouyang@sheffield.ac.uk

presence of collective amplitude damping, and give theoretical justifications for our numerical findings.

This article makes it possible to determine the optimal quantum sensing strategies for any quantum sensing problem that can be written as a channel estimation problem. This opens the door to investigate a plethora of quantum sensing problems, where the unknown parameter is embedded in the underlying quantum channel, for instance, in many quantum imaging problems[5]. In the following, we will refer to the Supplementary Information for details and proofs.

The question of using the optimal probe state in the single-parameter setting has been studied in various works[1,4,6–11]. Here, ref. 6 was the first study to consider the optimal channel estimation error in the single-parameter case. ref. 1 considers numerically finding the optimal probe states, and an explicit algorithm to find the optimal probe state was given in ref. [[4], Appendix F]. ref. 7 provides analytical bounds for the channel estimation error. References. [10,11] derived a semidefinite programming (SDP) form for the optimal precision in channel estimation. Namely,[10], focuses on finding optimal input state for multiple copies of channels while here we consider only a single channel. The references[8,9,12] also consider the question of finding optimal probe states for estimating multiple parameters embedded in a quantum channel. For probe states without ancilla assistance, ref. 9 numerically finds the optimal two-qubit probe state with the Holevo-Nagaoka (HN) bound when the channel models 3D-field sensing with independent and identical amplitude (i.i.d.) damping on the two qubits. ref. 8 considers uncorrelated measurements with unitary quantum channels and error-correctible noisy channels. This leaves open the questions of how to evaluate the optimal probe state and corresponding bounds for the general problem of quantum channel estimation with correlated or uncorrelated measurement strategies. ref. 12 finds the optimal probe state for channel estimation problems on one and two qubits.

In the channel estimation problem where we estimate the parameters embedded in the quantum channel, the set of parameters is continuous. If we discretize the channel estimation problem, we would obtain the problem of discriminating a discrete set of quantum channels[3,13–17]. Recently, the channel discrimination problem was formulated as a convex program, and this formulation made it possible to determine the optimal strategy to discriminate a pair of quantum channels[18]. However, it is unclear how to extend this result to the continuous parameter setting that we require in the channel estimation problem.

Results

Quantum state estimation

In quantum state estimation, we are given copies of an unknown state ρ_{θ_0} from the set of quantum states $\{\rho_\theta: \theta = (\theta^1, \ldots, \theta^d) \in \Theta\}$ parametrized by a continuous set $\Theta \subseteq \mathbb{R}^d$, and our aim is to construct an estimator $\hat{\theta}$ that estimates the true parameter θ_0.

We describe a measurement using a set of positive operators $\Pi = \{\Pi_x : x \in \mathcal{X}\}$ labeled by a set $\mathcal{X}$, where the completeness condition $\sum_{x\in\mathcal{X}} \Pi_x = I$ holds. By Born's rule, a measurement Π on a quantum state ρ_θ gives the classical label x and the state $\Pi_x\rho_\theta/\mathrm{Tr}(\Pi_x\rho_\theta)$ with probability $p_\theta(x) = \mathrm{Tr}(\Pi_x\rho_\theta)$. Given a function f of the classical label x, we denote $\mathbb{E}[f(x)|\Pi]$ as the expectation of $f(x)$, with probability distribution obtained according to Born's rule.

Given a measurement Π and an estimator $\hat{\boldsymbol{\theta}}$ that depends on the classical label x, we denote $\hat{\Pi} = (\Pi, \hat{\boldsymbol{\theta}})$ as an *estimator*. When the true parameter θ_0 is equal to θ, we define the mean-square error (MSE) matrix for the estimator $\hat{\Pi}$ as

$$V_{\boldsymbol{\theta}}[\hat{\Pi}] = \sum_{i,j=1}^{d} |i\rangle\langle j| \mathbb{E}_{\boldsymbol{\theta}}\left[(\hat{\theta}^i(x) - \theta^i)(\hat{\theta}^j(x) - \theta^j)|\Pi\right].$$

In multiparameter quantum metrology, the objective is to find an optimal estimator $\hat{\Pi} = (\Pi, \hat{\boldsymbol{\theta}})$ that minimizes $\mathrm{Tr}GV_{\boldsymbol{\theta}}[\hat{\Pi}]$, where a weight matrix G, a size d positive semidefinite matrix, quantifies the relative importance of the different parameters.

In the neighborhood of the true parameter θ_0, we define $D_j := \frac{\partial}{\partial\theta^j}\rho_\theta|_{\theta_0}$, and $\rho := \rho_{\theta_0}$. Our estimator $\hat{\Pi}$ is unbiased at $\theta_0 = \theta$ if for all $i = 1, \ldots, d$, the expectation of our estimator equals the true value of the parameter θ_0 when $\theta_0 = \theta$, that is

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\hat{\theta}^i(x)|\Pi\right] = \sum_{x\in\mathcal{X}} \hat{\theta}^i(x)\mathrm{Tr}\left[\rho_{\boldsymbol{\theta}}\Pi_x\right] = \theta^i. \tag{1}$$

Our estimator is globally unbiased if (1) holds for all $\theta \in \Theta$. Since globally unbiased estimators need not exist, we consider estimators that are unbiased in the neighborhood of the true parameter θ_0. Taking partial derivatives on both sides of (1), we get

$$\frac{\partial}{\partial\theta^j}\mathbb{E}_{\boldsymbol{\theta}}\left[\hat{\theta}^i(x)|\Pi\right] = \sum_{x\in\mathcal{X}} \hat{\theta}^i(x)\mathrm{Tr}D_j\Pi_x = \delta_i^j. \tag{2}$$

The estimator $\hat{\Pi}$ is locally unbiased if (1) holds for all $i = 1, \ldots, d$ for a fixed θ where $\theta_0 = \theta$, and when (2) holds for all $i, j = 1, \ldots, d$.

For any weight matrix $G = \sum_{i,j=1}^d g_{i,j}|i\rangle\langle j|$, the fundamental precision limit[19] is

$$C_{\boldsymbol{\theta}}[G] := \min_{\hat{\Pi}:\mathrm{l.u.at}\,\boldsymbol{\theta}} \mathrm{Tr}\left[GV_{\boldsymbol{\theta}}[\hat{\Pi}]\right],$$

where 'l.u. at $\boldsymbol{\theta}$' indicates our minimization over all possible estimators under the locally unbiasedness condition. Since this minimum is attained by $\hat{\Pi}$ satisfying (1) when we impose only the condition (2), it suffices to consider $C_{\boldsymbol{\theta}}[G]$ as a minimization with only the condition (2).

Cramer-Rao (CR) type bounds[19] describe any lower bound to the weighted trace of the MSE matrix. The fundamental precision limit $C_{\boldsymbol{\theta}}[G]$ is one such lower bound which is tight, and hence refered to as the *tight CR bound*[19]. Operationally, we may attain the tight CR bound using an uncorrelated measurement strategy in the asymptotic setting. The Holevo-Nagaoka (HN) bound[20–24] is a CR bound that describes the ultimate precision using correlated measurement strategies, and the HN bound can be strictly smaller than the tight CR bound[19].

We may efficiently approximate the tight CR bound using a semidefinite program (SDP)[19], with the precision of the approximation increasing with the complexity of the SDP. Other CR-type bounds are significantly more efficient to evaluate as simple SDPs, such as the Nagaoka-Hayashi (NH) bound, the HN bound, and the symmetric logarithmic derivative (SLD) bound. Recently, ref. 19 clarifies the relationship between these disparate CR-type bounds under a common conic proramming framework. For these conic programs, the optimization variable is an operator that can correspond to an estimator $\hat{\Pi} = (\Pi, \hat{\theta})$. Namely, the operator

$$X(\Pi, \hat{\theta}) := \sum_{x\in\mathcal{X}} \left(|0\rangle + \sum_{i=1}^{d} \hat{\theta}^i(x)|i\rangle\right)\left(\langle 0| + \sum_{i=1}^{d} \langle i|\hat{\theta}^i(x)\right) \otimes \Pi_x$$

acts on vectors in $\mathcal{R} \otimes \mathcal{H}$, where $\mathcal{R} = \mathbb{R}^{d+1}$ is spanned by basis vectors $|0\rangle, |1\rangle, \ldots, |d\rangle$ and $\mathcal{H}$ is the Hilbert space for Π.

The objective function of these conic programs is equal to the trace of the weighted MSE matrix $\mathrm{Tr}GV_{\boldsymbol{\theta}}[\hat{\Pi}]$, and can be written as a function of $X(\Pi, \hat{\theta})$. Namely,

$$\mathrm{Tr}GV_{\boldsymbol{\theta}}[\hat{\Pi}] = \mathrm{Tr}(G \otimes \rho)X(\Pi, \hat{\theta}). \tag{3}$$

Next, note that the completeness condition $\sum_{x\in\mathcal{X}} \Pi_x = I_{\mathcal{H}}$ using $X(\Pi, \hat{\theta})$ implies that

$$\mathrm{Tr}_{\mathcal{R}}(|0\rangle\langle 0| \otimes I_{\mathcal{H}})X(\Pi, \hat{\theta}) = I_{\mathcal{H}}, \tag{4}$$

where $\mathrm{Tr}_{\mathcal{R}}$ denotes the partial trace on system $\mathcal{R}$. Hence, we may interpret (4) as a rewriting of the completeness condition $\sum_{x\in\mathcal{X}} \Pi_x = I_{\mathcal{H}}$. Next, we

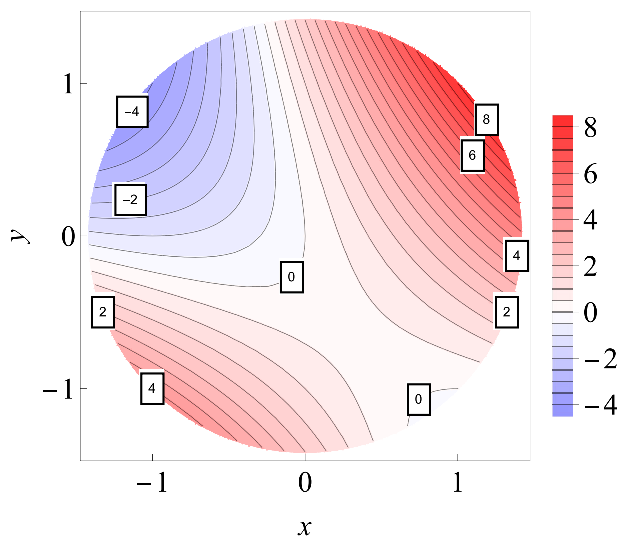

Fig. 1 | Biconvex function $f(x, y) = x^2 + y^2 + 4xy + 2x$. The function $f(x, y)$ is convex in x treating y as constant, and also convex in y treating x as constant. Minimizing $f(x, y)$ on the convex set $\{(x, y): x^2 + y^2 \le 2\}$ is not a convex optimization; there are two distinct local minima, each with a different value. Numerical methods that minimize iteratively in the variable x and y need not find the global minima. The local minima found depends on the optimization algorithm's initial starting point.

note that the condition (2) for a locally unbiased estimator guarantees

$$\mathrm{Tr}\left(\frac{1}{2}(|0\rangle\langle i| + |i\rangle\langle 0|) \otimes D_j\right) X(\Pi, \hat{\theta}) = \delta_{i,j}. \tag{5}$$

Hence, (5) reformulates the locally unbiased condition.

The operator $X(\Pi, \hat{\theta})$ has a tensor product structure; namely, $X(\Pi, \hat{\theta})$ is an element of a cone generated by separable states on $\mathcal{R} \otimes \mathcal{H}$. This cone $\mathcal{S}^1$ is a separable cone, which is the convex hull of operators that are a tensor product of a real positive semidefinite matrix on $\mathcal{R}$ and a complex positive semidefinite operators on $\mathcal{H}$ with bounded norm. Hence, we consider the minimization

$$S_1 := \min_{X \in \mathcal{S}^1} \{\mathrm{Tr}(G \otimes \rho) X | (4), (5) \text{ hold.}\},$$

which equals the tight CR type bound[19]. Minimization of $\mathrm{Tr}(G \otimes \rho)X$ subject to (4), (5) and over suitable cones that contain $\mathcal{S}^1$ can give conic programs with optimal value equal to the NH bound, the HN bound and the SLD bound.

Consider $\mathcal{B}$ as the vector space spanned by the tensor product of real symmetric matrices on $\mathcal{R}$ and bounded complex Hermitian matrices on $\mathcal{H}$. That is,

$$\mathcal{B} := \left\{ \sum_{j=0}^{d} \sum_{k=0}^{d} |k\rangle\langle j| \otimes X_{k,j} | X_{k,j} \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H}), X_{k,j} = X_{j,k} \right\},$$

where $\mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ denotes the set of self-adjoint (Hermitian) matrices on $\mathcal{H}$ with bounded norm. We extend the space $\mathcal{B}$ to

$$\mathcal{B}'' := \left\{ \sum_{j=0}^{d} \sum_{k=0}^{d} |k\rangle\langle j| \otimes X_{k,j} | X_{k,0} \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H}), X_{k,j} = (X_{j,k})^{\dagger} \right\}.$$

Based on the spaces $\mathcal{B}$ and $\mathcal{B}''$ we define cones over which we optimize X. For instance, we consider the cone

$$\mathcal{S}^2 := \{X \in \mathcal{B} | \langle v|X|v\rangle \ge 0 \text{ forall } |v\rangle \in \mathbb{C}^{d+1} \otimes \mathcal{H}\}.$$

We denote $\mathcal{R}_C = \mathbb{C}^{d+1}$. The subscript C denotes that $\mathcal{R}_C$ is the complexification of the real space $\mathcal{R} = \mathbb{R}^{d+1}$. We define $\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{\mathrm{PPT}}$ as the set of self-adjoint operators on $\mathcal{R}_C \otimes \mathcal{H}$ with positive partial transpose, and define the cone $\mathcal{S}^3$ as $\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{\mathrm{PPT}} \cap \mathcal{B}''$. Likewise, we define the set $\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{\mathrm{P}}$ as the set of positive semi-definite self-adjoint operators on $\mathcal{R}_C \otimes \mathcal{H}$, and define the cone $\mathcal{S}^4$ as $\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{\mathrm{P}} \cap \mathcal{B}''$. Then, for $k = 1, 2, 3, 4$, we define the conic programs

$$S_k := \min_{X \in \mathcal{S}^k} \{\mathrm{Tr}(G \otimes \rho) X | (4), (5) \text{ hold.}\}. \tag{6}$$

The relation $\mathcal{S}^1 \subset \mathcal{S}^2 \subset \mathcal{S}^3 \subset \mathcal{S}^4$ implies

$$S_1 \ge S_2 \ge S_3 \ge S_4. \tag{7}$$

In addition, we introduce a linear constraint to the operator $X \in \mathcal{B}''$ as

$$\mathrm{Tr} X((|j\rangle\langle i| - |i\rangle\langle j|) \otimes T) = 0 \tag{8}$$

for $i, j = 1, 2, \ldots, d$ and a trace-class self-adjoint operator T. We define the subset of $\mathcal{B}''$ that corresponds to this linear constraint as $\mathcal{B}''_T := \{X \in \mathcal{B}'' | (8) \text{ holds }.\}$. Next, given a density matrix ρ on $\mathcal{H}$, we define $\mathcal{S}^5(\rho)$ as $\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{\mathrm{P}} \cap \mathcal{B}''_\rho$. We consider the minimization:

$$S_5 := \min_{X \in \mathcal{S}^5(\rho)} \{\mathrm{Tr}(G \otimes \rho) X | (4), (5) \text{ hold }.\}.$$

The cones $\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^3, \mathcal{S}^4$ are independent of ρ, and only the cone $\mathcal{S}^5(\rho)$ depends on ρ. Since we have the relation $\mathcal{S}^2 \subset \mathcal{S}^5(\rho) \subset \mathcal{S}^4$, we have the following relations

$$S_4 \le S_5 \le S_2. \tag{9}$$

Since S_k depends on the model $(\rho, (D_j)_j)$, i.e., the probe state ρ and the partial derivatives of the probe state D_j for $k = 1, 2, 3, 4, 5$, we also write S_k as $S_k[\rho, (D_j)_j]$ to emphasize the CR-type bounds' dependence on ρ and D_j. Here S_2 equals the Nagaoka-Hayashi bound (NH bound) studied in references.[25–27], S_4 equals the SLD bound, and S_5 equals the HN bound[19]. In the single parameter ($d = 1$) case, the SLD bound is attainable, which implies that

$$S_1 = S_2 = S_3 = S_5 = S_4. \tag{10}$$

Channel estimation

In the channel estimation problem, we have a d-parameter channel family $\{\Lambda_\theta\}$, where the quantum channel Λ_θ that maps a probe state ρ on input system is $\mathcal{H}_A$ to output system is $\mathcal{H}_B$ also embeds the d parameters θ to be estimated. With copies of quantum states $\Lambda_\theta(\rho)$, we perform either correlated or uncorrelated measurements to obtain a probability distribution that depends on the parameters, from which we construct the best estimator for the parameters.

In channel estimation, we may purify the probe state ρ to a pure state on $\mathcal{H}_A \otimes \mathcal{H}_C$. Here, the Hilbert space $\mathcal{H}_C$, isomorphic to $\mathcal{H}_A$, is an ancillary system that the quantum channel has no access. Measurement strategies however may access the ancillary system. This setting follows the preceding paper[10] which studies the optimization of the one-parameter case under the assumption of the ancilla system's availability. Consideration of measurement strategies without access to this ancilla system is highly non-trivial, and is beyond the scope of our current study. In channel estimation under this setting, the CR-type bounds of interest are

$$\bar{S}_k[\rho_{AC}] := S_k\left[(\Lambda_\theta \otimes \iota_C)(\rho_{AC}), \left(\frac{\partial}{\partial \theta^j}(\Lambda_\theta \otimes \iota_C)(\rho_{AC})\right)_j\right], \tag{11}$$

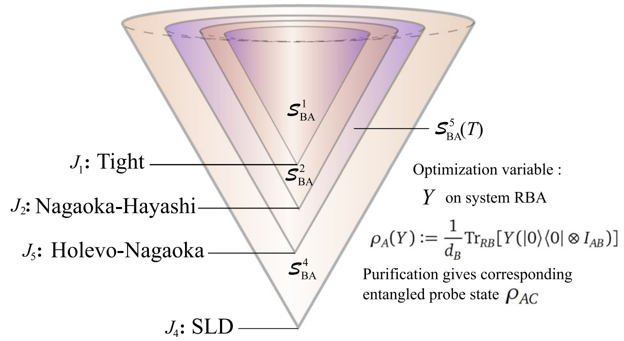

Fig. 2 | Relationship between J_1, J_2, J_5, J_4 and their associated cones $\mathcal{S}^1_{BA}, \mathcal{S}^2_{BA}, \mathcal{S}^5_{BA}(T), \mathcal{S}^4_{BA}$. The optimal values of these minimizations, are the tight bound, the NH bound, the HN bound, and the SLD bound respectively. We pictorally illustrate that $J_1 \geq J_2 \geq J_5 \geq J_4$. From the optimal solution Y^* of any conic program J_k, we can derive a corresponding $\rho_A(Y^*)$, whose purification to system AC yields the corresponding optimal probe state.

where ι_C denotes an identity channel on system $\mathcal{H}_C$. The ultimate precision bound using the optimal probe state is

$$\bar{S}_k := \min_{\rho_{AC}} \bar{S}_k[\rho_{AC}] \tag{12}$$

where the minimization is over all pure density operators on $\mathcal{H}_A \otimes \mathcal{H}_C$.

Direct solution of the optimization in (12) is challenging. Even if one solves the inner optimization for $\bar{S}_k[\rho_{AC}]$, it is unclear if the subsequent optimization in ρ_{AC} is a tractable optimization problem, such as a convex problem. For example, even if a function $f(x, y)$ is biconvex on a convex set, which means that it is convex one variable while treating the other variable as constant, the overall function need not be convex, and can have multiple local minima (see Fig. 1 for an illustrative example).

In this Article, we bypass these difficulties. Namely, we construct conic programs with optimal values equal to (12), which lets us find (12) using only a single optimization program. Moreover, from the solution of our conic program, we explicitly construct the optimal probe state (see Fig. 2).

Our formulation uses the isomorphism between the space $\mathcal{H}_C$ and the space $\mathcal{H}_A$, and optimizes over cones on $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_A$ rather than cones on $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Namely, instead of considering the cones $\mathcal{S}^k_{BC}$, we consider the cones $\mathcal{S}^k_{BA}$, where we obtain $\mathcal{S}^k_{BA}$ by replacing $\mathcal{H}$ with $\mathcal{H}_B \otimes \mathcal{H}_A$ in the definition of $\mathcal{S}^k$.

Our first tool is the Choi matrix[28] T_θ of a quantum channel Λ_θ. The idea is to rewrite $\Lambda_\theta(\rho)$ in terms of the Choi matrix T_θ and the input probe state ρ. Here, the Choi matrix T_θ of Λ_θ is an operator on $\mathcal{H}_B \otimes \mathcal{H}_A$ and is given by

$$T_\theta := (\Lambda_\theta \otimes \iota)(|I\rangle\langle I|),$$

where $|I\rangle := \sum_j |e_j\rangle|e_j\rangle$ is an unnormalized maximally entangled state and $\{|e_j\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$. Then, it follows that

$$\Lambda_\theta(\rho) = \mathrm{Tr}_A[T_\theta(I_B \otimes \rho)], \tag{13}$$

where I_B denotes the identity operator on $\mathcal{H}_B$. When the parameter θ is in the neighborhood of the true parameter θ_0, we denote the corresponding Choi matrix T_{θ_0} by T and its derivatives as $F_j := \frac{\partial}{\partial \theta^j} T_\theta|_{\theta=\theta_0}$.

The objective function of our conic programs are all equal to $\mathrm{Tr}[Y(G \otimes T)]$, where Y belongs to a cone in $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_A$.

We describe the constraints of our conic program using the following conditions.

(i) Given fixed Y on $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_A$, there exists a state ρ_A on $\mathcal{H}_A$ such that

$$\mathrm{Tr}_R[Y(|0\rangle\langle 0| \otimes I_{AB})] = I_B \otimes \rho_A. \tag{14}$$

(ii) For $j, j' = 1, \ldots, d$,

$$\frac{1}{2}\mathrm{Tr}[Y((|0\rangle\langle j'| + |j'\rangle\langle 0|) \otimes F_j)] = \delta_{j,j'}.$$

It turns out that condition (i) corresponds to the completeness condition for measurements, and (ii) corresponds to the locally unbiased condition. While the condition (ii) is a linear constraint, the condition (i) is not a linear constraint. However, note that condition (i) is equivalent to the following linear constraint.

(i') Let $\{|b\rangle\}$ be any orthonormal basis of $\mathcal{H}_B$. For $b \neq b'$, we have

$$\mathrm{Tr}_{RB}[Y(|0\rangle\langle 0| \otimes I_A \otimes |b\rangle\langle b'|)] = 0, \tag{15}$$

$$\begin{aligned} &\mathrm{Tr}_{RB}[Y(|0\rangle\langle 0| \otimes I_A \otimes |b\rangle\langle b|)] \\ &= \mathrm{Tr}_{RB}[Y(|0\rangle\langle 0| \otimes I_A \otimes |b'\rangle\langle b'|)] \end{aligned} \tag{16}$$

as operators on $\mathcal{H}_A$. Also,

$$\mathrm{Tr}[Y(|0\rangle\langle 0| \otimes I_A \otimes |b\rangle\langle b|)] = 1. \tag{17}$$

Note that when we impose the condition (i), the operator

$$\rho_A(Y) := \frac{1}{d_B}\mathrm{Tr}_{RB}[Y(|0\rangle\langle 0| \otimes I_{AB})] \tag{18}$$

is a density matrix, where d_B is the dimension of $\mathcal{H}_B$. This matrix $\rho_A(Y)$ helps us construct the optimal probe state.

Our new conic programs are the minimization of the objective function $\mathrm{Tr}[Y(G \otimes T)]$ subject to the linear constraints (i') and (ii) along with appropriate conic constraints. Namely, we define

$$J_k := \min_{Y \in \mathcal{S}^k_{BA}} \{\mathrm{Tr}[Y(G \otimes T)] | Y \text{ satisfies (i'), (ii).}\}$$

for $k = 1, 2, 3, 4$. For $k = 5$, we define $\mathcal{S}^5_{BA}(T) := \mathcal{S}(\mathcal{R}_C \otimes \mathcal{H}_{AB})_{\mathrm{P}} \cap \mathcal{B}''_T$, and define the conic program

$$J_5 := \min_{Y \in \mathcal{S}^5_{BA}(T)} \{\mathrm{Tr}[Y(G \otimes T)] | Y \text{ satisfies(i'), (ii).}\}.$$

We also use $J_k[T, (F_j)_j]$ to denote J_k when we need to clarify the dependence of J_k with the channel model $(T, (F_j)_j)$.

The main result of our paper is the following theorem.

Theorem 1. For $k = 1, 2, 3, 4, 5$, we have $J_k = \bar{S}_k$.

The proof of Theorem 1 involves drawing connections between the optimization in $\bar{S}_k$ and the optimization in J_k. Theorem 1 allows calculation of precision bounds for channel estimation optimized over probe states as given by $\bar{S}_k$ using the conic programs that correspond to J_k. Since the cones considered in J_k are analogous to the cones considered in S_k[19], we know how to solve J_k numerically. Furthermore for $k = 2, 3, 4, 5$, we can solve J_k via semidefinite programming (SDP). The calculation of J_1 is more challenging, because it requires the minimization over a certain separable cone on $\mathcal{R}_C \otimes$

$(\mathcal{H}_B \otimes \mathcal{H}_A)$ [ref. 19, Sec. IV]. Namely, J_1 expresses the ultimate precision when no correlated measurement is allowed. When correlated measurements are allowed across several output states, the bound J_5 expresses the ultimate precision. In the single parameter case, i.e., $d = 1$, the SLD bound is attainable, because (10) implies $J_1 = J_2 = J_3 = J_5 = J_4$.

The following algorithm calculates the optimal probe state given the optimal solution Y^* of J_k.

Algorithm 1. Find the optimal probe state for J_k.

1. Take as input the matrix Y^*, where Y^* is the optimal solution of J_k. Here, Y^* is a matrix on the system $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_A$.

2. Obtain the matrix $Z^* = Y^*(|0\rangle\langle 0| \otimes I_{AB})/d_B$, where $|0\rangle$ is a state on $\mathcal{R}_C$, I_{AB} is the identity matrix on system $\mathcal{H}_B \otimes \mathcal{H}_A$, and d_B is the dimension of $\mathcal{H}_B$.

3. Take the partial trace of Z^* over system $\mathcal{R}_C \otimes \mathcal{H}_B$. That is, obtain the matrix $\rho_A^* = \mathrm{Tr}_{RB}[Z^*]$.

4. Write the spectral decomposition of ρ_A^* as $\rho_A^* = \sum_{j=1}^{d_A} s_j|\phi_j\rangle\langle\phi_j|$, where d_A denotes the dimension of $\mathcal{H}_A$, $s_j \geq 0$ and $\{\phi_j: j = 1, \ldots, d_A\}$ is an orthonormal basis of $\mathcal{H}_A$.

5. Write the purification of ρ_A^* as $|\psi_{AC}\rangle = \sum_{j=1}^{d_A} \sqrt{s_j}|\phi_j\rangle \otimes U|\phi_j\rangle$, where U is any unitary map from $\mathcal{H}_A$ to $\mathcal{H}_C$. The optimal probe state is $\rho_{AC}^* = |\psi_{AC}\rangle\langle\psi_{AC}|$.

Most operationally significant are the conic programs J_5 and J_1, which help us find the optimal probe state using correlated and uncorrelated measurements respectively.

There are many situations where the maximally entangled probe state $|\Phi\rangle\langle\Phi|$ on $\mathcal{H}_A \otimes \mathcal{H}_C$ is the optimal probe state to use in channel estimation. In Section II D, we explore this possibility. Namely, we prove that $J_4 = \bar{S}_4[|\Phi\rangle\langle\Phi|]$ if and only if a particular operator-valued dual variable of J_4 is proportional to the identity operator. Furthermore, we prove that $J_5 = \bar{S}_5[|\Phi\rangle\langle\Phi|]$ if and only if a particular operator-valued dual variable of J_5 is proportional to the identity operator. These theoretical results allow us to find situations when the maximally entangled state is the optimal probe state for both the SLD bound and the HN bound.

Using this theory, we prove in the Supplemental Information that for channel estimation with a single embedded parameter, and where the channel is a depolarizing channel on a single qubit, the maximally entangled state is the optimal probe state, and furthermore, the tight bound, the NH bound, the SLD bound and the HN bound are all equivalent. When the channel applies generalized Paulis randomly on an input qudit probe state according an apriori determined probability distribution, and the channel estimation task is to estimate this probability distribution, we prove that the optimal probe state under the SLD bound is the maximally entangled state.

We also consider a quantum channel that is the mixture of unitary evolution according to the spin-j representation of SU(2) unitary evolution according to the spin-j representation of SU(2) and replacement by a completely mixed state. We also give equal weights to each of the three parameters that we estimate. We derive the analytical form for SLD bound. Furthermore, we prove that under purely SU(2) unitary evolution with zero noise, all the precision bounds $J_1, \ldots, J_5$ are equivalent, and the maximally entangled state is the optimal probe state for all the precision bounds.

In Section II G, we revisit the problem of quantum field sensing[29] in the multiparameter setting. In the noiseless setting, this is equivalent to the channel estimation problem for noisy SU(2) channels. Instead of a depolarizing type of channel, we consider noise introduced by collective amplitude damping [ref. 30, Eq. (7)]. The channel that we consider differs from[9] in two ways. First, we consider collective amplitude damping while[9] considers i.i.d. amplitude damping. Second, we model the channel using a master equation, considering collective amplitude damping that occurs during the SU(2) evolution, whereas the channel in[9] considers i.i.d. amplitude damping that occur after the unitary evolution.

Using MatLab computer code, we numerically determine the optimal probe state for the NH bound, the HN bound, and the SLD bound, and numerically evaluate the corresponding precision bounds. We numerically ascertain that in the noiseless setting, the maximally entangled state on the symmetric subspace is the optimal probe state.

When there is non-vanishing noise, we numerically ascertain that the SLD precision bound cannot be optimal. We furthermore prove this by calculating expectations of commutators of the symmetric logarithmic derivatives of different angular momentum generators.

To maximize the accessibility of our work, in Supplemental Information E, we formulate the mathematical optimization problems that correspond to J_2, J_3, J_4, J_5 as semidefinite programs to be used with the CVX package and provide the corresponding MatLab code. The outputs of these semidefinite programs are inputs to Algorithm 1 that calculates the optimal probe state.

On the efficiency of evaluating conic programs

Next we discuss the efficiency of evaluating the various bounds J_k. The conic programs J_k have the same structure as S_k, and hence algorithms used to solve S_k can solve J_k. When $k = 2, \ldots, 5$, the optimizations for J_j are efficiently solvable by SDPs.

When $k = 1$, the conic programing J_1 is challenging to solve. This is because optimizing J_1 involves a minimization over a separable cone, and such an minimization is as hard as the problem of deciding whether a given quantum state is entangled or separable, which is NP-hard[31]. Such a conic programming cannot be directly solved using SDP, and its optimization is much harder than using SDP to solve in J_2, J_3, J_4, J_5. .

In ref. 19, we discuss how one can approximate J_1 using SDP. For this we can choose a covering of a hypersphere in $\mathcal{R}$, and use a semidefinite program with number of variables that scales as the number of points in the covering of the hypersphere, and approximation error that depends on the covering radius of the covering.

Since such an approximation is numerically expensive, one can consider an alternative approximation of solving J_1, by employing the concept of symmetric extension. The symmetric extension was orginally introduced to consider the membership problem for separability[32,33]. We consider the systems $\mathbb{C}^{d+1^{\otimes n}} \otimes \mathcal{H}$, $\mathbb{C}^{d+1} \otimes \mathcal{H}^{\otimes n}$ and define

$$\begin{aligned} \mathcal{S}_n^1 &:= \{X \in \mathcal{B}(\mathbb{C}^{d+1^{\otimes n}}, \mathcal{H}) | \mathrm{Tr}_{j^c} X = \mathrm{Tr}_{1^c} X, X \geq 0\} \\ \tilde{\mathcal{S}}_n^1 &:= \{X \in \mathcal{B}(\mathbb{C}^{d+1}, \mathcal{H}^{\otimes n}) | \mathrm{Tr}_{j^c} X = \mathrm{Tr}_{1^c} X, X \geq 0\}, \end{aligned}$$

where Tr_{j^c} expresses the partial trace except for the j-th system on $\mathbb{C}^{d+1^{\otimes n}}$ or $\mathcal{H}^{\otimes n}$. The minimizer X_* of S_1 has a symmetric extension $X_{*,n}$ that belongs to $\mathcal{S}_n^1$ and $\tilde{\mathcal{S}}_n^1$ and satisfies $\mathrm{Tr}_{1^c} X_{*,n} = X_*$. Due to the condition $\mathrm{Tr}_{1^c} X_{*,n} = X_* \in S_1$, we have the following lower bounds of S_1

$$S_{1,n} := \min_{X \in \mathcal{S}_n^1} \{\mathrm{Tr}(G \otimes \rho)\mathrm{Tr}_{1^c} X | \mathrm{Tr}_{1^c} \text{ satisfies (4), (5).}\} \tag{19}$$

$$\tilde{S}_{1,n} := \min_{X \in \tilde{\mathcal{S}}_n^1} \{\mathrm{Tr}(G \otimes \rho)\mathrm{Tr}_{1^c} X | \mathrm{Tr}_{1^c} \text{ satisfies (4), (5).}\}, \tag{20}$$

which both can be calculated by SDP. Since an element X of $\mathcal{S}_n^1$ or $\tilde{\mathcal{S}}_n^1$ satisfies $\mathrm{Tr}_{1^c} X \in \mathcal{S}^2$, we have

$$S_1 \geq S_{1,n+1} \geq S_{1,n} \geq S_2, \quad S_1 \geq \tilde{S}_{1,n+1} \geq \tilde{S}_{1,n} \geq S_2.$$

Also,[33] showed that for any non-separable state ρ, there exists an integer n such that ρ does not belong to $S_{1,n}$. Since the speed of the convergence was studied in[34,35], we have

$$S_1 = \lim_{n\to\infty} S_{1,n} = \lim_{n\to\infty} \tilde{S}_{1,n}.$$

Hence the approximate calculation of J_1 by SDP follows similarly.

While we have SDP formulations that approximate J_1, because of the intrinsic hardness of exactly solving J_1, we should expect these SDP formulations to be numerically expensive to evaluate. Hence, we defer the numerical approximation of J_1 in the field sensing application for future work.

Optimality of the maximally entangled input state

There are many situations where the maximally entangled probe state is the optimal probe state for channel estimation. Here, we explore this possibility.

Denoting the maximally entangled state as $|\Phi\rangle\langle\Phi|$, for k = 1, 2, 3, 4, 5 we have

$$J_k[T,(F_j)_j] \le \bar{S}_k[T,(F_j)_j, |\Phi\rangle\langle\Phi|]. \tag{21}$$

This is because $|\Phi\rangle\langle\Phi|$ might be a suboptimal probe state. If the above inequalities are equalities, then $|\Phi\rangle\langle\Phi|$ would be an optimal probe state. Here, we derive necessary and sufficient conditions for the above inequalities to be equalities when k = 4, 5, addressing the optimality of $|\Phi\rangle\langle\Phi|$ for the SLD bound and the HN bound respectively. The proof for k = 4 and 5 crucially uses a particular inner product structure of the optimal values of S_4 and S_5, which are characterized by SLD Fisher information matrix, and RLD Fisher information matrix over an extended space [ref. 22, Theorem 4] and [ref. 36, Proposition 1] respectively. Since the optimal values of $S_1, \ldots, S_3$ are not known to have such a inner product structure, it is challenging to obtain similar results pertaining to the optimality of maximally entangled states for channel estimation for $J_1, \ldots, J_3$.

The key idea is that the normalized Choi state $T_N := \frac{1}{d_A}T$ and its derivatives $F_{j,N} := \frac{1}{d_A}F_j$ can be written as $T_N = \Lambda_\theta(|\Phi\rangle\langle\Phi|)$ and $F_{j,N} = \frac{\partial}{\partial\theta_j}\Lambda_\theta(|\Phi\rangle\langle\Phi|)$. We use the rescaled optimization variable Y_N: = $d_A Y$ to rewrite the constraints (i) and (ii) as

(i-N) Given fixed Y_N on $\mathcal{H}_R \otimes \mathcal{H}_B \otimes \mathcal{H}_A$, there exists a state ρ_A on $\mathcal{H}_A$ such that

$$\mathrm{Tr}_R[Y_N(|0\rangle\langle 0| \otimes I_{AB})] = I_B \otimes d_A\rho_A. \tag{22}$$

(ii-N) For $j, j' = 1, \ldots, d$,

$$\frac{1}{2}\mathrm{Tr}[Y_N((|0\rangle\langle j'| + |j'\rangle\langle 0|) \otimes F_{N,j})] = \delta_{j,j'}.$$

We rewrite (i-N) as the following linear constraint.

(i'-N) Let $\{|b\rangle\}_{b=1}^{d_B}$ be any orthonormal basis of $\mathcal{H}_B$. For $b \in \{1, \ldots, d_B - 1\}$ and $b' \in \{2, \ldots, d_B - 1\}$ with $b > b'$ we have the operator constraints

$$\mathrm{Tr}_{RB}[Y_N(|0\rangle\langle 0| \otimes I_A \otimes |b\rangle\langle b'|)] = 0, \tag{23}$$

$$\mathrm{Tr}_{RB}[Y_N(|0\rangle\langle 0| \otimes I_A \otimes (|b\rangle\langle b| - |b+1\rangle\langle b+1|)] = 0 \tag{24}$$

We also have the scalar constraint

$$\mathrm{Tr}[Y_N(|0\rangle\langle 0| \otimes I_A \otimes |1\rangle\langle 1|)] = d_A. \tag{25}$$

Using constraints (i'-N) and (ii-N), we rewrite J_k for k = 1, 2, 3, 4 as

$$J_k = \min_{Y_N \in \mathcal{S}^k_{BA}} \{\mathrm{Tr}[Y_N(G \otimes T_N)] | Y_N \text{ satisfies (i' − N), (ii − N).}\}, \tag{26}$$

and rewrite J_5 as

$$J_5 = \min_{Y \in \mathcal{S}^5_{BA}(T_N)} \{\mathrm{Tr}[Y_N(G \otimes T_N)] | Y_N \text{ satisfies (i' − N), (ii − N).}\}. \tag{27}$$

Since $T_N = \mathrm{Tr}_A[(T \otimes I_C)(I_B \otimes |\Phi\rangle\langle\Phi|)]$, by replacing the system $\mathcal{H}_C$ by $\mathcal{H}_A$, we can rewrite $\bar{S}_k[|\Phi\rangle\langle\Phi|]$ as

$$\begin{aligned}\bar{S}_k[|\Phi\rangle\langle\Phi|] &= \min_{Y_N \in \mathcal{S}^k_{BA}} \{\mathrm{Tr}[Y_N(G \otimes T_N)] | Y_N \text{ satisfies (i”), (ii − N).}\} \\ &= S_k[T_N, (F_{j,N})_j],\end{aligned} \tag{28}$$

$$\begin{aligned}\bar{S}_5[|\Phi\rangle\langle\Phi|] &= \min_{Y_N \in \mathcal{S}^5_{BA}(T_N)} \{\mathrm{Tr}[Y_N(G \otimes T_N)] | Y_N \text{ satisfies (i''), (ii − N).}\} \\ &= S_5[T_N, (F_{j,N})_j],\end{aligned} \tag{29}$$

for k = 1, 2, 3, 4, where the condition (i”) is defined as

(i”)

$$\mathrm{Tr}_R[Y_N(|0\rangle\langle 0| \otimes I_{AB})] = I_{AB}. \tag{30}$$

Now the conic programs J_k have constraints (i'-N) and (ii) while the conic programs $\bar{S}_k[|\Phi\rangle\langle\Phi|]$ have constraints (ii) and (ii-N). Therefore, the difference between the constraints (i'-N) and (i”) characterizes the difference between J_k and $\bar{S}_k[|\Phi\rangle\langle\Phi|]$.

Using the representation of $\bar{S}_k[|\Phi\rangle\langle\Phi|]$ as the quantum model $S_k[T_N, (F_{j,N})_j]$, we elaborate on the equivalence of J_k and $\bar{S}_k[|\Phi\rangle\langle\Phi|]$ for k = 4, 5 in subsequent subsections. This helps us determine when $|\Phi\rangle\langle\Phi|$ is the optimal probe state for both the SLD bound and the HN bound.

Equality condition for k = 4. Denoting $X \circ Y := \frac{1}{2}(XY + YX)$ as the symmetric product between operators X and Y, the SLD equations for quantum state estimation model $(\rho, (D_j)_j)$ are given as $\rho \circ L_j = D_j$. Here, we interpret L_j as a self-adjoint operator-valued solution to an SLD equation, and we call L_j as an SLD operator.

The use of SLD operators is natural when we discuss $S_4[|\Phi\rangle\langle\Phi]$. Namely, using the SLD operators, we can define the SLD Fisher information matrix J_{SLD} as $J_{\mathrm{SLD}} := \sum_{i,j}|i\rangle\langle j|\mathrm{Tr}[L_i\rho L_j]$. We define $L^i := \sum_{j=1}^{d}(J_{\mathrm{SLD}}^{-1})^{i,j}L_j$ as a linear combination of SLD operators that depend on the ith row of the inverse SLD Fisher information matrix. These operators L^i satisfy the constraint $\mathrm{Tr}D_jL^i = \delta^i_j$.

Since the operators L_i and L^i depend on the quantum state estimation model $(\rho, (D_j)_j)$, we also denote L_i and L^i as $L^i[\rho, (D_j)_j]$ and $L_i[\rho, (D_j)_j]$ respectively. We also denote J_{SLD} as $J_{\mathrm{SLD}}[\rho, (D_j)_j]$.

Interpreting $\bar{S}_k[|\Phi\rangle\langle\Phi|]$ as a quantum state estimation problem on the normalized Choi state T_N and its derivatives $F_{j,N}$, we denote the linear combination of corresponding SLD operators as $L^i_* := L^i[T_N, (F_{j,N})_j]$, and define the vector of such operators as $\vec{L}_* = (L^i_*)_i$. For a vector of Hermitian matrices $\vec{Z} = (Z^j)_j$, we define the operator

$$W_{\mathrm{SLD}}(T_N, \vec{Z}) := \sum_{1 \le i,j \le d} G_{j,i}Z^iT_NZ^j.$$

Then we have the following theorem.

Theorem 2. The following conditions are equivalent.

(A1) $J_4 = S_4[|\Phi\rangle\langle\Phi|]$.
(A2) $\mathrm{Tr}_B W_{\mathrm{SLD}}(T_N, \vec{L}_*)$ is proportional to the identity operator I_A.

In the Supplemental Information B1, we show that $W_{\mathrm{SLD}}(T_N, \vec{Z})$ corresponds to the operator-valued dual variable of the condition (i'-N) in the conic program J_4.

Equality condition for $k = 5$. The constraints for J_k are more complicated than for J_k for $k = 1, 2, 3, 4$, and hence it makes sense to write the dual variables of J_5 in as simple a form as possible. For this goal, in this subsection, given the weight matrix G, we choose the new parameter $\tilde{\theta} := \sqrt{G}\theta$. By using the estimator $\hat{\theta}$ of the parameter θ, the new parameter's estimator $\tilde{\theta}$ is given as $\sqrt{G}\hat{\theta}$. Hence, by using the covariance matrix $V_\theta[\hat{\Pi}]$ of the parameter θ, the covariance matrix $\tilde{V}_{\tilde{\theta}}[\hat{\Pi}]$ of the new parameter $\tilde{\theta}$ is given as $\sqrt{G}V_\theta[\hat{\Pi}]\sqrt{G}$. That is, we have

$$\mathrm{Tr}GV_\theta[\hat{\Pi}] = \mathrm{Tr}\tilde{V}_{\tilde{\theta}}[\hat{\Pi}].$$

In other words, under the new parameter $\tilde{\theta}$, the weight matrix is given as the identity matrix I so that the analysis on the weight matrix I can recover the case with a general weight matrix G. Therefore, without loss of generality, we can assume that the weight matrix is the identity matrix I.

Given a vector of Hermitian operators $\vec{Z} = (Z^1, \ldots, Z^d)$, we denote $\Pi(\vec{Z})$ as a block matrix $\vec{Z}$, where the components of $\Pi(\vec{Z})$ are given as $\Pi(\vec{Z})^{i,j} = Z^i Z^j$. That is, $\Pi(\vec{Z}) = \sum_{1\le i,j\le d}|i\rangle\langle j| \otimes (Z^i)^\dagger Z^j = (\sum_{i=1}^d |i\rangle \otimes (Z^i)^\dagger)(\sum_{i=1}^d \langle i| \otimes Z^i)$. Then, we define $\vec{Z}_*$ as the optimal solution of the minimization of a linear function of $\Pi(\vec{Z})$ subject to linear constraints on Z^i, namely,

$$\vec{Z}_* := \underset{\vec{Z}}{\mathrm{argmin}}\left\{\mathrm{Tr}\Pi(\vec{Z})(I\otimes T_N) | \mathrm{Tr}D_j Z^i = \delta^i_j\right\},$$

Based on $\vec{Z}_*$, we define the operator $V_*^{i,j} := \mathrm{Tr}Z_*^i (Z_*^j)^\dagger T_N$, and define $C_* := -\mathrm{Im}V_* |\mathrm{Im}V_*|^{-1}$, where $|A| := \sqrt{A^\dagger A}$ and the inverse $|\mathrm{Im}V_*|^{-1}$ acts only on the support of $|\mathrm{Im}V_*|$.

Next, we define

$$W_{\mathrm{HN}}(T_N, \vec{Z}) := \left(\sum_{i=1}^d Z^i T_N Z^i\right) - \sum_{1\le i,j\le d} \sqrt{-1}C_*^{i,j} Z^j T_N Z^i$$

which corresponds to a dual operator-valued variable for the condition (i'-N).

Theorem 3. The following conditions are equivalent.

(B1) $J_5 = S_5[|\Phi\rangle\langle\Phi|]$.
(B2) $\mathrm{Tr}_B W_{\mathrm{HN}}(T_N, \vec{Z}_*)$ is proportional to the identity operator I_A.

Examples: one-parameter case. Consider the qubit depolarizing channel

$$\Lambda_{0,p}(\rho) := (1-p)\rho + p\rho_{\mathrm{mix},B},$$

where $\rho_{\mathrm{mix},B} = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$. The channel Λ_θ is given as $\Gamma_{\theta,p}(\rho) := U_\theta \Lambda_{0,p}(\rho) U_\theta^\dagger$, where $U_\theta := \exp(i\theta\sigma_1)$ and $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$ is a bit-flip operator. Then, the following theorem holds.

Theorem 4. When G is 1, we have the following relation

$$\begin{aligned} J_k&\left[(\Gamma_{\theta,p}\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{d}{d\theta}\Gamma_{\theta,p}\otimes\iota\right)(|I\rangle\langle I|)\right] \\ &= \bar{S}_k\left[(\Gamma_{\theta,p}\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{d}{d\theta}\Gamma_{\theta,p}\otimes\iota\right)(|I\rangle\langle I|), |\Phi\rangle\langle\Phi|\right] \\ &= \frac{2-p}{8(1-p)^2} \end{aligned} \tag{31}$$

for $k = 1, 2, 3, 4, 5$.

Theorem 4 shows that the maximally entangled state is the optimal probe state for estimating θ for the channel $\Gamma_{\theta,p}$ that performs a rotation about the Pauli-X axis after applying a depolarizing channel with papameter p. Moreover, the precision bounds for correlated and uncorrelated measurement strategies are identical, and we furthermore have their precise analytical form.

We prove Theorem 4 in the Supplemental Information. The proof is an application of Theorem 2.

Examples: generalized Pauli channel

We consider the generalized Pauli channel on the qudit system $\mathcal{H} = \mathcal{H}_A = \mathcal{H}_B$, which is spanned by $\{|a\rangle\}_{a\in\mathbb{Z}_d}$. We define the operators $\mathsf{W}(a,b)$ for $a, b \in \mathbb{Z}_d$ as the following unitary matrices on $\mathcal{H}$;

$$\mathsf{X}(a) := \sum_{j\in\mathbb{Z}_d}|j+a\rangle\langle j|, \quad \mathsf{Z}(b) := \sum_{j\in\mathbb{Z}_d}\omega^{bj}|j\rangle\langle j|, \tag{32}$$

$$\mathsf{W}(a,b) := \mathsf{X}(a)\mathsf{Z}(b), \tag{33}$$

where $\omega := \exp(2\pi i/d)$. We introduce a distribution family p_θ over $\mathbb{Z}_d^2$. Then, we define the family of channels $\{\Lambda_\theta\}$ as

$$\Lambda_\theta(\rho) := \sum_{(a,b)\in\mathbb{Z}_d^2} p_\theta(a,b)\mathsf{W}(a,b)\rho\mathsf{W}(a,b)^\dagger. \tag{34}$$

We denote the Fisher information of the distribution family $\{P_\theta\}$. Then, as shown in[37,38], we have the following theorem.

Theorem 5. We have the following relations:

$$\begin{aligned} J_4&\left[(\Lambda_\theta\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{\partial}{\partial\theta^l}\Lambda_\theta\otimes\iota\right)(|I\rangle\langle I|))_l\right] \\ &= \bar{S}_4\left[(\Lambda_\theta\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{\partial}{\partial\theta^l}\Lambda_\theta\otimes\iota\right)(|I\rangle\langle I|))_l, |\Phi\rangle\langle\Phi|\right] \\ &= \mathrm{Tr}GJ_\theta^{-1}. \end{aligned} \tag{35}$$

Theorem 5 shows that the channel estimation problem of estimating a noisy generalized Pauli channel with noise parameters p_θ has the maximally entangled state as the optimal probe state with respect to the SLD precision bound.

We prove Theorem 5 in the Supplemental Information. The proof is again an application of Theorem 2.

Examples: Spin j representation of SU(2)

Next, we consider spin j representation of SU(2) over the Hilbert space $\mathcal{H}_j$. Here, $\sigma_{1,j}$, $\sigma_{2,j}$, and $\sigma_{3,j}$ are defined as the spin j representations of the generators of SU(2) on $\mathcal{H}_j$. We set $\mathcal{H}_A$ and $\mathcal{H}_B$ to be $\mathcal{H}_j$. Define the depolarizing channel

$$\Lambda_{0,p}(\rho) := (1-p)\rho + p\rho_{\mathrm{mix},B},$$

where $\rho_{\mathrm{mix},B}$ denotes the maximally mixed state on $\mathcal{H}_B$. Here, the channel Λ_θ is given as $\Lambda_{\theta,p}(\rho) := U_\theta\Lambda_{0,p}(\rho)U_\theta^\dagger$, where $U_\theta := \exp(i\sum_{k=1}^3 \theta^k\sigma_k)$.

Then, the following theorem holds.

Theorem 6. When the weight matrix G is chosen to be I, we have the following relations

$$\begin{aligned} J_4&\left[(\Lambda_{\theta,p}\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{\partial}{\partial\theta^l}\Lambda_{\theta,p}\otimes\iota\right)(|I\rangle\langle I|))_{l=1,2,3}\right] \\ &= \bar{S}_4\left[(\Lambda_{\theta,p}\otimes\iota)(|I\rangle\langle I|), \left(\tfrac{\partial}{\partial\theta^l}\Lambda_{\theta,p}\otimes\iota\right)(|I\rangle\langle I|))_{l=1,2,3}, |\Phi\rangle\langle\Phi|\right] \\ &= \frac{9\left(1-\dfrac{4j^2+4j-1}{(2j+1)^2}p\right)}{8j(j+1)(1-p)^2}. \end{aligned} \tag{36}$$

In particular, for $p = 0$, we have

$$
\begin{aligned}
&J_4\left[(\Lambda_{\theta,0}\otimes\iota)(|I\rangle\langle I|),\left(\tfrac{\partial}{\partial\theta^l}\Lambda_{\theta,0}\otimes\iota\right)(|I\rangle\langle I|)\right)_{l=1,2,3}\Big]\\
&=J_1\left[(\Lambda_{\theta,0}\otimes\iota)(|I\rangle\langle I|),\left(\tfrac{\partial}{\partial\theta^l}\Lambda_{\theta,0}\otimes\iota\right)(|I\rangle\langle I|)\right)_{l=1,2,3}\Big]\\
&=\bar{S}_1\left[(\Lambda_{\theta,0}\otimes\iota)(|I\rangle\langle I|),\left(\tfrac{\partial}{\partial\theta^l}\Lambda_{\theta,0}\otimes\iota\right)(|I\rangle\langle I|)\right)_{l=1,2,3},|\Phi\rangle\langle\Phi|\Big]\\
&=\frac{9}{8j(j+1)}.
\end{aligned}
\tag{37}
$$

Theorem 6 shows that the channel estimation problem of estimating the coefficients of the SU(2) generators that act on a partially depolarized state has the maximally entangled state as the optimal probe state with respect to the SLD precision bound. Furthermore, we have the precise analytical form of the corresponding SLD precision bound for all noise parameters p, and all spin values j.

We prove Theorem 6 in the Supplemental Information. The proof is again an application of Theorem 2.

The paper[39] studied a similar but different problem in the n-copy setting of the estimation of SU(D). That paper maximized the trace of SLD Fisher information matrix by varying the input state.

Application to field sensing

Model for field sensing. The canonical example that is widely considered in quantum metrology is 'field sensing', where a classical field interacts with an ensemble of qubits. When a 3D classical field interacts identically with n qubits, we can write the interaction Hamiltonian as

$$H_\theta = \theta_1 E^1 + \theta_2 E^2 + \theta_3 E^3$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ is a real vector that is proportional to the 3D field that we wish to estimate, and E^1, E^2, E^3 are angular momentum operators defined on n-qubits, given by

$$
\begin{aligned}
E^1 &= \tfrac{1}{2}\left(\tau_1^{(1)}+\cdots+\tau_1^{(n)}\right),\\
E^2 &= \tfrac{1}{2}\left(\tau_2^{(1)}+\cdots+\tau_2^{(n)}\right),\\
E^3 &= \tfrac{1}{2}\left(\tau_3^{(1)}+\cdots+\tau_3^{(n)}\right),
\end{aligned}
$$

$\tau_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\tau_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ are Pauli matrices that apply the bit-flip and phase-flip on a qubit, $\tau_2 = i\tau_1\tau_3$, and $\tau_j^{(k)}$ represents an n-qubit Pauli matrix that applies τ_j on the kth qubit, and identity operations everywhere else.

Amplitude damping operators A_j, which model energy loss, apply $\gamma|0\rangle\langle 1|$ on the jth qubit and the identity operator on other qubits. These operators arise because of a linear interaction between individual qubits and a Markovian zero-temperature bath[40]. Namely,

$$A_{j,\gamma} = \gamma I^{\otimes j-1}\otimes|0\rangle\langle 1|\otimes I^{\otimes(n-j)}.$$

The collective amplitude damping operator, also considered in [ref. 30, Eq. (7)], models collective energy loss, specifically because of a collective linear interaction between all n qubits and a Markovian zero-temperature bath, and is given by $\tilde{A}_\gamma := \sum_{j=1}^n A_{j,\gamma}$.

We model the evolution of an initial probe state using the master equation

$$\frac{d\rho}{dt} = \mathcal{L}_{\theta,\gamma}(\rho), \tag{38}$$

where t denotes time, and the operator $\mathcal{L}_\theta$ can be written as a linear operator, which is

$$\mathcal{L}_{\theta,\gamma}(\rho) = -i(H_\theta\rho - \rho H_\theta) + \tilde{A}_\gamma\rho\tilde{A}_\gamma^\dagger - \frac{1}{2}(\tilde{A}_\gamma^\dagger\tilde{A}_\gamma\rho + \rho\tilde{A}_\gamma^\dagger\tilde{A}_\gamma). \tag{39}$$

For a non-negative evolution time s, let $\tilde{\rho}_s$ denote the solution to the master equation (38). In particular, we can write $\tilde{\rho}_s = e^{s\mathcal{L}_{\theta,\gamma}}(\tilde{\rho}_0)$. This means that we can write $\tilde{\rho}_s$ as the Taylor series

$$\tilde{\rho}_s = \tilde{\rho}_0 + \sum_{k=1}^\infty \frac{s^k(\mathcal{L}_{\theta,\gamma})^k(\tilde{\rho}_0)}{k!}.$$

In our application, we set the evolution time as $s = 1$. Hence, the channel that we consider in our channel estimation problem is

$$\Lambda_{\theta,\gamma}(\rho) = e^{\mathcal{L}_{\theta,\gamma}}(\rho) = \rho + \sum_{k=1}^\infty \frac{\mathcal{L}_{\theta,\gamma}^k(\rho)}{k!}. \tag{40}$$

Using $\Lambda_{\theta,\gamma}$, we can calculate the corresponding Choi matrix $T_{\theta,\gamma}$, and its derivatives about the true paramater θ_0 are

$$F_{j,\gamma} = \frac{\partial}{\partial\theta^j}T_{\theta,\gamma}|_{\theta=\theta_0}$$

for $j = 1, 2, 3$.

Numerical results. Here, we set the true parameter as $\theta_0 = (0, 0, 0)$ and the number of qubits as $n = 2, 3, 4, 5$. We investigate our channel estimation problem when γ varies from 0 to 1. In our calculations, we set the weight matrix G as the identity matrix I.

We numerically evaluate the channel estimation precision bounds $J_{2,\gamma}$, $J_{4,\gamma}$, $J_{5,\gamma}$, where

$$J_{k,\gamma} = J_k[T_{\theta_0,\gamma}, (F_{1,\gamma}, F_{2,\gamma}, F_{3,\gamma})].$$

The precision bounds $J_{2,\gamma}$, $J_{4,\gamma}$, $J_{5,\gamma}$ are given by the optimal values of semidefinite programs with corresponding optimal solutions given by $Y_{2,\gamma}$, $Y_{4,\gamma}$, $Y_{5,\gamma}$, respectively. Using the optimal solutions $Y_{2,\gamma}$, $Y_{4,\gamma}$, $Y_{5,\gamma}$, we calculate the corresponding probe states on $\mathcal{H}_A$ given by $\rho_A(Y_{2,\gamma})$, $\rho_A(Y_{4,\gamma})$, $\rho_A(Y_{5,\gamma})$ respectively.

Let us denote $|\Phi\rangle$ as the maximally entangled state on the symmetric subspace, and consider the precision bounds that correspond to using $|\Phi\rangle$ as the input probe state for the channel estimation problem. These precision bounds are

$$S_{k,\gamma}^{\mathrm{sym}} := S_k\left[\Lambda_{\theta_0,\gamma}\otimes\iota_C(|\Phi\rangle\langle\Phi|),\left(\frac{\partial}{\partial\theta^j}\Lambda_{\theta,\gamma}\otimes\iota_C(|\Phi\rangle\langle\Phi|)|_{\theta=\theta_0}\right)_{j=1,2,3}\right].$$

We also numerically evaluate $S_{2,\gamma}^{\mathrm{sym}}$, $S_{4,\gamma}^{\mathrm{sym}}$, $S_{5,\gamma}^{\mathrm{sym}}$. When we have no access to an ancillary system $\mathcal{H}_C$, we may consider using the 3D-GHZ state[41]

$$|\psi_{\mathrm{3DGHZ}}\rangle = \frac{\sum_{j=0}^1|j\rangle^{\otimes n} + |+\rangle^{\otimes n} + |-\rangle^{\otimes n} + |+i\rangle^{\otimes n} + |-i\rangle^{\otimes n}}{N},$$

where $|\pm\rangle = \frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$ and $|\pm i\rangle = \frac{|0\rangle+\pm i|1\rangle}{\sqrt{2}}$, and N is the appropriate normalization factor. We denote the corresponding precision bounds for using the 3D-GHZ state without ancillary assistance as

$$S_{k,\gamma}^{3D} = S_k\left[\Lambda_{\theta_0}(\rho_{\mathrm{3DGHZ}}), \frac{\partial}{\partial\theta^j}\Lambda_{\theta,\gamma}(\rho_{\mathrm{3DGHZ}})|_{\theta=\theta_0}\right].$$

We also numerically calculate $S_{2,\gamma}^{3D}$, $S_{4,\gamma}^{3D}$, $S_{5,\gamma}^{3D}$.

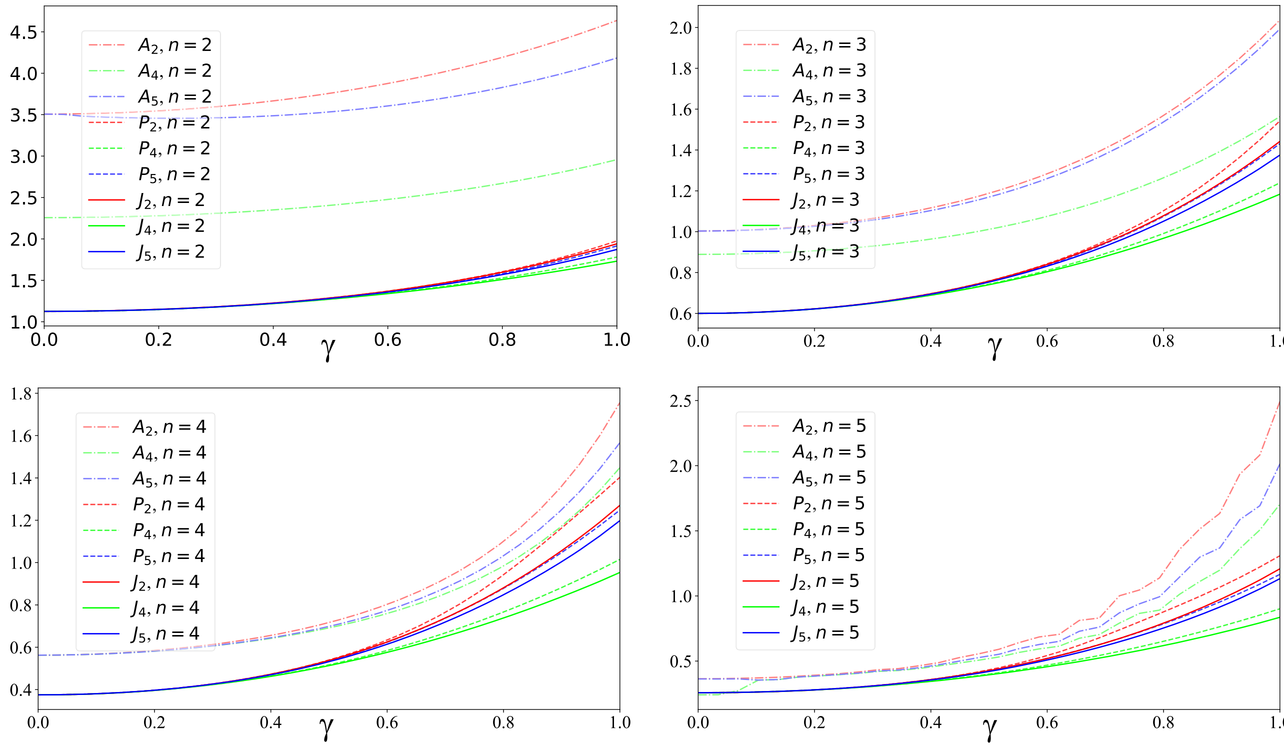

Fig. 3 | Plots of multiparameter Cramér-Rao bounds against increasing levels of noise γ. We numerically calculate the CR-bounds for field sensing when $\theta_1 = \theta_2 = \theta_3 = 0$ for fixed number of qubits $n = 2, 3, 4, 5$. Here, $J_k := J_{k,\gamma}$ denote the CR-bounds using the optimal probe state with ancilla assistance. In contrast, $P_k := S^{\rm sym}_{k,\gamma}$ denote the CR-bounds using the maximally entangled probe state on the symmetric subspace. We also use $A_k := S^{3D}_{k,\gamma}$ for the CR-type bounds using the 3D-GHZ state that do not require ancilla resistance. The vertical axis represents the CR-type lower bound on the sum of the variances of estimates on the three field components, which corresponds to a weight matrix G equal to the identity matrix. The horizontal axis denotes the amount of amplitude damping noise γ.

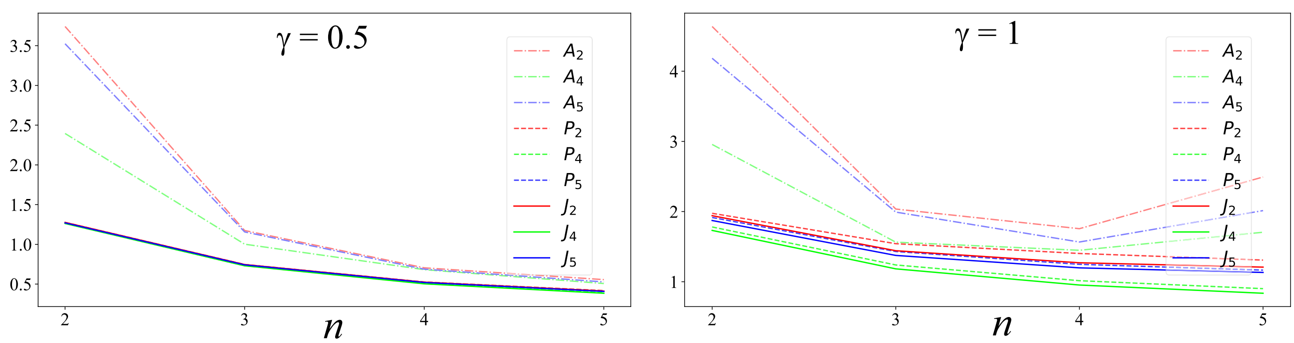

Fig. 4 | Plots of multiparameter Cramér-Rao bounds against increasing number of qubits. We represent the data in Fig. 3 differently, plotting the CR-bounds for fixed values of γ. The horizontal axis denotes the number of qubits n in the 3D-field sensing problem. The vertical axis is the sum of variances of estimates on the three field components corresponding to using the optimal probe state on the symmetric subspace (J_k), the maximally entangled state on the symmetric subspace (P_k), and the 3D-GHZ state without ancilla assistance (A_k). This plot shows the advantage of using the optimal probe state.

We numerically find that when $\gamma = 0$, we have

$$J_{2,0} = J_{4,0} = J_{5,0} = S^{\rm sym}_{2,0} = S^{\rm sym}_{4,0} = S^{\rm sym}_{5,0} = \frac{9}{n(n+2)}.$$

In fact, the analysis in Subsection II F showed that the same equality when the input state is limited to a state on the symmetric subspace. Our numerical analysis suggests that the support of the optimal input state is limited to the symmetric subspace.

Furthermore, when we solve the semidefinite programs corresponding to $J_{2,\gamma}$, $J_{4,\gamma}$, $J_{5,\gamma}$, we find that the corresponding optimal solutions $Y_{2,\gamma}$, $Y_{4,\gamma}$, $Y_{5,\gamma}$ have corresponding density matrices $\rho_A(Y_{2,\gamma})$, $\rho_A(Y_{4,\gamma})$, $\rho_A(Y_{5,\gamma})$ that are very close to the completely mixed state.

When $\gamma > 0$ we numerically find that

$$S^{3D}_2 > S^{\rm sym}_2 > J_2, \quad S^{3D}_4 > S^{\rm sym}_4 > J_4, \quad S^{3D}_5 > S^{\rm sym}_5 > J_5. \tag{41}$$

In Fig. 3, for different fixed values of $n = 2, 3, 4, 5$, we plot the precision bounds $S^{3D}_{k,\gamma}$, $S^{\rm sym}_{k,\gamma}$ and $J_{k,\gamma}$ on the vertical axis and $\gamma \in [0, 1]$ on the horizontal axies. In Fig. 4, for fixed $\gamma = 0.5$ or $\gamma = 1$, we plot the precision bounds A_k, P_k and J_k on the vertical axis, and the number of qubits n on the horizontal axis.

In this way, we confirm the suboptimality of the maximally entangled state by our numerical evaluation of $\rho_A(Y)$ using the optimal solution Y for the conic programs $J_{2,\gamma}$, $J_{4,\gamma}$ and $J_{5,\gamma}$ for γ. In particular, while $\rho_A(Y_{2,\gamma})$, $\rho_A(Y_{4,\gamma})$, and $\rho_A(Y_{5,\gamma})$ are still a diagonal matrices when $\gamma > 0$, they are not completely mixed states. Hence, their purifications cannot be the maximally entangled state. We can furthermore see the suboptimality of the maximally entangled state because $S^{\mathrm{sym}}_{k,\gamma} \geq J_{k,\gamma}$ for $k = 2, 4, 5$ when $\gamma > 0$. Curiously, from Fig. 3, the maximally entangled state on the symmetric subspace is nonetheless still quite close to optimal.

Motivating the above analysis, we prove the following theorem. When the support of the input state is limited to the symmetric subspace in the channel Λ_θ, we denote the obtained channel by $\Lambda^{\mathrm{sym}}_{\theta,\gamma}$. Using this limitation for the inputs, we define $T^{\mathrm{sym}}_{\theta,\gamma}$ and $F^{\mathrm{sym}}_{j,\gamma}$ in the same way.

Theorem 7. For $\gamma > 0$, we have

$$\begin{aligned} &J_4\left[T^{\mathrm{sym}}_{\theta_0,\gamma}, \left(F^{\mathrm{sym}}_{j,\gamma}\right)_{j=1,2,3}\right] \\ &\qquad\qquad < S^{\mathrm{sym}}_{k,\gamma} \\ &= S_4\left[\Lambda^{\mathrm{sym}}_{\theta_0,\gamma}\otimes\iota_C(|\Phi\rangle\langle\Phi|), \left(\tfrac{\partial}{\partial\theta^j}\Lambda^{\mathrm{sym}}_{\theta,\gamma}\otimes\iota_C(|\Phi\rangle\langle\Phi|)|_{\theta=\theta_0}\right)_{j=1,2,3}\right]. \end{aligned}$$

Since the relation $J_4[T_{\theta_0,\gamma}, (F_{j,\gamma})_{j=1,2,3}] \leq J_4[T^{\mathrm{sym}}_{\theta_0,\gamma}, (F^{\mathrm{sym}}_{j,\gamma})_{j=1,2,3}]$ holds, the above theorem means that the maximally entangled state on the symmetric subspace is not the optimal input state.

Discussion

We have unraveled the connection between the two-stage optimizations $\bar{S}_k$ and our conic programs J_k. The conic programs J_k are efficient to solve. While the two-stage optimizations $\bar{S}_k$ need not be efficient to solve, the conic programs J_k are efficient to solve. Using the optimal solutions of J_k, one easily finds the corresponding optimal probe state for the channel estimation problem. We illustrate the power of our conic programs with theoretical analysis on the scenario when the maximally entangled state is the optimal probe state, and also with numerical analysis for the often studied field sensing problem using quantum probe states.

Applications of theoretical findings extend far beyond the examples we explored. Indeed, any problem where we estimate multiple incompatible parameters embedded within a quantum channel using entangled probe states stands to benefit from our theory. This encompasses for example a plethora of applications in quantum imaging[5].

Recently, it was shown that field sensing using quantum probe states in the face of a linear rate of errors can approach the Heisenberg limit if we use finite rounds of quantum error correction[42] on appropriate permutation-invariant codes[43–45]. However, the corresponding question of what can be done using entangled probe states remains an interesting open problem.

Methods

We first give the main ideas of how to prove Theorem 1.

See the Supplementary Information for full details.

Equivalence of (i) and (i')

Condition (i) is not linear, and hence we like to show its equivalence to the set of linear constraints (i'). Since (i) ⇒ (i') is trivial, we show (i') ⇒ (i). Now assume that (i') holds. We denote $\mathrm{Tr}_B[\mathrm{Tr}_R[Y(|0\rangle\langle 0| \otimes I_{AB})](I_A \otimes |b\rangle\langle b|)] = \mathrm{Tr}_{RB}[Y(|0\rangle\langle 0| \otimes I_A \otimes |b\rangle\langle b|)]$ by $\rho_A(Y)$. Then, (17) guarantees that $\rho_A(Y)$ is a state. Also, (15) and (16) imply (14). Hence, we obtain (i), and this completes the proof.

Equality between the objective functions of $\bar{S}_k$ and $\bar{J}_k$

There are two optimization variables in $\bar{S}_k$, namely the probe state ρ_{AC} and an optimization variable X on $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. In contrast, the conic program J_k has only a single optimization variable Y on $\mathcal{R}_C \otimes \mathcal{H}_B \otimes \mathcal{H}_A$. Here, we clarify the connection between the optimization variable X and the optimization variable Y.

Note that the output state that corresponds to our input state ρ_{AC} at the true parameter value θ_0 is

$$(\Lambda_\theta\otimes\iota_C)(\rho_{AC})|_{\theta=\theta_0} = \mathrm{Tr}_A[(T \otimes I_C)(I_B \otimes \rho_{AC})], \tag{42}$$

and its derivatives are

$$\frac{\partial}{\partial\theta^j}(\Lambda_\theta\otimes\iota_C)(\rho_{AC})|_{\theta=\theta_0} = \mathrm{Tr}_A[(F_j \otimes I_C)(I_B \otimes \rho_{AC})]. \tag{43}$$

Here, the joint system $\mathcal{H}_B \otimes \mathcal{H}_C$ is accessible for our measurement for our estimation. Then, we write the CR-type bound $\bar{S}_k[\rho_{AC}]$ of channel estimation problem in (11) as

$$\bar{S}_k[\rho_{AC}] = S_k[\mathrm{Tr}_A[(T \otimes I_C)(I_B \otimes \rho_{AC})], (\mathrm{Tr}_A[(F_j \otimes I_C)(I_B \otimes \rho_{AC})])_j] \tag{44}$$

with $k = 1, 2, 3, 4, 5$. We call the pair $(T, (F_j)_j)$ as the channel model. In particular, we denote $\bar{S}_k[\rho_{AC}]$ by $\bar{S}_k[T, (F_j)_j, \rho_{AC}]$ when we need to clarify the dependence of the channel model.

In the optimization of (44), we can write the objective function as

$$\begin{aligned} &\mathrm{Tr}[G \otimes \mathrm{Tr}_A[(T \otimes I_C)(I_B \otimes \rho_{AC})]X] \\ &\quad = \mathrm{Tr}[G \otimes ((T \otimes I_C)(I_B \otimes \rho_{AC}))(I_A \otimes X)] \\ &\quad = \mathrm{Tr}[(G \otimes T)\mathrm{Tr}_C[(I_{RB} \otimes \rho_{AC})(I_A \otimes X)]]. \end{aligned}$$

Identifying Y as $\mathrm{Tr}_C[(I_{RB} \otimes \rho_{AC})(I_A \otimes X)]$, we see that we can write the objective function of (44) as $\mathrm{Tr}[(G \otimes T)Y]$, which is precisely the objective function of our conic programs J_k.

Sketch of the proof of Theorem 1

Now we sketch Theorem 1's proof. First we can establish an upper bound on J_k in terms of $\bar{S}_k$ in the following lemma.

Lemma 8. For $k = 1, 2, 3, 4, 5$, we have

$$J_k \leq \bar{S}_k. \tag{45}$$

To prove Lemma 8, we show that given any solution ρ_{AC} and X to $\bar{S}_k$, we can also construct a corresponding solution Y for J_k with the same value for the objective function. We prove the inequality opposite to (45), based on the discussion in Section II A, we rewrite the constraints for the completeness condition and the locally unbiased condition as

$$\mathrm{Tr}_R[X(|0\rangle\langle 0| \otimes I_{BC})] = I_{BC} \tag{46}$$

$$\frac{1}{2}\mathrm{Tr}[(I_A \otimes X)((|0\rangle\langle j'| + |j'\rangle\langle 0|) \otimes F_j \otimes I_C)(I_{RB} \otimes \rho_{AC})] = \delta_{j,j'}. \tag{47}$$

Then, we show the following lemma, which takes any solution Y of J_k, and constructs a corresponding probe state ρ_{AC} and solution X for $\bar{S}_k[\rho_{AC}]$, under the assumption that $\rho_A(Y)$ is full rank.

Lemma 9. For $k = 1, 2, 3, 4, 5$, we choose $Y \in \mathcal{S}^k_{BA}$ satisfying the conditions (i), (ii), and $\rho_A := \rho_A(Y) > 0$. We diagonalize ρ_A as $\sum_{j=1}^{d_A} s_j|\phi_j\rangle\langle\phi_j|$. We choose an orthonormal basis $\{\psi_j\}$ of $\mathcal{H}_C$ and the state ρ_{AC} as the pure state $\sum_{j=1}^{d_A}\sqrt{s_j}|\phi_j, \psi_j\rangle$, which is a purification of ρ_A. Then, we have

$$\mathrm{Tr}[Y(G \otimes T)] \geq \bar{S}_k[\rho_{AC}] \geq \bar{S}_k. \tag{48}$$

If the minimization in J_k is achieved by an operator Y for which $\rho_A(Y)$ is full rank, the combination of Lemmas 8 and 9 yields Theorem 1. However, the optimal operator Y might not admit a full rank $\rho_A(Y)$. Hence, we consider the following lemma.

Lemma 10. We have

$$J_k = \inf_{Y \in \mathcal{S}^k_{BA}} \left\{ \mathrm{Tr}[Y(G \otimes T)] \,\middle|\, \begin{array}{c} Y \text{ satisfies (i), (ii) and,} \\ \rho_A(Y) > 0, \textit{i.e.}, \\ \rho_A(Y) \text{ is full rank .} \end{array} \right\} \tag{49}$$

for $k = 1, 2, 3, 4$, and

$$J_5 = \inf_{Y \in \mathcal{S}^5_{BA}(T)} \left\{ \mathrm{Tr}[Y(G \otimes T)] \,\middle|\, \begin{array}{c} Y \text{ satisfies (i), (ii)and,} \\ \rho_A(Y) \text{ is full rank .} \end{array} \right\} \tag{50}$$

The combination of Lemma 9 and Lemma 10 yields $J_k \geq \bar{S}_k$, while Lemma 8 shows $J_k \leq \bar{S}_k$. Hence, $J_k = \bar{S}_k$, which proves Theorem 1. We supply the proofs of Lemmas 8, 9, and 10 in Supplemental Information A.

Methodology of numerical study of field sensing

Now we use $\rho' = \Lambda^{\approx}_{\theta,\gamma}(\rho)$ to approximate $\Lambda_{\theta,\gamma}(\rho)$ according to the following procedure: Given any input state ρ, we define

$$\bar{m}_*(\rho) = \min(100, \min\{j : \| (\mathcal{L}^j_{\theta,\gamma}(\rho)/j! \| < 10^{-12}\}).$$

Then we define $\Lambda^{\approx}_{\theta,\gamma}(\rho) := \sum_{j=0}^{m_*(\rho)} \mathcal{L}^j_{\theta,\gamma}(\rho)/j!$. On the state ρ_{AC}, we also define $\Lambda^{\approx}_{\theta,\gamma}(\rho_{AC}) := \sum_{j=0}^{m_*(\rho_{AC})} (\mathcal{L}^j_{\theta,\gamma} \otimes \iota_C)(\rho_{AC})/j!$. We obtain approximations of $T_{\theta,\gamma}$ with $T^{\approx}_{\theta,\gamma} = \Lambda^{\approx}_{\theta,\gamma}((n+1)|\Phi\rangle\langle\Phi|)$. We also obtain approximations of F_1, F_2, F_3 according to the formula

$$\begin{aligned} F^{\approx}_{1,\gamma} &= \frac{T^{\approx}_{(10^{-12},0,0),\gamma} - T^{\approx}_{(0,0,0),\gamma}}{10^{-12}}, \\ F^{\approx}_{2,\gamma} &= \frac{T^{\approx}_{(0,10^{-12},0),\gamma} - T^{\approx}_{(0,0,0),\gamma}}{10^{-12}}, \\ F^{\approx}_{3,\gamma} &= \frac{T^{\approx}_{(0,0,10^{-12}),\gamma} - T^{\approx}_{(0,0,0),\gamma}}{10^{-12}}. \end{aligned}$$

With the above, we approximate $J_{2,\gamma}, J_{4,\gamma}, J_{5,\gamma}$ with

$$J^{\approx}_{k,\gamma} = J_k\left[T^{\approx}_{(0,0,0),\gamma}, (F^{\approx}_{j,\gamma})_{j=1,2,3}\right].$$

We also approximate P_2, P_4, P_5 with

$$S^{\mathrm{sym},\approx}_{k,\gamma} = S_{k,\gamma}\left[T^{\approx}_{(0,0,0)/(n+1),\gamma}, (F^{\approx}_{j,\gamma}/(n+1))_{j=1,2,3}\right].$$

We approximate $\Lambda_{(0,0,0)}(\rho_{3\mathrm{DGHZ}})$ according to formula

$$Q_0 = \Lambda^{\approx}_{(0,0,0),\gamma}(\rho_{3\mathrm{DGHZ}})$$

and approximate $\frac{\partial}{\partial\theta^j}\Lambda_\theta(\rho_{3\mathrm{DGHZ}})|_{\theta=(0,0,0),\gamma}$ according to

$$\begin{aligned} Q^{\approx}_1 &= \frac{\Lambda^{\approx}_{(10^{-12},0,0),\gamma}(\rho_{3\mathrm{DGHZ}}) - \Lambda^{\approx}_{(0,0,0),\gamma}(\rho_{3\mathrm{DGHZ}})}{10^{-12}}, \\ Q^{\approx}_2 &= \frac{\Lambda^{\approx}_{(0,10^{-12},0),\gamma}(\rho_{3\mathrm{DGHZ}}) - \Lambda^{\approx}_{(0,0,0),\gamma}(\rho_{3\mathrm{DGHZ}})}{10^{-12}}, \\ Q^{\approx}_3 &= \frac{\Lambda^{\approx}_{(0,0,10^{-12}),\gamma}(\rho_{3\mathrm{DGHZ}}) - \Lambda^{\approx}_{(0,0,0),\gamma}(\rho_{3\mathrm{DGHZ}})}{10^{-12}}. \end{aligned}$$

This allows us to approximate $S^{3D}_{2,\gamma}, S^{3D}_{4,\gamma}, S^{3D}_{5,\gamma}$ according to

$$S^{3D,\approx}_{k,\gamma} = S_k[Q_0, (Q_1, Q_2, Q_3)].$$

In our numerical evaluations of $S^{3D,\approx}_{k,\gamma}, S^{\mathrm{sym},\approx}_{k,\gamma}, J^{\approx}_{k,\gamma}$, we evaluate the semidefinite programs according to the MatLab code given in Supplemental Information E.

Data availability

The data for the numerical plots are available from the authors upon reasonable request.

Code availability

The MatLab code is provided in the Supplemental Information.

Received: 19 April 2024; Accepted: 14 October 2024;
Published online: 07 November 2024

References

1. Escher, B., de Matos Filho, R. L. & Davidovich, L. General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology. *Nat. Phys.* **7**, 406 (2011).
2. Hayashi, M. Comparison between the Cramer-Rao and the mini-max approaches in quantum channel estimation. *Commun. Math. Phys.* **304**, 689 (2011).
3. Pirandola, S., Laurenza, R., Lupo, C. & Pereira, J. L. Fundamental limits to quantum channel discrimination. *npj Quantum Inf.* **5**, 50 (2019).
4. Zhou, S. & Jiang, L. Asymptotic theory of quantum channel estimation. *PRX Quantum* **2**, 010343 (2021).
5. Moreau, P.-A., Toninelli, E., Gregory, T. & Padgett, M. J. Imaging with quantum states of light. *Nat. Rev. Phys.* **1**, 367 (2019).
6. Fujiwara, A. & Imai, H. A fibre bundle over manifolds of quantum channels and its application to quantum statistics. *J. Phys. A: Math. Theor.* **41**, 255304 (2008).
7. Demkowicz-Dobrzański, R., Kołodyński, J. & Guţă, M. The elusive heisenberg limit in quantum-enhanced metrology. *Nat. Commun.* **3**, 1063 (2012).
8. Górecki, W., Zhou, S., Jiang, L. & Demkowicz-Dobrzański, R. Optimal probes and error-correction schemes in multi-parameter quantum metrology. *Quantum* **4**, 288 (2020).
9. Friel, J., Palittapongarnpim, P., Albarelli, F. & Datta, A. Attainability of the Holevo-Cramér-Rao bound for two-qubit 3d magnetometry. https://doi.org/10.48550/arXiv.2008.01502 (2020).
10. Liu, Q., Hu, Z., Yuan, H. & Yang, Y. Optimal strategies of quantum metrology with a strict hierarchy. *Phys. Rev. Lett.* **130**, 070803 (2023).
11. Altherr, A. & Yang, Y. Quantum metrology for non-markovian processes. *Phys. Rev. Lett.* **127**, 060501 (2021).
12. Conlon, L. O., Lam, P. K. & Assad, S. M. Multiparameter estimation with two-qubit probes in noisy channels. *Entropy* **25**, 1122 (2023).
13. Acín, A. Statistical distinguishability between unitary operations. *Phys. Rev. Lett.* **87**, 177901 (2001).
14. Sacchi, M. F. Optimal discrimination of quantum operations. *Phys. Rev. A* **71**, 062340 (2005).
15. Hayashi, M. Discrimination of two channels by adaptive methods and its application to quantum system. *IEEE Trans. Inf. Theory* **55**, 3807 (2009).
16. Zhuang, Q. & Pirandola, S. Ultimate limits for multiple quantum channel discrimination. *Phys. Rev. Lett.* **125**, 080505 (2020).
17. Wilde, M. M., Berta, M., Hirche, C. & Kaur, E. Amortized channel divergence for asymptotic quantum channel discrimination. *Lett. Math. Phys.* **110**, 2277 (2020).
18. Nakahira, K. & Kato, K. Generalized quantum process discrimination problems. *Phys. Rev. A* **103**, 062606 (2021).
19. Hayashi, M. & Ouyang, Y. Tight Cramér-Rao type bounds for multiparameter quantum metrology through conic programming. *Quantum* **7**, 1094 (2023).
20. Holevo, A. S. *Probabilistic and Statistical Aspects of Quantum Theory* (Edizioni della Normale, 2011).
21. Nagaoka, H. A new approach to Cramér-Rao bounds for quantum state estimation. *IEICE Tech. Rep.* **IT 89-42**, 9 (1989).

22. Hayashi, M. & Matsumoto, K. Asymptotic performance of optimal state estimation in qubit system. *J. Math. Phys.* **49**, 102101 (2008).
23. Albarelli, F., Friel, J. F. & Datta, A. Evaluating the Holevo Cramér-Rao bound for multiparameter quantum metrology. *Phys. Rev. Lett.* **123**, 200503 (2019).
24. Sidhu, J. S., Ouyang, Y., Campbell, E. T. & Kok, P. Tight bounds on the simultaneous estimation of incompatible parameters. *Phys. Rev. X* **11**, 011028 (2021).
25. Nagaoka, H. in *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (ed. Hayashi, M.) 133–149 (World Scientific, 2005).
26. Hayashi, M. On simultaneous measurement of noncommutative observables (in Japanese). *Surikaisekikenkyusho (RIMS) Kokyuroku (Development of Infinite-Dimensional Noncommutative Analysis)* 96 (1999).
27. Conlon, L. O., Suzuki, J., Lam, P. K. & Assad, S. M. Efficient computation of the Nagaoka–Hayashi bound for multiparameter estimation with separable measurements. *npj Quantum Inf.* **7**, 1 (2021).
28. Choi, M.-D. Completely positive linear maps on complex matrices. *Linear Algebra Appl.* **10**, 285 (1975).
29. Tóth, G. & Apellaniz, I. Quantum metrology from a quantum information science perspective. *J. Phys. A: Math. Theor.* **47**, 424006 (2014).
30. Duan, L.-M. & Guo, G.-C. Optimal quantum codes for preventing collective amplitude damping. *Phys. Rev. A* **58**, 3491 (1998).
31. Gurvits, L. Classical deterministic complexity of Edmonds' problem and quantum entanglement. In: *Proc. 35th Annual ACM Symposium on Theory of Computing*. ACM (Association for Computing Machinery) Publications, 10–19 (2003).
32. Doherty, A. C., Parrilo, P. A. & Spedalieri, F. M. Distinguishing separable and entangled states. *Phys. Rev. Lett.* **88**, 187904 (2002).
33. Doherty, A. C., Parrilo, P. A. & Spedalieri, F. M. Complete family of separability criteria. *Phys. Rev. A* **69**, 022308 (2004).
34. Navascués, M., Owari, M. & Plenio, M. B. Power of symmetric extensions for entanglement detection. *Phys. Rev. A* **80**, 052306 (2009).
35. Fawzi, H. The set of separable states has no finite semidefinite representation except in dimension 3×2. *Commun. Math. Phys.* **386**, 1319 (2021).
36. Yang, Y., Chiribella, G. & Hayashi, M. Attaining the ultimate precision limit in quantum state estimation. *Commun. Math. Phys.* **368**, 223 (2019).
37. Fujiwara, A. & Imai, H. Quantum parameter estimation of a generalized pauli channel. *J. Phys. A: Math. Gen.* **36**, 8093 (2003).
38. Hayashi, M. Private communication to A. Fujiwara (2003).
39. Imai, H. & Fujiwara, A. Geometry of optimal estimation scheme for su(d) channels. *J. Phys. A: Math. Theor.* **40**, 4391 (2007).
40. Chuang, I. L., Leung, D. W. & Yamamoto, Y. Bosonic quantum codes for amplitude damping. *Phys. Rev. A* **56**, 1114 (1997).
41. Baumgratz, T. & Datta, A. Quantum enhanced estimation of a multidimensional field. *Phys. Rev. Lett.* **116**, 030801 (2016).
42. Ouyang, Y. & Brennen, G. K. Quantum error correction on symmetric quantum sensors. https://doi.org/10.48550/arXiv.2212.06285 (2022).
43. Ouyang, Y. Permutation-invariant quantum codes. *Phys. Rev. A* **90**, 062317 (2014).
44. Ouyang, Y. Permutation-invariant quantum coding for quantum deletion channels. In: *2021 IEEE International Symposium on Information Theory (ISIT)*. The Insitute of Electrical and Electronics Engineers (IEEE), 1499–1503 (2021).
45. Hayashi, M. ed., *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (World Scientific, 2005).

Acknowledgements

M.H. is supported in part by the National Natural Science Foundation of China (Grant No. 62171212). Y.O. acknowledges support from EPSRC (Grant No. EP/W028115/1).

Author contributions

Both authors contributed to all parts of the paper.

Competing interests

The authors declare no competing interests.

Additional information

Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s41534-024-00905-x.

Correspondence and requests for materials should be addressed to Masahito Hayashi or Yingkai Ouyang.