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Robust projective measurements through measuring code-inspired observables

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Yingkai Ouyang

Quantum measurements are ubiquitous in quantum information processing tasks, but errors can render their outputs unreliable. Here, we present a scheme that implements a robust projective measurement through measuring code-inspired observables. Namely, given a projective POVM, a classical code, and a constraint on the number of measurement outcomes each observable can have, we construct commuting observables whose measurement is equivalent to the projective measurement in the noiseless setting. Moreover, we can correct *t* errors on the classical outcomes of the observables' measurement if the classical code corrects *t* errors. Since our scheme does not require the encoding of quantum data onto a quantum error correction code, it can help construct robust measurements for near-term quantum algorithms that do not use quantum error correction. Moreover, our scheme works for any projective POVM, and hence can allow robust syndrome extraction procedures in non-stabilizer quantum error correction codes.

Quantum measurements, ubiquitous in quantum information processing tasks, are basic building blocks used in all quantum algorithms, such as in quantum sampling[1–3], quantum learning[4–8], quantum channel estimation[9–12], quantum parameter estimation[13–20], or universal quantum computations[21–24]. However, errors in quantum measurements prevent these quantum algorithms from unlocking their full potential.

Quantum algorithms use either just the classical outputs of quantum measurements or both the classical outputs and the measured states. Near-term quantum algorithms such as quantum sampling, quantum learning, and quantum parameter estimation algorithms use primarily the classical outputs of quantum measurements. When errors afflict the classical outcomes these near-term quantum algorithms' measurements, the precision of these quantum algorithms' outputs suffers. Regarding near-term quantum algorithms, there has been a plethora of recent results on the topic of quantum error mitigation[25–30], where the goal is to reduce the statistical error of quantum measurements. This is achieved through repeated experiments and classical post-processing of the additional classical data obtained. However, the question of how to directly correct such measurement errors in these near-term algorithms without access to quantum error correction (QEC) is an open problem.

Universal quantum computations can use both quantum and classical outputs of measurements. Correction of both quantum and classical errors in measurements using stabilizer codes has been discussed in the context of data-syndrome codes[31–37], single-shot QEC[38–40], and fault-tolerant quantum computing[41]. However, the pertinent question of how to correct measurement errors for non-stabilizer codes, such as for bosonic codes[42–46], remains unanswered.

Here, we present a scheme that implements a robust projective measurement through measuring code-inspired observables. Namely, given a projective POVM, a classical code, and a constraint on the number of measurement outcomes each observable can have, we construct commuting observables whose measurement is equivalent to the noiseless projective measurement. Moreover, we can correct *t* errors on the classical outcomes of the observables' measurement if the classical code corrects *t* errors. The minimum number of commuting observables required depends on (1) the number of measurement outcomes for each commuting observable, (2) the number of measurement outcomes for the underlying projective measurement, and (3) the number of errors on classical outcomes that we wish to correct. We obtain bounds on the minimum number of commuting observables required based on bounds on the parameters of classical codes.

We suggest how to implement our scheme using ancillary coherent states. The requirements are modest. Namely, we need access to a linear coupling between the observables and ancillas, and the ability to perform homodyne measurement on the ancillas. Hence, using a modest amount of quantum control, we can in fact correct measurement errors, without need for QEC codes.

We explain how our scheme allows the correction of measurement errors in any QEC code that satisfies the Knill-Laflamme QEC criterion[47]. Namely, given any QEC code that corrects a set of errors $\mathfrak{K}$, we bound the minimum number of commuting observables $n_{\mathfrak{K},t}$ required to correctly perform the syndrome extraction stage in the Knill-Laflamme recovery procedure if there are up to *t* errors on the syndrome. Based on this, we give bounds on $n_{\mathfrak{K},t}$, and elucidate this bound for binary QEC codes and the binomial code.

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We give two ways to implement our scheme. The first approach involves a dispersive coupling of the quantum system to an ancillary bosonic mode which allows us to measure our constructed observables using geometric phase gates and homodyne detection. The second approach is specialized to robust stabilizer measurements for qubit stabilizer codes, and we give a protocol to measure our observables via unitary control on the extended system and measurement of the first qubit (see Fig. 2). We illustrate the potential of our scheme numerically where we compare the performance of our scheme with a (1) baseline scheme that measures a projective POVM with 16 elements via the measuring of four observables and (2) a simple repetition scheme (see Figs. 3 and 4).

We envision our scheme to complement existing quantum error mitigation techniques, and thereby enhance the performance of near-term quantum algorithms. In the longer term, our scheme can also enhance the design of fault-tolerant quantum computations on non-stabilizer codes, such as those reliant on bosonic codes[46,48].

Results

Measurements

We can describe a measurement as a POVM[49], which is a set of positive operators that sum to the identity operator. Without loss of generality, we can always focus on projective POVMs, where the positive operators are furthermore pairwise orthogonal projectors. This is because Naimark's theorem ensures that for any POVM, we can always perform a projective POVM on an extended Hilbert space[50].

From the Born rule, measuring a projective POVM $P: = \{P_1,\ldots,P_M\}$ with pairwise orthogonal projectors on an input state ρ yields the *post-measurement state*$\rho_k := P_k\rho P_k/\mathrm{tr}[\rho P_j]$ with probability $p_k := \mathrm{tr}[\rho P_k]$. We denote the measurement's output as (ρ_k, k) where k is the measurement's *classical outcome* that allows us to uniquely identify the post-measurement state ρ_k.

Mathematically, an observable is a Hermitian operator. Consider an observable $O = \sum_k \lambda_k P_k$, where λ_k are distinct real numbers for different values of k. Measurement of O on ρ gives an output (ρ_k, λ_k) comprising of a post-measurement state and some eigenvalue of O. According to the Born rule, we obtain (ρ_k, λ_k) with probability p_k. Since there exists a function that maps λ_k back to k, the measurement of O is the same as the measurement of P.

Errors affect a measurement's output in two different ways. First, errors can corrupt the classical outcome k. Such errors can lead us to mistakenly conclude that the post-measurement state is ρ_v for $v \neq k$ when the true post-measurement state is in fact ρ_k. Second, errors can corrupt the post-measurement state ρ_k. Here, we propose a measurement scheme that allows correction of errors on classical outcomes.

Now, let q be an integer where $q \geq 2$, and let us define a Hermitian operator with q distinct eigenvalues as a q-observable. Operationally, the integer q counts the number of possible measurement outcomes of each observable.

Commuting observables from classical codes

In the observable O, the integers $1,\ldots,M$ label M distinct measurement outcomes. Consider a classical code C comprising of M distinct codewords. When C is a q-ary code of length n, each codeword is a vector in $\{0, 1,\ldots,q-1\}^n$. We denote

$$\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$$

as the kth codeword of C, and we can write $C = \{\mathbf{x}^{(k)}: k = 1,\ldots,M\}$.

Each integer $1,\ldots,M$ labels exactly one codeword in C. The encoder E_C of C is a bijective map from the classical labels in $\{1,\ldots,M\}$ to codewords in C. Namely, $E_C(k) = \mathbf{x}^{(k)}$. Without errors on the components of $\mathbf{x}^{(k)}$, a decoder of C performs the inverse map of E_C, and maps the codeword $\mathbf{x}^{(k)}$ back to the label k.

In the measurement of P, errors could afflict its classical outcome. To address this, we propose the measurement of n commuting q-observables $Q_1,\ldots,Q_n$ that encode redundant information about P. We denote the classical outcome of Q_j's measurement as y_j and denote the output of the measurements of $Q_1,\ldots,Q_n$ as $(\tau, \mathbf{y})$ where τ denotes the post-measurement state and $\mathbf{y} = (y_1, \ldots, y_n)$. We want the q-observables to be *consistent* with P, in the sense that measurement of the q-observables performs the same measurement as P in the noiseless setting. Hence we give the following definition.

Definition 1. Let P be a projective POVM and $Q_1,\ldots,Q_n$ be commuting observables. The observables $Q_1,\ldots,Q_n$ are consistent with P if there exists a function f such that for any output $(\tau, \mathbf{y})$ of the measurement of $Q_1,\ldots,Q_n$ on ρ, we have $\tau = \rho_{f(\mathbf{y})}$.

We propose to construct q-observables using information about a projective POVM P and a classical q-ary code C. Namely, for $j = 1, \ldots, n$ we define the q-observables as

$$Q_j(C, P) := \sum_{k=1}^{M} x_j^{(k)} P_k. \quad (1)$$

When the context is clear, we use Q_j to denote $Q_j(C, P)$. From the orthogonality of the projectors P_k, the observables $Q_1,\ldots,Q_n$ are pairwise commuting, which allows us to measure $Q_1,\ldots,Q_n$ in any order.

In our construction, the correctibility of errors on the measurement outcomes of our q-observables depends on the minimum distance of C, given by

$$d(C) := \min_{\mathbf{y} \neq \mathbf{z} \in C} d_H(\mathbf{y}, \mathbf{z}),$$

where $d_H(\mathbf{y}, \mathbf{z}): = |\{i: y_i \neq z_i\}|$ is the Hamming distance between tuples $\mathbf{y}$ and $\mathbf{z}$. Namely, we can correct any $t(C): = \lfloor (d(C) - 1)/2 \rfloor$ errors on the classical outcomes of $Q_1,\ldots,Q_n$.

A decoder of a classical code can correct up to $t(C)$ measurement errors on $\mathbf{y}$. This is because a noiseless $\mathbf{y}$ must be a codeword in C. Here, a decoder $\mathcal{D}$ of a code C is a function $\mathcal{D} : \{0, \ldots, q-1\}^n \to \{1, \ldots, M\}$ which maps an n-tuple to an index that labels the codewords. Given some non-negative integer a, we say that $\mathcal{D}$ is an a-decoder of C, if for all $k = 1, \ldots, M$, and for all $\mathbf{y}$ such that $d(\mathbf{y}, \mathbf{x}^{(k)}) \leq a$, we have

$$\mathcal{D}(\mathbf{y}) = k. \quad (2)$$

An a-decoder corrects a errors. When $d(C) = d$, then there is a $t(C)$-decoder for C. Our main result is the following.

Theorem 1. Let C be a q-ary code of length n, and let $\mathcal{D}$ be a $t(C)$-decoder for C. We measure $Q_1(C, P), \ldots, Q_n(C, P)$ on a quantum state ρ, and obtain the classical outcome $\mathbf{y} = (y_1, \ldots, y_n)$ along with the post-measurement state τ. Suppose that at most $t(C)$ components of $\mathbf{y}$ have been corrupted. Then $\tau = \rho_{\mathcal{D}(\mathbf{y})}$.

In our proof of Theorem 1, we show that in the noiseless setting, the n-tuple of classical outcomes is a codeword of C. When there are at most $t(C)$ errors on the classical outcomes, the decoder $\mathcal{D}$ corrects these errors. Hence, the observables $Q_1(C, P),\ldots,Q_n(C, P)$ are consistent with P, even in the presence of some errors on the classical outcomes.

As an example, consider a scheme that uses the shortened Hamming code C_6 and a projective POVM $P = \{P_1,\ldots,P_8\}$ to define the six binary observables to measure both in the noiseless setting. In this example, the parameters of the code are $q = 2$, $n = 6$, $d(C_6) = 3$, and $M = 8$. Now the code C_6 is a linear code generated by binary vectors $a_1 = \mathtt{100011}$, $a_2 = \mathtt{010101}$, and $a_3 = \texttt{001110}$, and has eight codewords given by

$$\begin{aligned}
\mathbf{x}^{(1)} &= \texttt{000000},\\
\mathbf{x}^{(2)} &= \texttt{100011} = a_1,\\
\mathbf{x}^{(3)} &= \texttt{010101} = a_2,\\
\mathbf{x}^{(4)} &= \texttt{001110} = a_3,\\
\mathbf{x}^{(5)} &= \texttt{110110} = a_1 + a_2,\\
\mathbf{x}^{(6)} &= \texttt{101101} = a_1 + a_3,\\
\mathbf{x}^{(7)} &= \texttt{011011} = a_2 + a_3,\\
\mathbf{x}^{(8)} &= \texttt{111000} = a_1 + a_2 + a_3.
\end{aligned} \tag{3}$$

Applying the definition (1) along with the form for the codewords in (3), the corresponding binary observables are

$$\begin{aligned}
Q_1(C_6, P) &= P_2 + P_5 + P_6 + P_8\\
Q_2(C_6, P) &= P_3 + P_5 + P_7 + P_8\\
Q_3(C_6, P) &= P_4 + P_6 + P_7 + P_8\\
Q_4(C_6, P) &= P_3 + P_4 + P_5 + P_6\\
Q_5(C_6, P) &= P_2 + P_4 + P_5 + P_7\\
Q_6(C_6, P) &= P_2 + P_3 + P_6 + P_7.
\end{aligned}$$

We illustrate these binary observables in Fig. 1. Now consider no errors on classical outcomes. When we measure $Q_1(C_6, P)$ and obtain the classical outcome 0, the state must be on the support of $I - Q_1(C_6, P) = P_1 + P_3 + P_4 + P_7$, where I denotes the identity operator. If we measure $Q_2(C_6, P)$ and obtain the classical outcome 1, then the state is on the support of $P_1 + P_3 + P_4 + P_7$ and $P_3 + P_5 + P_7 + P_8$. Hence the state is on the support of $P_3 + P_7$. If we measure $Q_3(C_6, P)$ and obtain the classical outcome 1, then the state is on the support of $P_3 + P_7$ and $P_4 + P_6 + P_7 + P_8$. Hence the state is on the support of P_7. Further measurements of the observables $Q_4(C_6, P)$, $Q_5(C_6, P)$, $Q_6(C_6, P)$ give redundant information about where the state is projected on, and we obtain the codeword $\mathbf{x}^{(7)}$ as the classical outcome.

In Fig. 1 we illustrate the measurement of the binary observables $Q_1(C_6, P),\ldots,Q_n(C_6, P)$ when an error afflicts the classical outcome of $Q_5(C_6, P)$.

Implications

Combinatorics. *What is the minimum number of q-observables required to correct t errors on the classical outcome of a projective POVM with M projectors*? We answer this question in the following.

Corollary 2. Let P be a projective POVM with M projectors. Let $n_q(M, d)$ to be the shortest n such that there exists a code of length n and with at least M codewords and distance at least d. Let $n_{q,t,M}$ be the smallest integer such that there exist observables $Q_1,\ldots,Q_n$ consistent with P, even after any t errors occur on the classical outcomes of $Q_1,\ldots,Q_n$. Then $n_{q,t,M} = n_q(M, 2t + 1)$.

Proof. From Theorem 1, we know that the condition for $Q_1,\ldots,Q_n$ to be consistent with P after t errors occur on the classical outcomes is equivalent to the condition that a q-ary classical code C has length n, distance at least $2t + 1$, and has M codewords.

The combinatorics of $n_q(M, d)$ directly relates to the combinatorics of $A_q(n, d)$, where $A_q(n, d)$ is the maximum number of codewords in a q-ary code with Hamming distance d and with codewords having n components. Note that $n_{q,t,q} = 2t + 1$ through the use of a q-ary repetition code. Using results on the combinatorics of $A_q(n, d)$ and $n_q(M, d)$[51], we illustrate the values of $n_{q,t,M}$, in Table 1 for $q = 2$, $t = 1, 2, 3$ and $2 \le M \le 40$.

When the number M of projectors in P is very large, we can bound M in terms of the volume of a q-ary Hamming ball of radius t, which we denote as $V_{q,n}(t) := \sum_{j=0}^{t} \binom{n}{j} (q \vdash 1)^j$. Namely,

$$q^n / V_{q,n}(2t) \le M \le q^n / V_{q,n}(t), \tag{4}$$

where the upper and lower bounds are the Hamming bound and Gilbert-Varshamov bound respectively[52]. Bounds such as Johnson's bound[53] or linear programming bounds for classical codes[54,55] can tighten the upper bound in (4).

Application (quantum error correction). We can describe the recovery channel of any QEC code as a two-stage process[47]. In the first stage, we measure a carefully chosen projective measurement with POVM Π'. Upon measuring Π', we get a classical outcome and a quantum output. The classical output labels the subspace that the quantum output resides in. In the second stage, a unitary operation dependent on the classical outcome brings the quantum output back to the codespace.

The projectors in Π' depend on the QEC code and the set of operators $\mathfrak{K}$ to be corrected. Since the number of correctible spaces of the code is at most $|\mathfrak{K}|$, and at most one projector corresponds to an uncorrectible space, we have $|\Pi'| \le |\mathfrak{K}| + 1$. For a distance p-ary QEC code on m qudits that corrects k errors, we can choose $\mathfrak{K}$ so that $|\mathfrak{K}| = V_{p^2,m}(k)$. From[47], $|\Pi'| \le |\mathfrak{K}|$. Hence, for an m qubit QEC code that corrects a single error (has distance 3), we have $|\Pi'| \le 2 + 3m$.

As an example, consider the optimal non-additive nine-qubit binary QEC code that has codespace of dimension 12, and with distance 3[56]. In this case $|\Pi'| \le 29$. From Table 1, deploying our scheme with 10 binary observables allows the correction of up to one error on the classical outcome of Π'. In contrast, the noiseless decoding of this non-additive nine-qubit code in ref. 56 requires five binary observables, and repeating these measurements thrice to allow the correction of one error necessitates the use of 15 binary observables, which is greater than the 10 binary observables our scheme requires.

As another example, we consider the binomial code[44], which is a bosonic code on a single mode that corrects gain errors, loss errors and phase errors. Here, loss errors, gain errors and phase errors are monomials of a, $a^\dagger$ and $a^\dagger a$ respectively where a denotes the mode's lowering operator. Namely, a binomial code that corrects g_1 gain errors, g_0 loss errors, and k phase errors has as its set of correctible errors $\mathfrak{K} = \{a^j : j = 0, \ldots, g_0\} \cup \{(a^\dagger)^j : j = 0, \ldots, g_1\} \cup \{(a^\dagger a)^j : j = 0, \ldots, k\}$. Clearly, $|\mathfrak{K}| = g_0 + g_1 + k + 1$. Such a binomial code has two parameters, the gap $g = g_0 + g_1 + 1$, and $N = \max\{g_0, g_1, 2k\}$ and encodes one logical qubit, and is defined by the logical codewords in [ref. 44, Eq. (7)]. For such a binomial code where $g_0 = g_1 = k$, we have $|\Pi'| \le 3k + 2$. In Table 2, we present the minimum number of binary observables that are consistent with Π' after the occurrence of up to a single error on the classical outcome of their measurement.

Implementation (dispersive coupling with a bosonic mode). Similarly to refs. 57,58, we can couple our quantum state to n bosonic modes initialized as coherent states $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$ and measure the modes to implement our scheme. Let $\hat{n}_j$ be the number operator on the jth mode, and suppose that $2\pi|\alpha_j|^2 \gg q$. The interaction Hamiltonians

$$W_j = \gamma Q_j \otimes \hat{n}_j \tag{5}$$

model a dispersive coupling between the quantum system and the ancillary bosonic modes.

Now let $|\phi\rangle$ be a state for which $Q_j|\phi\rangle = z_j|\phi\rangle$ for all $j = 1,\ldots,n$. Then $W_j|\phi\rangle|\alpha_j\rangle = |\phi\rangle(\gamma z_j \hat{n}_j|\alpha_j\rangle)$. Hence $e^{-iW\theta}|\phi\rangle|\alpha_j\rangle = |\phi\rangle e^{-i\theta\gamma z_j \hat{n}_j}|\alpha_j\rangle = |\phi\rangle |e^{-i\theta\gamma z_j}\alpha_j\rangle$. With $\theta = 2\pi/(q\gamma)$, the initial phase space distribution of the jth mode with radius $|\alpha_j|^2$ and standard deviation $1/\sqrt{2}$ maps to up to q different equiangular rotations in the complex plane. Using balanced homodyne detection[59] we can measure the quadratures of the output bosonic fields. Because we chose $2\pi|\alpha_j|^2 \gg q$, the distributions for different z_j will be distinguishable. Hence we project onto the eigenspaces of Q_j in a non-destructive way. Repeating the procedure for $j = 1,\ldots,n$ allows us to obtain

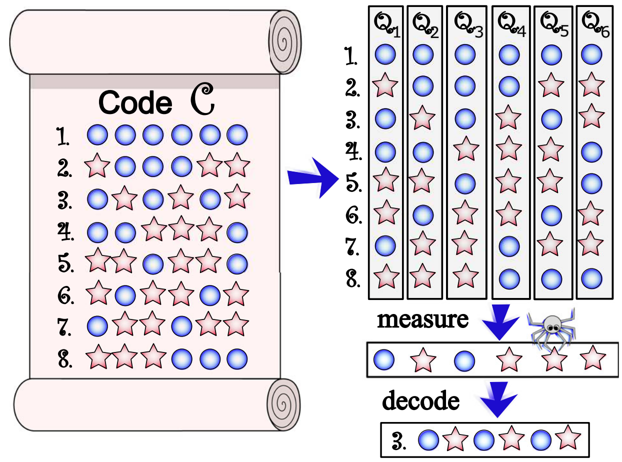

Fig. 1 | Our scheme. Suppose that a projective measurement P projects a quantum state into one of 8 orthogonal subspaces. We label each subspace with a codeword of a classical code C. We depict a shortened Hamming code with 8 codewords. Each codeword is a 6-bit string, and the code corrects one error. Then, we illustrate the commuting q-observables $Q_1,\ldots,Q_6$ as columns on the right side of the diagram. Each q-observable is an appropriate linear combination of projectors in P. Here, $q = 2$, corresponding to binary outcome observables. A spider illustrates an error that occurs on the measurement outcome of Q_5. We can correct this error using the decoder of C, recovering the correct measurement outcome. We conclude that the quantum state has been projected to the third subspace.

Table 1 | Some values of $n_{2,t,M}$

t\M	2	4	6	8	12	16	20	38–40
1	3	6	7	7	8	8	9	10
2	5	9	10	11	11	12	12	14
3	7	12	14	14	15	15	16	18

Table 2 | Some values of the minimum number of binary observables consistent with Π' needed to correct up to one error on the classical outcomes of their measurement n, and furthermore correct k gain, loss and phase errors

k	1	2	3	4	5	6	7	8
$\lvert\Pi'\rvert$	5	8	11	14	17	20	23	26
n	7	7	8	8	9	9	10	10

the classical outcome $(z_1,\ldots,z_n)$ in the noiseless setting. From Theorem 1, we can correct up to t errors on $(z_1,\ldots,z_n)$ using a classical decoder.

Implementation (robust stabilizer code measurements). We consider measuring the stabilizers for an $[[m, k, d]]$ qubit stabilizer code. Measuring $m-k$ stabilizer generators corresponds to performing a projective measurement with 2^{m-k} possible measurement outcomes. Implementing our robust measurement scheme for these measurement outcomes corresponds to the implementation of stabilizer measurements for data syndrome codes[31–37]. For example, the correction of a t measurement errors requires the use of a classical $[n, m-k, 2t+1]$ code. Consider a 17-qubit distance 3 surface code that encodes a single logical qubit[60,61], and where we want to correct a single measurement error ($t = 1$). In this case, the shortest length linear code we use in our protocol is a classical [21, 16, 3] code[51,62]. Hence, performing robust stabilizer code measurements according to the idea of data-syndrome codes involves measuring 21 binary observables.

We can reduce the number of binary observables to measure by performing QEC according to the Knill-Laflamme recovery procedure[47]. Given

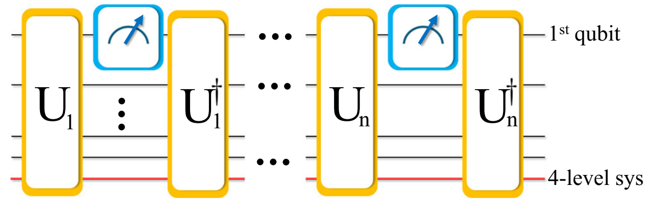

Fig. 2 | Scheme to robustly measure n binary observables on an m-qubit system. For this, we extend the last qubit to a four-level system, apply appropriate unitaries U_j and $U_j^\dagger$, and measure the first qubit in the computational basis.

an $[[m, k, d]]$ qubit stabilizer code, this procedure involves a projective measurement onto $V_{4,m}(t) = \sum_{l=0}^{t}\binom{m}{l}3^l$ correctible spaces and one uncorrectible space with POVM Π' where $|\Pi'| = V_{4,m}(t)+1$. For a 17-qubit distance 3 surface code, we would have $|\Pi'| = V_{4,17}(1)+1 = 53$. Selecting codewords within a [12,8,3] binary linear code[62] allows us to obtain a (12, 53, ≥3) binary code. When used in our protocol, this code gives 12 binary observables. The measurement of these observables allows us to perform a projection according to Π' and the correction of a single measurement error. This example shows that the number of binary observables required for a robust measurement in a QEC scheme can be significantly fewer than those obtained from a data syndrome codes scheme.

We may measure the observables $Q_1,\ldots,Q_n$ according to the scheme depicted in Fig. 2. By construction, the observable Q_j is an operator on m qubits, with $2^m/2 + a_j$ zero eigenvalues and $2^m/2 - a_j$ one eigenvalues where $|a_j| \le 2^{m/2}$. We can write the spectral decomposition of Q_j as

$$Q_j = \sum_{k=1}^{2^{m/2}-a_j} \lambda_0 |\psi_{j,k}\rangle\langle\psi_{j,k}| + \sum_{l=2^{m/2}-a_j+1}^{2^m} \lambda_1 |\psi_{j,l}\rangle\langle\psi_{j,l}| \tag{6}$$

where $\lambda_0 = 0, \lambda_1 = 1$ and $\{|\psi_{j,k}\rangle : k = 1,\ldots,2^m\}$ is an orthonormal basis for the m-qubit Hilbert space. Extending the dimension of the last qubit to a four-level system and considering an orthonormal basis $\{|\psi_{j,k}\rangle : k = 1,\ldots,2^{m+1}\}$ for this extended space, we construct an observable Q_j' which has 2^{m+1} eigenvalues with the following spectral decomposition

$$\begin{aligned} Q_j' = & \sum_{k=1}^{2^{m/2}-a_j} \lambda_0 |\psi_{j,k}\rangle\langle\psi_{j,k}| + \sum_{l=2^{m/2}-a_j+1}^{2^m} \lambda_1 |\psi_{j,l}\rangle\langle\psi_{j,l}| \\ & + \sum_{k=2^m+1}^{2^m+2^{m/2}+a_j} \lambda_0 |\psi_{j,k}\rangle\langle\psi_{j,k}| + \sum_{l=2^m+2^{m/2}+a_j+1}^{2^{m+1}} \lambda_1 |\psi_{j,l}\rangle\langle\psi_{j,l}|. \end{aligned} \tag{7}$$

By construction, Q_j' has (1) the same number of zero and one eigenvalues, and (2) we have $Q_j' Q_j = Q_j$. Let $\mathrm{d}(k)$ be a bijective function that maps a decimal integer $k \in \{1,\ldots,2^m\}$ to a length $m-1$ vector in $\mathbb{Z}_2^{\otimes m-2} \otimes \mathbb{Z}_4$, and let $(y, \mathrm{d}(k))$ denote the length m vector $\mathrm{d}(k)$ in $\mathbb{Z}_2^{\otimes m-1} \otimes \mathbb{Z}_4$ with the $y \in \{0,1\}$ as its first component. Consider a unitary operator on the extended system given by

$$\begin{aligned} U_j = & \sum_{k=1}^{2^{m/2}-a_j} |(0,\mathrm{d}(k))\rangle\langle\psi_{j,k}| \\ & + \sum_{k=1}^{2^{m/2}+a_j} |(0,\mathrm{d}(2^m-k))\rangle\langle\psi_{j,2^m+k}| \\ & + \sum_{k=1}^{2^{m/2}+a_j} |(1,\mathrm{d}(2^m+k))\rangle\langle\psi_{j,2^m-k}| \\ & + \sum_{k=1}^{2^{m/2}-a_j} |(1,\mathrm{d}(2^{m+1}-k))\rangle\langle\psi_{j,2^m+k}|. \end{aligned} \tag{8}$$

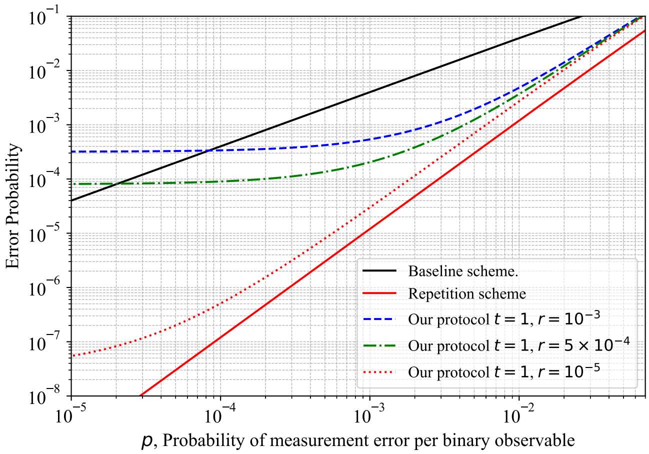

Fig. 3 | We consider a baseline scheme, which is the measurement of four binary observables, each with error probability *p*. Next we consider a repetition scheme, where we measure each of the four binary observables thrice and take a majority vote. Last, we consider our protocol that measures 8 consecutive binary observables and corrects $t = 1$ measurement errors for different values of degradation in measurement accuracy r with each measurement of a binary observable in our protocol.

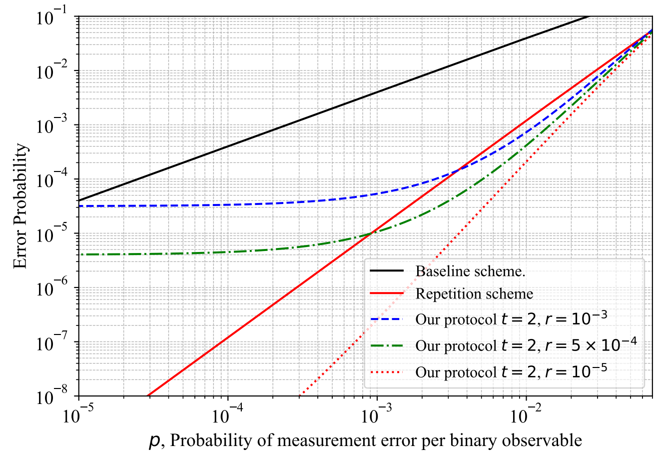

Fig. 4 | Our scheme versus repetition schemes and a baseline scheme. We plot the error probabilities (vertical axis) of schemes considered in Fig. 3 and of our protocol that measures 12 consecutive biinary observables and corrects $t = 2$ measurement errors for different values of degradation in measurement accuracy r with each measurement of a binary observable in our protocol. The probability of measurement error per binary observable here is p (horizontal axis).

Measuring the binary observable Q'_j involves (1) applying the unitary U_j, (2) measuring the first qubit in the computational basis $\{|0\rangle, |1\rangle\}$, and (3) applying the unitary $U_j^\dagger$. When the initial state has no leakage into the extended space, a measurement outcome of $|j\rangle$ on the first qubit corresponds to obtaining an eigenvalue of j for both the observables Q'_j and Q_j. Figure 2 illustrates these steps in the measurements of $Q_1,\ldots,Q_n$ for robust stabilizer code measurements in the limit where measurement errors dominate over gate errors.

Comparison of our protocol with a baseline scheme. Consider the measurement of a POVM with 16 measurement outcomes via the measurement of four binary observables. Here, each measurement's classical outcome has an error with probability p. The overall error probability of four repeated binary observable measurements for this baseline scheme is $1-(1-p)^4$.

We may also consider a simple repetition scheme. For each binary observable, we can repeat its measurement thrice and use a majority vote on the classical measurement outcomes. Using this method, the error probability of each binary observable measurement becomes $p_1 = 3p^2(1-p)+p^3$. Then the overall error probability of four repeated binary observable measurements by measuring a total of 12 binary observables is $1-(1-p_1)^4$.

We construct binary observables $Q_1,\ldots,Q_n$ according to our protocol for the correction of t errors on the 16 classical measurement outcomes. In particular, for $t = 1, 2$ we have $n = 8, 12$ according to Table 1. We can measure these observables according to our scheme in Fig. 2, and model the error probability of the classical measurement outcome of Q_j as $p+(j-1)r$, where $r > 0$ quantifies the degradation in measurement accuracy with each consecutive measurement. Under this model, the probability of no errors in the classical measurement outcomes of $Q_1,\ldots,Q_n$ is $q_0 = (1-p)(1-p-r)\ldots(1-p-(n-1)r)$. The probability of one error in the classical measurement outcomes of $Q_1,\ldots,Q_n$ is $q_1 = q_0(p/(1-p)+(p+r)/(1-p-r) \;+\cdots\; +(p+(n-1)r)/(1-p-(n-1)r))$. We similarly calculate q_2, the probability of two errors in the classical measurement outcomes of $Q_1,\ldots,Q_n$. Then the error probability of our protocol that corrects 1 and 2 errors is $1-q_0-q_1$ and $1-q_0-q_1-q_2$ respectively. We numerically evaluate the performance of the baseline protocol, the repetition protocol and our protocol for different values of r and t in Figs. 3 and 4. Figure 3 which considers $t = 1$ shows that the repetition scheme that uses 12 binary observables slightly outperforms our protocol which uses 8 binary observables. Figure 4 shows that our protocol which uses 12 binary observables can outperform the repetition protocol that uses 12 binary observables when the degradation in consecutive measurements r is sufficiently small and when the per binary observable error rate p is not too small.

Discussions

We proposed a set of commuting q-observables whose measurement is consistent with a given projective measurement, even after some errors corrupt the classical outcomes of the measurement of the observables. Hence, measuring these commuting observables effectively implements a robust projective measurement.

There is potential to study how near-term quantum algorithms that do not rely on QEC can be improved using our scheme in realistic settings. Moreover, it would be interesting to explore the implementation of our scheme with other non-stabilizer codes, such as concatenated cat codes[42,63], rotation-invariant codes[46], permutation-invariant codes[45,64–69], codeword-stabilized codes[70], error-avoiding codes[71–73], and certain codes that lie within the ground space of local Hamiltonians[74].

Methods

Proof. Proof of Theorem 1. Let $\mathbf{z} = (z_1,\ldots,z_n)$ denote the classical outcome if no errors occurred. Then, $\mathbf{z}-\mathbf{y}$ has a Hamming weight of at most t. Furthermore, we have

$$\mathcal{D}(\mathbf{z}) = \mathcal{D}(\mathbf{y}). \tag{9}$$

Case 1: No errors on measurement outcomes. When we measure the observable $Q_j(C, P)$ and obtain the classical outcome z_j, the resultant state must be on the support of the projector

$$P_{j,z_j} = \sum_{k: z_j = x_j^{(k)}} P_k. \tag{10}$$

After measuring the observables $Q_1(C, P),\ldots,Q_n(C, P)$, we obtain the classical outcomes $z_1,\ldots,z_n$. Then the state τ is on the support of

$$\prod_{j=1}^{n} P_{j,z_j} = \sum_{k: z_j = x_j^{(k)}, j=1,\ldots,n} P_k = \sum_{k: \mathbf{z} = \mathbf{x}^{(k)}} P_k. \tag{11}$$

From (11), $\mathbf{z}$ must belong to C. Since there are no repeated codewords in C, there is a unique k for which $\mathbf{z}=\mathbf{x}^{(k)}$. Together with the fact that $t(C)\geq 0$, it follows that

$$\prod_{j=1}^{n} P_{j,z_j} = P_{\mathcal{D}(\mathbf{z})}. \tag{12}$$

The state τ must be on the support of $P_{\mathcal{D}(\mathbf{z})}$, which means that $\tau=\rho_{\mathcal{D}(\mathbf{z})}$.

Case 2: At most $t(C)$ errors on classical outcomes. From case 1, we know that $\tau=\rho_{\mathcal{D}(\mathbf{z})}$. Since $\mathcal{D}(\mathbf{y})=\mathcal{D}(\mathbf{z})$, we have $\tau=\rho_{\mathcal{D}(\mathbf{y})}$.

Data availability
Data is available upon request.

Code availability
Codes are available upon request.

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Y.O., contributed to all parts of this work.

Competing interests
The author declares no competing interests.

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