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5d 2-Chern-Simons Theory and 3d Integrable Field Theories

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Abstract: The 4-dimensional semi-holomorphic Chern-Simons theory of Costello and Yamazaki provides a gauge-theoretic origin for the Lax connection of 2-dimensional integrable field theories. The purpose of this paper is to extend this framework to the setting of 3-dimensional integrable field theories by considering a 5-dimensional semi-holomorphic higher Chern-Simons theory for a higher connection (A, B) on $\mathbb{R}^3 \times \mathbb{C}P^1$. The input data for this theory are the choice of a meromorphic 1-form ω on $\mathbb{C}P^1$ and a strict Lie 2-group with cyclic structure on its underlying Lie 2-algebra. Integrable field theories on \mathbb{R}^3 are constructed by imposing suitable boundary conditions on the connection (A, B) at the 3-dimensional defects located at the poles of ω and choosing certain admissible meromorphic solutions of the bulk equations of motion. The latter provides a natural notion of higher Lax connection for 3-dimensional integrable field theories, including a 2-form component B which can be integrated over Cauchy surfaces to produce conserved charges. As a first application of this approach, we show how to construct a generalization of Ward's $(2 + 1)$ -dimensional integrable chiral model from a suitable choice of data in the 5-dimensional theory.

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1. Introduction

Although there is no universally agreed definition of integrability in the context of field theories, a hallmark of integrability in this infinite dimensional setting is the existence of

infinitely many independent conserved charges. One very general and powerful framework allowing for the systematic construction of these conserved charges is the Lax formalism [Lax], in which the field equations are expressed as the consistency condition for an overdetermined system of linear partial differential equations. An important incarnation of this formalism is in the context of 2-dimensional field theories where a *Lax connection* is defined as an on-shell flat connection that depends meromorphically on an auxiliary Riemann surface C , which is usually taken to be the Riemann sphere $\mathbb{C}P^1$. If a 2-dimensional field theory admits a Lax connection, then its holonomy along curves of constant time, which depends analytically on C , serves as a generating function for infinitely many conserved charges. Unfortunately, a suitable Lax connection for a given 2-dimensional field theory is typically found by clever guesswork, making its origin quite obscure.

In their seminal paper [CY], Costello and Yamazaki gave a very elegant gauge-theoretic origin for the Lax connection in 2-dimensional integrable field theories which is based on a 4-dimensional semi-holomorphic variant of Chern-Simons theory [Nek, Cos1, Cos2, Wit, CWY1, CWY2]. In this approach, integrable field theories on a 2-dimensional manifold Σ , with Lax connection depending meromorphically on a Riemann surface C , arise as specific solutions to this 4-dimensional semi-holomorphic Chern-Simons theory on $\Sigma \times C$. The Lagrangian of the latter is given by $\omega \wedge \text{CS}(A)$, where ω is a fixed meromorphic 1-form on C and $\text{CS}(A)$ is the Chern-Simons 3-form for a \mathfrak{g} -valued 1-form A on $\Sigma \times C$. A 2-dimensional integrable field theory is then determined by the choice of 1-form ω and of boundary conditions imposed on the gauge field A at the surface defects $\Sigma \times \{x\} \subset \Sigma \times C$ located at each pole x of the 1-form ω . Importantly, the Lax connection emerges naturally as a meromorphic solution of the bulk equations of motion for the gauge field A , with pole structure determined by the zeros of ω and satisfying the chosen boundary conditions.

This relatively recent gauge-theoretic approach to 2-dimensional integrable field theories has already established itself as a very powerful tool for constructing new 2-dimensional classical integrable field theories, leading to the discovery of vast new families of examples. This framework also serves as a very efficient organizational tool for navigating the ever expanding zoo of 2-dimensional integrable field theories. See for instance [DLMV, Sch, BL, FSY1, HL, LaV, CSV, FSY2, HTC, FSY3, FSY4, LiV, BP, LW].

In stark contrast to this extremely rich 2-dimensional setting, there are currently very few known examples of integrable field theories in higher dimensions. Perhaps the most well-known examples are the Kadomtsev-Petviashvili (KP) equation in 3 dimensions and the anti-self-dual Yang-Mills (ASDYM) equation in 4 dimensions. Another less well-known example, which will be particularly relevant for us later, is Ward's equation [War2, War3] which describes a non-relativistic modification of the non-integrable 3-dimensional chiral model that was obtained as a reduction of the ASDYM equation. The KP equation admits a more exotic Lax formalism which roughly speaking encodes the third dimension using the language of pseudo-differential operators, see for instance [BBT] for an extensive review. The Lax formalism used to encode the ASDYM equation, on the other hand, is based on a partial flatness condition for a certain connection on \mathbb{R}^4 . The latter lies at the heart of the Penrose-Ward correspondence [War1] which relates solutions of the ASDYM equation on \mathbb{R}^4 to certain holomorphic vector bundles on twistor space $\mathbb{P}\mathbb{T} = \mathbb{C}P^3 \setminus \mathbb{C}P^1$.

Inspired by the Penrose-Ward correspondence, and following a proposal of Costello, it was shown by Bittleston and Skinner in [BS] (see also the related works [Pen, CCHLT]) that various known Lagrangians for the ASDYM equation can be derived starting from

a 6-dimensional holomorphic Chern-Simons theory on twistor space $\mathbb{P}\mathbb{T}$. More specifically, the Lagrangian for the latter is given by $\Omega \wedge \mathfrak{hCS}(\mathcal{A})$, where Ω is a fixed meromorphic $(3, 0)$ -form on $\mathbb{P}\mathbb{T}$ and $\mathfrak{hCS}(\mathcal{A})$ is the holomorphic Chern-Simons $(0, 3)$ -form for a \mathfrak{g} -valued $(0, 1)$ -form \mathcal{A} on $\mathbb{P}\mathbb{T}$. The inevitable presence of poles in Ω leads to a violation of gauge invariance which can nevertheless be restored by imposing suitable boundary conditions on \mathcal{A} at these poles. The ASDYM equation then emerges from the simplest choice of Ω , with a pair of double poles and without zeros, in much the same way as 2-dimensional integrable field theories emerge from 4-dimensional semi-holomorphic Chern-Simons theory. Interestingly, it was also shown in [BS] that applying the same procedure starting from more general $(3, 0)$ -forms Ω leads to other 4-dimensional actions with field equations generalizing the ASDYM equation and which are again manifestly integrable by the Penrose-Ward correspondence.

It is therefore tempting to regard 6-dimensional holomorphic Chern-Simons theory on $\mathbb{P}\mathbb{T}$ as playing an analogous role to 4-dimensional semi-holomorphic Chern-Simons theory for describing 4-dimensional integrable field theories. However, although the ASDYM equation is certainly a 4-dimensional integrable field theory – it is exactly solvable by the ADHM construction [ADHM] which is rooted in the Penrose-Ward correspondence – its Lax formalism lacks certain features one would expect of a higher-dimensional integrable field theory.

Recall that the defining properties of the Lax connection for a 2-dimensional integrable field theory ensure that its holonomy along curves of constant time is both conserved and depends analytically on an auxiliary Riemann surface C . For a field theory on a $(d + 1)$ -dimensional spacetime with $d \geq 1$, a conserved charge should be given by the integral of a d -form over a d -dimensional submanifold representing a constant time slice. One should therefore expect an adequate notion of Lax connection for a $(d + 1)$ -dimensional integrable field theory to involve a d -form. This observation was already made nearly 30 years ago in [AFS] and further explored in subsequent works, see for instance [GMS, ASW]. However, in these works, although the right notion of higher connections was used, a meromorphic dependence on an auxiliary Riemann surface was never considered. For a recent review of existing descriptions of classical integrable field theories in 3 dimensions see [GM].

The goal of this paper is to initiate the exploration of integrable field theories in higher dimensions using higher gauge-theoretic methods. Specifically, we will focus in this paper on the problem of constructing 3-dimensional integrable field theories. Our proposal is to start from a 5-dimensional higher gauge theory variant of the 4-dimensional semi-holomorphic Chern-Simons theory from [CY] which is defined on a product manifold $X = M \times C$ with M representing a 3-dimensional spacetime and C a Riemann surface. The structure group G from the ordinary approach is generalized to a strict Lie 2-group which we describe explicitly in terms of a crossed module of Lie groups (G, H, t, α) , see e.g. [SW, FMP, Wal1, Wal2, BF] and also Sect. 2 for a review. A connection (or gauge field) in this context is given by a pair $(A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h})$ consisting of both a 1-form and a 2-form taking values in the underlying Lie 2-algebra of the structure Lie 2-group. Such higher connections have 1-dimensional and also 2-dimensional holonomies (see e.g. [SW, FMP, Wal2]), which provides additional flexibility for the construction of conserved charges in a 3-dimensional integrable field theory, see also Sect. 2.3. The Lagrangian of our theory takes the form $\omega \wedge \text{CS}(A, B)$, where ω is a fixed meromorphic 1-form on C and $\text{CS}(A, B)$ is the 2-Chern-Simons 4-form for the higher connection (A, B) which is associated with the choice of a suitable non-degenerate invariant pairing on the Lie 2-algebra, see e.g. [JMRSW, Sect. 5.2] and

also Sect. 2 for a review. In analogy to the case of 4-dimensional semi-holomorphic Chern-Simons theory, the action of our theory is not automatically gauge-invariant so that one has to impose suitable boundary conditions for the connection (A, B) on the 3-dimensional defects $M \times \{x\} \subset X$ located at each pole x of ω . Meromorphic solutions to the Euler-Lagrange equations of this action naturally provide flat higher connections (A, B) on M which depend meromorphically on C , i.e. candidates for Lax connections for 3-dimensional integrable field theories on M . In our approach, an integrable field theory is specified by the choice of 1.) a structure Lie 2-group with non-degenerate invariant pairing on its Lie 2-algebra, 2.) a meromorphic 1-form ω on C , and 3.) suitable boundary conditions for (A, B) at the defects located at each pole x of ω .

We will now describe in more detail our results by outlining the content of this paper. In Sect. 2, we provide a brief introduction to higher gauge theory. The reader can find more details in the articles [SW, FMP, Wal1, Wal2] and the review [BF]. This includes a quick recap of crossed modules of Lie groups and Lie algebras (Sect. 2.1), higher gauge fields and their gauge transformations (Sect. 2.2), higher parallel transports (Sect. 2.3) and the construction of the 2-Chern-Simons 4-form (Sect. 2.4). The material presented in this section is rather standard and well-known, probably with the exception of the gauge transformation property of the 2-Chern-Simons 4-form in Proposition 2.7, which was derived earlier in [Zuc] only under additional assumptions on the crossed module. We would like to emphasize that it is important for us to consider also connections (A, B) which *do not* necessarily satisfy the so-called fake-flatness condition $dA + \frac{1}{2}[A, A] - t_*(B) = 0$ since otherwise our action functional would degenerate. This pushes us out of the standard framework for higher connections developed in [SW, FMP, Wal1, Wal2], which as a consequence prevents us from considering also 2-gauge transformations between gauge transformations, see Remark 2.3. (More concisely, this means that our non-fake-flat higher connections only form a groupoid and not a 2-groupoid.) There are recent developments towards a theory of non-fake-flat higher connections through so-called adjusted connections, see e.g. [RSW, KS, Tel], but the additional adjustment data seem to be incompatible with the type of boundary conditions we would like to impose on our connections, see Sects. 3.3 and 3.4. These inconveniences associated with non-fake-flat connections disappear once we go on-shell since solutions of our 5-dimensional semi-holomorphic 2-Chern-Simons theory are fully flat connections, and hence in particular fake-flat. This means that the construction of 2-dimensional holonomies from [SW, FMP, Wal1, Wal2], which are needed in our approach to generate conserved charges, is directly applicable in our context once we go on-shell.

In Sect. 3, we study in detail our 5-dimensional semi-holomorphic 2-Chern-Simons theory in the special case where the meromorphic 1-form ω has only simple poles. In Sect. 3.1 we spell out concretely the action functional for this theory and in Subsection 3.2 we analyze the properties of this action under gauge transformations. It is shown that there are violations to gauge invariance which are localized at the 3-dimensional defects $M \times \{x\} \subset X$ located at the poles x of ω , see Proposition 3.1. In Sect. 3.3 we restore gauge invariance by imposing suitable boundary conditions which are determined by the choice of an isotropic crossed submodule $(G^\diamond, H^\diamond, t^z, \alpha^z) \subseteq (G^z, H^z, t^z, \alpha^z)$ of the crossed module associated with the defect. We will show in Subsection 3.4 that these boundary conditions admit an equivalent (homotopical) interpretation in terms of edge mode fields living on the defect, which take the form of pairs (k, κ) with $k \in C^\infty(M, G^z)$ a group-valued function and $\kappa \in \Omega^1(M, \mathfrak{h}^z)$ a Lie algebra valued 1-form on the 3-dimensional spacetime M . While the group-valued edge modes k are familiar from 4-dimensional semi-holomorphic Chern-Simons theory, see e.g. [BSV], the 1-form edge

modes κ are a novel feature of our higher gauge theoretic approach to 3-dimensional integrable field theories. Our approach results in an explicit action functional (3.22) for the edge modes on M which gives them their dynamics. In Sect. 3.5 we derive the Euler-Lagrange equations of our extended bulk+defect action functional (3.22) and find solutions which describe flat connections on M that are meromorphic on C , as required for a Lax connection.

In Sect. 4, we generalize the results from Sect. 3 to the case where ω has poles of arbitrary order. Using the concept of regularized integrals from [LZ] and [BSV], this is easily achievable and does not pose any additional challenges. The main new feature of the higher-order pole case is that the defect crossed module $(G^{\hat{z}}, H^{\hat{z}}, t^{\hat{z}}, \alpha^{\hat{z}})$ consists of products of jet groups associated with (G, H, t, α) .

The aim of Sect. 5 is to apply our approach to construct explicit examples of 3-dimensional integrable field theories. The key concept which enables these constructions is that of admissible connections from [BSV], which we generalize to our present context of higher gauge theory. In Sect. 5.1 we introduce a suitable concept of maximality for isotropic crossed submodules under which it can be expected that the Lax connection (A, B) can be expressed uniquely in terms of the edge mode fields (k, κ) . In Sect. 5.2 we present a toy-model to illustrate our proposed construction of 3-dimensional integrable field theories by focusing on one of the simplest choices of ω given by a meromorphic 1-form with a single zero, a simple pole and a double pole. The resulting 3-dimensional field theory in this case is given by Chern-Simons theory. This is integrable in the sense that the Chern-Simons equations of motion, i.e. flatness of the connection, arise as the flatness of the associated Lax connection, but the Lax connection in this toy-example is rather trivial from the perspective integrable field theory since it is constant in the complex coordinate $z \in C$.

A more interesting example is presented in Sect. 5.3, where the meromorphic 1-form ω is taken to have four simple zeros and three double poles. By choosing a suitable isotropic crossed submodule, we derive equations of motion for the edge mode fields and observe that they are related to Ward's equation, also known as the integrable chiral model [War2, War3]. In particular, our approach provides a direct derivation of Ward's consistency assumption that the distinguished vector in Ward's equation is normalized and spacelike. The Ward equation was originally obtained from a particular choice of gauge in the Yang-Mills-Higgs system on $\mathbb{R}^{2,1}$, which itself arises as a symmetry reduction of the ASDYM equation on $\mathbb{R}^{2,2}$ by the action of a one-parameter group of non-null translations. It was shown in [BS] that the same symmetry reduction applied to 6-dimensional holomorphic Chern-Simons theory on twistor space $\mathbb{P}\mathbb{T}$ leads to a partially holomorphic 5-dimensional variant of Chern-Simons theory on the quotient $\mathbb{P}\mathbb{N}$ of $\mathbb{P}\mathbb{T}$ by this group of translations. Moreover, it was mentioned in [BS] that Ward's model can be naturally obtained from this 5-dimensional partially holomorphic Chern-Simons theory. We stress, however, that there is no immediate relationship between our 5-dimensional 2-Chern-Simons theory, which is based on higher gauge fields $A \in \Omega^1(X, \mathfrak{g})$ and $B \in \Omega^2(X, \mathfrak{h})$, and the 5-dimensional theory considered in [BS], which is based on an ordinary gauge field $A' \in \Omega^1(\mathbb{P}\mathbb{N}, \mathfrak{g})$. In fact, the model we construct in Sect. 5.3 is a considerable generalization of the Ward model which involves three 1-form fields valued in \mathfrak{h} as well as the G -valued field of the original Ward model. We expect that the presence of \mathfrak{h} -valued 1-form fields is a general feature of 3-dimensional integrable field theories constructed via our approach, which is ultimately connected to the existence of a 2-form component $B \in \Omega^2(X, \mathfrak{h})$ of the higher Lax connection.

2. Preliminaries on Higher Gauge Theory

In this section we recall some basic aspects of higher gauge theory, i.e. the theory of connections on higher-categorical analogues of principal bundles in which the structure group is generalized from a Lie group to a Lie 2-group. We consider only strict Lie 2-groups, which we describe in terms of crossed modules of Lie groups, and globally trivial principal 2-bundles. More details about higher gauge theory can be found in the articles [SW,FMP,Wal1,Wal2] and the review [BF].

2.1. Crossed modules of Lie groups and Lie algebras. A convenient and computationally efficient model for strict Lie 2-groups is given by the following

Definition 2.1. A *crossed module of Lie groups* is a tuple (G, H, t, α) consisting of two Lie groups G and H , a Lie group homomorphism $t : H \rightarrow G$, and a smooth action $\alpha : G \times H \rightarrow H$ of G on H in terms of Lie group automorphisms, such that

$$t(\alpha(g, h)) = g t(h) g^{-1}, \tag{2.1a}$$

$$\alpha(t(h), h') = h h' h^{-1}, \tag{2.1b}$$

for all $g \in G$ and $h, h' \in H$.

Associated to each crossed module of Lie groups (G, H, t, α) is a *crossed module of Lie algebras* $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$ which models the Lie 2-algebra of the corresponding strict Lie 2-group. Here \mathfrak{g} and \mathfrak{h} denote the Lie algebras of, respectively, G and H . The Lie algebra homomorphism

$$t_* := dt|_{1_H} : \mathfrak{h} \longrightarrow \mathfrak{g} \tag{2.2}$$

is the differential of the Lie group homomorphism t at the identity $1_H \in H$ and the Lie algebra homomorphism

$$\alpha_* := d\alpha|_{1_G} : \mathfrak{g} \longrightarrow \text{Der}(\mathfrak{h}) \tag{2.3a}$$

is the differential of the adjunct $\alpha : G \rightarrow \text{Aut}(H)$ of α at $1_G \in G$. One can equivalently regard α_* as a linear map

$$\alpha_* : \mathfrak{g} \otimes \mathfrak{h} \longrightarrow \mathfrak{h}, \tag{2.3b}$$

for which the Lie derivation property reads as

$$\alpha_*(x, [y, y']) = [\alpha_*(x, y), y'] + [y, \alpha_*(x, y')], \tag{2.4}$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. The two properties (2.1) differentiate to

$$t_*(\alpha_*(x, y)) = [x, t_*(y)], \tag{2.5a}$$

$$\alpha_*(t_*(y), y') = [y, y'], \tag{2.5b}$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$.

There are two additional differentials of the smooth action $\alpha : G \times H \rightarrow H$ which play an important role in higher gauge theory. With a slight abuse of notation, we shall denote all these differentials by the same symbol α_* . First, for every $g \in G$, one has a

Lie group homomorphism $\alpha_g := \alpha(g, \cdot) : H \rightarrow H$ of which one can take the differential $d\alpha_g|_{1_H} : \mathfrak{h} \rightarrow \mathfrak{h}$ at $1_H \in H$. Allowing $g \in G$ to vary, one obtains the map

$$\alpha_* := d\alpha(\cdot)|_{1_H} : G \times \mathfrak{h} \longrightarrow \mathfrak{h}. \tag{2.6}$$

Second, for every $h \in H$, one has a smooth map $\tilde{\alpha}_h := \alpha(\cdot, h) h^{-1} : G \rightarrow H$ which preserves the identity elements, i.e. $\tilde{\alpha}_h(1_G) = 1_H$, but not necessarily the group multiplications. Taking the differential at $1_G \in G$ defines a linear map $d\tilde{\alpha}_h|_{1_G} : \mathfrak{g} \rightarrow \mathfrak{h}$. Allowing $h \in H$ to vary, one obtains the map

$$\alpha_* := d\tilde{\alpha}(\cdot)|_{1_G} : \mathfrak{g} \times H \longrightarrow \mathfrak{h}. \tag{2.7}$$

Note that the three maps in (2.3), (2.6) and (2.7) can be distinguished from their source, so denoting all of them by the same symbol will likely cause no confusion.

2.2. Higher gauge fields and gauge transformations. Given any manifold X and crossed module of Lie groups (G, H, t, α) , we will always consider the corresponding trivial principal 2-bundle over X . The following definition is from [Wal2, Appendix A.1].

Definition 2.2. Let X be a manifold and (G, H, t, α) crossed module of Lie groups.

- (a) A *connection* is a pair (A, B) consisting of a \mathfrak{g} -valued 1-form $A \in \Omega^1(X, \mathfrak{g})$ and an \mathfrak{h} -valued 2-form $B \in \Omega^2(X, \mathfrak{h})$.
- (b) A *gauge transformation* is a pair (g, γ) consisting of a G -valued smooth function $g \in C^\infty(X, G)$ and an \mathfrak{h} -valued 1-form $\gamma \in \Omega^1(X, \mathfrak{h})$. It transforms a connection (A, B) to the connection $^{(g, \gamma)}(A, B)$ specified by

$$^{(g, \gamma)}A := g A g^{-1} - dg g^{-1} - t_*(\gamma), \tag{2.8a}$$

$$^{(g, \gamma)}B := \alpha_*(g, B) - F(\gamma) - \alpha_*(^{(g, \gamma)}A, \gamma), \tag{2.8b}$$

where $F(\gamma) := d\gamma + \frac{1}{2}[\gamma, \gamma] \in \Omega^2(X, \mathfrak{h})$.

- (c) We denote by $\text{Con}_{(G, H, t, \alpha)}(X)$ the groupoid whose objects are all connections (A, B) and whose morphisms $(g, \gamma) : (A, B) \rightarrow ^{(g, \gamma)}(A, B)$ are all gauge transformations between connections. The composition of two morphisms $(g_1, \gamma_1) : (A, B) \rightarrow ^{(g_1, \gamma_1)}(A, B)$ and $(g_2, \gamma_2) : ^{(g_1, \gamma_1)}(A, B) \rightarrow ^{(g_2, \gamma_2)}(A, B)$ is defined by

$$(g_2, \gamma_2) (g_1, \gamma_1) := (g_2 g_1, \gamma_2 + \alpha_*(g_2, \gamma_1)), \tag{2.9}$$

for all $g_1, g_2 \in C^\infty(X, G)$ and $\gamma_1, \gamma_2 \in \Omega^1(X, \mathfrak{h})$, and the identity morphisms are $(1_G, 0) : (A, B) \rightarrow (A, B)$. The inverse of a morphism $(g, \gamma) : (A, B) \rightarrow ^{(g, \gamma)}(A, B)$ is given explicitly by

$$(g, \gamma)^{-1} = (g^{-1}, -\alpha_*(g^{-1}, \gamma)). \tag{2.10}$$

Remark 2.3. For the purpose of our paper, it is important that we do *not* demand the fake-flatness condition $\text{fcurv}(A, B) := F(A) - t_*(B) := dA + \frac{1}{2}[A, A] - t_*(B) = 0$ because this would degenerate the action functionals studied in the later sections. The theory of non-fake-flat connections is unfortunately not yet well understood, especially when it comes to their 2-categorical aspects. In the fake-flat case, there also exists a concept of 2-gauge transformations $a : (g, \gamma) \Rightarrow {}^a(g, \gamma)$ between gauge transformations, which are

parametrized by H -valued smooth functions $a \in C^\infty(X, H)$ and transform according to

$${}^a g := t(a) g, \tag{2.11a}$$

$${}^a \gamma := a \gamma a^{-1} - da a^{-1} - \alpha_*({}^{(g,\gamma)} A, a). \tag{2.11b}$$

These 2-gauge transformations are only well-defined in the non-fake-flat case provided that the rather unnatural and highly restrictive constraint $\alpha_*(\text{fcurv}(A, B), a) = 0$ holds true. Because of these issues, we will neglect such 2-gauge transformations in our constructions and consider only the ordinary groupoid $\text{Con}_{(G,H,t,\alpha)}(X)$ of connections and gauge transformations instead of a potential 2-groupoid refinement involving also 2-gauge transformations.

2.3. Higher parallel transport. Given any connection (A, B) as in Definition 2.2, one can define parallel transports along 1-dimensional paths $\gamma : [0, 1] \rightarrow X, s \mapsto \gamma(s)$ and also along 2-dimensional surfaces $\Gamma : [0, 1]^2 \rightarrow X, (s, u) \mapsto \Gamma(s, u)$. The parallel transport along a 1-dimensional path γ is given by evaluating the path-ordered exponential

$$\mathfrak{g}_\gamma(s) = \mathcal{P} \exp \left(\int_0^s A \left(\frac{d}{ds'} \gamma(s') \right) ds' \right) \in G \tag{2.12a}$$

at $s = 1$, where $\frac{d}{ds'} \gamma(s') \in T_{\gamma(s')} X$ denotes the tangent vector. Recall that the path-ordered exponential is defined as the solution of the differential equation

$$\frac{d}{ds} \mathfrak{g}_\gamma(s) = \mathfrak{g}_\gamma(s) A \left(\frac{d}{ds} \gamma(s) \right) \tag{2.12b}$$

for the initial condition $\mathfrak{g}_\gamma(0) = 1_G$. The parallel transport along a 2-dimensional surface Γ is given by evaluating the surface-ordered exponential

$$\mathfrak{h}_\Gamma(s, u) = \mathcal{S} \exp \left(\int_0^u \int_0^s B \left(\frac{\partial}{\partial s'} \Gamma(s', u'), \frac{\partial}{\partial u'} \Gamma(s', u') \right) ds' du' \right) \in H \tag{2.13a}$$

at $(s, u) = (1, 1)$. The latter is defined as the solution of the differential equation

$$\frac{\partial}{\partial u} \mathfrak{h}_\Gamma(s, u) = \mathfrak{h}_\Gamma(s, u) \int_0^s \alpha_* \left(\mathfrak{g}_{\gamma^0}(u) \mathfrak{g}_{\gamma_u}(s'), B \left(\frac{\partial}{\partial s'} \Gamma(s', u), \frac{\partial}{\partial u} \Gamma(s', u) \right) \right) ds' \tag{2.13b}$$

for the initial condition $\mathfrak{h}_\Gamma(s, 0) = 1_H$, for all $s \in [0, 1]$, where we used the notation $\Gamma(s, u) = \gamma^s(u) = \gamma_u(s)$.

In [SW,FMP,Wal2], it is shown that in the case where the connection (A, B) is fake-flat, i.e. the fake-curvature $F(A) - t_*(B) = 0$ vanishes, the 1- and 2-dimensional parallel transports assemble into a 2-functor from the path 2-groupoid of the manifold X to the classifying space of the structure Lie 2-group (G, H, t, α) . This means that parallel transports satisfy suitable 1- and 2-dimensional composition properties, which allow one to compute them also along non-trivial shapes such as 2-spheres or 2-tori.

In our context of 3-dimensional integrable field theories, as discussed later in Sect. 5, the 2-dimensional parallel transport of a fully flat connection (A, B) on a 3-dimensional spacetime $X = \mathbb{R}^3$ gives rise to conserved charges which are localized on fixed-time

surfaces $\{t\} \times \mathbb{R}^2 \subset X$. Let us explain this crucial point more explicitly. In order to ensure that the relevant integrals exist, we assume that the connection (A, B) has spacelike compact support on X , i.e. its restriction $(A, B)|_t$ to each fixed-time surface $\{t\} \times \mathbb{R}^2 \subset X$ is compactly supported. Denoting by $\Gamma_t : [0, 1]^2 \rightarrow \{t\} \times \mathbb{R}^2 \subset X$ any fixed-time 2-dimensional surface which contains the support of $(A, B)|_t$, we consider the 2-dimensional parallel transport

$$h_{\Gamma_t} = \mathcal{S} \exp \left(\int_0^1 \int_0^1 B \left(\frac{\partial}{\partial s'} \Gamma_t(s', u'), \frac{\partial}{\partial u'} \Gamma_t(s', u') \right) ds' du' \right) \in H. \tag{2.14}$$

In the case where (A, B) is fully flat, i.e. the fake-curvature $F(A) - t_*(B) = 0$ and the 3-form curvature $dB + \alpha_*(A, B) = 0$ both vanish, this parallel transport is time-independent, i.e.

$$h_{\Gamma_t} = h_{\Gamma_{t'}} \tag{2.15}$$

for all $t, t' \in \mathbb{R}$. (This statement follows from the same argument which proves thin homotopy invariance of 2-dimensional parallel transports, see e.g. [SW, Appendix A.3] or [FMP, Theorem 2.30].) This implies that $h_{\Gamma_t} \in H$ is a conserved charge. In the case of a 3-dimensional integrable field theory (see Sect. 5), the fully flat higher Lax connection (A, B) further depends meromorphically on an auxiliary Riemann surface C , and hence so does the conserved charge h_{Γ_t} . Laurent expansion then leads to an infinite tower of conserved charges which are localized on the fixed-time surfaces $\{t\} \times \mathbb{R}^2 \subset X$.

Remark 2.4. We would like to point out a potential relationship between the higher parallel transports from this subsection and the concept of generalized (higher-form) symmetries from [GKSW, BCH]. The conserved charges (2.14) constructed from 2-dimensional parallel transports of a fully flat connection (A, B) are localized in codimension 1 of spacetime $X = \mathbb{R}^3$, so they correspond to 0-form symmetries according the terminology of generalized symmetries. Considering loops in a fixed-time surface, i.e. 1-dimensional paths $\gamma : [0, 1] \rightarrow \{t\} \times \mathbb{R}^2 \subset X$ such that $\gamma(0) = \gamma(1)$, we can consider also 1-dimensional parallel transports $\mathfrak{g}_\gamma \in G$ which are localized in codimension 2 of spacetime. It is important to stress that the fake-flatness condition $F(A) - t_*(B) = 0$ does in general *not* imply that these codimension 2 charges are conserved under deformations of the loop, so they are in general *not* 1-form symmetries in the usual sense of generalized symmetries. However, the violation of the conservation of $\mathfrak{g}_\gamma \in G$ under deformations of the loop is controlled by 2-dimensional parallel transports since the curvature $F(A) = t_*(B)$ of the 1-form A is determined via fake-flatness in terms of the 2-form B . This seems to indicate that we are dealing with a system of 0-form and 1-form symmetries which have a non-trivial interplay with each other. It would be interesting to make the latter point more precise by identifying the algebraic structures underlying this interplay. We would like to note that for a crossed module with $t_* = 0$, fake-flatness implies that $F(A) = 0$, hence the codimension 2 charges $\mathfrak{g}_\gamma \in G$ decouple from the codimension 1 charges $h_{\Gamma_t} \in H$ and, as a consequence, are strictly conserved.

2.4. The 2-Chern-Simons 4-form. Let X be a manifold of dimension $n \geq 4$ and (G, H, t, α) a crossed module of Lie groups. Suppose that we are given a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{h} \longrightarrow \mathbb{C} \tag{2.16a}$$

on the underlying crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$ which is G -invariant, i.e.

$$\langle g x g^{-1}, \alpha_*(g, y) \rangle = \langle x, y \rangle, \tag{2.16b}$$

for all $g \in G, x \in \mathfrak{g}$ and $y \in \mathfrak{h}$, and satisfies the symmetry property

$$\langle t_*(y), y' \rangle = \langle t_*(y'), y \rangle, \tag{2.16c}$$

for all $y, y' \in \mathfrak{h}$. Note that the G -invariance property (2.16b) differentiates to the \mathfrak{g} -invariance property

$$\langle [x, x'], y \rangle + \langle x', \alpha_*(x, y) \rangle = 0, \tag{2.17}$$

for all $x, x' \in \mathfrak{g}$ and $y \in \mathfrak{h}$.

We will now construct from these data a 4-form on X which is a higher-dimensional generalization of the usual Chern-Simons 3-form in ordinary gauge theory. As a first step, let us observe that there exists a double complex

$$\begin{array}{ccccccc} \Omega^{(0,0)}(X, \mathfrak{g}) & \xrightarrow{d} & \Omega^{(0,1)}(X, \mathfrak{g}) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{(0,n)}(X, \mathfrak{g}) \\ \uparrow t_* & & \uparrow t_* & & & & \uparrow t_* \\ \Omega^{(-1,0)}(X, \mathfrak{h}) & \xrightarrow{d} & \Omega^{(-1,1)}(X, \mathfrak{h}) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{(-1,n)}(X, \mathfrak{h}) \end{array} \tag{2.18}$$

whose horizontal differential is the de Rham differential d and vertical differential is the linear map t_* from the crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$. The bi-degrees are as indicated in the parentheses. Totalizing this double complex leads to the cochain complex

$$L := \left(\begin{array}{ccccccc} 0 & \xrightarrow{d_L} & \Omega^0(X, \mathfrak{g}) & \xrightarrow{d_L} & \dots & \xrightarrow{d_L} & \Omega^{n-1}(X, \mathfrak{g}) & \xrightarrow{d_L} & \Omega^n(X, \mathfrak{g}) \\ \oplus & & \oplus & & & & \oplus & & \oplus \\ \Omega^0(X, \mathfrak{h}) & & \Omega^1(X, \mathfrak{h}) & & \dots & & \Omega^n(X, \mathfrak{h}) & & 0 \end{array} \right) \tag{2.19}$$

concentrated in degrees $\{-1, 0, \dots, n\}$. More explicitly, a cochain of degree p in L is a pair $\ell = \omega \oplus \eta \in L^p = \Omega^p(X, \mathfrak{g}) \oplus \Omega^{p+1}(X, \mathfrak{h})$ of Lie algebra-valued differential forms and the differential d_L reads explicitly as

$$d_L \ell := d_L(\omega \oplus \eta) := (d\omega + (-1)^p t_*(\eta)) \oplus d\eta. \tag{2.20}$$

Note that the degree 1 cochains $\mathcal{A} = A \oplus B \in L^1 = \Omega^1(X, \mathfrak{g}) \oplus \Omega^2(X, \mathfrak{h})$ in this complex are precisely the connections on the trivial principal 2-bundle associated with (G, H, t, α) from Definition 2.2.

Using the additional structures from the crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$, one can endow the cochain complex L with the structure of a dg-Lie algebra. The Lie bracket reads explicitly as

$$[\ell, \ell']_L := [\omega, \omega'] \oplus \left(\alpha_*(\omega, \eta') - (-1)^{pp'} \alpha_*(\omega', \eta) \right), \tag{2.21}$$

for all $\ell = \omega \oplus \eta \in L^p$ and $\ell' = \omega' \oplus \eta' \in L^{p'}$. Finally, using also the non-degenerate pairing (2.16), one obtains a differential form-valued cyclic structure

$$\langle \cdot, \cdot \rangle_L : L \otimes L \longrightarrow \Omega^{\bullet+1}(X) \tag{2.22}$$

of degree 1 on the dg-Lie algebra L . This reads explicitly as

$$\langle \ell, \ell' \rangle_L := \langle \omega \oplus \eta, \omega' \oplus \eta' \rangle_L := \langle \omega, \eta' \rangle + (-1)^{pp'} \langle \omega', \eta \rangle, \tag{2.23}$$

for all $\ell = \omega \oplus \eta \in L^p$ and $\ell' = \omega' \oplus \eta' \in L^{p'}$.

The following definition is motivated by the construction of higher Chern-Simons actions in terms of Maurer-Cartan theory, see e.g. [JMRSW, Sect. 5.2].

Definition 2.5. The 2-Chern-Simons 4-form associated to a connection $\mathcal{A} = A \oplus B \in L^1 = \Omega^1(X, \mathfrak{g}) \oplus \Omega^2(X, \mathfrak{h})$ is defined as

$$\text{CS}(\mathcal{A}) := \text{CS}(A, B) := \langle \mathcal{A}, \frac{1}{2}d_L \mathcal{A} + \frac{1}{3!}[\mathcal{A}, \mathcal{A}]_L \rangle_L \in \Omega^4(X). \tag{2.24}$$

Lemma 2.6. The 2-Chern-Simons 4-form (2.24) can be expanded in components as

$$\text{CS}(A, B) = \langle F(A) - \frac{1}{2}t_*(B), B \rangle - \frac{1}{2}d \langle A, B \rangle, \tag{2.25}$$

where we recall that $F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(X, \mathfrak{g})$.

Proof. We have

$$\begin{aligned} \text{CS}(A, B) &= \langle A \oplus B, \frac{1}{2}((dA - t_*(B)) \oplus dB) + \frac{1}{3!}([A, A] \oplus 2\alpha_*(A, B)) \rangle_L \\ &= \frac{1}{2} \langle A, dB \rangle + \frac{1}{3} \langle A, \alpha_*(A, B) \rangle + \frac{1}{2} \langle dA - t_*(B), B \rangle + \frac{1}{3!} \langle [A, A], B \rangle \\ &= \frac{1}{2} \langle A, dB \rangle + \frac{1}{2} \langle dA + [A, A] - t_*(B), B \rangle \\ &= \langle dA + \frac{1}{2}[A, A] - \frac{1}{2}t_*(B), B \rangle - \frac{1}{2}d \langle A, B \rangle, \end{aligned} \tag{2.26}$$

where in the third step we used the \mathfrak{g} -invariance property (2.17) and the fact that A is a 1-form to write $\langle A, \alpha_*(A, B) \rangle = \langle [A, A], B \rangle$, and in last step we used $\langle A, dB \rangle = \langle dA, B \rangle - d \langle A, B \rangle$. □

We will require later the following result about the transformation behavior of the 2-Chern-Simons 4-form under the gauge transformations from Definition 2.2.

Proposition 2.7. The 2-Chern-Simons 4-form (2.25) transforms under gauge transformations $(g, \gamma) : (A, B) \rightarrow {}^{(g,\gamma)}(A, B)$, for $g \in C^\infty(X, G)$ and $\gamma \in \Omega^1(X, \mathfrak{h})$, as

$$\begin{aligned} \text{CS}({}^{(g,\gamma)}(A, B)) &= \text{CS}(A, B) - \frac{1}{2}d \left(\langle gAg^{-1}, F(\gamma) \rangle + \langle t_*(\gamma), d\gamma + \frac{1}{3}[\gamma, \gamma] \rangle \right. \\ &\quad \left. - \langle dg g^{-1} + t_*(\gamma), \alpha_*(g, B) + F(\gamma) \rangle \right), \end{aligned} \tag{2.27}$$

where we recall that $F(\gamma) = d\gamma + \frac{1}{2}[\gamma, \gamma] \in \Omega^2(X, \mathfrak{h})$.

Proof. This is a lengthy but straightforward computation, so we just highlight the main steps. For the first term of the 2-Chern-Simons 4-form (2.25), one uses the explicit form of the gauge transformations (2.8a) and (2.8b) to show that $F^{(g,\gamma)}A - \frac{1}{2}t_*(^{(g,\gamma)}B) = g(F(A) - \frac{1}{2}t_*(B))g^{-1} - \frac{1}{2}t_*(F(\gamma) + \alpha_*(^{(g,\gamma)}A, \gamma))$. From this it then follows that

$$\begin{aligned} \langle F^{(g,\gamma)}A - \frac{1}{2}t_*(^{(g,\gamma)}B), ^{(g,\gamma)}B \rangle &= \langle F(A) - \frac{1}{2}t_*(B), B \rangle \\ &\quad - \langle F^{(g,\gamma)}A + \frac{1}{2}t_*(F(\gamma) + \alpha_*(^{(g,\gamma)}A, \gamma)), F(\gamma) + \alpha_*(^{(g,\gamma)}A, \gamma) \rangle. \end{aligned} \tag{2.28}$$

The second term on the right-hand side is found to be exact by making repeated use of the properties (2.5) and (2.17), and the Jacobi identities for \mathfrak{g} and \mathfrak{h} . Explicitly, one finds

$$\begin{aligned} \langle F^{(g,\gamma)}A - \frac{1}{2}t_*(^{(g,\gamma)}B), ^{(g,\gamma)}B \rangle &= \langle F(A) - \frac{1}{2}t_*(B), B \rangle \\ &\quad - \frac{1}{2}d\left(\langle ^{(g,\gamma)}A, 2F(\gamma) + \alpha_*(^{(g,\gamma)}A, \gamma) \rangle + \langle t_*(\gamma), d\gamma + \frac{1}{3}[\gamma, \gamma] \rangle\right). \end{aligned} \tag{2.29}$$

For the second term of the 2-Chern-Simons 4-form (2.25), one finds

$$\begin{aligned} -\frac{1}{2}d\langle ^{(g,\gamma)}A, ^{(g,\gamma)}B \rangle &= -\frac{1}{2}d\langle A, B \rangle \\ &\quad + \frac{1}{2}d\left(\langle dg g^{-1} + t_*(\gamma), \alpha_*(g, B) \rangle + \langle ^{(g,\gamma)}A, F(\gamma) + \alpha_*(^{(g,\gamma)}A, \gamma) \rangle\right). \end{aligned} \tag{2.30}$$

The result now follows by combining the above transformation properties of the first and second term in the 2-Chern-Simons 4-form (2.25). \square

3. 5d2-Chern-Simons Theory with Simple Poles

In this section we define and analyze a 5-dimensional generalization of 4-dimensional semi-holomorphic Chern-Simons theory [Cos2, CWY1, CWY2, CY]. Our notations and conventions follow [BSV]. In order to simplify our presentation, we consider in this section first the special case where ω is a meromorphic 1-form on the Riemann sphere $\mathbb{C}P^1$ which has only simple poles and postpone the more involved case of higher-order poles to Sect. 4. We denote by $z \subset \mathbb{C}P^1$ the set of poles and by $\zeta \subset \mathbb{C}P^1$ the set of zeros of ω . We assume that ω has at least one zero, i.e. $|\zeta| \geq 1$.

The field theory we consider is defined on the 5-dimensional manifold

$$X := M \times C, \tag{3.1}$$

where $M = \mathbb{R}^3$ is the 3-dimensional Cartesian space, which we interpret as spacetime, and $C := \mathbb{C}P^1 \setminus \zeta$ is the Riemann sphere with all zeros of ω removed. We choose a global coordinate $z : C \rightarrow \mathbb{C}$ on the factor C , which exists because it is assumed that $|\zeta| \geq 1$.

3.1. Action. Let (G, H, t, α) be a crossed module of Lie groups endowed with a non-degenerate invariant pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathbb{C}$ as in (2.16). We assume throughout the whole paper that both G and H are connected and simply-connected. Using the 2-Chern-Simons 4-form from Definition 2.5, we define the action

$$S_\omega(A, B) := \frac{i}{2\pi} \int_X \omega \wedge CS(A, B) \tag{3.2}$$

on the set of connections $(A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h})$, where with a slight abuse of notation we denote the pullback of the meromorphic 1-form ω along the projection $X = M \times C \rightarrow C$ by the same symbol.

Note that the components of A and B with a leg along dz do not contribute to the action (3.2) because $\omega = \varphi dz$ is a meromorphic 1-form. To remove these non-dynamical fields from the theory, we consider in what follows connections which are modeled on the quotient

$$\overline{\Omega^\bullet}(X) := \Omega^\bullet(X) / (dz) \tag{3.3a}$$

of the de Rham calculus on X , endowed with the differential

$$\bar{d} := d_M + \bar{\partial} \tag{3.3b}$$

given by the sum of the differential d_M along the factor M and the Dolbeault differential $\bar{\partial}$ along the factor C of the product manifold $X = M \times C$. We denote by

$$\overline{\text{Con}}_{(G,H,t,\omega)}(X) \tag{3.4}$$

the analogue of the groupoid from Definition 2.2 where connections and gauge transformations are modeled on the quotient de Rham calculus $(\overline{\Omega^\bullet}(X), \bar{d})$.

We observe that the integrand of the action (3.2) is singular at the 3-dimensional defect

$$D := M \times z = \bigsqcup_{x \in z} (M \times \{x\}) \subset X \tag{3.5}$$

which is localized at the poles $x \in z$ of ω . In the present case of simple poles, it follows by the same argument as in [BSV, Lemma 2.1] that these singularities are locally integrable near each component $M \times \{x\} \subset X$ of the defect, hence the action (3.2) is well-defined. (In the case of higher-order poles, the integral requires a regularization, see Sect. 4 for the details.)

3.2. Gauge transformations. In analogy to the 4-dimensional case, the action (3.2) of 5-dimensional semi-holomorphic 2-Chern-Simons theory is not invariant under arbitrary gauge transformations $(g, \gamma) : (A, B) \rightarrow {}^{(g,\gamma)}(A, B)$ as in Definition 2.2 (with the differential d replaced by \bar{d} from the quotient de Rham calculus (3.3)), where $g \in C^\infty(X, G)$ and $\gamma \in \overline{\Omega}^1(X, \mathfrak{h})$. Indeed, using Proposition 2.7, we compute

$$\begin{aligned} S_\omega({}^{(g,\gamma)}(A, B)) &= S_\omega(A, B) - \frac{i}{4\pi} \int_X \omega \wedge \bar{d} \left(\langle gAg^{-1}, \bar{F}(\gamma) \rangle + \langle t_*(\gamma), \bar{d}\gamma + \frac{1}{3}[\gamma, \gamma] \right. \\ &\quad \left. - \langle \bar{d}g g^{-1} + t_*(\gamma), \alpha_*(g, B) + \bar{F}(\gamma) \rangle \right), \end{aligned} \tag{3.6}$$

where $\bar{F}(\gamma) := \bar{d}\gamma + \frac{1}{2}[\gamma, \gamma] \in \overline{\Omega}^2(X, \mathfrak{h})$ denotes the curvature with respect to the quotient de Rham calculus (3.3), and note that the second term does in general not vanish. The violation of gauge invariance is due to the poles of ω and hence it is localized at the defect (3.5). To state this observation precisely, we have to introduce some more notations and terminology. Let us denote the inclusions of the defect into X by

$$\iota_x : M \times \{x\} \hookrightarrow X, \quad \iota := \bigsqcup_{x \in z} \iota_x : D \hookrightarrow X. \tag{3.7}$$

Pulling back functions or differential forms along these inclusions defines maps

$$t^* : C^\infty(X, N) \longrightarrow C^\infty(D, N) \cong C^\infty(M, N^z), \tag{3.8a}$$

$$t^* : \overline{\Omega}^q(X, V) \longrightarrow \Omega^q(D, V) \cong \Omega^q(M, V^z), \tag{3.8b}$$

where N is any smooth manifold (e.g. one of the Lie groups G or H) and V is any vector space (e.g. one of the Lie algebras \mathfrak{g} or \mathfrak{h}). Here $(\cdot)^z := \prod_{x \in z} (\cdot)$ denotes the product over all poles. We extend the pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathbb{C}$ to these products by setting

$$\langle \cdot, \cdot \rangle_\omega : \mathfrak{g}^z \otimes \mathfrak{h}^z \longrightarrow \mathbb{C}, \quad \mathcal{X} \otimes \mathcal{Y} \longmapsto \langle \mathcal{X}, \mathcal{Y} \rangle_\omega := \sum_{x \in z} k^x \langle \mathcal{X}^x, \mathcal{Y}^x \rangle, \tag{3.9a}$$

where $\mathcal{X} = (\mathcal{X}^x)_{x \in z} \in \mathfrak{g}^z, \mathcal{Y} = (\mathcal{Y}^x)_{x \in z} \in \mathfrak{h}^z$ and the coefficients

$$k^x := \text{Res}_x(\omega) \in \mathbb{C} \tag{3.9b}$$

are the residues of ω at its poles $x \in z$.

Proposition 3.1. *Under a gauge transformation $(g, \gamma) : (A, B) \rightarrow {}^{(g, \gamma)}(A, B)$, with $g \in C^\infty(X, G)$ and $\gamma \in \overline{\Omega}^1(X, \mathfrak{h})$, the action (3.2) transforms as*

$$\begin{aligned} S_\omega({}^{(g, \gamma)}(A, B)) &= S_\omega(A, B) + \frac{1}{2} \int_M \left(\left\langle t^*(g) t^*(A) t^*(g)^{-1}, F_M(t^*(\gamma)) \right\rangle_\omega \right. \\ &\quad + \left\langle t^*(t_*(\gamma)), d_M t^*(\gamma) + \frac{1}{3} [t^*(\gamma), t^*(\gamma)] \right\rangle_\omega \\ &\quad \left. - \left\langle d_M t^*(g) t^*(g)^{-1} + t^*(t_*(\gamma)), t^*(\alpha_*(g, B)) + F_M(t^*(\gamma)) \right\rangle_\omega \right), \end{aligned} \tag{3.10}$$

where $F_M(t^*(\gamma)) := d_M t^*(\gamma) + \frac{1}{2} [t^*(\gamma), t^*(\gamma)] \in \Omega^2(M, \mathfrak{h}^z)$ denotes the curvature and d_M the de Rham differential on the 3-manifold M .

Proof. This is a direct consequence of (3.6) and the Cauchy-Pompeiu integral formula. See [BSV, Lemma 2.2] for more details. \square

3.3. Boundary conditions at the defect. The result in Proposition 3.1 suggests that one has to impose suitable boundary conditions at the defect $D \subset X$ in order to obtain a gauge-invariant action. Note that such boundary conditions must be imposed on both the connections (A, B) and their gauge transformations $(g, \gamma) : (A, B) \rightarrow {}^{(g, \gamma)}(A, B)$. We shall focus on a simple class of boundary conditions which are determined by the choice of a crossed submodule of $(G^z, H^z, t^z, \alpha^z)$ that is isotropic with respect to the pairing $\langle \cdot, \cdot \rangle_\omega$ in (3.9). More precisely, this is given by the choice of two Lie subgroups

$$G^\diamond \subseteq G^z = \prod_{x \in z} G, \quad H^\diamond \subseteq H^z = \prod_{x \in z} H, \tag{3.11a}$$

such that the two structure maps $t^z = \prod_{x \in z} t$ and $\alpha^z = \prod_{x \in z} \alpha$ restrict to

$$t^z : H^\diamond \longrightarrow G^\diamond, \quad \alpha^z : G^\diamond \times H^\diamond \longrightarrow H^\diamond. \tag{3.11b}$$

Denoting the associated crossed module of Lie algebras by $(\mathfrak{g}^\diamond, \mathfrak{h}^\diamond, \iota_*^z, \alpha_*^z)$, the requirement of isotropy means that the restricted pairing vanishes, i.e.

$$\langle \cdot, \cdot \rangle_\omega \Big|_{\mathfrak{g}^\diamond \otimes \mathfrak{h}^\diamond} = 0. \tag{3.11c}$$

We will refer to such a crossed submodule $(G^\diamond, H^\diamond, t^z, \alpha^z)$ of $(G^z, H^z, t^z, \alpha^z)$ as being isotropic.

Definition 3.2. The *groupoid of boundary conditioned fields* for the isotropic crossed submodule (3.11) is defined as the subgroupoid $\mathcal{F}^\diamond \subseteq \overline{\text{Con}}_{(G,H,t,\alpha)}(X)$ of the groupoid of connections and gauge transformations from (3.4) which is specified by the following data:

- An object in \mathcal{F}^\diamond is a connection $(A, B) \in \overline{\Omega}^1(X, \mathfrak{g}) \times \overline{\Omega}^2(X, \mathfrak{h})$ which satisfies the boundary condition $\iota^*(A, B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond)$.
- A morphism in \mathcal{F}^\diamond is a gauge transformation $(g, \gamma) : (A, B) \rightarrow (g \cdot \gamma)(A, B)$, with $g \in C^\infty(X, G)$ and $\gamma \in \overline{\Omega}^1(X, \mathfrak{h})$, which satisfies the boundary condition $\iota^*(g, \gamma) \in C^\infty(M, G^\diamond) \times \Omega^1(M, \mathfrak{h}^\diamond)$.

Proposition 3.3. For any choice of isotropic crossed submodule $(G^\diamond, H^\diamond, t^z, \alpha^z)$ as in (3.11), the action (3.2) restricts to a gauge-invariant function $S_\omega : \mathcal{F}^\diamond \rightarrow \mathbb{C}$ on the subgroupoid $\mathcal{F}^\diamond \subseteq \overline{\text{Con}}_{(G,H,t,\alpha)}(X)$ of boundary conditioned fields.

Proof. This follows directly from Proposition 3.1 and the isotropy condition (3.11c) with respect to $\langle \cdot, \cdot \rangle_\omega$. □

3.4. Edge modes. The groupoid of boundary conditioned fields from Definition 3.2 arises as the *strict pullback*

$$\begin{array}{ccc}
 \mathcal{F}^\diamond & \dashrightarrow & \overline{\text{Con}}_{(G,H,t,\alpha)}(X) \\
 \downarrow & & \downarrow \iota^* \\
 \text{Con}_{(G^\diamond, H^\diamond, t^z, \alpha^z)}(M) & \hookrightarrow & \text{Con}_{(G^z, H^z, t^z, \alpha^z)}(M)
 \end{array} \tag{3.12}$$

in the category of groupoids. Since groupoids naturally form a 2-category, it is more appropriate to consider a *homotopy pullback* instead of a strict one. We shall now spell out an explicit model for the homotopy pullback of the diagram (3.12) and explain how this admits an interpretation in terms of edge modes living on the defect $D \subset X$.

Proposition 3.4. A model for the homotopy pullback $\mathcal{F}_{\text{ho}}^\diamond$ of the diagram (3.12) is given by the groupoid which is specified by the following data:

- An object in $\mathcal{F}_{\text{ho}}^\diamond$ is a tuple $((A, B), (k, \kappa))$ consisting of a connection $(A, B) \in \overline{\Omega}^1(X, \mathfrak{g}) \times \overline{\Omega}^2(X, \mathfrak{h})$ on X and a gauge transformation $(k, \kappa) \in C^\infty(M, G^z) \times \Omega^1(M, \mathfrak{h}^z)$ on M , such that

$${}^{(k,\kappa)}\iota^*(A, B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond). \tag{3.13}$$

• A morphism in $\mathcal{F}_{\text{ho}}^\diamond$ is a tuple

$$((g, \gamma), (g^\diamond, \gamma^\diamond)) : ((A, B), (k, \kappa)) \longrightarrow ((g, \gamma)(A, B), (g^\diamond, \gamma^\diamond)(k, \kappa) \iota^*(g, \gamma)^{-1}) \tag{3.14}$$

consisting of a gauge transformation $(g, \gamma) \in C^\infty(X, G) \times \overline{\Omega}^1(X, \mathfrak{h})$ on X and a gauge transformation $(g^\diamond, \gamma^\diamond) \in C^\infty(M, G^\diamond) \times \Omega^1(M, \mathfrak{h}^\diamond)$ on M . Using the composition and inversion formulas for gauge transformations from Definition 2.2, the second component of the target of this morphism reads explicitly as

$$(g^\diamond, \gamma^\diamond)(k, \kappa) \iota^*(g, \gamma)^{-1} = \left(g^\diamond k \iota^*(g)^{-1}, \gamma^\diamond + \alpha_*^\sharp(g^\diamond, \kappa) - \alpha_*^\sharp(g^\diamond k \iota^*(g)^{-1}, \iota^*(\gamma)) \right). \tag{3.15}$$

Proof. This follows immediately by applying the usual homotopy pullback construction for groupoids, see e.g. [MMST, Appendix A], to the present example. \square

Remark 3.5. This result is analogous to the case of 4-dimensional semi-holomorphic Chern-Simons theory from [BSV, Sect. 4.2]. The gauge transformation component $(k, \kappa) \in C^\infty(M, G^z) \times \Omega^1(M, \mathfrak{h}^z)$ of an object $((A, B), (k, \kappa))$ in $\mathcal{F}_{\text{ho}}^\diamond$ can be interpreted as an edge mode living on the defect whose role is to witness, through its induced transformation, the boundary condition from Definition 3.2 for the connection $(A, B) \in \overline{\Omega}^1(X, \mathfrak{g}) \times \overline{\Omega}^2(X, \mathfrak{h})$.

Theorem 3.6. *The canonical functor*

$$\begin{aligned} \Phi : \mathcal{F}^\diamond &\longrightarrow \mathcal{F}_{\text{ho}}^\diamond, \\ (A, B) &\longmapsto ((A, B), (1_{G^z}, 0)), \\ (g, \gamma) &\longmapsto ((g, \gamma), \iota^*(g, \gamma)), \end{aligned} \tag{3.16}$$

from the strict to the homotopy pullback is an equivalence of groupoids.

Proof. We have to verify that Φ is essentially surjective on objects and fully faithful on morphisms.

Essential surjectivity: Consider any object $((A, B), (k, \kappa))$ in $\mathcal{F}_{\text{ho}}^\diamond$. Then the gauge transformation $(k, \kappa) \in C^\infty(M, G^z) \times \Omega^1(M, \mathfrak{h}^z) \cong C^\infty(D, G) \times \Omega^1(D, \mathfrak{h})$ can be extended along the defect inclusion $D \subset X$, which yields an element $(\tilde{k}, \tilde{\kappa}) \in C^\infty(X, G) \times \overline{\Omega}^1(X, \mathfrak{h})$ satisfying $\iota^*(\tilde{k}, \tilde{\kappa}) = (k, \kappa)$. This defines a gauge transformation $(\tilde{k}, \tilde{\kappa}) : (A, B) \rightarrow (\tilde{k}, \tilde{\kappa})(A, B)$ in $\overline{\text{Con}}_{(G, H, \iota, \alpha)}(X)$ such that $\iota^*(\tilde{k}, \tilde{\kappa})(A, B) = (k, \kappa) \iota^*(A, B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond)$, i.e. $(\tilde{k}, \tilde{\kappa})(A, B)$ defines an object in the strict pullback \mathcal{F}^\diamond . Essential surjectivity on objects is then proven by considering the morphism

$$((\tilde{k}, \tilde{\kappa}), (1_{G^\diamond}, 0)) : ((A, B), (k, \kappa)) \longrightarrow ((\tilde{k}, \tilde{\kappa})(A, B), (1_{G^z}, 0)) \tag{3.17}$$

in $\mathcal{F}_{\text{ho}}^\diamond$.

Fully faithfulness: Consider two objects (A, B) and (A', B') in the strict pullback \mathcal{F}^\diamond . Then a morphism

$$((g, \gamma), (g^\diamond, \gamma^\diamond)) : ((A, B), (1_{G^z}, 0)) \longrightarrow ((A', B'), (1_{G^z}, 0)) \tag{3.18}$$

between their images in $\mathcal{F}_{\text{ho}}^\diamond$ exists if and only if

$$(1_{G^z}, 0) = (g^\diamond, \gamma^\diamond)(1_{G^z}, 0)\iota^*(g, \gamma)^{-1} = (g^\diamond, \gamma^\diamond)\iota^*(g, \gamma)^{-1}, \tag{3.19}$$

i.e. $(g^\diamond, \gamma^\diamond) = \iota^*(g, \gamma)$. This is equivalent to the statement that the morphism lies in the image of the functor Φ , which proves fully faithfulness on morphisms. \square

The gauge-invariant action $S_\omega : \mathcal{F}^\diamond \rightarrow \mathbb{C}$ from Proposition 3.3 can be transferred along the equivalence $\Phi : \mathcal{F}^\diamond \rightarrow \mathcal{F}_{\text{ho}}^\diamond$ from Theorem 3.6 to a gauge-invariant action $S_\omega^{\text{ext}} : \mathcal{F}_{\text{ho}}^\diamond \rightarrow \mathbb{C}$ on the groupoid $\mathcal{F}_{\text{ho}}^\diamond$ in which the edge modes are manifestly included. This equivalent point of view will be useful later to identify an integrable field theory on the 3-dimensional manifold M . The value of the extended action $S_\omega^{\text{ext}}((A, B), (k, \kappa))$ on an object in $\mathcal{F}_{\text{ho}}^\diamond$ is determined as follows: In the proof of essential surjectivity in Theorem 3.6, we have constructed a gauge transformation $(\tilde{k}, \tilde{\kappa})$ from $((A, B), (k, \kappa))$ to the image under $\Phi : \mathcal{F}^\diamond \rightarrow \mathcal{F}_{\text{ho}}^\diamond$ of an object $(\tilde{k}, \tilde{\kappa})(A, B)$ in \mathcal{F}^\diamond . We then define

$$\begin{aligned} S_\omega^{\text{ext}}((A, B), (k, \kappa)) &:= S_\omega((\tilde{k}, \tilde{\kappa})(A, B)) = \frac{i}{2\pi} \int_X \omega \wedge \langle \overline{F}(A) - \frac{1}{2}t_*(B), B \rangle \\ &+ \frac{1}{2} \int_M \left(\left\langle \iota^*(A), \alpha_*^z(k, \iota^*(B)) + F_M(\kappa) \right\rangle_\omega + \left\langle t_*^z(\kappa), d_M \kappa + \frac{1}{3}[\kappa, \kappa] \right\rangle_\omega \right), \end{aligned} \tag{3.20}$$

where we used Proposition 3.1 and the explicit expression (2.25) for the 2-Chern-Simons 4-form. Recalling also (2.8), which in our present context reads as

$${}^{(k, \kappa)}\iota^*(A) = k \iota^*(A) k^{-1} - d_M k k^{-1} - t_*^z(\kappa) \in \Omega^1(M, \mathfrak{g}^\diamond), \tag{3.21a}$$

$${}^{(k, \kappa)}\iota^*(B) = \alpha_*^z(k, \iota^*(B)) - F_M(\kappa) - \alpha_*^z({}^{(k, \kappa)}\iota^*(A), \kappa) \in \Omega^2(M, \mathfrak{h}^\diamond), \tag{3.21b}$$

we can rewrite the extended action further by solving the second identity for $\alpha_*^z(k, \iota^*(B))$. Using also that $\left\langle \iota^*(A), {}^{(k, \kappa)}\iota^*(B) \right\rangle_\omega = 0$ as a consequence of isotropy, we then obtain

$$\begin{aligned} S_\omega^{\text{ext}}((A, B), (k, \kappa)) &= \frac{i}{2\pi} \int_X \omega \wedge \langle \overline{F}(A) - \frac{1}{2}t_*(B), B \rangle \\ &+ \frac{1}{2} \int_M \left(\left\langle \iota^*(A), \alpha_*^z({}^{(k, \kappa)}\iota^*(A), \kappa) + 2F_M(\kappa) \right\rangle_\omega + \left\langle t_*^z(\kappa), d_M \kappa + \frac{1}{3}[\kappa, \kappa] \right\rangle_\omega \right). \end{aligned} \tag{3.22}$$

Note that the defect action on M is a function of the edge mode $(k, \kappa) \in C^\infty(M, G^z) \times \Omega^1(M, \mathfrak{h}^z)$ and the pullback $\iota^*(A, B) \in \Omega^1(M, \mathfrak{g}^z) \times \Omega^2(M, \mathfrak{h}^z)$ of the connection.

3.5. Equations of motion. We shall now derive the Euler-Lagrange equations of the extended action (3.22), which will yield bulk equations of motion on X and also defect equations of motion on the 3-dimensional manifold M . In anticipation of our applications to integrable field theories, we shall restrict as in [BSV, Sect. 5] the extended action to the full subgroupoid

$$\mathcal{F}_{\text{ho}}^{\diamond, 0} \subseteq \mathcal{F}_{\text{ho}}^\diamond \tag{3.23}$$

whose objects $((A, B), (k, \kappa))$ are such that the connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ does not have legs along $d\bar{z}$. (Recall from (3.3) that the $d\bar{z}$ legs are already quotiented out since they do not contribute to the action. Hence, connections in $\mathcal{F}_{\text{ho}}^{\circ,0}$ have only legs along the factor M of the product manifold $X = M \times C$.) Ideally, one would like to interpret this restriction as a gauge choice, but it is currently not clear to us if the inclusion (3.23) defines an equivalence of groupoids, see also [BSV, Remark 5.1] for a similar issue in the 4-dimensional case. Restricting the extended action (3.22) to the full subgroupoid (3.23) leads to a further simplification

$$\begin{aligned}
 S_{\omega}^{\text{ext}}((A, B), (k, \kappa)) &= \frac{i}{2\pi} \int_X \omega \wedge (\bar{\partial}A, B) \\
 &+ \frac{1}{2} \int_M \left(\left\langle \left\langle (k, \kappa) \iota^*(A), \alpha_*^z((k, \kappa) \iota^*(A), \kappa) + 2F_M(\kappa) \right\rangle \right\rangle_{\omega} + \left\langle \left\langle t_*^z(\kappa), d_M \kappa + \frac{1}{3}[\kappa, \kappa] \right\rangle \right\rangle_{\omega} \right),
 \end{aligned}
 \tag{3.24}$$

where we recall that $\bar{\partial}$ is the Dolbeault differential on the factor C of the product manifold $X = M \times C$.

Varying the action (3.24) is slightly non-trivial because the individual components of an object $((A, B), (k, \kappa))$ in $\mathcal{F}_{\text{ho}}^{\circ,0}$ are constrained by the condition that $(k, \kappa) \iota^*(A, B) \in \Omega^1(M, \mathfrak{g}^{\diamond}) \times \Omega^2(M, \mathfrak{h}^{\diamond})$, see Proposition 3.4. A suitable way to parametrize such variations is by $(A^{\epsilon}, B^{\epsilon}) := (A + \epsilon \mathfrak{a}, B + \epsilon \mathfrak{b})$ and $(k^{\epsilon}, \kappa^{\epsilon}) := (e^{\epsilon \chi} k, \kappa + \epsilon \rho)$, for ϵ a small parameter. Using (3.21) and performing a 1st-order Taylor expansion in ϵ , we obtain the induced variations

$$\delta^{(k, \kappa)} \iota^*(A) = [\chi, k \iota^*(A) k^{-1} - d_M k k^{-1}] - d_M \chi + k \iota^*(\mathfrak{a}) k^{-1} - t_*^z(\rho), \tag{3.25a}$$

$$\begin{aligned}
 \delta^{(k, \kappa)} \iota^*(B) &= \alpha_*^z(\chi, \alpha_*^z(k, \iota^*(B))) + \alpha_*^z(k, \iota^*(\mathfrak{b})) \\
 &- d_M \rho - [\kappa, \rho] - \alpha_*^z(\delta^{(k, \kappa)} \iota^*(A), \kappa) - \alpha_*^z((k, \kappa) \iota^*(A), \rho)
 \end{aligned}
 \tag{3.25b}$$

for the combinations of fields entering the defect action. Note that the constraint on the variations is then fulfilled to 1st-order in ϵ if and only if $\delta^{(k, \kappa)} \iota^*(A) \in \Omega^1(M, \mathfrak{g}^{\diamond})$ and $\delta^{(k, \kappa)} \iota^*(B) \in \Omega^2(M, \mathfrak{h}^{\diamond})$.

One can now work out the variation of the extended action (3.24) and one finds after a simplification using (3.21) and (3.25) that

$$\begin{aligned}
 \delta S_{\omega}^{\text{ext}} &= \frac{i}{2\pi} \int_X \omega \wedge \left(\langle \mathfrak{a}, \bar{\partial}B \rangle + \langle \bar{\partial}A, \mathfrak{b} \rangle \right) \\
 &+ \int_M \left(\left\langle \left\langle k^{-1} \chi k, \iota^*(d_M B + \alpha_*(A, B)) \right\rangle \right\rangle_{\omega} + \left\langle \left\langle \iota^*(F_M(A) - t_*(B)), \alpha_*^z(k^{-1}, \rho) \right\rangle \right\rangle_{\omega} \right),
 \end{aligned}
 \tag{3.26}$$

where we use that under the pullback ι^* only the d_M component of the differential $\bar{d} = d_M + \bar{\partial}$ on the quotient de Rham calculus $\overline{\Omega}^{\bullet}(X)$ from (3.3) survives. Let us also note that the term $-\left\langle \left\langle \delta^{(k, \kappa)} \iota^*(A), (k, \kappa) \iota^*(B) \right\rangle \right\rangle_{\omega}$ on M which one finds in this calculation vanishes manifestly as a consequence of isotropy. We summarize this result in the following

Proposition 3.7. *The Euler-Lagrange equations of the extended action S_ω^{ext} in (3.24) on the full subgropoid (3.23) are given by the bulk equations of motion on X*

$$\omega \wedge \bar{\partial} B = 0, \quad \omega \wedge \bar{\partial} A = 0, \tag{3.27}$$

and by the defect equations of motion on M

$$t^*(d_M B + \alpha_*(A, B)) = 0, \quad t^*(F_M(A) - t_*(B)) = 0. \tag{3.28}$$

Proof. To derive the bulk equations, one uses (3.26) for all variations $(A^\epsilon, B^\epsilon) := (A + \epsilon \mathbf{a}, B + \epsilon \mathbf{b})$ and $(k^\epsilon, \kappa^\epsilon) := (k, \kappa)$ with $\mathbf{a} \in \bar{\Omega}^{1,0}(X, \mathfrak{g})$ and $\mathbf{b} \in \bar{\Omega}^{2,0}(X, \mathfrak{h})$ supported on the complement $X \setminus D$ of the defect $D \subset X$. Note that such variations manifestly satisfy the constraints $\delta^{((k,\kappa))} t^*(A) \in \Omega^1(M, \mathfrak{g}^\diamond)$ and $\delta^{((k,\kappa))} t^*(B) \in \Omega^2(M, \mathfrak{h}^\diamond)$ on the induced variations.

To derive the defect equations, we consider any variation $(k^\epsilon, \kappa^\epsilon) := (e^\epsilon k, \kappa + \epsilon \rho)$ of the edge modes with $\chi \in C^\infty(M, \mathfrak{g}^z)$ and $\rho \in \Omega^1(M, \mathfrak{h}^z)$. From the explicit form of the induced variations (3.25), one finds that the conditions $\delta^{((k,\kappa))} t^*(A) = 0$ and $\delta^{((k,\kappa))} t^*(B) = 0$, which in particular imply the constraint, can be solved uniquely for $t^*(\mathbf{a}) \in \Omega^1(M, \mathfrak{g}^z) \cong \Omega^1(D, \mathfrak{g})$ and $t^*(\mathbf{b}) \in \Omega^2(M, \mathfrak{h}^z) \cong \Omega^2(D, \mathfrak{h})$. Choosing any extensions $\mathbf{a} \in \bar{\Omega}^{1,0}(X, \mathfrak{g})$ and $\mathbf{b} \in \bar{\Omega}^{2,0}(X, \mathfrak{h})$ of these forms along the defect inclusion $D \subset X$ gives a compatible variation of the connection $(A^\epsilon, B^\epsilon) := (A + \epsilon \mathbf{a}, B + \epsilon \mathbf{b})$. The result then follows from (3.26). \square

4. 5d 2-Chern-Simons Theory with Higher Poles

All constructions and results from Sect. 3 can be generalized to the case where the meromorphic 1-form ω on $\mathbb{C}P^1$ has higher-order poles by using the concept of *regularized integrals* from [LZ] and [BSV]. We shall now briefly state the relevant results and refer the reader to [BSV, Section 3] for more details and complete proofs in the similar case of 4-dimensional semi-holomorphic Chern-Simons theory.

As in the previous section, we consider the 5-dimensional manifold $X = M \times C$ with $M = \mathbb{R}^3$ the 3-dimensional Cartesian space, interpreted as spacetime, and $C = \mathbb{C}P^1 \setminus \zeta$ the Riemann sphere with all zeros $\zeta \subset \mathbb{C}P^1$ of ω removed. We can and will choose a global coordinate $z : C \rightarrow \mathbb{C}$ on C since by our hypotheses $|\zeta| \geq 1$. Using this coordinate, we can write the meromorphic 1-form as

$$\omega = \sum_{x \in \mathbf{z}} \sum_{p=0}^{n_x-1} \frac{k_p^x}{(z-x)^{p+1}} dz, \tag{4.1}$$

where $n_x \in \mathbb{Z}_{\geq 1}$ denotes the order of the pole $x \in \mathbf{z}$ and $k_p^x \in \mathbb{C}$ are constants.

Let us denote by $n := \max(n_x)_{x \in \mathbf{z}}$ the maximal order among all poles of ω . We introduce the corresponding Weil algebra

$$T^n := \mathbb{C}[\epsilon]/(\epsilon^n) \tag{4.2}$$

of order n and define the holomorphic $(n - 1)$ -jet prolongation

$$j_X^* : \bar{\Omega}^q(X) \longrightarrow \bar{\Omega}^q(X) \otimes T^n, \quad \eta \longmapsto \sum_{p=0}^{n-1} \frac{1}{p!} \partial_z^p \eta \otimes \epsilon^p, \tag{4.3}$$

where we recall that $(\overline{\Omega^\bullet(X)}, \bar{d})$ denotes the quotient de Rham calculus from (3.3). The generalization of the action (3.2) to the case of higher-order poles is then given by the regularized integral

$$S_\omega(A, B) := \frac{i}{2\pi} - \int_X \omega \wedge \text{CS}(A, B) := \frac{i}{2\pi} \int_X (\omega \wedge j_X^* \text{CS}(A, B))_{\text{reg}}, \tag{4.4a}$$

where

$$(\omega \wedge j_X^* \text{CS}(A, B))_{\text{reg}} := \sum_{x \in \mathbf{z}} \sum_{p=0}^{n_x-1} \frac{k_p^x}{z-x} dz \wedge \frac{1}{p!} \partial_z^p \text{CS}(A, B). \tag{4.4b}$$

The key property of this regularization construction (see [BSV, Lemma 3.2]) is that the 5-form $(\omega \wedge j_X^* \text{CS}(A, B))_{\text{reg}}$ is locally integrable near all poles $x \in \mathbf{z}$ of ω and that it agrees with the ordinary wedge product $\omega \wedge \text{CS}(A, B) = (\omega \wedge j_X^* \text{CS}(A, B))_{\text{reg}} + d\psi$ up to an exact term which is singular and non-integrable near the poles of ω . Note that the action coincides with (3.2) in the case where ω has only simple poles, i.e. $n_x = 1$ for all $x \in \mathbf{z}$.

The generalization of the defect (3.5) to the higher-order pole case is given by the formal manifold

$$\widehat{D} := \bigsqcup_{x \in \mathbf{z}} (M \times \ell T_x^{n_x}), \tag{4.5}$$

where $\ell T_x^{n_x}$ denotes the locus (in the sense of synthetic geometry [Koc]) of the Weil algebra $T_x^{n_x} = \mathbb{C}[\epsilon_x]/(\epsilon_x^{n_x})$ of order n_x given by the order of the pole $x \in \mathbf{z}$. One should interpret $\ell T_x^{n_x}$ as an infinitesimally thickened point. The formal defect (4.5) embeds

$$j : \widehat{D} \hookrightarrow X \tag{4.6}$$

into X , which induces pullback maps

$$j^* : C^\infty(X, N) \longrightarrow C^\infty(\widehat{D}, N) \cong C^\infty(M, N^{\hat{\mathbf{z}}}) \tag{4.7a}$$

$$j^* : \overline{\Omega}^q(X, V) \longrightarrow \Omega^q(\widehat{D}, V) \cong \Omega^q(M, V^{\hat{\mathbf{z}}}) \tag{4.7b}$$

generalizing (3.8), where

$$N^{\hat{\mathbf{z}}} := \prod_{x \in \mathbf{z}} C^\infty(\ell T_x^{n_x}, N) \tag{4.8}$$

denotes the product of $(n_x - 1)$ -jet manifolds over the manifold N and

$$V^{\hat{\mathbf{z}}} := \prod_{x \in \mathbf{z}} (V \otimes T_x^{n_x}). \tag{4.9}$$

Explicitly, the pullback maps (4.7) are pullbacks along the inclusions $\iota_x : M \times \{x\} \hookrightarrow X$ of holomorphic jet prolongations, i.e.

$$j^*(\cdot) = \left(\sum_{p=0}^{n_x-1} \frac{1}{p!} \iota_x^* (\partial_z^p(\cdot)) \otimes \epsilon_x^p \right)_{x \in \mathbf{z}}. \tag{4.10}$$

The generalization of the pairing (3.9) to the higher-order pole case is given by

$$\langle\langle \cdot, \cdot \rangle\rangle_\omega : \mathfrak{g}^{\hat{z}} \otimes \mathfrak{h}^{\hat{z}} \longrightarrow \mathbb{C}, \quad \mathcal{X} \otimes \mathcal{Y} \longmapsto \langle\langle \mathcal{X}, \mathcal{Y} \rangle\rangle_\omega := \sum_{x \in \mathbb{Z}} \sum_{p, q=0}^{n_x-1} k_{p+q}^x \langle \mathcal{X}_p^x, \mathcal{Y}_q^x \rangle, \tag{4.11}$$

where $\mathcal{X} = (\sum_{p=0}^{n_x-1} \mathcal{X}_p^x \otimes \epsilon_x^p)_{x \in \mathbb{Z}} \in \mathfrak{g}^{\hat{z}}, \mathcal{Y} = (\sum_{q=0}^{n_x-1} \mathcal{Y}_q^x \otimes \epsilon_x^q)_{x \in \mathbb{Z}} \in \mathfrak{h}^{\hat{z}}$ and the coefficients k_{p+q}^x are determined from ω , see (4.1). (Our convention is that $k_{p+q}^x = 0$ for all $p + q > n_x - 1$.)

The following result is the generalization of Proposition 3.1 to the case of higher-order poles.

Proposition 4.1. *Under a gauge transformation $(g, \gamma) : (A, B) \rightarrow {}^{(g, \gamma)}(A, B)$, with $g \in C^\infty(X, G)$ and $\gamma \in \overline{\Omega}^1(X, \mathfrak{h})$, the regularized action (4.4) transforms as*

$$\begin{aligned} S_\omega({}^{(g, \gamma)}(A, B)) &= S_\omega(A, B) + \frac{1}{2} \int_M \left(\langle\langle \mathbf{J}^*(g) \mathbf{J}^*(A) \mathbf{J}^*(g)^{-1}, F_M(\mathbf{J}^*(\gamma)) \rangle\rangle_\omega \right. \\ &\quad + \langle\langle \mathbf{J}^*(t_*(\gamma)), d_M \mathbf{J}^*(\gamma) + \frac{1}{3} [\mathbf{J}^*(\gamma), \mathbf{J}^*(\gamma)] \rangle\rangle_\omega \\ &\quad \left. - \langle\langle d_M \mathbf{J}^*(g) \mathbf{J}^*(g)^{-1} + \mathbf{J}^*(t_*(\gamma)), \mathbf{J}^*(\alpha_*(g, B)) + F_M(\mathbf{J}^*(\gamma)) \rangle\rangle_\omega \right). \end{aligned} \tag{4.12}$$

Proof. This is a direct consequence of Proposition 2.7 and the same arguments as in [BSV, Lemma 3.3 and Proposition 3.4]. \square

To impose boundary conditions, we choose an isotropic crossed submodule

$$(G^\diamond, H^\diamond, t^{\hat{z}}, \alpha^{\hat{z}}) \subseteq (G^{\hat{z}}, H^{\hat{z}}, t^{\hat{z}}, \alpha^{\hat{z}}) \tag{4.13}$$

with respect to the pairing (4.11), where as a consequence of (4.8) the ambient crossed module consists of products of jet groups

$$G^{\hat{z}} = \prod_{x \in \mathbb{Z}} C^\infty(\ell T_x^{n_x}, G), \quad H^{\hat{z}} = \prod_{x \in \mathbb{Z}} C^\infty(\ell T_x^{n_x}, H), \tag{4.14}$$

see also [Viz] for a more explicit description of such jet groups. The construction of the groupoid $\mathcal{F}_{\text{ho}}^\diamond$ of boundary conditioned fields with edge modes from Proposition 3.4 generalizes in the evident way: One simply replaces the crossed module $(G^z, H^z, t^z, \alpha^z)$ of product groups by the crossed module $(G^{\hat{z}}, H^{\hat{z}}, t^{\hat{z}}, \alpha^{\hat{z}})$ of products of jet groups, and further replaces the pullback maps t^* by the maps \mathbf{j}^* from (4.7). Following the same steps as in Sect. 3, one then arrives at the extended action

$$\begin{aligned} S_\omega^{\text{ext}}((A, B), (k, \kappa)) &= \frac{i}{2\pi} - \int_X \omega \wedge \langle \overline{F}(A) - \frac{1}{2} t_*(B), B \rangle \\ &\quad + \frac{1}{2} \int_M \left(\langle\langle {}^{(k, \kappa)} \mathbf{j}^*(A), \alpha_*^{\hat{z}}({}^{(k, \kappa)} \mathbf{j}^*(A), \kappa) + 2F_M(\kappa) \rangle\rangle_\omega + \langle\langle t_*^{\hat{z}}(\kappa), d_M \kappa + \frac{1}{3} [\kappa, \kappa] \rangle\rangle_\omega \right), \end{aligned} \tag{4.15}$$

which generalizes (3.22) to the higher-order pole case. Restricting as in (3.23) to the full subgroupoid

$$\mathcal{F}_{\text{ho}}^{\diamond,0} \subseteq \mathcal{F}_{\text{ho}}^{\diamond} \tag{4.16}$$

whose objects $((A, B), (k, \kappa))$ are such that the connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ does not have legs along $d\bar{z}$, we obtain as in (3.24) a further simplification

$$\begin{aligned} S_{\omega}^{\text{ext}}((A, B), (k, \kappa)) &= \frac{i}{2\pi} - \int_X \omega \wedge \langle \bar{\partial}A, B \rangle \\ &+ \frac{1}{2} \int_M \left(\left\langle \left\langle (k, \kappa) \mathbf{j}^*(A), \alpha_*^{\hat{z}}(k, \kappa) \mathbf{j}^*(A), \kappa \right\rangle \right\rangle_{\omega} + \left\langle \left\langle t_*^{\hat{z}}(\kappa), d_M \kappa + \frac{1}{3}[\kappa, \kappa] \right\rangle \right\rangle_{\omega} \right). \end{aligned} \tag{4.17}$$

In complete analogy to Proposition 3.7, one can work out the variation of this action, which yields the bulk equations of motion on X

$$\omega \wedge \bar{\partial}B = 0, \quad \omega \wedge \bar{\partial}A = 0, \tag{4.18}$$

and the defect equations of motion on M

$$\mathbf{j}^*(d_M B + \alpha_*(A, B)) = 0, \quad \mathbf{j}^*(F_M(A) - t_*(B)) = 0, \tag{4.19}$$

where we again use that under the pullback \mathbf{j}^* only the d_M component of the differential $\bar{d} = d_M + \bar{\partial}$ survives.

5. Construction of 3d Integrable Field Theories

With our preparations from Sects. 3 and 4, we are now ready to construct 3-dimensional integrable field theories on M . The key observation which makes this endeavor possible is that the bulk equations of motion (4.18) imply that the connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ is holomorphic away from the zeros $\zeta \subset \mathbb{C}P^1$ of the meromorphic 1-form ω and that the defect equations of motion (4.19) are the pullback to the defect of the M -relative flatness condition for the connection (A, B) . These are almost the properties which one requires for a higher Lax connection. However, as in the context of 4-dimensional semi-holomorphic Chern-Simons theory, see [BSV, Sect. 5], the following crucial points need further attention:

1. To qualify as a Lax connection, the connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ must not only be holomorphic away from the zeros of ω , which is implied by the bulk equations of motion (4.18), but it further must be meromorphic on all of $\mathbb{C}P^1$.
2. The flatness conditions implied by the defect equations of motion (4.19) must lift along \mathbf{j}^* to the M -relative flatness conditions $d_M B + \alpha_*(A, B) = 0$ and $F_M(A) - t_*(B) = d_M A + \frac{1}{2}[A, A] - t_*(B) = 0$ on X .
3. The boundary conditions $(k, \kappa) \mathbf{j}^*(A, B) \in \Omega^1(M, \mathfrak{g}^{\diamond}) \times \Omega^2(M, \mathfrak{h}^{\diamond})$ should admit a unique solution for the Lax connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ in terms of the edge mode fields $(k, \kappa) \in C^{\infty}(M, G^{\hat{z}}) \times \Omega^1(M, \mathfrak{h}^{\hat{z}})$, making the latter the only degrees of freedom of the 3-dimensional integrable field theory on M .

Although the uniqueness assumption in the third point is not strictly necessary, it holds in the vast majority of examples in the context of 4-dimensional semi-holomorphic Chern-Simons theory, see for instance [LaV]. We will thus focus in the present work on the situation where the boundary conditions admit a unique solution and come back to this in Sect. 5.1 below.

The first two issues can be successfully solved by considering a special class of solutions $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ of the bulk equations of motion (4.18) which have a specific behavior towards the zeros of ω . The following definition originated in [BSV, Sect. 5].

Definition 5.1. (a) Let V be a vector space and consider the vector space $\overline{\Omega}^{q,0}(X, V)$ of V -valued $(q, 0)$ -forms on the product manifold $X = M \times C$. We define $\overline{\Omega}_\omega^{q,0}(X, V) \subseteq \overline{\Omega}^{q,0}(X, V)$ as the subspace consisting of all V -valued $(q, 0)$ -forms which are meromorphic on $\mathbb{C}P^1$ with poles at each zero $y \in \zeta$ of ω of order at most that of the zero.

(b) A connection $(A, B) \in \overline{\Omega}_\omega^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}_\omega^{2,0}(X, \mathfrak{h})$ is called *admissible* if its M -relative curvature

$$\text{curv}_M(A, B) := \left(F_M(A) - t_*(B), d_M B + \alpha_*(A, B) \right) \in \overline{\Omega}_\omega^{2,0}(X, \mathfrak{g}) \times \overline{\Omega}_\omega^{3,0}(X, \mathfrak{h}) \tag{5.1}$$

lies in the subspaces from item (a).

The following key result has been proven in [BSV, Lemma 5.5 and Proposition 5.6].

Proposition 5.2. Let $((A, B), (k, \kappa)) \in \mathcal{F}_{\text{ho}}^{\diamond,0}$ be any object in the groupoid (4.16) such that $(A, B) \in \overline{\Omega}_\omega^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}_\omega^{2,0}(X, \mathfrak{h})$ is an admissible connection. Then $((A, B), (k, \kappa))$ solves the bulk equations of motion (4.18) and the defect equations of motion (4.19) are equivalent to the M -relative flatness conditions

$$d_M B + \alpha_*(A, B) = 0, \quad F_M(A) - t_*(B) = 0 \tag{5.2}$$

on the 5-dimensional manifold X .

5.1. A notion of maximality for isotropic crossed submodules. In the setting of 4-dimensional semi-holomorphic Chern-Simons theory, a powerful approach to solving the boundary conditions for the gauge field in terms of the edge modes was developed in [LaV]. An important requirement in the construction of [LaV] is that the isotropic subalgebra entering the boundary conditions is *maximal*, namely that its dimension should be half that of the defect Lie algebra. A sufficient additional condition was then identified for the existence of a *unique* solution to these boundary conditions, see [LaV, Lemma 2.2 and Section 4.4].

Of course, it would be very desirable to generalize the full construction of [LaV] to the present 3-dimensional setting. Our more modest goal in this section is to identify a suitable notion of *maximality* for the isotropic crossed submodule $(G^\diamond, H^\diamond, t^{\hat{z}}, \alpha^{\hat{z}}) \subseteq (G^{\hat{z}}, H^{\hat{z}}, t^{\hat{z}}, \alpha^{\hat{z}})$, for which it can be expected that the boundary conditions admit a unique solution for the Lax connection $(A, B) \in \overline{\Omega}^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}^{2,0}(X, \mathfrak{h})$ in terms of the edge modes $(k, \kappa) \in C^\infty(M, G^{\hat{z}}) \times \Omega^1(M, \mathfrak{h}^{\hat{z}})$. Specifically, guided by the

4-dimensional semi-holomorphic Chern-Simons case as an analogy, we will make a particular choice of ansatz for the meromorphic connection (A, B) and give a simple counting argument to determine the expected dimensions of \mathfrak{g}^\diamond and \mathfrak{h}^\diamond for the unique solvability of the boundary conditions. In the next subsections we will then present concrete examples which fit within this proposed scheme. For our argument we assume the following specific properties of the meromorphic 1-form $\omega \in \Omega^1(\mathbb{C}P^1)$.

Assumption 5.3. The meromorphic 1-form $\omega \in \Omega^1(\mathbb{C}P^1)$ from (4.1) only has simple zeros and its total number of poles (counting multiplicities) is divisible by 3, i.e.

$$|z| = \sum_{x \in z} n_x \in 3\mathbb{Z}_{\geq 1}. \tag{5.3}$$

Let us pick any zero $y_0 \in \zeta$ of ω and denote by $\zeta' := \zeta \setminus \{y_0\}$ the set of the remaining zeros. As a consequence of Assumption 5.3 and the identity $|z| = |\zeta| + 2$ for the total numbers of poles and zeros of any meromorphic 1-form on the Riemann sphere $\mathbb{C}P^1$, we observe that $|\zeta'| \in 3\mathbb{Z}_{\geq 0}$ is either 0 (in the case where ω has only one zero) or divisible by 3. This allows us to choose a decomposition

$$\zeta' = \zeta_1 \sqcup \zeta_2 \sqcup \zeta_3 \tag{5.4}$$

into three subsets of the same size $|\zeta_1| = |\zeta_2| = |\zeta_3|$. The labels 1, 2, 3 will correspond to a choice of coordinates u_i , for $i = 1, 2, 3$, on the 3-dimensional spacetime $M = \mathbb{R}^3$.

Using the above choices, we consider the following ansatz

$$A = \sum_{i=1}^3 \left(A_c^i + \sum_{y \in \zeta_i} \frac{A_y^i}{z - y} \right) du_i \in \overline{\Omega}_\omega^{1,0}(X, \mathfrak{g}), \tag{5.5a}$$

$$B = \sum_{i,j=1}^3 \left(B_c^{ij} + \sum_{y \in \zeta_i \sqcup \zeta_j \sqcup \{y_0\}} \frac{B_y^{ij}}{z - y} \right) du_i \wedge du_j \in \overline{\Omega}_\omega^{2,0}(X, \mathfrak{h}) \tag{5.5b}$$

for a connection $(A, B) \in \overline{\Omega}_\omega^{1,0}(X, \mathfrak{g}) \times \overline{\Omega}_\omega^{2,0}(X, \mathfrak{h})$, where $A_c^i, A_y^i \in C^\infty(M, \mathfrak{g})$ and $B_c^{ij}, B_y^{ij} \in C^\infty(M, \mathfrak{h})$ are arbitrary coefficient functions depending only on M . Note that the connection (5.5) is admissible in the sense of Definition 5.1. The number of independent degrees of freedom of this connection, which we count as scalars on M , is given by

$$\text{dof}(A) = \sum_{i=1}^3 (|\zeta_i| + 1) \times \dim(\mathfrak{g}) = (|\zeta| + 2) \times \dim(\mathfrak{g}), \tag{5.6a}$$

$$\text{dof}(B) = \sum_{i>j} (|\zeta_i| + |\zeta_j| + 2) \times \dim(\mathfrak{h}) = 2 \times (|\zeta| + 2) \times \dim(\mathfrak{h}). \tag{5.6b}$$

These degrees of freedom are constrained by the boundary conditions

$${}^{(k,\kappa)} \mathbf{j}^*(A) = k \mathbf{j}^*(A) k^{-1} - d_M k k^{-1} - t_*^{\hat{z}}(\kappa) \in \Omega^1(M, \mathfrak{g}^\diamond), \tag{5.7a}$$

$${}^{(k,\kappa)} \mathbf{j}^*(B) = \alpha_*^{\hat{z}}(k, \mathbf{j}^*(B)) - F_M(\kappa) - \alpha_*^{\hat{z}}({}^{(k,\kappa)} \mathbf{j}^*(A), \kappa) \in \Omega^2(M, \mathfrak{h}^\diamond). \tag{5.7b}$$

Counting the number of boundary conditions (again as scalars on M), one finds

$$\text{bdy}(A) = 3 \times (\dim(\mathfrak{g}^{\hat{z}}) - \dim(\mathfrak{g}^{\diamond})) = 3 \times (|z| \times \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{\diamond})), \quad (5.8a)$$

$$\text{bdy}(B) = 3 \times (\dim(\mathfrak{h}^{\hat{z}}) - \dim(\mathfrak{h}^{\diamond})) = 3 \times (|z| \times \dim(\mathfrak{h}) - \dim(\mathfrak{h}^{\diamond})). \quad (5.8b)$$

For the unique solvability for (A, B) of the boundary conditions (5.7) one requires that there are as many boundary conditions as there are degrees of freedom, i.e. $\text{dof}(A) = \text{bdy}(A)$ and $\text{dof}(B) = \text{bdy}(B)$. Using also the identity $|z| = |\zeta| + 2$ we conclude that a necessary condition for the unique solvability of (5.7), assuming the ansatz (5.5), is given by

$$\dim(\mathfrak{g}^{\diamond}) = \frac{2}{3} \dim(\mathfrak{g}^{\hat{z}}), \quad \dim(\mathfrak{h}^{\diamond}) = \frac{1}{3} \dim(\mathfrak{h}^{\hat{z}}) = \frac{1}{3} \dim(\mathfrak{g}^{\hat{z}}), \quad (5.9)$$

where in the last step we used that $\dim(\mathfrak{h}) = \dim(\mathfrak{g})$ as a consequence of the non-degenerate pairing (2.16). The condition (5.9) implies in particular that the isotropic crossed submodule $(G^{\diamond}, H^{\diamond}, t^{\hat{z}}, \alpha^{\hat{z}}) \subseteq (G^{\hat{z}}, H^{\hat{z}}, t^{\hat{z}}, \alpha^{\hat{z}})$ must be *maximal* in the sense that its total dimension $\dim(\mathfrak{g}^{\diamond}) + \dim(\mathfrak{h}^{\diamond}) = \dim(\mathfrak{g}^{\hat{z}})$ is half of the total dimension $\dim(\mathfrak{g}^{\hat{z}}) + \dim(\mathfrak{h}^{\hat{z}}) = 2 \dim(\mathfrak{g}^{\hat{z}})$ of the ambient crossed module.

5.2. Toy-example: 3-dimensional Chern-Simons theory. In this subsection we show how one can recover the usual 3-dimensional Chern-Simons theory as a defect theory of our 5-dimensional semi-holomorphic 2-Chern-Simons theory. For this we consider the crossed module of Lie groups $(G, G, \text{id}, \text{Ad})$ with $t = \text{id} : G \rightarrow G$, $g \mapsto g$ the identity map and $\alpha = \text{Ad} : G \times G \rightarrow G$, $(g, g') \mapsto g g' g^{-1}$ the adjoint action. The associated crossed module of Lie algebras is given by $(\mathfrak{g}, \mathfrak{g}, \text{id}, \text{ad})$ with $\text{ad} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, $(x, x') \mapsto [x, x']$ the Lie algebra adjoint action. For the non-degenerate pairing in (2.16) we take any non-degenerate Ad-invariant symmetric pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ on the Lie algebra \mathfrak{g} . For the meromorphic 1-form $\omega \in \Omega^1(\mathbb{C}P^1)$ we choose

$$\omega = \frac{1-z}{z} dz = \frac{dz}{z} - dz, \quad (5.10)$$

which has a simple zero at $z = 1$, a simple pole at $z = 0$ and a double pole at $z = \infty$. Note that Assumption 5.3 is satisfied. To avoid confusion, let us highlight that we choose for convenience in this and the next example a coordinate z on $\mathbb{C}P^1$ in which ∞ is a pole of ω , while in Sects. 3 and 4 the coordinate was chosen such that ∞ corresponds to a zero of ω .

The associated crossed module of jet groups (4.14) is given by $(G^{\hat{z}}, G^{\hat{z}}, \text{id}, \text{Ad})$ with

$$G^{\hat{z}} = G \times (G \ltimes \tilde{\mathfrak{g}}), \quad (5.11a)$$

where the factor G corresponds to the simple pole at $z = 0$ and the semi-direct product $G \ltimes \tilde{\mathfrak{g}}$ corresponds to the double pole at $z = \infty$. The notation $\tilde{\mathfrak{g}}$ is used to distinguish between the Lie algebra \mathfrak{g} and the Abelian Lie group $\tilde{\mathfrak{g}} := \mathfrak{g}$ with group operation $+$ and identity element $0 \in \mathfrak{g}$. The group structure reads explicitly as

$$(g_0, (g_{\infty}, x_{\infty})) (g'_0, (g'_{\infty}, x'_{\infty})) = (g_0 g'_0, (g_{\infty} g'_{\infty}, x_{\infty} + g_{\infty} x'_{\infty} g_{\infty}^{-1})), \quad (5.11b)$$

for all $(g_0, (g_\infty, x_\infty)), (g'_0, (g'_\infty, x'_\infty)) \in G^{\hat{z}}$, and the identity element is $1_{G^{\hat{z}}} = (1_G, (1_G, 0))$. The corresponding crossed module of Lie algebras is given by $(\mathfrak{g}^{\hat{z}}, \mathfrak{g}^{\hat{z}}, \text{id}, \text{ad})$ with

$$\mathfrak{g}^{\hat{z}} = \mathfrak{g} \times (\mathfrak{g} \ltimes \mathfrak{g}_{\text{ab}}), \tag{5.12a}$$

where \mathfrak{g}_{ab} denotes the Abelian Lie algebra given by the vector space \mathfrak{g} and the trivial Lie bracket. The Lie algebra structure reads explicitly as

$$[(x_0, (x_\infty, y_\infty)), (x'_0, (x'_\infty, y'_\infty))] = ([x_0, x'_0], ([x_\infty, x'_\infty], [x_\infty, y'_\infty] + [y_\infty, x'_\infty])), \tag{5.12b}$$

for all $(x_0, (x_\infty, y_\infty)), (x'_0, (x'_\infty, y'_\infty)) \in \mathfrak{g}^{\hat{z}}$.

The pairing $\langle \cdot, \cdot \rangle_\omega : \mathfrak{g}^{\hat{z}} \otimes \mathfrak{g}^{\hat{z}} \rightarrow \mathbb{C}$ from (4.11) reads in the present example as

$$\langle (x_0, (x_\infty, y_\infty)), (x'_0, (x'_\infty, y'_\infty)) \rangle_\omega = \langle x_0, x'_0 \rangle - \langle x_\infty, x'_\infty \rangle + \langle x_\infty, y'_\infty \rangle + \langle y_\infty, x'_\infty \rangle, \tag{5.13}$$

for all $(x_0, (x_\infty, y_\infty)), (x'_0, (x'_\infty, y'_\infty)) \in \mathfrak{g}^{\hat{z}}$. A possible choice for an isotropic crossed submodule $(G^\diamond, H^\diamond, \text{id}, \text{Ad}) \subseteq (G^{\hat{z}}, G^{\hat{z}}, \text{id}, \text{Ad})$ is given by

$$G^\diamond = G \times (\{1_G\} \ltimes \tilde{\mathfrak{g}}), \quad H^\diamond = \{1_G\} \times (\{1_G\} \ltimes \tilde{\mathfrak{g}}). \tag{5.14}$$

Note that this choice satisfies our maximality condition (5.9).

The ansatz (5.5) for the Lax connection specializes in the present example to

$$A = A_c = \sum_{i=1}^3 A_c^i du_i, \tag{5.15a}$$

$$B = B_c + \frac{B_1}{z-1} = \sum_{i,j=1}^3 \left(B_c^{ij} + \frac{B_1^{ij}}{z-1} \right) du_i \wedge du_j. \tag{5.15b}$$

Our goal is to determine the forms $A_c \in \Omega^1(M, \mathfrak{g})$ and $B_c, B_1 \in \Omega^2(M, \mathfrak{g})$ by solving the boundary conditions (5.7). Let us consider a general edge mode field $(k, \kappa) \in C^\infty(M, G^{\hat{z}}) \times \Omega^1(M, \mathfrak{g}^{\hat{z}})$ in this model, which we can write more explicitly as

$$k = (k_0, (k_\infty, l_\infty)), \quad \kappa = (\kappa_0, (\kappa_\infty, \lambda_\infty)). \tag{5.16}$$

This can be simplified considerably by using the gauge transformations in (3.15), with ι replaced by \mathbf{j} since we are in the context of higher-order poles, in order to gauge fix the edge modes. For the transformation parameters $(g^\diamond, \gamma^\diamond) \in C^\infty(M, G^\diamond) \times \Omega^1(M, \mathfrak{h}^\diamond)$ and $(g, \gamma) \in C^\infty(M, G) \times \Omega^1(M, \mathfrak{g})$ which are constant along $\mathbb{C}P^1$, the component k transforms as

$$\begin{aligned} k' &= g^\diamond k \mathbf{j}^*(g)^{-1} = (g_0^\diamond, (1_G, x_\infty^\diamond)) (k_0, (k_\infty, l_\infty)) (g^{-1}, (g^{-1}, 0)) \\ &= (g_0^\diamond k_0 g^{-1}, (k_\infty g^{-1}, x_\infty^\diamond + l_\infty)), \end{aligned} \tag{5.17}$$

which becomes the identity $k' = (1_G, (1_G, 0))$ when choosing $g = k_\infty, g_0^\diamond = k_\infty k_0^{-1}$ and $x_\infty^\diamond = -l_\infty$. This allows us to fix without loss of generality the gauge in which

$k = 1_{G\hat{z}} = (1_G, (1_G, 0))$ is the identity. Under residual gauge transformations, which are characterized by $g^\diamond = 1_{G^\diamond}$ and $g = 1_G$, the component κ transforms as

$$\begin{aligned} \kappa' &= \gamma^\diamond + \kappa - \mathbf{j}^*(\gamma) = (0, (0, \gamma_\infty^\diamond)) + (\kappa_0, (\kappa_\infty, \lambda_\infty)) - (\gamma, (\gamma, 0)) \\ &= (\kappa_0 - \gamma, (\kappa_\infty - \gamma, \gamma_\infty^\diamond + \lambda_\infty)), \end{aligned} \tag{5.18}$$

which becomes $\kappa' = (0, (\kappa_\infty - \kappa_0, 0))$ when choosing $\gamma = \kappa_0$ and $\gamma_\infty^\diamond = -\lambda_\infty$. Hence, the general form of the gauge fixed edge mode is

$$k = 1_{G\hat{z}} = (1_G, (1_G, 0)), \quad \kappa = (0, (\kappa_\infty, 0)). \tag{5.19}$$

Working out the first boundary condition (5.7) for the ansatz (5.15) and the gauge fixed edge mode yields

$${}^{(k,\kappa)}\mathbf{j}^*(A) = \mathbf{j}^*(A) - \kappa = (A_c, (A_c - \kappa_\infty, 0)) \in \Omega^1(M, \mathfrak{g}^\diamond), \tag{5.20a}$$

from which we deduce using also (5.14) that $A_c = \kappa_\infty$. For the second boundary condition (5.7) we then find

$$\begin{aligned} {}^{(k,\kappa)}\mathbf{j}^*(B) &= \mathbf{j}^*(B) - F_M(\kappa) - \text{ad}({}^{(k,\kappa)}\mathbf{j}^*(A), \kappa) \\ &= (B_c - B_1, (B_c - F_M(\kappa_\infty), B_1)) \in \Omega^1(M, \mathfrak{h}^\diamond), \end{aligned} \tag{5.20b}$$

from which we deduce using also (5.14) that $B_c = F_M(\kappa_\infty)$ and $B_1 = B_c = F_M(\kappa_\infty)$. The candidate Lax connection for this model is thus given by

$$(A, B) = \left(\kappa_\infty, \frac{z}{z-1} F_M(\kappa_\infty) \right) \in \overline{\Omega}_\omega^{(1,0)}(X, \mathfrak{g}) \times \overline{\Omega}_\omega^{(2,0)}(X, \mathfrak{g}). \tag{5.21}$$

Inserting this result into the action (4.17) yields the defect action

$$S_M(\kappa_\infty) = - \int_M \left\langle \kappa_\infty, \frac{1}{2} d_M \kappa_\infty + \frac{1}{3!} [\kappa_\infty, \kappa_\infty] \right\rangle, \tag{5.22}$$

which coincides in this example with the usual 3-dimensional Chern-Simons action for the edge mode $\kappa_\infty \in \Omega^1(M, \mathfrak{g})$. The equation of motion is the flatness condition $F_M(\kappa_\infty) = 0$, which implies that the Lax connection (5.21) is M -relative flat $\text{curv}_M(A, B) = 0$ when going on-shell. We further observe that the defect action (5.22) is gauge-invariant under additional gauge transformations which take the usual form ${}^s\kappa_\infty = g \kappa_\infty g^{-1} - d_M g g^{-1}$, for $g \in C^\infty(M, G)$. We expect that this is a remnant of the 2-categorical nature of higher connections (see Remark 2.3), but since we currently do not know how to consistently include these aspects in our approach we cannot give a precise argument or proof for this claim.

From the point of view of integrable field theory, the example determined by the action (5.22) and the corresponding Lax connection (5.21) is somewhat trivial. When going on-shell, the Lax connection simplifies to $(A, B)|_{\text{on-shell}} = (\kappa_\infty, 0)$, i.e. it does not have any z -dependence. So the conserved charges, which one may construct by taking holonomies of the Lax connection, are simply ordinary Wilson loops for the flat Chern-Simons gauge field κ_∞ . The more involved example which we will study in the next subsection will have more interesting integrability features.

5.3. Example: Ward equation. In this subsection we derive and study an example of a 3-dimensional integrable field theory which is related to the Ward equation [War2, War3]. For this we consider the shifted tangent crossed module of Lie groups $T[1]G := (G, \tilde{\mathfrak{g}}, 1_G, \text{Ad})$, where we use again the notation $\tilde{\mathfrak{g}}$ to distinguish between the Lie algebra \mathfrak{g} and the Abelian Lie group $\tilde{\mathfrak{g}} := \mathfrak{g}$ with group operation $+$ and identity element $0 \in \mathfrak{g}$. The map $t = 1_G : \tilde{\mathfrak{g}} \rightarrow G, x \mapsto 1_G$ is constantly assigning the identity element $1_G \in G$ and $\alpha = \text{Ad} : G \times \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}, (g, x) \mapsto gxg^{-1}$ is the adjoint action. For the non-degenerate pairing (2.16) we take any non-degenerate Ad-invariant symmetric pairing on the Lie algebra \mathfrak{g} . For the meromorphic 1-form $\omega \in \Omega^1(\mathbb{C}P^1)$ we choose

$$\omega = \frac{z \prod_{i=1}^3 (z - a_i)}{(z - r)^2 (z - s)^2} dz = \left(\frac{\ell_r^1}{(z - r)^2} + \frac{\ell_r^0}{z - r} + \frac{\ell_s^1}{(z - s)^2} + \frac{\ell_s^0}{z - s} + 1 \right) dz, \tag{5.23}$$

which has four simple zeros at $z = 0, a_1, a_2, a_3$ and three double poles at $z = r, s, \infty$, hence Assumption 5.3 is satisfied. As in the previous example of Sect. 5.2, we use again a coordinate z on $\mathbb{C}P^1$ in which ∞ is a pole of ω . (Note that the constants $\ell_r^1, \ell_r^0, \ell_s^1, \ell_s^0$ in the second expression are fixed in terms of the constants a_1, a_2, a_3, r, s in the first one.)

The associated crossed module of jet groups (4.14) is given by $(G^{\hat{z}}, \tilde{\mathfrak{g}}^{\hat{z}}, 1_{G^{\hat{z}}}, \text{Ad})$ with

$$G^{\hat{z}} = (G \times \tilde{\mathfrak{g}}) \times (G \times \tilde{\mathfrak{g}}) \times (G \times \tilde{\mathfrak{g}}), \tag{5.24a}$$

$$\tilde{\mathfrak{g}}^{\hat{z}} = (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \times (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \times (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}), \tag{5.24b}$$

where the three factors correspond to the three double poles at r, s, ∞ . The group structures on the individual factors read explicitly as

$$(g, x)(g', x') = (g g', x + g x' g^{-1}), \quad (x, y)(x', y') = (x + x', y + y'), \tag{5.25}$$

for all $(g, x), (g', x') \in G \times \tilde{\mathfrak{g}}$ and $(x, y), (x', y') \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$, and the identity elements of the individual factors are $1_{G \times \tilde{\mathfrak{g}}} = (1_G, 0)$ and $1_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}} = (0, 0)$. The action $\text{Ad} : G^{\hat{z}} \times \tilde{\mathfrak{g}}^{\hat{z}} \rightarrow \tilde{\mathfrak{g}}^{\hat{z}}$ reads on each factor as

$$\text{Ad}((g, x), (x', y')) = (g x' g^{-1}, g y' g^{-1} + [x, g x' g^{-1}]), \tag{5.26}$$

for all $(g, x) \in G \times \tilde{\mathfrak{g}}$ and $(x', y') \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$.

The corresponding crossed module of Lie algebras is given by $(\mathfrak{g}^{\hat{z}}, \mathfrak{g}_{\text{ab}}^{\hat{z}}, 0, \text{ad})$ with

$$\mathfrak{g}^{\hat{z}} = (\mathfrak{g} \times \mathfrak{g}_{\text{ab}}) \times (\mathfrak{g} \times \mathfrak{g}_{\text{ab}}) \times (\mathfrak{g} \times \mathfrak{g}_{\text{ab}}), \tag{5.27a}$$

$$\mathfrak{g}_{\text{ab}}^{\hat{z}} = (\mathfrak{g}_{\text{ab}} \times \mathfrak{g}_{\text{ab}}) \times (\mathfrak{g}_{\text{ab}} \times \mathfrak{g}_{\text{ab}}) \times (\mathfrak{g}_{\text{ab}} \times \mathfrak{g}_{\text{ab}}). \tag{5.27b}$$

The Lie algebra structure on \mathfrak{g}_{ab} is the trivial one and on the individual factors of $\mathfrak{g}^{\hat{z}}$ the Lie bracket reads as

$$[(x, y), (x', y')] = ([x, x'], [x, y'] + [y, x']), \tag{5.28}$$

for all $(x, y), (x', y') \in \mathfrak{g} \times \mathfrak{g}_{\text{ab}}$. The map $t_*^{\hat{z}} = 0$ is trivial and $\text{ad} : \mathfrak{g}^{\hat{z}} \otimes \mathfrak{g}_{\text{ab}}^{\hat{z}} \rightarrow \mathfrak{g}_{\text{ab}}^{\hat{z}}$ is given on each factor by

$$\text{ad}((x, y), (x', y')) = ([x, x'], [x, y'] + [y, x']), \tag{5.29}$$

for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}_{\text{ab}}$ and $(x', y') \in \mathfrak{g}_{\text{ab}} \times \mathfrak{g}_{\text{ab}}$.

The pairing $\langle \cdot, \cdot \rangle_\omega : \mathfrak{g}^{\hat{z}} \otimes \mathfrak{g}_{\text{ab}}^{\hat{z}} \rightarrow \mathbb{C}$ from (4.11) reads in the present example as

$$\begin{aligned} & \left\langle \left((x_r, y_r), (x_s, y_s), (x_\infty, y_\infty) \right), \left((x'_r, y'_r), (x'_s, y'_s), (x'_\infty, y'_\infty) \right) \right\rangle_\omega \\ &= \ell_r^0 \langle x_r, x'_r \rangle + \ell_r^1 \left(\langle x_r, y'_r \rangle + \langle y_r, x'_r \rangle \right) \\ &+ \ell_s^0 \langle x_s, x'_s \rangle + \ell_s^1 \left(\langle x_s, y'_s \rangle + \langle y_s, x'_s \rangle \right) \\ &- \left(\ell_r^0 + \ell_s^0 \right) \langle x_\infty, x'_\infty \rangle - \langle x_\infty, y'_\infty \rangle - \langle y_\infty, x'_\infty \rangle, \end{aligned} \quad (5.30)$$

for all $((x_r, y_r), (x_s, y_s), (x_\infty, y_\infty)) \in \mathfrak{g}^{\hat{z}}$ and $((x'_r, y'_r), (x'_s, y'_s), (x'_\infty, y'_\infty)) \in \mathfrak{g}_{\text{ab}}^{\hat{z}}$. A possible choice for an isotropic crossed submodule $(G^\diamond, H^\diamond, 1_{G^{\hat{z}}}, \text{Ad}) \subseteq (G^{\hat{z}}, \mathfrak{g}^{\hat{z}}, 1_{G^{\hat{z}}}, \text{Ad})$ is given by

$$G^\diamond = (\{1_G\} \times \tilde{\mathfrak{g}}) \times (G \times \tilde{\mathfrak{g}}) \times (\{1_G\} \times \tilde{\mathfrak{g}}), \quad (5.31a)$$

$$H^\diamond = (\{0\} \times \tilde{\mathfrak{g}}) \times (\{0\} \times \{0\}) \times (\{0\} \times \tilde{\mathfrak{g}}). \quad (5.31b)$$

Note that this choice satisfies our maximality condition (5.9).

Choosing $z = 0$ as the distinguished zero of ω , the ansatz (5.5) for the Lax connection specializes in the present example to

$$A = \sum_{i=1}^3 \left(A_c^i + \frac{A_{a_i}^i}{z - a_i} \right) du_i, \quad (5.32a)$$

$$B = \sum_{i,j=1}^3 \left(B_c^{ij} + \frac{B_{a_i}^{ij}}{z - a_i} + \frac{B_{a_j}^{ij}}{z - a_j} + \frac{B_0^{ij}}{z} \right) du_i \wedge du_j. \quad (5.32b)$$

We now determine the coefficient functions $A_c^i, A_{a_i}^i \in C^\infty(M, \mathfrak{g})$ and $B_c^{ij}, B_{a_i}^{ij}, B_{a_j}^{ij}, B_0^{ij} \in C^\infty(M, \mathfrak{g}_{\text{ab}})$ by solving the boundary conditions (5.7). For this it is again convenient to use the gauge transformations in (3.15), with ι replaced by \mathbf{j} since we are in the context of higher-order poles, in order to gauge fix the edge modes $(k, \kappa) \in C^\infty(M, G^{\hat{z}}) \times \Omega^1(M, \mathfrak{g}_{\text{ab}}^{\hat{z}})$ according to

$$k = ((k_r, 0), (1_G, 0), (1_G, 0)), \quad \kappa = ((\kappa_r, 0), (0, \lambda_s), (\kappa_\infty, 0)). \quad (5.33)$$

Working out the first boundary condition in (5.7) for the ansatz (5.32) and the gauge fixed edge mode yields

$$A|_{z=r} = k_r^{-1} d_M k_r, \quad A|_{z=\infty} = 0, \quad (5.34a)$$

and the second boundary condition yields

$$B|_{z=r} = k_r^{-1} (d_M \kappa_r) k_r, \quad B|_{z=\infty} = d_M \kappa_\infty, \quad (5.34b)$$

$$B|_{z=s} = 0, \quad \partial_z B|_{z=s} = \Lambda_s := d_M \lambda_s + [A|_{z=s}, \lambda_s]. \quad (5.34c)$$

The system of equations (5.34) can be solved for the coefficient functions appearing in the ansatz (5.32), which gives

$$A_c^i = 0, \tag{5.35a}$$

$$A_{a_i}^i = (r - a_i) k_r^{-1} \partial_{u_i} k_r, \tag{5.35b}$$

$$B_c^{ij} = (d_M \kappa_\infty)^{ij}, \tag{5.35c}$$

$$B_{a_i}^{ij} = \frac{(r-a_i)(s-a_i)^2}{a_i(a_i-a_j)} \left(\frac{r(r-a_j)}{(r-s)^2} k_r^{-1} (d_M \kappa_r)^{ij} k_r - \frac{s(s-a_j)}{r-s} \Lambda_s^{ij} - (d_M \kappa_\infty)^{ij} \right), \tag{5.35d}$$

$$B_{a_j}^{ij} = \frac{(r-a_j)(s-a_j)^2}{a_j(a_j-a_i)} \left(\frac{r(r-a_i)}{(r-s)^2} k_r^{-1} (d_M \kappa_r)^{ij} k_r - \frac{s(s-a_i)}{r-s} \Lambda_s^{ij} - (d_M \kappa_\infty)^{ij} \right), \tag{5.35e}$$

$$B_0^{ij} = \frac{rs^2}{a_i a_j} \left(\frac{(r-a_i)(r-a_j)}{(r-s)^2} k_r^{-1} (d_M \kappa_r)^{ij} k_r - \frac{(s-a_i)(s-a_j)}{r-s} \Lambda_s^{ij} - (d_M \kappa_\infty)^{ij} \right). \tag{5.35f}$$

Hence, we have uniquely solved the boundary conditions for the connection (A, B) from (5.32) in terms of the edge mode (k, κ) .

Inserting this result into the action (4.17), and noting that the Chern-Simons term vanishes since $t_*^{\hat{z}} = 0$ in the present example, yields the defect action

$$S_M(k_r, \kappa_r, \lambda_s, \kappa_\infty) = \int_M \left(\ell_s^1 \langle A|_{z=s}, d_M \lambda_s + \frac{1}{2} [A|_{z=s}, \lambda_s] \right. \\ \left. + \ell_r^1 \langle \partial_z A|_{z=r}, k_r^{-1} (d_M \kappa_r) k_r \rangle - \langle \partial_{z^{-1}} A|_{z=\infty}, d_M \kappa_\infty \rangle \right), \tag{5.36}$$

where the values of A and its z derivative at the various poles $z = r, s, \infty$ are determined in terms of the edge mode by (5.32) and (5.35). The corresponding Lax connection (A, B) for this theory is given by inserting (5.35) into (5.32). As a consequence of Proposition 5.2, the Euler-Lagrange equations for the action (5.36) are equivalent to the M -relative flatness conditions for the Lax connection, which in our present example read as

$$d_M B + [A, B] = 0, \quad d_M A + \frac{1}{2} [A, A] = 0 \tag{5.37}$$

since $t_* = 0$.

By a slightly lengthy computation, the system of equations (5.37) can be worked out component-wise by inserting (5.32) together with the explicit coefficient functions given in (5.35). One then finds that the 2-form equation $d_M A + \frac{1}{2} [A, A] = 0$ is equivalent to the flatness condition

$$d_M A|_{z=s} + \frac{1}{2} [A|_{z=s}, A|_{z=s}] = 0 \tag{5.38}$$

for the connection $A|_{z=s} = k_r^{-1} d_M^{1,-1} k_r \in \Omega^1(M, \mathfrak{g})$ which is expressed here in terms of the edge mode $k_r \in C^\infty(M, G)$ and the weighted derivative defined for general $m, n \in \mathbb{Z}$ as $d_M^{m,n} := \sum_{i=1}^3 (r - a_i)^m (s - a_i)^n du_i \wedge \partial_{u_i}$. The top-form equation $d_M B + [A, B] = 0$ is equivalent to

$$(s - r) \ell_s^1 d_M^{0,-1} \left(d_M \lambda_s + [A|_{z=s}, \lambda_s] \right) + P = 0, \tag{5.39a}$$

where we have introduced the short-hand notation

$$P := d_M^{1,0}(\mathfrak{d}_M \kappa_\infty) + [\partial_{z^{-1}} A|_{z=\infty}, \mathfrak{d}_M \kappa_\infty] + \ell_r^1 d_M^{-1,0}(k_r^{-1}(\mathfrak{d}_M \kappa_r) k_r). \quad (5.39b)$$

The top-form equation of motion (5.39) is rather non-transparent and hence difficult to analyze in full generality. We will now show that, restricting to a special class of solutions, it is related to the Ward equation. For this we assume that the edge mode fields $\kappa_r, \kappa_\infty \in \Omega^1(M, \mathfrak{g}_{\text{ab}})$ are de Rham closed, i.e. $\mathfrak{d}_M \kappa_r = \mathfrak{d}_M \kappa_\infty = 0$, and that $\lambda_s = \sum_{i=1}^3 \lambda_s^i du_i$ has constant coefficient functions, i.e. $\partial_{u_i} \lambda_s^j = 0$ for all $i, j \in \{1, 2, 3\}$. It then follows that $P = 0$, $\mathfrak{d}_M \lambda_s = 0$ and $d_M^{0,-1} \lambda_s = 0$, so the top-form equation (5.39) simplifies to

$$[d_M^{0,-1} A|_{z=s}, \lambda_s] = 0. \quad (5.40)$$

Inserting $A|_{z=s} = k_r^{-1} d_M^{1,-1} k_r$ and working out the weighted derivatives, one finds that this is equivalent to the equation

$$\begin{aligned} & \left[(r - a_3) \partial_{u_2} (k_r^{-1} \partial_{u_3} k_r) - (r - a_2) \partial_{u_3} (k_r^{-1} \partial_{u_2} k_r), (s - a_1) \lambda_s^1 \right] \\ & + \left[(r - a_1) \partial_{u_3} (k_r^{-1} \partial_{u_1} k_r) - (r - a_3) \partial_{u_1} (k_r^{-1} \partial_{u_3} k_r), (s - a_2) \lambda_s^2 \right] \\ & + \left[(r - a_2) \partial_{u_1} (k_r^{-1} \partial_{u_2} k_r) - (r - a_1) \partial_{u_2} (k_r^{-1} \partial_{u_1} k_r), (s - a_3) \lambda_s^3 \right] = 0. \end{aligned} \quad (5.41)$$

If we now choose the constants λ_s^i such that $\eta_s := (s - a_i) \lambda_s^i$, for all $i = 1, 2, 3$, we obtain

$$\left[\sum_{i,j=1}^3 N_{ij} \partial_{u_i} (k_r^{-1} \partial_{u_j} k_r), \eta_s \right] = 0, \quad (5.42a)$$

where N_{ij} are the entries of the matrix

$$N = \begin{pmatrix} 0 & r - a_2 & a_3 - r \\ a_1 - r & 0 & r - a_3 \\ r - a_1 & a_2 - r & 0 \end{pmatrix}. \quad (5.42b)$$

Decomposing $N = g + \tau$ into its symmetric part g and antisymmetric part τ gives

$$g = \begin{pmatrix} 0 & \frac{a_1 - a_2}{2} & \frac{a_3 - a_1}{2} \\ \frac{a_1 - a_2}{2} & 0 & \frac{a_2 - a_3}{2} \\ \frac{a_3 - a_1}{2} & \frac{a_2 - a_3}{2} & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & r - \frac{a_1 + a_2}{2} & \frac{a_1 + a_3}{2} - r \\ \frac{a_1 + a_2}{2} - r & 0 & r - \frac{a_2 + a_3}{2} \\ r - \frac{a_1 + a_3}{2} & \frac{a_2 + a_3}{2} - r & 0 \end{pmatrix}. \quad (5.43)$$

Considering the trace and the determinant of the symmetric part g , one deduces that 1.) the eigenvalues of g are all non-zero since by hypothesis $a_i \neq a_j$, for all $i \neq j$, and 2.) one eigenvalue has the opposite sign of the other two eigenvalues. This means that g defines a Lorentzian metric. Using the freedom to multiply the equation of motion (5.42) by -1 , we can assume without loss of generality that g has signature $(- + +)$. We also observe that the coordinates u_i which were used in describing the admissible pole structure of the connection (A, B) in (5.32) all turn out to be null with respect to the metric (5.43). Concerning the antisymmetric part τ , we apply the Hodge operator $*$ associated with g and an arbitrary choice of orientation of $M = \mathbb{R}^3$ and find that the

resulting covector $v = *(\tau)$ has squared norm $\|v\|_g^2 = 1$ with respect to the Lorentzian metric g , hence it defines a normalized spacelike covector.

In order to make the relationship between the equation of motion (5.42) and Ward’s equation [War2, War3] more explicit, we can transform the null coordinates (u_1, u_2, u_3) to new coordinates (t, x, y) in which the metric takes the standard Minkowski form $\text{diag}(-1, 1, 1)$. In these coordinates the equation of motion (5.42) then reads as

$$\begin{aligned} & \left[-\partial_t(k_r^{-1} \partial_t k_r) + \partial_x(k_r^{-1} \partial_x k_r) + \partial_y(k_r^{-1} \partial_y k_r) \right. \\ & \quad + a \left(\partial_x(k_r^{-1} \partial_y k_r) - \partial_y(k_r^{-1} \partial_x k_r) \right) \\ & \quad + b \left(\partial_y(k_r^{-1} \partial_t k_r) - \partial_t(k_r^{-1} \partial_y k_r) \right) \\ & \quad \left. + c \left(\partial_t(k_r^{-1} \partial_x k_r) - \partial_x(k_r^{-1} \partial_t k_r) \right), \eta_s \right] = 0, \end{aligned} \tag{5.44}$$

where (a, b, c) are the components of the normalized spacelike covector v in this choice of coordinates, i.e. they satisfy $-a^2 + b^2 + c^2 = 1$. The expression in the first entry of the Lie bracket in (5.44) is precisely the left-hand side of Ward’s equation, including Ward’s normalization condition for the covector v . Any solution $k_r \in C^\infty(M, G)$ to Ward’s equation is thus a solution of our top-form equation of motion $d_M B + [A, B] = 0$, provided that the other edge mode fields κ_r, κ_∞ are chosen to be de Rham closed and that $\lambda_s = \sum_{i=1}^3 \lambda_s^i du_i$ has constant coefficient functions such that $(s - a_i) \lambda_s^i = (s - a_j) \lambda_s^j$, for all $i, j \in \{1, 2, 3\}$. We would like to note that our equation of motion (5.44) also captures solutions to the inhomogeneous Ward equation with right-hand side given by a current $j \in \Omega^1(M, \mathfrak{g})$ which lies in the kernel of $[\cdot, \eta_s]$, i.e. $[j, \eta_s] = 0$.

It remains to investigate the 2-form equation (5.38). Working in our original set of null coordinates (u_1, u_2, u_3) , the three components of this equation can be written explicitly as

$$(r - a_j) \partial_{u_i}(k_r^{-1} \partial_{u_j} k_r) - (r - a_i) \partial_{u_j}(k_r^{-1} \partial_{u_i} k_r) = 0, \tag{5.45}$$

for $i, j \in \{1, 2, 3\}$ with $i < j$. Note that these are precisely the individual summands entering the Ward equation (5.41). One possible mechanism to solve both equations of motion (5.45) and (5.42), as required for the full flatness of the Lax connection (A, B) , is to consider solutions $k_r \in C^\infty(M, G)$ of the Ward equation which are constant along one of the null coordinates u_i of spacetime. The solutions one obtains in this way would then be ‘chiral’ in this chosen null direction. For such chiral solutions of the Ward equation, one can then construct families of conserved charges by taking both 1-dimensional and 2-dimensional holonomies [SW, FMP, Wal2] of the associated fully flat Lax connection (A, B) , see also Sect. 2.3.

We would like to conclude this section by observing that, even without enforcing the very restrictive 2-form equation (5.45), our approach leads to a family of conserved charges for solutions to the top-form equation $d_M B + [A, B] = 0$, and hence in particular for general solutions to the Ward equation. The origin of these conserved charges lies in the fact that in the present example the Lie group $H = \tilde{\mathfrak{g}}$ is Abelian with group operation $+$, hence the exponential map and 2-dimensional holonomy simplify drastically. This allows us to build a conserved charge for every homogeneous ad-invariant polynomial $p \in (\text{Sym}^n \mathfrak{g}^*)^{\mathfrak{g}}$ of degree n on the Lie algebra \mathfrak{g} : Consider the product manifold $M^n =$

$M \times \cdots \times M$ and denote by $\text{pr}_i : M^n \rightarrow M$ the projection onto the i -th factor. From these data we can define the differential form

$$p(B) := p(\text{pr}_1^*(B) \wedge \cdots \wedge \text{pr}_n^*(B)) \in \Omega^{2n}(M^n), \quad (5.46)$$

which as a consequence of the top-form equation $d_M B + [A, B] = 0$ is closed. Indeed, from the Leibniz rule for d_{M^n} and the Lie bracket, one observes that

$$\begin{aligned} d_{M^n} p(B) &= p\left(d_{M^n}\left(\text{pr}_1^*(B) \wedge \cdots \wedge \text{pr}_n^*(B)\right) + \left[\sum_{i=1}^n \text{pr}_i^*(A), \text{pr}_1^*(B) \wedge \cdots \wedge \text{pr}_n^*(B)\right]\right) \\ &= \sum_{i=1}^n p\left(\text{pr}_1^*(B) \wedge \cdots \wedge \text{pr}_i^*(d_M B + [A, B]) \wedge \cdots \wedge \text{pr}_n^*(B)\right) = 0. \end{aligned} \quad (5.47)$$

Picking any family of Cauchy surfaces $\Sigma_1, \dots, \Sigma_n \subset M$, one obtains a multi-local conserved charge

$$Q_p(B) := \int_{\Sigma_1 \times \cdots \times \Sigma_n} p(B) \quad (5.48)$$

which depends meromorphically on the spectral parameter $z \in \mathbb{C}P^1$.

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Data Availability All data generated or analyzed during this study are contained in this document.

Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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