Article:

Malbos, P., Massacrier, T. and Struth, G. (2024) Single-set cubical categories and their formalisation with a proof assistant. Journal of Automated Reasoning, 68 (4). 20. ISSN 0168-7433

https://doi.org/10.1007/s10817-024-09710-9

Single-set cubical categories and their formalisation with a proof assistant (extended version)

Philippe Malbos - Tanguy Massacrier - Georg Struth

Abstract – We introduce a single-set axiomatisation of cubical ω-categories, including connections and inverses. We justify these axioms by establishing a series of equivalences between the category of single-set cubical ω-categories, and their variants with connections and inverses, and the corresponding cubical ω-categories. We also report on the formalisation of cubical ω-categories with the Isabelle/HOL proof assistant, which has been instrumental in developing the single-set axiomatisation.

Keywords – Cubical ω-categories, formalised mathematics, Isabelle/HOL, higher-dimensional rewriting.

M.S.C. 2020 – 18N30, 68V15, 03B35, 68Q42.

1. Introduction

Cubical sets and categories are fundamental structures widely used in mathematics and theoretical computer science. Several lines of research have shaped their axioms. Cubical sets provide abstract descriptions of higher-dimensional cubes and their faces. They were first introduced in mathematics for modelling homotopy types [42, 55]. Their algebraic and categorical descriptions were subsequently obtained via topological cubical complexes and similar structures [16, 44]. Cubical categories, which equip cubical sets with compositions along faces of higher-dimensional cubes, were introduced by Brown and Higgins for their generalisation of van Kampen's theorem to higher dimensions [12, 14]. These articles also introduce a notion of connection on cubical sets, essentially an operation of rotation of neighbouring faces. More recently, Lucas [45] has added a notion of inversion for cubes that imposes a groupoid stucture on parts of the cubical structure. See [37] for a discussion of additional structure on cubical sets.

Formally, a cubical set is a family of sets $(K_n)_{n\in\mathbb{N}}$ equipped with face maps $\partial^{\alpha}_{n,i} : K_n \longrightarrow K_{n-1}$ and degeneracy maps $\varepsilon_{n,i} : K_{n-1} \longrightarrow K_n$, for $1 \leqslant i \leqslant n$ and $\alpha \in \{+,-\}$. The former attach faces to higher dimensional cubes; the latter represent lower dimensional cubes as degenerate higher dimensional ones. The cubical structure is imposed by the cubical relations

$$\partial^{\alpha}_{n-1,i}\partial^{\beta}_{n,j} = \partial^{\beta}_{n-1,j-1}\partial^{\alpha}_{n,i} \quad (i < j), \qquad \varepsilon_{n+1,i}\varepsilon_{n,j} = \varepsilon_{n+1,j+1}\varepsilon_{n,i} \quad (i \leqslant j),$$

$$\partial^{\alpha}_{n,i}\varepsilon_{n,j} = \varepsilon_{n-1,j-1}\partial^{\alpha}_{n-1,i} \quad (i < j), \qquad \partial^{\alpha}_{n,i}\varepsilon_{n,j} = id \quad (i = j), \qquad \partial^{\alpha}_{n,i}\varepsilon_{n,j} = \varepsilon_{n-1,j}\partial^{\alpha}_{n-1,i-1} \quad (i > j).$$

In a cubical category, compositions of cubes along their faces are defined for each direction i in a way compatible with face and degeneracy maps. Adding connections and inverses to cubical sets and categories imposes further axioms, as expected.

The category of cubical sets with connections and structure-preserving maps between them forms a strict test category *à la* Grothendieck [51], which makes it suitable for studying homotopy [15, 62]. Compared to simplicial models, they facilitate the handling of products. Further, Al-Agl, Brown and Steiner have shown that categories of cubical categories with connections and those of globular categories (strict ω-categories), another kind of higher categories, are equivalent [2].

In computer science, some fundamental models of homotopy type theory are based on cubical sets [9, 21]; see [3] for an overview. They support a constructive approach to Kan fibrations in the simplicial set model of homotopy type theory [43], several properties of which are undecidable [8]. This prevents a computational interpretation of Voevodsky's univalence axiom, which is possible in the cubical model [10].

A second application of cubical sets in computer science lies in geometrical and topological models of concurrency [26]. A prominent example are higher-dimensional-automata [54, 63]. Here, n-cells represent transitions of a concurrent system where n concurrent events are active, and the cubical cell structure comes from the fact that each concurrent event in an n-dimensional cell can either be active or inactive in each of its $2n$ $(n-1)$-dimensional faces. Higher-dimensional automata subsume many other models of concurrency [63]. They have been studied from homological [33, 34, 41], homotopical [28, 32, 36], language theoretic [24] and algorithmic [27] points of views.

Finally, at the interface of mathematics and computing, cubical categories have recently been proposed as a tool for higher-dimensional rewriting [4, 46], a categorification of term rewriting [61] with applications in categorical algebra. Diagrammatic statements and proofs of abstract rewriting results, such as Newman's lemma or the Church-Rosser theorem, use indeed cubical shapes; confluence diagrams associated with critical pairs, triples and n-tuples form squares, cubes and n-cubes, respectively. A categorical description leads to the notion of polygraphic resolution, which allows the study of homotopical properties of rewriting systems [4, 39, 40]. Explicit constructions of such resolutions lend themselves naturally to a formalisation in cubical categories [45, 46].

The ubiquity of cubical sets and cubical categories alone merits a formalisation with a proof assistant to support reasoning with these highly combinatorial structures in applications (the axioms for cubical ω-categories with connections and inverses in Subsection 3.1 below, for instance, cover about two pages). Yet instead of merely typing an extant axiomatisation into a prover and checking some well known properties by machine, we use the Isabelle/HOL proof assistant [53] to develop an alternative axiomatisation for cubical categories. It is based on single-set categories [47], where only arrows are modelled explicitly, while objects remain implicit via their one-to-one correspondence with identity arrows. Single-set approaches have a long history in category theory [48]; they feature in well-known textbooks [29, 30, 47] and form the basis of three encyclopaedic formalisations of category theory with Isabelle [56–58]. Formally, a single-set category is a set $\mathcal{S}$ with source and target maps $\delta^-, \delta^+ : \mathcal{S} \to \mathcal{S}$

and a composition $\circ$, a partial operation such that $x \circ y$ is defined if and only if $\delta^+ x = \delta^- y$ for $x, y \in \mathcal{S}$, which satisfy

$$\delta^-(x \circ y) = \delta^- x, \qquad \delta^+(x \circ y) = \delta^+ y, \qquad x \circ \delta^+ x = x, \qquad \delta^- x \circ x = x,$$
$$(x \circ y) \circ z = x \circ (y \circ z), \qquad \delta^- = \delta^+ \delta^-, \qquad \delta^+ = \delta^- \delta^+.$$

Identity arrows arise as fixed points of δ^- or equivalently those of δ^+. Single-set categories are therefore algebraically simpler than their classical siblings defined via objects and arrows. Functors and natural transformations are simply functions [29]. Single-set higher categories may thus be more suitable for symbolic reasoning and automated proof search than their classical counterparts. Indeed, single-set globular categories are used widely [2, 13, 47, 59] and have been formalised with Isabelle [19, 20]. Yet single-set cubical categories remain to be defined.

This is not entirely straightforward. We had to introduce symmetry maps that relate the sets of fixed points modelling higher identities in different directions as replacements of the traditional degeneracy maps. Initially, this led to an unwieldy number of axioms, which would have been tedious to use and would have inflated the categorical equivalence proof, which justifies them relative to their classical counterparts.

Isabelle has been instrumental in taming these axioms due to its powerful support for proof automation and counterexample search, which sets it apart from other proof assistants. Its proof automation comes from internal simplification and proof procedures and external proof search tools – so-called hammers – for first-order logic. Counterexample search uses SAT solvers and decision procedures, for instance for linear arithmetic. This combination supports not only a natural mathematical workflow with proofs and refutations, it also allows checking axiom systems for redundancy (via deduction) and irredundancy (via counterexamples) rapidly and effectively. It has already proved its worth for developing other algebraic axiomatisations [19, 23, 31]. Here, Isabelle has helped us to bring the single-set axiomatisation for cubical categories to a manageable size without compromising its structural coherence, to simplify candidate axioms and to analyse candidate axioms that emerged during our development rapidly. Starting from an around 40 initial candidate axioms, we have used Isabelle in an iterative process, simplifying candidate axioms and removing redundant ones, then attempting an equivalence proof, and adding new axioms if that failed. Without our confidence in Isabelle's automated proof support, we might not have attempted this research.

The single-set axiomatisation for cubical categories thus forms the main conceptual contribution in this article. Our main technical contribution consists in the proofs of categorical equivalence mentioned, and our main engineering contribution is the formalisation of a mathematical component for cubical categories with Isabelle. In combination, these results constitute a case study in innovative, not merely reconstructive formalised mathematics.

The overall structure of this article is simple: our axioms for single-set cubical categories are introduced in Section 2, the proofs that the resulting categories are essentially the same as their classical counterparts are given in Section 3, our Isabelle formalisation and the workflow leading to our axioms are discussed in Section 4. Finally, we summarise our results and present some avenues for future work in Section 5.

More specifically, we recall the variant of single-set categories [22, 25, 60], on which our axioms for single-set cubical categories are based, in Subsection 2.1. Subsection 2.2 introduces single-set cubical ω-categories, axiomatised as a set $\mathcal{S}$ equipped with families of maps indexed by directions $i \in \mathbb{N}_+$: face maps δ_i^- and δ_i^+, composition maps $\circ_i$, symmetry maps s_i and reverse symmetry maps $\widetilde{s_i}$. We show how single-set cubical n-categories appear as truncations. We define the category SCub_ω with single-set cubical ω-categories as objects and functions corresponding to functors of classical ω-categories as morphisms. We also list some structural properties of these categories. In Subsection 2.3 we add connections, in Subsection 2.4 we further add inverses in each dimension greater than p. This yields the categories $\mathsf{SCub}_\omega^\gamma$ and $\mathsf{SCub}_{(\omega,p)}^\gamma$ as well as truncated variants for each dimension n. Inverses are relevant to constructive proofs in homotopy type theory and higher-dimensional rewriting.

In Subsection 3.1 we recall the classical axioms for cubical ω- and n-categories, including connections and inverses. This leads to the categories Cub_ω, $\mathsf{Cub}_\omega^\Gamma$ and $\mathsf{Cub}_{(\omega,p)}^\Gamma$ with classical cubical ω-categories as objects and functors as morphisms. We then present proofs of the equivalences $\mathsf{SCub}_\omega \simeq \mathsf{Cub}_\omega$ in Theorem 3.2.1, $\mathsf{SCub}_\omega^\gamma \simeq \mathsf{Cub}_\omega^\Gamma$ in Theorem 3.3.1 and $\mathsf{SCub}_{(\omega,p)}^\gamma \simeq \mathsf{Cub}_{(\omega,p)}^\Gamma$ in Theorem 3.4.1 in Subsections 3.2, 3.3 and 3.4 respectively. Straightforward modifications yield similar equivalences between n-categories, which we do not list explicitly.

Subsection 4.1 contains a brief overview of Isabelle, Subsection 4.2 recalls the formalisation of single-set categories with Isabelle [60], which underlies our formalisation of cubical ω-categories with and without connections in Subsection 4.3. For technical reasons, we do not formalise cubical (ω, p)-categories. But we show how a non-trivial proof about $(\omega, 0)$-categories (of Proposition 2.4.8) can be formalised with our axiomatisation at the same level of granularity.

While this article can be read as an exercise in formalised mathematics, we did not aim to formalise all our results, as it would distract from our main goal: to showcase the unique benefits of Isabelle's proof automation in the analysis of higher categories. Alternatively, disregarding Section 4, it can be read as a mathematical paper with contributions beyond Isabelle. While the calculational lemmas in Section 2 have been checked by machine, and proofs therefore been omitted, the categorical equivalences in Section 3 have not been formalised, though that would have been possible at least in parts. Once again: our main use case for Isabelle in this work has been the development of the axioms in SCub_ω, $\mathsf{SCub}_\omega^\gamma$ and $\mathsf{SCub}_{(\omega,p)}^\gamma$. Further work with proof assistants on higher categories, higher rewriting and higher automata is left for future work. Our Isabelle components for cubical categories, including a PDF proof document, can be found in the Archive of Formal Proofs [52].

2. Single-set cubical categories

In this section we introduce our axiomatisation of single-set cubical categories. In Subsection 2.1 we recall a previous axiomatisation of single-set categories. In Subsection 2.2 we introduce single-set cubical ω-categories and n-categories. Extensions of these categories with connections and inverses are presented in Subsections 2.3 and 2.4.

2.1. Single-set categories

We start with recalling the definition and basic properties of single-set categories. While any axiomatisation would work for our purposes, we have chosen one in which the partiality of arrow composition is captured by a multioperation that maps pairs of elements to sets of elements, including the empty set [22, 25]. It is already well-supported by Isabelle components [60] and has previously served as a basis for formalising globular single-set ω-categories [19, 20].

2.1.1. A *single-set category* $(\mathcal{S}, \delta^-, \delta^+, \odot)$ consists of the following data:

- a set $\mathcal{S}$ of *cells*,
- *face maps* $\delta^\alpha : \mathcal{S} \to \mathcal{S}$ for $\alpha \in \{-, +\}$, which are extended to $\mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ by taking images,
- a *composition map* $\odot : \mathcal{S} \times \mathcal{S} \to \mathcal{P}(\mathcal{S})$, which is extended to $\mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ as

$$X \odot Y = \bigcup_{x \in X, y \in Y} x \odot y, \qquad \text{for all } X, Y \subseteq \mathcal{S}\,.$$

It satisfies, for all $x, y, z \in \mathcal{S}$,

(i) *associativity*: $\{x\} \odot (y \odot z) = (x \odot y) \odot \{z\}$,

(ii) *units*: $x \odot \delta^+ x = \{x\}$ and $\delta^- x \odot x = \{x\}$,

(iii) *locality*: $x \odot y \neq \emptyset \Leftrightarrow \delta^+ x = \delta^- y$,

(iv) *functionality*: $\forall z, z' \in x \odot y,\ z = z'$.

The cells of single-set categories correspond to arrows of classical categories. The face maps δ^- and δ^+ send each cell in $\mathcal{S}$ to its *source cell* and *target cell*, respectively, which are *identity cells*. These are in bijective correspondence with objects of classical categories.

Henceforth we tacitly assume that upper indices such as α in all face maps δ^α range over $\{-, +\}$. We also write $\delta^{-\alpha}$ to indicate that values of α are exchanged relative to an occurrence of δ^α. Further, in order to avoid lengthy technical terms, we henceforth refer to single-set categories simply as categories wherever possible.

2.1.2. Remark. Omitting the functionality axiom and the right-to-left direction of the locality axiom from the definition of single-set categories yields axioms for *catoids*. Removing locality, functionality and the unit axioms yields *multisemigroups*. See [25] for details.

2.1.3. Composition as a partial operation. Two cells x, y of a category $\mathcal{S}$ are *composable* if $x \odot y \neq \emptyset$, in which case we write $\Delta(x, y)$ for short. Functionality makes $\odot$ a partial operation $\circ : \Delta \hookrightarrow \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, which sends each $(x, y) \in \mathcal{S} \times \mathcal{S}$ to the unique $z \in x \odot y$ whenever $\Delta(x, y)$. As we can recover

$$x \odot y = \begin{cases} \{x \circ y\} & \text{if } \Delta(x, y), \\ \emptyset & \text{otherwise} \end{cases}$$

from δ^-, δ^+ and $\circ$, we henceforth write $(\mathcal{S}, \delta^-, \delta^+, \circ)$ instead of $(\mathcal{S}, \delta^-, \delta^+, \odot)$ and work with $\circ$ instead of $\odot$.

2.1.4. A *morphism* $f : \mathcal{S} \to \mathcal{S}'$ *of categories* $\mathcal{S}$ and $\mathcal{S}'$ is a map satisfying, for all $x, y \in \mathcal{S}$,

$$f \delta^\alpha = \delta'^\alpha f \qquad \text{and} \qquad \Delta(x, y) \ \Rightarrow\ f(x \circ y) = f(x) \circ' f(y).$$

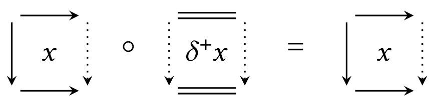
Such morphisms correspond to functors between classical categories.

2.1.5. Example. With $\circ$, the associativity axiom of categories becomes

$$\Delta(x, y \circ z) \wedge \Delta(y, z) \Leftrightarrow \Delta(x, y) \wedge \Delta(x \circ y, z),$$
$$\Delta(x, y \circ z) \wedge \Delta(y, z) \Rightarrow x \circ (y \circ z) = (x \circ y) \circ z.$$

By the first law, the left-hand side of the associativity law is defined if and only if its right-hand side is. By the second law, the two sides of this law are equal if either side is defined. Likewise, the unit axioms simplify to $\Delta(x, \delta^+x)$, $\Delta(\delta^-x, x)$, $x \circ \delta^+x = x$ and $\delta^-x \circ x = x$.

2.1.6. Example. In preparation for the cubical categories below, suppose that the cells of a category are formed by squares that can be composed horizontally, for instance the commuting diagrams in an arrow category. The right unit axiom $x \circ \delta^+x = x$ above can then be illustrated as



The upper and lower faces of δ^+x are drawn as equality arrows to indicate that the left and right faces of this cell, shown as dotted arrows, are equal. We assign a more precise semantics to such cubes below.

We frequently need the following laws in calculations. They have been verified with Isabelle [60].

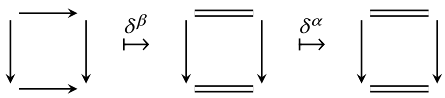
2.1.7. Lemma. *Let $\mathcal{S}$ be a category. Then*

(i) $\delta^\alpha\delta^\beta = \delta^\beta$,

(ii) $\delta^-x = x \Leftrightarrow \delta^+x = x$ *for all* $x \in \mathcal{S}$,

(iii) $\delta^-(x \circ y) = \delta^-x$ *and* $\delta^+(x \circ y) = \delta^+y$ *for all* $x, y \in \mathcal{S}$ *such that* $\Delta(x, y)$.

2.1.8. Example. We can illustrate Lemma 2.1.7(i) as



2.1.9. Fixed points. Lemma 2.1.7(i) and (ii) imply that the sets of fixed points of δ^- and δ^+ in a category $\mathcal{S}$ are equal and also equal to the sets of all left and right identities in $\mathcal{S}$. We write $\mathcal{S}^\delta$ for the resulting set of all identities. It corresponds to the set of objects in (small) classical categories. In our examples, identities are illustrated by degenerated cubes, in which some opposite arrows are equality arrows.

2.2. Single-set cubical categories

Our single-set axiomatisation of cubical ω-categories is based on a family of categories $(\mathcal{S}, \delta_i^-, \delta_i^+, \circ_i)$ for each $i \in \mathbb{N}_+ = \mathbb{N}\setminus\{0\}$. We thus equip our previous notation with indices. In particular we write $\mathcal{S}^i$ for the set of fixed points of δ_i^α.

2.2.1. A *single-set cubical ω-category* consists of a family of single-set categories $(\mathcal{S}, \delta_i^-, \delta_i^+, \circ_i)_{i\in\mathbb{N}_+}$ with *symmetry maps* $s_i : \mathcal{S} \to \mathcal{S}$ and *reverse symmetry maps* $\widetilde{s}_i : \mathcal{S} \to \mathcal{S}$ for each $i \in \mathbb{N}_+$. These satisfy, for all $w, x, y, z \in \mathcal{S}$ and $i, j \in \mathbb{N}_+$,

(i) $\delta_i^\alpha \delta_j^\beta = \delta_j^\beta \delta_i^\alpha$ if $i \neq j$,

(ii) $\delta_i^\alpha (x \circ_j y) = \delta_i^\alpha x \circ_j \delta_i^\alpha y$ if $i \neq j$ and $\Delta_j(x, y)$,

(iii) $(w \circ_i x) \circ_j (y \circ_i z) = (w \circ_j y) \circ_i (x \circ_j z)$ if $i \neq j$, $\Delta_i(w, x)$, $\Delta_i(y, z)$, $\Delta_j(w, y)$ and $\Delta_j(x, z)$,

(iv) $s_i(\mathcal{S}^i) \subseteq \mathcal{S}^{i+1}$ and $\widetilde{s}_i(\mathcal{S}^{i+1}) \subseteq \mathcal{S}^i$,

(v) $\widetilde{s}_i s_i x = x$ and $s_i \widetilde{s}_i y = y$ if $x \in \mathcal{S}^i$ and $y \in \mathcal{S}^{i+1}$,

(vi) $\delta_j^\alpha s_j x = s_j \delta_{j+1}^\alpha x$ and $\delta_i^\alpha s_j x = s_j \delta_i^\alpha x$ if $i \neq j, j+1$ and $x \in \mathcal{S}^j$,

(vii) $s_i(x \circ_{i+1} y) = s_i x \circ_i s_i y$ and $s_i(x \circ_j y) = s_i x \circ_j s_i y$ if $j \neq i, i+1$, $x, y \in \mathcal{S}^i$ and $\Delta_j(x, y)$,

(viii) $s_i x = x$ if $x \in \mathcal{S}^i \cap \mathcal{S}^{i+1}$,

(ix) $s_i s_j x = s_j s_i x$ if $|i - j| \geqslant 2$ and $x \in \mathcal{S}^i \cap \mathcal{S}^j$,

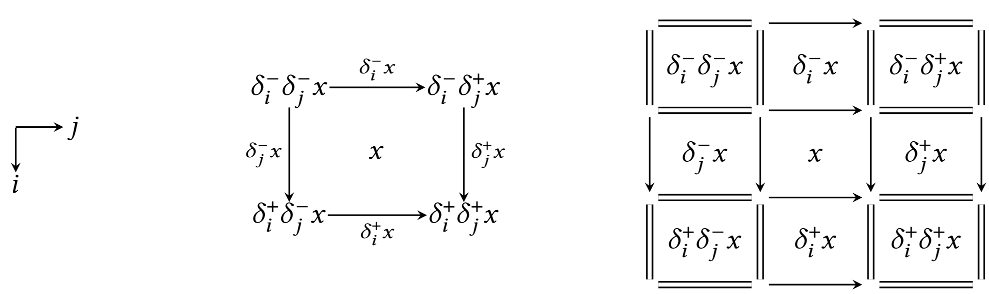
(x) $\exists k \in \mathbb{N}\ \forall i \geqslant k+1,\ x \in \mathcal{S}^i$.

A *single-set cubical n-category*, for $n \in \mathbb{N}$, is defined by the same data, but the index $i \in \mathbb{N}_+$ is restricted to $1 \leqslant i \leqslant n$, and the s_i and $\widetilde{s}_i$ to $1 \leqslant i \leqslant n-1$. Likewise, all ω-axioms are restricted to these ranges, and Axiom (x) is omitted, as it is now entailed. All results for ω-categories in this article restrict to n-categories. As the ω-categories or n-categories considered in this article are usually cubical, we drop this adjective whenever possible. Hence we often refer to single-set cubical ω-categories simply as ω-categories and likewise for n-categories.

As previously, we call the elements of ω-categories *cells*. The face maps δ_i^- and δ_i^+ attach *lower* and *upper faces in direction i* to them. The symmetry maps s_i and reverse symmetry maps $\widetilde{s}_i$ rotate identities for $\circ_i$ to identities for $\circ_{i+1}$ and vice versa.

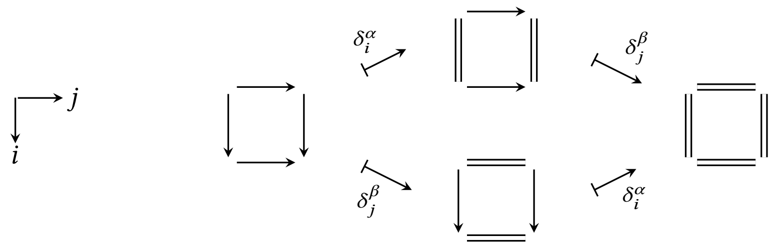
2.2.2. Explanation of axioms. The cells of categories model higher-dimensional cubes, possibly with degenerate faces, which can be composed by gluing them together along their faces. While the examples in Subsection 2.1 provide some intuition for squares in low dimensions, we can illustrate higher dimensional cells and their compositions only through projections.

A cell x and its faces in the directions i and j can be illustrated as

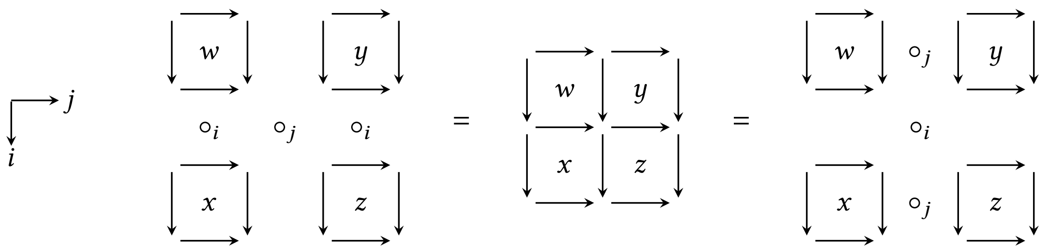


The arrows on the left indicate the directions i and j. The diagram on the right shows the faces of x as degenerate cells. They are identities of the composition in the same direction. Many of the axioms ω-categories can be illustrated by such diagrams.

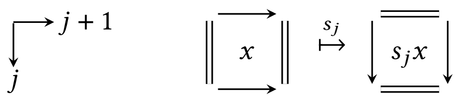
- Axiom (i) determines the cubical cell shape. It can be depicted as



- Axioms (i) and (ii) make face maps morphisms and hence functors with respect to the underlying categories in any other direction.
- The *interchange law* in Axiom (iii) makes composition in any direction bifunctorial with respect to the compositions in any other direction. It can be depicted as



- The *typing axioms* in (iv) restrict s_i to $\mathcal{S}^i \to \mathcal{S}^{i+1}$ and $\widetilde{s_i}$ to $\mathcal{S}^{i+1} \to \mathcal{S}^i$. Though the s_i and $\widetilde{s_i}$ are defined as total maps on $\mathcal{S}$, only their typed versions matter.
- Axiom (v) states that each $s_i : \mathcal{S}^i \to \mathcal{S}^{i+1}$ and $\widetilde{s_i} : \mathcal{S}^{i+1} \to \mathcal{S}^i$ forms a bijective pair.
- Axiom (vi) captures the action of a symmetry map s_j on a cell $x \in \mathcal{S}^j$: it rotates it together with its faces into a cell in $\mathcal{S}^{j+1}$:



- Axioms (vi) and (vii) state that the symmetry maps s_i are morphisms, hence functors, with respect to the categories in any direction $j \neq i, i+1$.
- The *dimensionality* axiom (viii) imposes that s_i is an identity map on $\mathcal{S}^i \cap \mathcal{S}^{i+1}$.
- Axiom (ix) is a *braiding* axiom for symmetries. An illustration would require higher-dimensional cubes.

– Axiom (x) imposes that every cell has finite dimension, as defined below. It is important for the proof of equivalence between single-set cubical categories and their classical counterparts as well as for the coherence of the entire approach.

A more structural explanation of symmetries and their reverses is given in the following subsections.

2.2.3. Lattice of fixed points. Let $\mathcal{S}$ be an ω-category. It is easy to see that the set of all $\mathcal{S}^I = \bigcap_{i\in I} \mathcal{S}^i$, for $I \subseteq \mathbb{N}_+$, forms a lattice with respect to set inclusion. In fact, $\mathcal{S}^I \subseteq \mathcal{S}^J$ if and only if $I \supseteq J$. In this lattice, the S^I with co-finite I model sets of cells of finite dimension.

Formally, a cell $x \in \mathcal{S}$ *has dimension at most* $k \in \mathbb{N}_+$ if $x \in \mathcal{S}^I$ for some $I \subseteq \mathbb{N}_+$ with $|\mathbb{N}_+ \setminus I| = k$; it *has dimension* k if k is the least positive integer for which it has dimension at most k.

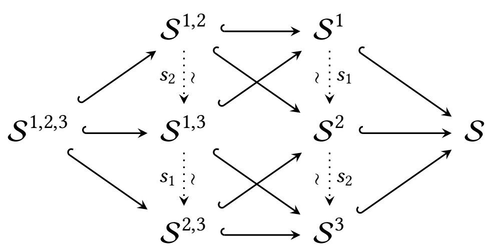
If the complements of the co-finite sets $I, J \subseteq \mathbb{N}_+$ have the same (finite) cardinality, then the restrictions of symmetries, their reverses and their compositions induce bijections $\mathcal{S}^I \simeq \mathcal{S}^J$. These identify cells of the same dimension; they respect face maps and compositions.

For finite $I = \{i_1, \ldots, i_k\}$, we write $\mathcal{S}^{i_1,\ldots,i_k}$. In this case, Axiom (2.2.1) (i) implies that $x \in \mathcal{S}^{i_1,\ldots,i_k}$ if and only if $\delta_{i_1}^{\alpha_1} \ldots \delta_{i_k}^{\alpha_k} x = x$ for all $\alpha_1, \ldots, \alpha_k$.

Finally, we write $\mathcal{S}^{>n}$ for $\mathcal{S}^I$ when $I = \{i \in \mathbb{N} \mid i > n\}$. All cells in $\mathcal{S}^{>n}$ have dimension at most n. Axiom (x) implies that $\mathcal{S} = \bigcup_{n\geqslant 0} \mathcal{S}^{>n}$. Hence every cell $x \in \mathcal{S}$ has indeed some finite dimension, which is crucial for the constructions of the equivalence proof relating single-set cubical categories with their classical counterparts in Section 3. In particular, Axiom (x) thus imposes that the gradation of $\mathcal{S}$ in terms of $\mathcal{S}^{>n}$ corresponds to the gradation of the cubical sets K_n shown in the introduction.

2.2.4. Remark. The definition of symmetry in (2.2.1) differs from the notions of symmetric cubical monoid and category, introduced by Grandis and Mauri [37] and Grandis [35], respectively.

2.2.5. Example. The lattice of fixed points for the 3-category $\mathcal{S}$ is given by the Hasse diagram



All cells in $\mathcal{S}^{1,2,3}$ have dimension 0, those in $\mathcal{S}^{1,2}$, $\mathcal{S}^{1,3}$ and $\mathcal{S}^{2,3}$ have dimension at most 1, those in $\mathcal{S}^1$, $\mathcal{S}^2$ and $\mathcal{S}^3$ dimension at most 2, and those in $\mathcal{S}$ dimension at most 3.

2.2.6. Categories of cubical categories. A *morphism* $f : \mathcal{S} \to \mathcal{S}'$ *of* ω*-categories* $\mathcal{S}$ and $\mathcal{S}'$ is a morphism of the underlying categories, for each $i \in \mathbb{N}_+$, which preserves symmetries restricted to their types. Hence, for $i \geqslant 1$ and $x \in \mathcal{S}^i$, $f s_i x = s'_i f x$. This defines the category SCub_ω of single-set cubical ω-categories.

Owing to the definition in (2.1.4), a morphism $f : \mathcal{S} \to \mathcal{S}'$ of ω-categories restricts to a map $\mathcal{S}^I \to \mathcal{S}'^I$ for each $I \subseteq \mathbb{N}_+$. These restrictions also commute with the symmetries.

A *morphism of* n*-categories* is defined as above, but with symmetries ranging over $1 \leqslant i \leqslant n-1$. This defines the category SCub_n of single-set cubical n-categories.

2.2.7. Lemma. *Morphisms in* SCub_ω *preserve typed reverse symmetry maps:* $f \tilde{s}_i x = \tilde{s}'_i f x$*, for all* $f : \mathcal{S} \to \mathcal{S}'$*,* $i \in \mathbb{N}_+$ *and* $x \in \mathcal{S}^{i+1}$.

2.2.8. Truncations. There is a truncation functor $U^n_m : \mathrm{SCub}_n \to \mathrm{SCub}_m$, for all $1 \leqslant m \leqslant n$, which forgets the k-dimensional structure for each $k > m$. It sends each n-category $\mathcal{S}$ to the m-category on the set $\mathcal{S}^{m+1,\dots,n}$ and restricts the ranges of face maps and compositions to $1 \leqslant i \leqslant m$, as well as that of the symmetry maps to $1 \leqslant i \leqslant m-1$.

There is also a truncation functor $U_m : \mathrm{SCub}_\omega \to \mathrm{SCub}_m$, which sends each ω-category $\mathcal{S}$ to the m-category with set $\mathcal{S}^{>m}$ and restricts face and symmetry maps as well as compositions as above.

2.2.9. Properties of symmetries. The following facts provide further structural properties of symmetries and reverse symmetries. They have been proved with Isabelle.

2.2.10. Lemma. *Let $\mathcal{S}$ be an ω-category. For all $i, j \in \mathbb{N}_+$,*

- **(i)** $\delta^\alpha_{j+1} s_j x = s_j \delta^\alpha_j x$ *if* $x \in \mathcal{S}^j$,
- **(ii)** $s_i(x \circ_i y) = s_i x \circ_{i+1} s_i y$ *if* $x, y \in \mathcal{S}^i$ *and* $\Delta_j(x, y)$,
- **(iii)** *Yang-Baxter:* $s_i s_{i+1} s_i x = s_{i+1} s_i s_{i+1} x$ *if* $x \in \mathcal{S}^{i,i+1}$.

An automatic proof of Lemma 2.2.10(ii), called *sym-func1* in our Isabelle component, is shown in Section 4.3.

Using Axiom 2.2.1(v), we can further derive properties for $\widetilde{s}$ that are dual to the symmetry axioms.

2.2.11. Lemma. *Let $\mathcal{S}$ be an ω-category. For all $i, j \in \mathbb{N}_+$,*

- **(i)** *if* $x \in \mathcal{S}^{j+1}$, *then*

$$\delta^\alpha_i \widetilde{s}_j x = \begin{cases} \widetilde{s}_j \delta^\alpha_{j+1} x & \text{if } i = j, \\ \widetilde{s}_j \delta^\alpha_j x & \text{if } i = j+1, \\ \widetilde{s}_j \delta^\alpha_i x & \text{otherwise.} \end{cases}$$

- **(ii)** *if* $x, y \in \mathcal{S}^{i+1}$ *and* $\Delta_j(x, y)$, *then*

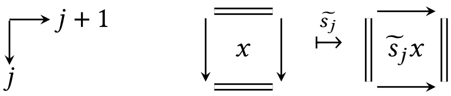
$$\widetilde{s}_i(x \circ_j y) = \begin{cases} \widetilde{s}_i x \circ_{i+1} \widetilde{s}_i y & \text{if } j = i, \\ \widetilde{s}_i x \circ_i \widetilde{s}_i y & \text{if } j = i+1, \\ \widetilde{s}_i x \circ_j \widetilde{s}_i y & \text{otherwise,} \end{cases}$$

- **(iii)** $\widetilde{s}_i x = x$ *if* $x \in \mathcal{S}^{i,i+1}$,
- **(iv)** *if* $|i - j| \geqslant 2$, $x \in \mathcal{S}^{i,j+1}$, $y \in \mathcal{S}^{i+1,j}$ *and* $z \in \mathcal{S}^{i+1,j+1}$, *then*

$$s_i \widetilde{s}_j x = \widetilde{s}_j s_i x, \qquad \widetilde{s}_i s_j y = s_j \widetilde{s}_i y, \qquad \widetilde{s}_i \widetilde{s}_j z = \widetilde{s}_j \widetilde{s}_i z,$$

- **(v)** $\widetilde{s}_i \widetilde{s}_{i+1} \widetilde{s}_i x = \widetilde{s}_{i+1} \widetilde{s}_i \widetilde{s}_{i+1} x$ *if* $x \in \mathcal{S}^{i+1,i+2}$.

Lemma 2.2.11(i) can be depicted, for $x \in \mathcal{S}^{j+1}$, as



An interactive proof of the third case in this part of Lemma 2.2.11, called *inv-sym-face* in our Isabelle component, is also shown in Subsection 4.3.

2.3. Connections

We now add connections to cubical categories, translating the approach of Al-Agl, Brown and Steiner [2].

2.3.1. *Connection maps* for a cubical ω-category $\mathcal{S}$ are maps $\gamma_i^\alpha : \mathcal{S} \to \mathcal{S}$, for $i \in \mathbb{N}_+$ and $\alpha \in \{-,+\}$, satisfying, for all $i, j \in \mathbb{N}_+$,

(i) $\delta_j^\alpha \gamma_j^\alpha x = x$, $\delta_{j+1}^\alpha \gamma_j^\alpha x = s_j x$ and $\delta_i^\alpha \gamma_j^\beta x = \gamma_j^\beta \delta_i^\alpha x$ if $i \neq j, j+1$ and $x \in \mathcal{S}^j$,

(ii) if $j \neq i, i+1$ and $x, y \in \mathcal{S}^i$, then

$$\begin{aligned} \Delta_{i+1}(x,y) &\Rightarrow \gamma_i^+(x \circ_{i+1} y) = (\gamma_i^+ x \circ_{i+1} s_i x) \circ_i (x \circ_{i+1} \gamma_i^+ y), \\ \Delta_{i+1}(x,y) &\Rightarrow \gamma_i^-(x \circ_{i+1} y) = (\gamma_i^- x \circ_{i+1} y) \circ_i (s_i y \circ_{i+1} \gamma_i^- y), \\ \Delta_j(x,y) &\Rightarrow \gamma_i^\alpha(x \circ_j y) = \gamma_i^\alpha x \circ_j \gamma_i^\alpha y, \end{aligned}$$

(iii) $\gamma_i^\alpha x = x$ if $x \in \mathcal{S}^{i,i+1}$,

(iv) $\gamma_i^+ x \circ_{i+1} \gamma_i^- x = x$ and $\gamma_i^+ x \circ_i \gamma_i^- x = s_i x$ if $x \in \mathcal{S}^i$,

(v) $\gamma_i^\alpha \gamma_j^\beta x = \gamma_j^\beta \gamma_i^\alpha x$ if $|i-j| \geqslant 2$ and $x \in \mathcal{S}^{i,j}$,

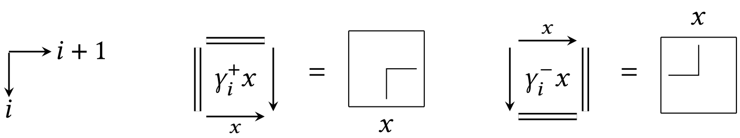
(vi) $s_{i+1} s_i \gamma_{i+1}^\alpha x = \gamma_i^\alpha s_{i+1} x$ if $x \in \mathcal{S}^{i,i+1}$.

Connection maps for a cubical n-category $\mathcal{S}$ are maps $\gamma_i^\alpha : \mathcal{S} \to \mathcal{S}$ in the range $1 \leqslant i \leqslant n-1$ satisfying the above axioms within appropriate ranges.

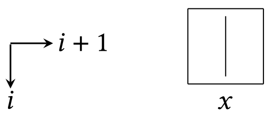
As for symmetries, we henceforth assume that upper indices of connection maps range over $\{-,+\}$.

Connection maps γ_i^α restrict to the type $\mathcal{S}^i \to \mathcal{S}$, and we restrict our attention to these types.

2.3.2. Explanation of axioms. Henceforth we illustrate connection maps as

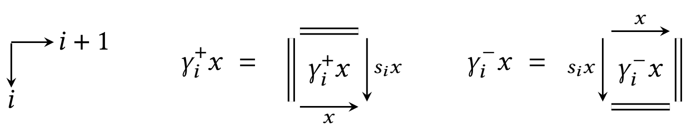


The two diagrams on the left of the equations follow the style of previous sections, those on the right are standard in the literature [2, 14, 17]. Fixed points in direction i are shown as

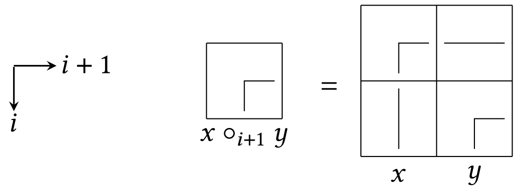


Using these diagrams, we can explain some of the axioms with connections.

– Axiom (i) determines the cubical shape of $\gamma_i^\alpha x$:



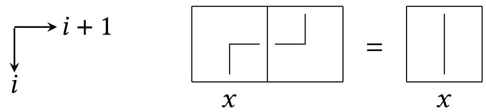
– The first two equations of the *corner axiom* (ii) can be shown as



– Axioms (i) and (ii) impose that connection maps are morphisms and hence functors with respect to the underlying categories in any direction $j \neq i, i+1$.

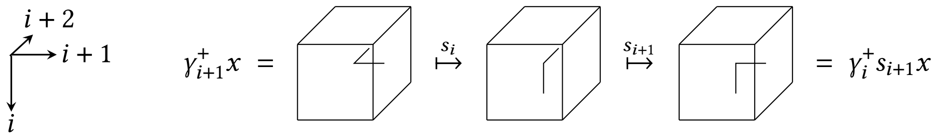
– By Axiom (iii), γ_i^α is an identity map on $\mathcal{S}^{i,i+1}$.

– The *zigzag axiom*, (iv) is depicted as



– The *braiding axiom* (v) is hard to illustrate: four dimensions would be required for drawing it.

– Finally, Axiom (vi) relates connections in different directions:



2.3.3. Category of ω-categories with connections. A *morphism* $f : \mathcal{S} \to \mathcal{S}'$ *of ω-categories with connections* is a morphism of ω-categories that preserves connections: $f\gamma_i^\alpha x = \gamma'^\alpha_i fx$, for all $i \in \mathbb{N}_+$ and $x \in \mathcal{S}^i$. This defines the category $\mathsf{SCub}_\omega^\gamma$ of single-set cubical ω-categories with connections. In a same way, we define the category SCub_n^γ of n-categories with connections.

For $1 \leqslant m \leqslant n$, there is a truncation functor $U_m^n : \mathsf{SCub}_n^\gamma \to \mathsf{SCub}_m^\gamma$ that forgets the k-dimensional structure for $k > m$. It sends any n-category with connections $\mathcal{S}$ to the m-category with connections with set $\mathcal{S}^{m+1,\dots,n}$, keeping only the face maps and compositions up to m and the symmetries and connections up to $m-1$.

There is also a truncation functor $U_m : \mathsf{SCub}_\omega^\gamma \to \mathsf{SCub}_m^\gamma$ that forgets the k-dimensional structure for $k > m$. It sends any ω-category with connections $\mathcal{S}$ to the m-category with connections with set $\mathcal{S}^{>m}$, keeping only the face maps and compositions indexed up to m and the symmetries and connections indexed up to $m-1$.

The following properties of connections have been proved using Isabelle.

2.3.4. Lemma. *Let $\mathcal{S}$ be an ω-category with connections. For all $i, j \in \mathbb{N}_+$,*

(i) $\delta^{\alpha}_{j}\gamma^{-\alpha}_{j}x = \delta^{\alpha}_{j+1}x$ *and* $\delta^{\alpha}_{j+1}\gamma^{-\alpha}_{j}x = \delta^{\alpha}_{j+1}x$ *if* $x \in \mathcal{S}^{i}$,

(ii) *if* $j \neq i, i+1$, $x, y \in \mathcal{S}^{i}$ *and* $\Delta_{i+1}(x, y)$, *then*

$$\gamma^{+}_{i}(x \circ_{i+1} y) = (\gamma^{+}_{i}x \circ_{i} x) \circ_{i+1} (s_{i}x \circ_{i} \gamma^{+}_{i}y), \quad \text{and} \quad \gamma^{-}_{i}(x \circ_{i+1} y) = (\gamma^{-}_{i}x \circ_{i} s_{i}y) \circ_{i+1} (y \circ_{i} \gamma^{-}_{i}y),$$

(iii) $\gamma^{\alpha}_{i}s_{j}x = s_{j}\gamma^{\alpha}_{i}x$ *and* $\gamma^{\alpha}_{i}\widetilde{s}_{j}y = \widetilde{s}_{j}\gamma^{\alpha}_{i}y$ *if* $|i - j| \geqslant 2$, $x \in \mathcal{S}^{i,j}$ *and* $y \in \mathcal{S}^{i,j+1}$,

(iv) $\widetilde{s}_{i}\widetilde{s}_{i+1}\gamma^{\alpha}_{i}x = \gamma^{\alpha}_{i+1}\widetilde{s}_{i+1}x$ *if* $x \in \mathcal{S}^{i,i+2}$.

2.4. Inverses

Applications in higher rewriting and homotopy type theory require cubical (ω, p)-categories, where cells of dimension strictly greater than p are invertible. More specifically, $p = 0$ is needed for homotopy type theory [9] and the categorification of abstract rewriting, while the categorification of string rewriting requires $p = 1$, and that of term and diagram rewriting $p = 2$ [4]. Here, we translate an approach introduced by Lucas [45] to single-sets, following his notation closely.

2.4.1. Single-set cubical categories with inverses. A cell x of an ω-category with connections $\mathcal{S}$ is r_i*-invertible*, for $i \in \mathbb{N}_{+}$, if there exists a cell $y \in \mathcal{S}$ such that

$$\Delta_{i}(x, y), \qquad x \circ_{i} y = \delta^{-}_{i}x, \qquad \Delta_{i}(y, x), \qquad y \circ_{i} x = \delta^{+}_{i}x.$$

A cell $x \in \mathcal{S}^{>n}$ *has an* r_i*-invertible* $n-1$*-shell*, for $k, i \in \mathbb{N}_{+}$, if $\delta^{\alpha}_{j}x$ is r_i-invertible for each $1 \leqslant j \leqslant n$ with $j \neq i$.

A *single-set cubical* (ω, p)*-category (with connections)* is a single-set cubical (ω, p)-category (with connections) $\mathcal{S}$ such that every cell in $\mathcal{S}^{>n}$ with an r_i-invertible $n-1$-shell is r_i-invertible for all $n \geqslant p+1$ and $1 \leqslant i \leqslant n$.

We have shown with Isabelle that the r_i-inverse of any x is uniquely defined. We therefore write $r_i x$ for the r_i-inverse of x.

We restrict these definitions to n-categories with connections as usual, removing the indices outside the range $1 \leqslant i \leqslant n$. In particular, a *single-set cubical* (n, p)*-category* is a single-set cubical n-category $\mathcal{S}$ such that every cell in $\mathcal{S}^{k+1,\dots,n}$ with an r_i-invertible $k-1$-shell is r_i-invertible for all $p+1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant k$,

2.4.2. Category of (ω, p)-categories. A *morphism of* (ω, p)*-categories* is simply a morphism in $\mathsf{SCub}^{\gamma}_{\omega}$. Inverses are preserved because $\Delta_{i}(x, r_{i}x)$ implies $\Delta_{i}(f(x), f(r_{i}x))$ and $f(x) \circ_{i} f(r_{i}x) = \delta^{-}_{i}f(x)$, and likewise for the opposite order of composition. Thus $f(x)$ is r_i-invertible with inverse $f(r_{i}x)$. The situation for (n, p)-categories is similar. This defines the categories $\mathsf{SCub}^{\gamma}_{(\omega,p)}$ and $\mathsf{SCub}^{\gamma}_{(n,p)}$.

2.4.3. Explanation of inverse maps. A $(1, 0)$-category $\mathcal{S}$ is a category with a map $r_1 : \mathcal{S} \to \mathcal{S}$ such that $\Delta_{1}(x, r_{1}x)$, $\Delta_{1}(r_{1}x, x)$, $x \circ_{1} r_{1}x = \delta^{-}_{1}x$ and $r_{1}x \circ_{1} x = \delta^{+}_{1}x$, for every $x \in \mathcal{S}$. This defines a groupoid. The cell x is sent by r_1 to the backward arrow in the following diagram

$$r_{1}\left(\delta^{-}_{1}x \xrightarrow{x} \delta^{+}_{1}x\right) \;=\; \delta^{+}_{1}x \leftarrow \delta^{-}_{1}x.$$

A $(2,0)$-category C is a 2-category with maps $r_1, r_2 : \mathcal{S} \to \mathcal{S}$ such that $\Delta_i(x, r_i x)$, $\Delta_i(r_i x, x)$, $x \circ_i r_i x = \delta_i^- x$, $r_i x \circ_i x = \delta_i^+ x$, for $i \in \{1,2\}$ and $x \in \mathcal{S}$. The inverses of x are depicted as

$$r_1\left(\begin{array}{ccc} & \xrightarrow{\delta_1^- x} & \\ \delta_2^- x \big\downarrow & x & \big\downarrow \delta_2^+ x \\ & \xrightarrow[\delta_1^+ x]{} & \end{array}\right) = \begin{array}{ccc} & \xrightarrow{\delta_1^+ x} & \\ r_1\delta_2^- x \big\uparrow & r_1 x & \big\uparrow r_1\delta_2^+ x \\ & \xrightarrow[\delta_1^- x]{} & \end{array}, \qquad r_2\left(\begin{array}{ccc} & \xrightarrow{\delta_1^- x} & \\ \delta_2^- x \big\downarrow & x & \big\downarrow \delta_2^+ x \\ & \xrightarrow[\delta_1^+ x]{} & \end{array}\right) = \begin{array}{ccc} & \xleftarrow{r_2\delta_1^- x} & \\ \delta_2^+ x \big\downarrow & r_2 x & \big\downarrow \delta_2^- x \\ & \xleftarrow[r_2\delta_1^+ x]{} & \end{array}.$$

2.4.4. Truncation. The truncation functors $U_m^n : \mathsf{SCub}_n^\gamma \to \mathsf{SCub}_m^\gamma$ and $U_m : \mathsf{SCub}_\omega^\gamma \to \mathsf{SCub}_m^\gamma$ induce functors $U_m^n : \mathsf{SCub}_{(n,p)}^\gamma \to \mathsf{SCub}_{(m,p)}^\gamma$ and $U_m : \mathsf{SCub}_{(\omega,p)}^\gamma \to \mathsf{SCub}_{(m,p)}^\gamma$ for $p \leqslant m \leqslant n$.

2.4.5. Remark. For $i \in \mathbb{N}_+$, every cell in $\mathcal{S}^i$ is its own r_i-inverse.

2.4.6. Lemma. *Let $\mathcal{S}$ be an (ω, p)-category with connections. For every $i, j \in \mathbb{N}_+$ and every r_i-invertible $x, y \in \mathcal{S}$,*

- **(i)** *$\delta_i^\alpha r_i x = \delta_i^{-\alpha} x$ and $\delta_j^\alpha r_i x = r_i \delta_j^\alpha x$ if $j \neq i$,*
- **(ii)** *if $j \neq i$, then $r_i(x \circ_i y) = r_i y \circ_i r_i x$ if $\Delta_i(x, y)$, and $r_i(x \circ_j y) = r_i x \circ_j r_i y$ if $\Delta_j(x, y)$,*
- **(iii)** *$r_i s_{i-1} x = s_{i-1} x$ and $r_i s_j y = s_j r_i y$ if $j \neq i-1$, $x \in \mathcal{S}^{i-1}$ and $y \in \mathcal{S}^j$,*
- **(iv)** *$r_i \widetilde{s}_i x = \widetilde{s}_i x$ and $r_i \widetilde{s}_j y = \widetilde{s}_j r_i y$ if $j \neq i$, $x \in \mathcal{S}^{i+1}$ and $y \in \mathcal{S}^{j+1}$,*
- **(v)** *$r_i \gamma_j^\alpha x = \gamma_j^\alpha r_i x$ if $i \neq j, j+1$ and $x \in \mathcal{S}^j$.*

Proof. We prove the first item of (i) as an example. By r_i-invertibility of x, $\Delta_i(x, r_i x)$ and $\Delta_i(r_i x, x)$. Hence $\delta_i^+ x = \delta_i^- r_i x$ and $\delta_i^+ r_i x = \delta_i^- x$. Suppose $j \neq i$. Then $z = \delta_j^\alpha r_i x$ satisfies

$$\Delta_i(\delta_j^\alpha x, z), \qquad \delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x, \qquad \Delta_i(z, \delta_j^\alpha x), \qquad z \circ_i \delta_j^\alpha x = \delta_i^+ \delta_j^\alpha x.$$

Thus $z = r_i \delta_j^\alpha x$, because r_i-inverses are unique. The remaining proofs are similar. □

2.4.7. Lemma. *Let $\mathcal{S}$ be an (ω, p)-category. If $x \in \mathcal{S}^{>n}$ is r_i-invertible, then $r_i x \in \mathcal{S}^{>n}$.*

Proof. Suppose x is r_i-invertible. For any $m \geqslant n+1$, apply δ_m^α to $x \circ_i r_i x = \delta_i^- x$ and $x \circ_i r_i x = \delta_i^- x$. This yields $\delta_m^\alpha r_i x \circ_i x = \delta_i^+ x$. Uniqueness of $r_i x$ then implies that $\delta_m^\alpha r_i x = r_i x$. □

2.4.8. Proposition. *Every cell in an $(\omega, 0)$-category is r_i-invertible for each $i \in \mathbb{N}_+$.*

Proof. By Axiom (x), $\mathcal{S} = \bigcup_{n \geqslant 0} \mathcal{S}^{>n}$. We show by induction on the dimension n of cells that every cell $x \in \mathcal{S}^{>n}$ is r_i-invertible for all $i \in \mathbb{N}_+$.

For $n = 0$, suppose $x \in \mathcal{S}^{>0}$. Then x is its own r_i-inverse for all $i \in \mathbb{N}_+$.

Suppose the property holds for $n-1$. Let $i \in \mathbb{N}_+$ and $x \in \mathcal{S}^{>n}$. If $i \geqslant n+1$, then x is its own r_i-inverse, so suppose $i \leqslant n$. Let $1 \leqslant j \leqslant n$ with $i \neq j$. We have $\delta_j^\alpha x \in \mathcal{S}^{j,n+1,n+2,\ldots}$ and thus $s_{n-1} \ldots s_j \delta_j^\alpha x \in \mathcal{S}^{>n}$ using Axioms (iv) and (vi) of Definition 2.2.1. There are two cases depending on the value of i.

– If $j < i \leqslant n$, then $s_{n-1} \dots s_j \delta_j^\alpha x$ has an r_{i-1}-inverse y by the induction hypothesis. Writing $z = \widetilde{s}_j \dots \widetilde{s}_{n-1} y$ and using property (ii) of Lemma 2.2.11,

$$\begin{aligned} z \circ_i \delta_j^\alpha x &= \widetilde{s}_j \dots \widetilde{s}_{n-1} y \circ_i \widetilde{s}_j \dots \widetilde{s}_{n-1} s_{n-1} \dots s_j \delta_j^\alpha x \\ &= \widetilde{s}_j \dots \widetilde{s}_{n-1} (y \circ_{i-1} s_{n-1} \dots s_j \delta_j^\alpha x) \\ &= \widetilde{s}_j \dots \widetilde{s}_{n-1} \delta_{i-1}^+ s_{n-1} \dots s_j \delta_j^\alpha x \\ &= \delta_i^+ \delta_j^\alpha x. \end{aligned}$$

Similarly $\delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x$, $\Delta_i(z, \delta_j^\alpha x)$ and $\Delta_i(\delta_j^\alpha x, z)$. So z is the r_i-inverse of $\delta_j^\alpha x$.

– If $i < j$ then $s_{n-1} \dots s_j \delta_j^\alpha x$ has an r_i-inverse y by the induction hypothesis. Using the same abbreviation and lemma as before,

$$\begin{aligned} z \circ_i \delta_j^\alpha x &= \widetilde{s}_j \dots \widetilde{s}_{n-1} y \circ_i \widetilde{s}_j \dots \widetilde{s}_{n-1} s_{n-1} \dots s_j \delta_j^\alpha x \\ &= \widetilde{s}_j \dots \widetilde{s}_{n-1} (y \circ_i s_{n-1} \dots s_j \delta_j^\alpha x) \\ &= \widetilde{s}_j \dots \widetilde{s}_{n-1} \delta_i^+ s_{n-1} \dots s_j \delta_j^\alpha x \\ &= \delta_i^+ \delta_j^\alpha x, \end{aligned}$$

using property (ii) of Lemma 2.2.11. Likewise, $\delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x$, $\Delta_i(z, \delta_j^\alpha x)$ and $\Delta_i(\delta_j^\alpha x, z)$, and it follows again that z is the r_i-inverse of $\delta_j^\alpha x$.

This shows that x has an r_i-invertible $n-1$-shell and is therefore r_i-invertible. □

A formalisation of this proof with Isabelle is shown in Subsection 4.6.

3. Equivalence with classical cubical categories

We now present our main theorems: a series of equivalences which justify our single-set axioms relative to the classical ones. We begin in Subsection 3.1 by recalling the classical axioms for cubical categories. For cubical categories with connection, we use the axioms of Al-Agl, Brown and Steiner [2]; for inverses, we follow Lucas [45]. These categories, with appropriate functors between them, form the categories Cub_ω without connections, $\mathsf{Cub}_\omega^\Gamma$ with connections and $\mathsf{Cub}_{(\omega,p)}^\Gamma$ with inverses. In Subsection 3.2, we prove an equivalence of categories $\mathsf{SCub}_\omega \simeq \mathsf{Cub}_\omega$, in Subsection 3.3 we extend it to $\mathsf{SCub}_\omega^\gamma \simeq \mathsf{Cub}_\omega^\Gamma$ and in Subsection 3.4 further to $\mathsf{SCub}_{(\omega,p)}^\gamma \simeq \mathsf{Cub}_{(\omega,p)}^\Gamma$.

Straightforward modifications of these proofs lead to equivalences between the corresponding n-categories. We do not list them explicitly.

3.1. Cubical categories

First we recall the classical definitions of cubical categories as cubical sets with cell compositions.

3.1.1. A *(cubical) ω-category* is a family $C = (C_n)_{n\in\mathbb{N}}$ of sets of *n-cells* with *face maps* $\partial^\alpha_{n,i} : C_n \to C_{n-1}$, *degeneracy maps* $\varepsilon_{n,i} : C_{n-1} \to C_n$ and *compositions* $\star_{n,i} : C_n \times_{n,i} C_n \to C_n$, for $1 \leqslant i \leqslant n$, where $C_n \times_{n,i} C_n$ denotes the pullback of the cospan $\partial^+_{n,i} : C_n \to C_{n-1} \leftarrow C_n : \partial^-_{n,i}$. These satisfy, for all $1 \leqslant i, j \leqslant n$,

(i) $a \star_{n,i} (b \star_{n,i} c) = (a \star_{n,i} b) \star_{n,i} c$ if either side is defined,

(ii) $a \star_{n,i} \varepsilon_{n,i}\partial^+_{n,i} a = \varepsilon_{n,i}\partial^-_{n,i} a \star_{n,i} a = a$,

(iii) $\partial^\alpha_{n-1,i}\partial^\beta_{n,j} = \partial^\beta_{n-1,j-1}\partial^\alpha_{n,i}$ if $i < j$,

(iv) if a, b are $\star_{n,j}$-composable then

$$\partial^\alpha_{n,i}(a \star_{n,j} b) = \begin{cases} \partial^\alpha_{n,i} a \star_{n,j-1} \partial^\alpha_{n,i} b & \text{if } i < j, \\ \partial^-_{n,i} a & \text{if } i = j \text{ and } \alpha = -, \\ \partial^+_{n,i} b & \text{if } i = j \text{ and } \alpha = +, \\ \partial^\alpha_{n,i} a \star_{n,j} \partial^\alpha_{n,i} b & \text{if } i > j, \end{cases}$$

(v) if a, b are $\star_{n,i}$-composable, c, d are $\star_{n,i}$-composable, a, c are $\star_{n,j}$-composable and b, d are $\star_{n,j}$-composable, then $(a \star_{n,i} b) \star_{n,j} (c \star_{n,i} d) = (a \star_{n,j} c) \star_{n,i} (b \star_{n,j} d)$,

(vi)

$$\partial^\alpha_{n,i}\varepsilon_{n,j} = \begin{cases} \varepsilon_{n-1,j-1}\partial^\alpha_{n-1,i} & \text{if } i < j, \\ id_{C_{n-1}} & \text{if } i = j, \\ \varepsilon_{n-1,j}\partial^\alpha_{n-1,i-1} & \text{if } i > j, \end{cases}$$

(vii) if a, b are $\star_{n,j}$-composable then

$$\varepsilon_{n+1,i}(a \star_{n,j} b) = \begin{cases} \varepsilon_{n+1,i} a \star_{n+1,j+1} \varepsilon_{n+1,i} b & \text{if } i \leqslant j, \\ \varepsilon_{n+1,i} a \star_{n+1,j} \varepsilon_{n+1,i} b & \text{if } i > j, \end{cases}$$

(viii) $\varepsilon_{n+1,i}\varepsilon_{n,j} = \varepsilon_{n+1,j+1}\varepsilon_{n,i}$ if $i \leqslant j$.

A *(cubical) n-category* is a family $C = (C_k)_{k\leqslant n}$ of sets of *k-cells* with *face maps* $\partial^\alpha_{k,i} : C_k \to C_{k-1}$, *degeneracy maps* $\varepsilon_{k,i} : C_{k-1} \to C_k$ and *composition maps* $\star_{k,i} : C_k \times_{k,i} C_k \to C_k$, for $1 \leqslant i \leqslant k \leqslant n$. These satisfy the above axioms, removing those involving face maps, compositions and degeneracies outside the appropriate ranges.

Each degeneracy map $\varepsilon_{n,i}$ yields identities for $\star_{n,i}$.

3.1.2. A *(cubical) ω-category with connections* C is an ω-category with *connection maps* $\Gamma^\alpha_{n,i} : C_{n-1} \to C_n$ for $1 \leqslant i < n$, such that

(i)

$$\partial^\alpha_{n,i}\Gamma^\beta_{n,j} = \begin{cases} \Gamma^\beta_{n-1,j-1}\partial^\alpha_{n-1,i} & \text{if } i < j, \\ id_{C_{n-1}} & \text{if } i = j, j+1 \text{ and } \alpha = \beta, \\ \varepsilon_{n-1,j}\partial^\alpha_{n-1,j} & \text{if } i = j, j+1 \text{ and } \alpha = -\beta, \\ \Gamma^\beta_{n-1,j}\partial^\alpha_{n-1,i-1} & \text{if } i > j+1, \end{cases}$$

(ii) if a, b are $\star_{n,j}$-composable then

$$\Gamma^\alpha_{n+1,i}(a \star_{n,j} b) = \begin{cases} \Gamma^\alpha_{n+1,i} a \star_{n+1,j+1} \Gamma^\alpha_{n+1,i} b & \text{if } i < j, \\ (\Gamma^-_{n+1,i} a \star_{n+1,i} \varepsilon_{n+1,i+1} b) \star_{n+1,i+1} (\varepsilon_{n+1,i} b \star_{n+1,i} \Gamma^-_{n+1,i} b) & \text{if } i = j \text{ and } \alpha = -, \\ (\Gamma^+_{n+1,i} a \star_{n+1,i} \varepsilon_{n+1,i} a) \star_{n+1,i+1} (\varepsilon_{n+1,i+1} a \star_{n+1,i} \Gamma^+_{n+1,i} b) & \text{if } i = j \text{ and } \alpha = +, \\ \Gamma^\alpha_{n+1,i} a \star_{n+1,j} \Gamma^\alpha_{n+1,i} b & \text{if } i > j, \end{cases}$$

(iii) $\Gamma^+_{n,i} a \star_{n,i} \Gamma^-_{n,i} a = \varepsilon_{n,i+1} a$ and $\Gamma^+_{n,i} a \star_{n,i+1} \Gamma^-_{n,i} a = \varepsilon_{n,i} a$,

(iv)

$$\Gamma^\alpha_{n+1,i} \varepsilon_{n,j} = \begin{cases} \varepsilon_{n+1,j+1} \Gamma^\alpha_{n,i} & \text{if } i < j, \\ \varepsilon_{n+1,i} \varepsilon_{n,i} & \text{if } i = j, \\ \varepsilon_{n+1,j} \Gamma^\alpha_{n,i-1} & \text{if } i > j, \end{cases}$$

(v)

$$\Gamma^\alpha_{n+1,i} \Gamma^\beta_{n,j} = \begin{cases} \Gamma^\beta_{n+1,j+1} \Gamma^\alpha_{n,i} & \text{if } i < j, \\ \Gamma^\alpha_{n+1,i+1} \Gamma^\alpha_{n,i} & \text{if } i = j \text{ and } \alpha = \beta. \end{cases}$$

A *(cubical) n-category with connections* is an n-category with *connection maps* $\Gamma^\alpha_{k,i} : C_{k-1} \to C_k$ for $1 \leqslant i < k \leqslant n$, satisfying axioms with the appropriate index restrictions.

There is some index shifting in the above axioms. For instance in the axiom describing the faces of compositions 3.1.1(iv) and the one describing the faces of connections 3.1.2(i), there is a $j-1$ index appearing in the case where $j > i$, which does not appear in the other cases. Furthermore, there are two indices, for the dimension k and the direction i. Their single-set counterparts 2.2.1(ii) and 2.3.1(i) do not have such explicit index shifting and the dimension index is missing.

3.1.3. Cubical categories with inverses. Let $1 \leqslant i \leqslant n$. An n-cell a of an ω-category with connections C is *$R_{n,i}$-invertible* if there is a n-cell b such that

$$a \star_{n,i} b = \varepsilon_{n,i} \partial^-_{n,i} a \quad \text{and} \quad b \star_{n,i} a = \varepsilon_{n,i} \partial^+_{n,i} a.$$

Each $R_{n,i}$-inverse of an n-cell a is unique and denoted $R_{n,i} a$. A n-cell a *has an $R_{n-1,i}$-invertible shell* if the cells $\partial^\alpha_{n,j} a$ are $R_{n-1,i-1}$-invertible for all $1 \leqslant j < i$, and the cells $\partial^\alpha_{n,j} a$ are $R_{n-1,i}$-invertible for all $i < j \leqslant n$.

A *(cubical) (ω, p)-category (with connections) C*, for $p \in \mathbb{N}$, is an ω-category with connections in which, for all $n > p$ and $1 \leqslant i \leqslant n$, every n-cell with an $R_{n-1,i}$-invertible shell is $R_{n,i}$-invertible.

The above definitions of invertibility and invertible shell extend to n-categories with connections, by removing the $R_{k,i}$ with indices (k, i) whenever $k > n$. In particular, for $0 \leqslant p \leqslant n$, a *(cubical) (n, p)-category (with connections) C* is an n-category with connections in which, for all $p+1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant k$, every k-cell with an $R_{k-1,i}$-invertible shell is $R_{k,i}$-invertible.

3.1.4. Categories of cubical categories. A *functor $F : C \to \mathcal{D}$ of ω-categories* is a family of maps $(F_n : C_n \to \mathcal{D}_n)_{n\in\mathbb{N}}$ that preserve all face, degeneracy and composition maps:

$$F_{n-1}\partial^\alpha_{n,i} = \partial^\alpha_{n,i}F_n, \qquad F_n(a \star_{n,i} b) = F_n a \star_{n,i} F_n b, \qquad F_n\varepsilon_{n,i} = \varepsilon_{n,i}F_{n-1},$$

for all $1 \leqslant i \leqslant n$ and $\star_{n,i}$-composable $a, b \in C_n$. Cubical ω-categories and their functors form the category Cub_ω.

Further, a *functor of ω-categories with connections* is a functor between the underlying ω-categories that preserves the connection maps: $F_n\Gamma^\alpha_{n,i} = \Gamma^\alpha_{n,i}F_{n-1}$, for all $1 \leqslant i < n$. Cubical ω-categories with connections and their functors form the category $\mathsf{Cub}^\Gamma_\omega$.

Finally, a *functor of (ω, p)-categories $F : C \to \mathcal{D}$* is a functor between the underlying ω-categories with connections. As in the single-set case, inverses are preserved by such functors. Cubical (ω, p)-categories and their functors form the category $\mathsf{Cub}^\Gamma_{(\omega,p)}$.

The categories Cub_n, Cub^Γ_n and $\mathsf{Cub}^\Gamma_{(n,p)}$ of cubical n-categories and their morphisms are defined by truncation as usual.

3.2. Equivalence for cubical ω-categories

We are now prepared for the following result.

3.2.1. Theorem. *There is an equivalence of categories*

$$(-)^c : \mathsf{SCub}_\omega \rightleftarrows \mathsf{Cub}_\omega : (-)^s.$$

We develop the proof of $\mathsf{SCub}_\omega \simeq \mathsf{Cub}_\omega$ in the remainder of this subsection. The functors $(-)^c$ and $(-)^s$ are defined in (3.2.2) and (3.2.4), the natural isomorphisms in (3.2.6) and (3.2.8) below. We henceforth refer to classical and single-set ω-categories to distinguish between objects in Cub_ω and SCub_ω.

3.2.2. The functor $(-)^c$. For each category $(\mathcal{S}, \delta, \circ, s)$ in SCub_ω, we define the category $(\mathcal{S}, \delta, \circ, s)^c = (\mathcal{S}^c, \partial, \varepsilon, \star)$ in Cub_ω with

- **(i)** sets of n-cells $\mathcal{S}^c_n = \mathcal{S}^{>n}$ for all $n \in \mathbb{N}$,
- **(ii)** $\partial^\alpha_{n,i} : \mathcal{S}^c_n \to \mathcal{S}^c_{n-1}$ such that $\partial^\alpha_{n,i} = s_{n-1} \dots s_i \delta^\alpha_i$ for all $1 \leqslant i \leqslant n$,
- **(iii)** $\varepsilon_{n,i} : \mathcal{S}^c_{n-1} \to \mathcal{S}^c_n$ such that $\varepsilon_{n,i} = \widetilde{s}_i \dots \widetilde{s}_{n-1}$ for all $1 \leqslant i \leqslant n$,
- **(iv)** compositions $\star_{n,i}$ such that $x \star_{n,i} y = x \circ_i y$ if $\Delta_i(x, y)$ and undefined otherwise, for all $1 \leqslant i \leqslant n$ and $x, y \in \mathcal{S}^c_n$.

With each morphism $f : \mathcal{S} \to \mathcal{S}'$ in SCub_ω, $(-)^c$ associates the functor $f^c : \mathcal{S}^c \to \mathcal{S}'^c$ on n-cells in Cub_ω as the restriction of f to a map from $\mathcal{S}^c_n = \mathcal{S}^{>n}$ to $\mathcal{S}'^c_n = \mathcal{S}'^{>n}$, for each $n \in \mathbb{N}$.

3.2.3. Lemma. *The functor $(-)^c$ is well-defined.*

Proof. We need to show that $(-)^c : \mathsf{SCub}_\omega \to \mathsf{Cub}_\omega$ sends each category in SCub_ω to a category in Cub_ω and each morphism in the former category to a functor in the latter, and that it is itself a functor.

First we show that $(\mathcal{S}^c, \partial, \varepsilon, \star)$ is indeed a classical ω-category, that is, we check the axioms in 3.1.1. Here we only consider some representative examples. Proofs for the remaining axioms shown in Appendix A.1. Suppose $1 \leqslant i, j \leqslant n$ and $a, b, c, d \in (\mathcal{S}^c)_n$.

(i) If $a \star_{n,i} (b \star_{n,i} c)$ is defined, then $a \star_{n,i} (b \star_{n,i} c) = a \circ_i (b \circ_i c) = (a \circ_i b) \circ_i c = (a \star_{n,i} b) \star_{n,i} c$ because $\Delta_i(a, b \star_{n,i} c)$ and $\Delta_i(b, c)$, using Axioms 2.1.1(i), (iii), (iv) among others. The same holds if $(a \star_{n,i} b) \star_{n,i} c$ is defined.

(ii) The compositions $a \star_{n,i} \varepsilon_{n,i} \partial^+_{n,i} a$ and $\varepsilon_{n,i} \partial^-_{n,i} a \star_{n,i} a$ are defined because $\Delta_i(a, \varepsilon_{n,i}\partial^+_{n,i} a)$ and $\Delta_i(\varepsilon_{n,i}\partial^-_{n,i} a, a)$, using Axioms 2.1.1(ii), (iii), (iv) among others. It follows that $a \star_{n,i} \varepsilon_{n,i}\partial^+_{n,i} a = a \circ_i \delta^+_i a = a$ and $\varepsilon_{n,i}\partial^-_{n,i} a \star_{n,i} a = \delta^-_i a \circ_i a = a$ because $\varepsilon_{n,i}\partial^\alpha_{n,i} a = \delta^\alpha_i a$.

(iii) If $i < j$, then Axiom 2.2.1(i) and other facts imply that

$$\begin{aligned}
\partial^\beta_{n-1,j-1}\partial^\alpha_{n,i} &= s_{n-2} \dots s_{j-1}\delta^\beta_{j-1} s_{n-1} \dots s_i \delta^\alpha_i \\
&= s_{n-2} \dots s_{j-1} s_{n-1} \dots s_j \delta^\beta_{j-1} s_{j-1} \dots s_i \delta^\alpha_i \\
&= s_{n-2} \dots s_{j-1} s_{n-1} \dots s_j \delta^\beta_j s_{j-2} \dots s_i \delta^\alpha_i \\
&= s_{n-2} \dots s_i \delta^\alpha_i s_{n-1} \dots s_j \delta^\beta_j \\
&= \partial^\alpha_{n-1,i}\partial^\beta_{n,j}.
\end{aligned}$$

Next we show that f^c is a morphism in Cub_ω. Suppose $1 \leqslant i \leqslant n$ and $a, b \in \mathcal{S}^c_n$. Then

(i) $f^c_{n-1}\partial^\alpha_{n,i} = f s_{n-1} \dots s_i \delta^\alpha_i = s_{n-1} \dots s_i \delta^\alpha_i f = \partial^\alpha_{n,i} f^c_n$,

(ii) if $\partial^+_{n,i} a = \partial^-_{n,i} b$ then $\Delta_i(a, b)$ and $f^c_n(a \star_{n,i} b) = f(a \circ_i b) = f(a) \circ_i f(b) = f^c_n a \star_{n,i} f^c_n b$,

(iii) $f^c_n \varepsilon_{n,i} = f \widetilde{s}_i \dots \widetilde{s}_{n-1} = \widetilde{s}_i \dots \widetilde{s}_{n-1} f = \varepsilon_{n,i} f^c_{n-1}$.

Further, it is clear from the definition that f^c preserves sources, targets and compositions of morphisms in Cub_ω. □

All calculations in this proof are performed within SCub_ω. Their formalisation with our Isabelle component seems therefore routine.

3.2.4. The functor $(-)^s$. Next, we define the category $(C, \partial, \varepsilon, \star)^s = (\mathcal{S}, \delta, \circ, s)$ in SCub_ω for each cubical $(C, \partial, \varepsilon, \star)$ in Cub_ω

(i) The underlying set is the following colimit in the category Set:

$$\mathcal{S} = \mathrm{colim}(C_0 \xrightarrow{\varepsilon_{1,1}} C_1 \xrightarrow{\varepsilon_{2,2}} C_2 \xrightarrow{\varepsilon_{3,3}} \dots) = \bigsqcup_{n \in \mathbb{N}} C_n / {\sim},$$

where $a \in C_m$ and $b \in C_n$ with $m \leqslant n$ are equivalent if and only if $b = \varepsilon_{n,n} \dots \varepsilon_{m+1,m+1} a$ by injectivity of the degeneracy maps. We write $\varphi_n : C_n \longrightarrow \mathcal{S}$ for the maps forming a cocone to the colimit. They sends each n-cell a to its equivalence class in the above quotient.

(ii) The $\delta^\alpha_i : \mathcal{S} \longrightarrow \mathcal{S}$, for $i \in \mathbb{N}_+$, are the unique morphisms in Set induced by the cocone $(\varphi_n \varepsilon_{n,i} \partial^\alpha_{n,i})_{n \geqslant i}$. They send the equivalence class of $a \in C_n$, $n \geqslant i$, to the set $\delta^\alpha_i [a]_\sim = [\varepsilon_{n,i}\partial^\alpha_{n,i} a]_\sim$.

(iii) The $s_i : \mathcal{S} \longrightarrow \mathcal{S}$, for $i \in \mathbb{N}_+$, are the unique morphisms in Set induced by the cocone $(\varphi_n \varepsilon_{n,i+1} \partial^-_{n,i})_{n \geqslant i+1}$. They send the equivalence class of $a \in C_n$ to $s_i [a]_\sim = [\varepsilon_{n,i+1}\partial^-_{n,i} a]_\sim$.

(iv) The $\widetilde{s_i} : \mathcal{S} \to \mathcal{S}$, for $i \in \mathbb{N}_+$, are the unique morphisms in Set induced by the cocone $(\varphi_n \varepsilon_{n,i} \partial^-_{n,i+1})_{n \geqslant i+1}$. They send the equivalence class of $a \in C_n$ to $\widetilde{s_i}[a]_\sim = [\varepsilon_{n,i} \partial^-_{n,i+1} a]_\sim$.

(v) The $\circ_i : \mathcal{S} \times_{\Delta_i} \mathcal{S} \to \mathcal{S}$, for $i \in \mathbb{N}_+$, send the equivalence classes of $a, b \in C_n$, to $[a]_\sim \circ_i [b]_\sim = [a \star_{n,i} b]_\sim$.

For each functor $g : C \to C'$ in Cub_ω we define the morphism $g^s : C^s \to C'^s$ in SCub_ω as the unique morphism in Set induced by the cocone $(\varphi'_n \circ g_n : C_n \to C'^s)_{n \in \mathbb{N}}$, where the φ'_n are the inclusion maps $C'_n \to C'^s$. It sends the equivalence class of $a \in C_n$ to $g^s([a]_\sim) = [g(a)]_\sim$ in C'_n.

In the following, we do not distinguish between equivalence classes and their representatives.

3.2.5. Lemma. *The functor $(-)^s$ is well-defined.*

Proof. The proof is similar to that of Lemma 3.2.3. First we check that $(\mathcal{S}, \delta, \circ, s)$ is a category in SCub_ω, verifying the axioms in 2.2.1. Once again we only present some cases and refer to Appendix A.2 for the remaining ones. Suppose $i, j \in \mathbb{N}_+$ and $w, x, y, z \in \mathcal{S}$ with representatives in C_n for $n \geqslant i, j$.

(i) $\Delta_i(x, y) \Leftrightarrow \partial^+_{n,i} x = \partial^-_{n,i} y$, hence $\Delta_i(x, y \circ_i z)$ and $\Delta_i(y, z)$ if and only $\Delta_i(x \circ_i y, z)$ and $\Delta_i(x, y)$. It follows that $x \circ_i (y \circ_i z) = x \star_{n,i} (y \star_{n,i} z) = (x \star_{n,i} y) \star_{n,i} z = (x \circ_i y) \circ_i z$.

(ii) $x \circ_i \delta^+_i x = x \star_{n,i} \varepsilon_{n,i} \partial^+_{n,i} x = x$ and $\delta^-_i x \circ_i x = \varepsilon_{n,i} \partial^-_{n,i} x \star_{n,i} x = x$.

(iii)
- If $i < j$ then
$$\begin{aligned}
\delta^\alpha_i \delta^\beta_j x &= \varepsilon_{n,i} \partial^\alpha_{n,i} \varepsilon_{n,j} \partial^\beta_{n,j} x \\
&= \varepsilon_{n,i} \varepsilon_{n-1,j-1} \partial^\alpha_{n-1,i} \partial^\beta_{n,j} x \\
&= \varepsilon_{n,j} \varepsilon_{n-1,i} \partial^\beta_{n-1,j-1} \partial^\alpha_{n,i} x \\
&= \varepsilon_{n,j} \partial^\beta_{n,j} \varepsilon_{n,i} \partial^\alpha_{n,i} x \\
&= \delta^\beta_j \delta^\alpha_i x,
\end{aligned}$$
- if $i > j$ then
$$\begin{aligned}
\delta^\alpha_i \delta^\beta_j x &= \varepsilon_{n,i} \partial^\alpha_{n,i} \varepsilon_{n,j} \partial^\beta_{n,j} x \\
&= \varepsilon_{n,i} \varepsilon_{n-1,j} \partial^\alpha_{n-1,i-1} \partial^\beta_{n,j} x \\
&= \varepsilon_{n,j} \varepsilon_{n-1,i-1} \partial^\beta_{n-1,j} \partial^\alpha_{n,i} x \\
&= \varepsilon_{n,j} \partial^\beta_{n,j} \varepsilon_{n,i} \partial^\alpha_{n,i} x \\
&= \delta^\beta_j \delta^\alpha_i x.
\end{aligned}$$

Next we show that g^s is a morphism in SCub_ω. For all $1 \leqslant i \leqslant n$ and $x, y \in \mathcal{S}$ with representatives a, b in C_n,

(i) $g^s \delta^\alpha_i x = [g \varepsilon_{n,i} \partial^\alpha_{n,i} x]_\sim = [\varepsilon'_{n,i} \partial'^\alpha_{n,i} g x]_\sim = \delta'^\alpha_i g^s x$,

(ii) if $\Delta_i(x, y)$, then $\partial^+_{n,i} a = \partial^-_{n,i} b$, so $g^s(x \circ_i y) = [g(a \star_{n,i} b)]_\sim = [g(a) \star_{n,i} g(b)]_\sim = g^s(x) \circ_i g^s(y)$,

(iii) if $n \geqslant i + 1$, then $g^s s_i x = [g \varepsilon_{n,i+1} \partial^-_{n,i} x]_\sim = [\varepsilon'_{n,i+1} \partial'^-_{n,i} g x]_\sim = s'_i g^s x$.

Finally, g^s is indeed a morphism: it is clear from its definition that it preserves sources and targets, while reservation of compositions of morphisms in SCub_ω follows from properties of colimits. □

This time, the calculations in the proof above are performed in Cub_ω. Their formalisation with Isabelle is beyond the scope of this work.

3.2.6. Natural isomorphism $(-)^s \circ (-)^c \Rightarrow id$. Let $(\mathcal{S}, \delta, \circ, s)$ be a category of SCub_ω and $(C, \partial, \varepsilon, \star) = (\mathcal{S}, \delta, \circ, s)^c$ as previously. The category $(\mathcal{S}', \delta', \circ', s') := ((\mathcal{S}, \delta, \circ, s)^c)^s$ in SCub_ω is computed as follows. Let $i \in \mathbb{N}_+$.

(i) $\mathcal{S}' = \mathrm{colim}(C_0 \overset{\varepsilon_{1,1}}{\to} C_1 \overset{\varepsilon_{2,2}}{\to} \dots) = \mathrm{colim}(\mathcal{S}^{>0} \overset{id}{\to} \mathcal{S}^{>1} \overset{id}{\to} \dots)$. Axiom 2.2.1(x) then implies that

$$\mathcal{S}' = \bigsqcup_{n \in \mathbb{N}} \mathcal{S}^{>n} / \sim \; = \bigcup_{n \in \mathbb{N}} \mathcal{S}^{>n} = \mathcal{S}\,.$$

(ii) The face maps are $\delta'^{\alpha}_i = \delta^{\alpha}_i$, because they send each $x \in \mathcal{S}^{>n}$ with $n \geqslant i$ to $\delta'^{\alpha}_i x = \varepsilon_{n,i} \partial^{\alpha}_{n,i} x = \delta^{\alpha}_i x$. Thus in particular $\mathcal{S}'^i = \mathcal{S}^i$.

(iii) The symmetries and reverse symmetries are $s'_i = s_i \delta^-_i$ and $\widetilde{s}'_i = \widetilde{s}_i \delta^-_{i+1}$, because they send $x \in \mathcal{S}^{>n}$ with $n \geqslant i+1$ to $s'_i x = \varepsilon_{n,i+1} \partial^-_{n,i} x = s_i \delta^-_i x$ and $\widetilde{s}'_i x = \varepsilon_{n,i} \partial^-_{n,i+1} x = \widetilde{s}_i \delta^-_{i+1}$. Therefore, $s'_i = s_i$ on $\mathcal{S}^i$ and $\widetilde{s}'_i = \widetilde{s}_i$ on $\mathcal{S}^{i+1}$.

(iv) The compositions are $\circ'_i = \circ_i$, because for all $x, y \in \mathcal{S}^{>n}$ with $n \geqslant i$ such that $\Delta'_i(x, y)$, we have $\Delta_i(x, y)$ and the compositions $\circ'_i$ send x, y to $x \circ'_i y = x \star_{n,i} y = x \circ_i y$.

Further, any morphism $f : \mathcal{S} \to \mathcal{T}$ in SCub_ω satisfies $(f^c)^s = f$.

3.2.7. Lemma. *The maps $id_{\mathcal{S}} : \mathcal{S}' \to \mathcal{S}$ induce a natural isomorphism $\mu : (-)^s \circ (-)^c \Rightarrow id$.*

Proof. First, $id \delta'^{\alpha}_i = \delta^{\alpha}_i id$ and $\Delta_i(x, y)$ implies $id(x \circ'_i y) = id(x) \circ_i id(y)$. Second, $id(s'_i x) = s_i id(x)$ for all $x \in \mathcal{S}'^i$. So, as $\mathcal{S}' = \mathcal{S}$, $\mu_{\mathcal{S}} = id : \mathcal{S}' \to \mathcal{S}$ is a morphism in SCub_ω.

Naturality of the family μ and the invertibility of each component $\mu_{\mathcal{S}}$ are clear. □

3.2.8. Natural isomorphism $id \Rightarrow (-)^c \circ (-)^s$. Let $(C, \partial, \varepsilon, \star)$ be a category in Cub_ω and let $(\mathcal{S}, \delta, \circ, s) = (C, \partial, \varepsilon, \star)^s$ as before. The category $(C', \partial', \varepsilon', \star') = ((C, \partial, \varepsilon, \star)^s)^c$ in Cub_ω is computed as follows. Let $1 \leqslant i \leqslant n$.

(i) The sets of n-cells are

$$C'_n = \mathcal{S}^{>n} = \left(\bigsqcup_{m \in \mathbb{N}} C_m / \sim \right)^{>n}.$$

(ii) The face maps $\partial'^{\alpha}_{n,i}$ send each $a \in C'_n$, represented by some $a_0 \in C_m$ with $m \geqslant n$, to

$$\partial'^{\alpha}_{n,i} a = s_{n-1} \dots s_i \delta^{\alpha}_i a = [\varepsilon_{m,n} \partial^-_{m,n-1} \dots \varepsilon_{m,i+1} \partial^-_{m,i} \varepsilon_{m,i} \partial^{\alpha}_{m,i} a_0]_\sim = [\varepsilon_{m,n} \partial^{\alpha}_{m,i} a_0]_\sim .$$

(iii) The degeneracies $\varepsilon'_{n,i}$ send each $a \in C'_n$, represented by some $a_0 \in C_m$ with $m \geqslant n$, to

$$\varepsilon'_{n,i} a = \widetilde{s}_i \dots \widetilde{s}_{n-1} a = [\varepsilon_{m,i} \partial^-_{m,i+1} \dots \varepsilon_{m,n-1} \partial^-_{m,n} a_0]_\sim = [\varepsilon_{m,i} \partial^-_{m,n} a_0]_\sim .$$

(iv) The compositions $a \star'_{n,i} b$, where $a, b \in C'_n$ are represented by some $a_0, b_0 \in C_m$ with $m \geqslant n$, satisfy $a \star'_{n,i} b = [a_0 \star_{m,i} b_0]_\sim$ whenever $\Delta_i(a, b)$, that is, when $a_0 \star_{n,i} b_0$ is defined, and are undefined otherwise.

Further, for any morphism $g : C \to \mathcal{D}$ in Cub_ω, $(g^s)^c$ sends $[a]_\sim$ in $(C^s)^c_n$ to $[g(a)]_\sim$ in $(\mathcal{D}^s)^c_n$.

Suppose $\varphi_n : C_n \to \mathcal{S}$ is the morphism to the colimit, which sends any n-cell a to its equivalence class in $\bigsqcup_{m\in\mathbb{N}} C_m/\sim$. Its image is included in C'_n because $a \sim \varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a = a'$ for each $a \in C_n$ and $i \geqslant n+1$. Thus $\delta^-_i \varphi_n a = [\varepsilon_{i,i}\partial^-_{i,i} a']_\sim = \varphi_n a$. Let $(\eta_C)_n : C_n \to C'_n$ be the induced map and $\eta_C : C \to C'$ the family $((\eta_C)_n)_{n\in\mathbb{N}}$. Let further $(\overline{\eta}_C)_n : C'_n \to C_n$ be the map that sends b, represented by some $a \in C_m$ with $m \geqslant n$, to $\partial^-_{n+1,n+1} \dots \partial^-_{m,m} a$. It is well-defined because the image of b does not depend on the choice of the representative: indeed $a \sim a'$ and $a' \in C_l$ with $l \geqslant m$ imply

$$\partial^-_{n+1,n+1} \dots \partial^-_{l,l} a' = \partial^-_{n+1,n+1} \dots \partial^-_{l,l} \varepsilon_{l,l} \dots \varepsilon_{m+1,m+1} a = \partial^-_{n+1,n+1} \dots \partial^-_{m,m} a.$$

Finally, we write $\overline{\eta}_C : C' \to C$ for the family $\big((\overline{\eta}_C)_n\big)_{n\in\mathbb{N}}$.

3.2.9. Lemma. *The maps $\eta_C : C \to C'$ induce a natural isomorphism $\eta : id \Rightarrow (-)^c \circ (-)^s$.*

Proof. We need to show that the maps $\eta_C : C \leftrightarrows C' : \overline{\eta}_C$ are morphisms in Cub_ω, which are natural and inverses of each other. Let $a, b \in C_n$ and $c \in C_{n-1}$. Then

$$\begin{aligned}
(\eta_C)_{n-1}\partial^\alpha_{n,i} a &= [\partial^\alpha_{n,i} a]_\sim = [\varepsilon_{n,n}\partial^\alpha_{n,i} a]_\sim = \partial'^\alpha_{n,i}[a]_\sim = \partial'^\alpha_{n,i}(\eta_C)_n a,\\
(\eta_C)_n(a \star_{n,i} b) &= [a \star_{n,i} b]_\sim = [a]_\sim \star'_{n,i} [b]_\sim = (\eta_C)_n a \star'_{n,i} (\eta_C)_n b,\\
(\eta_C)_n \varepsilon_{n,i} c &= [\varepsilon_{n,i} c]_\sim = [\varepsilon_{n,i}\partial^-_{n,n}\varepsilon_{n,n} c]_\sim = \varepsilon'_{n,i}[\varepsilon_{n,n} c]_\sim = \varepsilon'_{n,i}[c]_\sim = \varepsilon'_{n,i}(\eta_C)_{n-1} c.
\end{aligned}$$

The η_C are therefore morphisms in Cub_ω. The proof that the $\overline{\eta}_C$ are morphisms in Cub_ω is similar.

The maps η_C and $\overline{\eta}_C$ are inverses. Indeed, $\eta_C(\overline{\eta}_C(b)) = [\partial^-_{n+1,n+1} \dots \partial^-_{m,m} a]_\sim = b$ holds for every $b \in C'_n$ represented by some $a \in C_m$ with $m \geqslant n$, and likewise for the other composition.

Finally, the family η is natural because $\eta_C g(a) = [g(a)]_\sim = (g^s)^c[a]_\sim = g(\overline{\eta}_C a)$ for every functor $g : C \to \mathcal{D}$. □

3.2.10. Remark. All axioms of single-set ω-categories have been used in one direction of the proof of Theorem 3.2.1, and they have been derived in the other one. Single-set ω-categories and their classical counterparts are therefore essentially the same.

Interestingly, in this proof, Axiom 2.2.1(x) has only been used to establish the natural ismorphism μ, more precisely for showing that $\mathcal{S} = \mathcal{S}'$ in the colimit construction. It is unnecessary for a proof of equivalence between SCub_n and Cub_n, where the colimit construction simplifies; see also 2.2. The proofs of all other properties work uniformly for n and ω.

3.3. Equivalence for connections

In this subsection we extend the equivalence $\mathsf{SCub}_\omega \simeq \mathsf{Cub}_\omega$ to connections.

3.3.1. Theorem. *There is an equivalence of categories*

$$(-)^{c\Gamma} : \mathsf{SCub}^\gamma_\omega \rightleftarrows \mathsf{Cub}^\Gamma_\omega : (-)^{s\gamma}.$$

In the proof, we go through the same steps as before.

3.3.2. The functor $(-)^{c\Gamma}$. For every category $(\mathcal{S}, \delta, \circ, \gamma)$ in $\mathrm{SCub}^{\gamma}_{\omega}$, the category $(C, \partial, \varepsilon, \star, \Gamma) := (\mathcal{S}, \delta, \circ, \gamma)^{c\Gamma}$ in $\mathrm{Cub}^{\Gamma}_{\omega}$ is defined as follows:

(i) the underlying ω-category in Cub_{ω} is $(C, \partial, \varepsilon, \star) = (\mathcal{S}, \delta, \circ, s)^{c}$,

(ii) the connections are the restrictions $\Gamma^{\alpha}_{n,i} : C_{n-1} \to C_n$ of $\Gamma^{\alpha}_{n,i} = \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{n-1}$ for $1 \leqslant i < n$.

For each morphism $f : \mathcal{S} \to \mathcal{S}'$ in $\mathrm{SCub}^{\gamma}_{\omega}$ we define the morphisms $f^{c\Gamma} : \mathcal{S}^{c\Gamma} \to \mathcal{S}'^{c\Gamma}$ as f^c on n-cells in $\mathrm{Cub}^{\Gamma}_{\omega}$ for each $n \in \mathbb{N}$.

3.3.3. Lemma. *The functor $(-)^{c\Gamma}$ is well-defined.*

Proof. As before, we first check that $(\mathcal{S}, \partial, \varepsilon, \star, \Gamma)^{c\Gamma}$ defines a category in $\mathrm{Cub}^{\Gamma}_{\omega}$. We only need to consider the connection axioms. We show selected axioms only and refer to Appendix A.3 for the others. Let $1 \leqslant i, j < n$ and $a, b, c, d \in \mathcal{S}^{c\Gamma}_n$.

(i) – If $i < j$, then Axiom 2.3.1(i) and others imply that

$$\begin{aligned}
\partial^{\alpha}_{n,i}\Gamma^{\beta}_{n,j} &= s_{n-1} \dots s_j s_{j-1} \gamma^{\beta}_j s_{j-2} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{n-1} \delta^{\alpha}_i \\
&= s_{n-1} \dots s_{j+1} \gamma^{\beta}_{j-1} s_j s_{j-2} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{n-1} \delta^{\alpha}_i \\
&= \gamma^{\beta}_{j-1} s_{j-2} \dots s_i \delta^{\alpha}_i \\
&= \Gamma^{\beta}_{n-1,j-1} \partial^{\alpha}_{n-1,i},
\end{aligned}$$

– $\partial^{\alpha}_{n,i}\Gamma^{\alpha}_{n,i} = s_{n-1} \dots s_i \delta^{\alpha}_i \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{n-1} = id.$

(ii) If a, b are $\star_{n,j}$-composable, then Axiom 2.3.1(ii) and others imply that,

– if $i < j$ then

$$\begin{aligned}
\Gamma^{\alpha}_{n+1,i}(a \star_{n,j} b) &= \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_j (\widetilde{s}_{j+1} \dots \widetilde{s}_n a \circ_j \widetilde{s}_{j+1} \dots \widetilde{s}_n b) \\
&= \gamma^{\alpha}_i (\widetilde{s}_i \dots \widetilde{s}_n a \circ_{j+1} \widetilde{s}_i \dots \widetilde{s}_n b) \\
&= \Gamma^{\alpha}_{n+1,i} a \star_{n+1,j+1} \Gamma^{\alpha}_{n+1,i} b,
\end{aligned}$$

– if $i = j$ then

$$\begin{aligned}
\Gamma^{-}_{n+1,i}(a \star_{n,i} b) &= \gamma^{-}_i (\widetilde{s}_i \dots \widetilde{s}_n a \circ_{i+1} \widetilde{s}_i \dots \widetilde{s}_n b) \\
&= (\gamma^{-}_i \widetilde{s}_i \dots \widetilde{s}_n a \circ_i s_i \widetilde{s}_i \dots \widetilde{s}_n b) \circ_{i+1} (\widetilde{s}_i \dots \widetilde{s}_n b \circ_i \gamma^{-}_i \widetilde{s}_i \dots \widetilde{s}_n b) \\
&= (\Gamma^{-}_{n+1,i} a \star_{n+1,i} \varepsilon_{n+1,i+1} b) \star_{n+1,i+1} (\varepsilon_{n+1,i} b \star_{n+1,i} \Gamma^{-}_{n+1,i} b).
\end{aligned}$$

(iv) If $i < j$ then $\Gamma^{\alpha}_{n+1,i}\varepsilon_{n,j} = \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{j-1} \widetilde{s}_{j+1} \dots \widetilde{s}_n \widetilde{s}_j \dots \widetilde{s}_{n-1} = \widetilde{s}_{j+1} \dots \widetilde{s}_n \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{n-1} = \varepsilon_{n+1,j+1} \Gamma^{\alpha}_{n,i}$.

It remains to be shown that $f^{c\Gamma}$ is a morphism in $\mathrm{Cub}^{\Gamma}_{\omega}$. Indeed, for each $1 \leqslant i \leqslant n$ and $a, b \in \mathcal{S}^{c\Gamma}_n$,

$$f^{c\Gamma}\Gamma^{\alpha}_{n,i} a \;=\; f \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{n-1} a \;=\; \gamma^{\alpha}_i \widetilde{s}_i \dots \widetilde{s}_{n-1} f a \;=\; \Gamma^{\alpha}_{n,i} f^{c\Gamma} a.$$

The claim then follows from Lemma 3.2.3. □

3.3.4. The functor $(-)^{s\gamma}$. For $(C, \partial, \varepsilon, \star, \Gamma)$ in $\mathrm{Cub}^{\Gamma}_{\omega}$, the category $(C, \partial, \varepsilon, \star, \Gamma)^{s\gamma} := (\mathcal{S}, \delta, \odot, s, \gamma)$ in $\mathrm{SCub}^{\gamma}_{\omega}$ is defined as follows:

(i) The underlying single-set cubical ω-category in SCub_{ω} is $(C, \partial, \varepsilon, \star)^{s} = (\mathcal{S}, \delta, \odot, s)$.

(ii) The $\gamma^{\alpha}_{i} : \mathcal{S} \to \mathcal{S}$, for $i \geqslant 1$, are the unique morphisms in Set induced by the cocone $(\varphi_n \Gamma^{\alpha}_{n,i} \partial^{\alpha}_{n,i})_{n>i}$. They send the equivalence class of each $a \in C_n$ to $\gamma^{\alpha}_{i}[a]_{\sim} = [\Gamma^{\alpha}_{n,i}\partial^{\alpha}_{n,i} a]_{\sim}$.

For each morphism $g : C \to C'$ in Cub_{ω}, the morphism $g^{s\gamma} : C^{s\gamma} \to C'^{s\gamma}$ in $\mathrm{SCub}^{\gamma}_{\omega}$ is defined as g^{s}.

3.3.5. Lemma. *The functor $(-)^{s\gamma}$ is well-defined.*

Proof. As usual, we start with checking that $(\mathcal{S}, \delta, \odot, s, \gamma)$ is a category in $\mathrm{SCub}^{\gamma}_{\omega}$ It remains to consider the single-set axioms for connections. As usual, we only show some cases and refer to Appendix A.4 for the remaining ones. Let $i, j \in \mathbb{N}_{+}$ and $x, y \in \mathcal{S}$ with representatives a, b in C_n with $n > i, j$.

(i) If $i \neq j, j+1$ and $x \in \mathcal{S}^{j}$, then $\partial^{-}_{n,j} x = \partial^{+}_{n,j} x$ so $\delta^{\alpha}_{j}\gamma^{\alpha}_{j} x = \varepsilon_{n,j}\partial^{\alpha}_{n,j}\Gamma^{\alpha}_{n,j}\partial^{\alpha}_{n,j} x = \delta^{\alpha}_{j} x = x$.

(ii) If $j \neq i, i+1$, $x, y \in \mathcal{S}^{i}$ and $\Delta_{i+1}(x, y)$ then

$$\begin{aligned}\gamma^{+}_{i}(x \circ_{i+1} y) &= \Gamma^{+}_{n,i}(\partial^{+}_{n,i} x \star_{n,i} \partial^{+}_{n,i} y)\\ &= (\Gamma^{+}_{n,i}\partial^{+}_{n,i} x \star_{n+1,i+1} \varepsilon_{n,i+1}\partial^{+}_{n,i} x) \star_{n+1,i} (\varepsilon_{n,i}\partial^{+}_{n,i} x \star_{n+1,i+1} \Gamma^{+}_{n,i}\partial^{+}_{n,i} y)\\ &= (\gamma^{+}_{i} x \circ_{i+1} s_i x) \circ_i (x \circ_{i+1} \gamma^{+}_{i} y).\end{aligned}$$

(v) If $x \in \mathcal{S}^{i,j}$ and $i < j-1$ then

$$\gamma^{\alpha}_{i}\gamma^{\beta}_{j} x = \Gamma^{\alpha}_{n,i}\Gamma^{\beta}_{n-1,j-1}\partial^{\alpha}_{n-1,i}\partial^{\beta}_{n,j} x = \Gamma^{\beta}_{n,j}\Gamma^{\alpha}_{n-1,i}\partial^{\beta}_{n-1,j-1}\partial^{\alpha}_{n,i} x = \gamma^{\beta}_{j}\gamma^{\alpha}_{i} x.$$

(vi) If $x \in \mathcal{S}^{i,i+1}$ then $\partial^{-}_{n,i} a = \partial^{+}_{n,i} a$ and $\partial^{-}_{n,i+1} a = \partial^{+}_{n,i+1} a$ so

$$s_{i+1} s_i \gamma^{\alpha}_{i+1} x = \varepsilon_{n,i+2}\Gamma^{\alpha}_{n-1,i}\partial^{-}_{n-1,i}\partial^{\alpha}_{n,i+1} x = \Gamma^{\alpha}_{n,i}\varepsilon_{n-1,i+1}\partial^{\alpha}_{n-1,i}\partial^{-}_{n,i+1} x = \gamma^{\alpha}_{i} s_{i+1} x.$$

It remains to show that $g^{s\gamma}$ is a morphism in $\mathrm{SCub}^{\gamma}_{\omega}$. For $i \geqslant 1$ and $x \in \mathcal{S}^{i}$,

$$g^{s\gamma}\gamma^{\alpha}_{i} x = [g\Gamma^{\alpha}_{n,i}\partial^{\alpha}_{n,i} x]_{\sim} = [\Gamma^{\alpha}_{n,i}\partial^{\alpha}_{n,i} g x]_{\sim} = \gamma^{\alpha}_{i} g^{s\gamma} x.$$

The claim then follows from Lemma 3.2.5. □

3.3.6. Natural isomorphism $(-)^{s\gamma} \circ (-)^{c\Gamma} \Rightarrow$ *id*. Let $(\mathcal{S}, \delta, \circ, s, \gamma)$ be a category in $\mathrm{SCub}^{\gamma}_{\omega}$ and let $(C, \partial, \varepsilon, \star, \Gamma) = (\mathcal{S}, \delta, \circ, s, \gamma)^{c\Gamma}$. The category $(\mathcal{S}', \delta', \circ', s', \gamma') := ((\mathcal{S}, \delta, \circ, s, \gamma)^{c\Gamma})^{s\gamma}$ in $\mathrm{SCub}^{\gamma}_{\omega}$ is determined as follows. For $i \in \mathbb{N}_{+}$,

(i) the underlying ω-category in SCub_{ω} is $(\mathcal{S}', \delta', \circ', s') = ((\mathcal{S}, \delta, \circ, s)^{c})^{s}$

(ii) the connections are $\gamma'^{\alpha}_{i} = \gamma^{\alpha}_{i}\delta^{\alpha}_{i}$; they send $x \in \mathcal{S}^{>n}$ with $n > i$ to $\gamma'^{\alpha}_{i} x = \Gamma^{\alpha}_{n,i}\partial^{\alpha}_{n,i} x = \gamma^{\alpha}_{i}\delta^{\alpha}_{i} x$. Hence in particular $\gamma'^{\alpha}_{i} = \gamma^{\alpha}_{i}$ on $\mathcal{S}^{i}$.

Also, for each morphism $f : \mathcal{S} \to \mathcal{T}$ in $\mathrm{SCub}^{\gamma}_{\omega}$, $(f^{c\Gamma})^{s\gamma} = (f^{c})^{s}$, so that $(f^{c\Gamma})^{s\gamma} = f$.

3.3.7. Lemma. *The maps $\mu_{\mathcal{S}} : \mathcal{S}' \to \mathcal{S}$ induce a natural isomorphism $\mu : (-)^{s\gamma} \circ (-)^{c\Gamma} \Rightarrow id$.*

Proof. Relative to Lemma 3.2.7 it remains to show that $\mu_{\mathcal{S}}$ is a morphism in $\mathrm{SCub}^{\gamma}_{\omega}$. Indeed, $id\gamma'^{\alpha}_{i} x = \gamma^{\alpha}_{i} id x$ for all $x \in \mathcal{S}'^{i}$. □

3.3.8. Natural isomorphism $id \Rightarrow (-)^{c\Gamma} \circ (-)^{s\gamma}$. Let $(C, \partial, \varepsilon, \star, \Gamma)$ be a category in $\mathrm{Cub}^{\Gamma}_{\omega}$ and let $(\mathcal{S}, \delta, \circ, s, \gamma) := (C, \partial, \varepsilon, \star, \Gamma)^{s\gamma}$ as previously. In this case, the category $(C', \partial', \varepsilon', \star', \Gamma') := ((C, \partial, \varepsilon, \star, \Gamma)^{s\gamma})^{c\Gamma}$ in $\mathrm{Cub}^{\Gamma}_{\omega}$ is determined as follows. For $1 \leqslant i < n$,

- **(i)** the underlying ω-category in Cub_{ω} is $(C', \partial', \varepsilon', \star') = ((C, \partial, \varepsilon, \star)^{s})^{c}$.
- **(ii)** the connections are $\Gamma'^{\alpha}_{n,i}$; they send $a \in C'_n$ represented by some $a_0 \in C_m$ with $m \geqslant n$ to

$$\Gamma'^{\alpha}_{n,i} a = \gamma^{\alpha}_{i} \widetilde{s}_i \dots \widetilde{s}_{n-1} a = [\Gamma^{\alpha}_{m,i} \partial^{\alpha}_{m,i} \varepsilon_{m,i} \partial^{-}_{m,i+1} \dots \varepsilon_{m,n-1} \partial^{-}_{m,n} a_0] = [\Gamma^{\alpha}_{m,i} \partial^{-}_{m,n} a_0].$$

Moreover, for each morphism $g : C \to \mathcal{D}$ in Cub_{ω}, $(g^{s\gamma})^{c\Gamma} = (g^{s})^{c}$ sends $[a]_{\sim}$ in $(C^{s\gamma})^{c\Gamma}_{n}$ to $[g(a)]$ in $(\mathcal{D}^{s\gamma})^{c\Gamma}_{n}$.

3.3.9. Lemma. *The maps $\eta_C : C \to C'$ induce a natural isomorphism $\eta : id \Rightarrow (-)^{c\Gamma} \circ (-)^{s\gamma}$.*

Proof. We must show, relative to Lemma 3.2.9, that η_C and $\overline{\eta}_C$ are morphisms in $\mathrm{Cub}^{\Gamma}_{\omega}$. Indeed, for all $1 \leqslant i < n$, $a \in C_{n-1}$ and $b \in C'_{n-1}$ represented by some $b_0 \in C_{m-1}$,

$$\begin{aligned} &\Gamma'^{\alpha}_{n,i} \eta_C a = \Gamma'^{\alpha}_{n,i} [\varepsilon_{n,n} a] = [\Gamma^{\alpha}_{n,i} a] = \eta_C \Gamma^{\alpha}_{n,i} a, \\ &\overline{\eta}_C \Gamma'^{\alpha}_{n,i} b = \partial^{-}_{n+1,n+1} \dots \partial^{-}_{m,m} \Gamma^{\alpha}_{m,i} \partial^{-}_{m,n} b_0 = \Gamma^{\alpha}_{n,i} \partial^{-}_{n,n} \dots \partial^{-}_{m,m} b_0 = \Gamma^{\alpha}_{n,i} \overline{\eta}_C b. \end{aligned}$$

□

3.4. Equivalence for inverses

Finally, we extend the equivalence $\mathrm{SCub}^{\gamma}_{\omega} \simeq \mathrm{Cub}^{\Gamma}_{\omega}$ to the case (ω, p).

3.4.1. Theorem. *The categories $\mathrm{SCub}^{\gamma}_{(\omega,p)}$ and $\mathrm{Cub}^{\Gamma}_{(\omega,p)}$ are equivalent.*

Proof. More specifically, we show that the functors $(-)^{c\Gamma} : \mathrm{SCub}^{\gamma}_{\omega} \simeq \mathrm{Cub}^{\Gamma}_{\omega} : (-)^{s\gamma}$ from Theorem 3.3.1 extend to $\mathrm{SCub}^{\gamma}_{(\omega,p)} \simeq \mathrm{Cub}^{\Gamma}_{(\omega,p)}$. First, suppose $\mathcal{S}$ is a category in $\mathrm{SCub}^{\gamma}_{(\omega,p)}$. Let $C = \mathcal{S}^{c\Gamma}$ and, for all $n > p$ and $1 \leqslant i \leqslant n$, pick an n-cell c in C_n with an $R_{n-1,i}$-invertible shell.

- Then $\partial^{\alpha}_{n,j} c$ has an $R_{n-1,i-1}$-inverse, d say, for each $1 \leqslant j < i$, by the hypothesis. Therefore

$$\begin{aligned} \partial^{\alpha}_{n,j} c \star_{n-1,i-1} d &= \varepsilon_{n-1,i-1} \partial^{-}_{n-1,i-1} \partial^{\alpha}_{n,j} c, \\ s_{n-1} \dots s_j \delta^{\alpha}_{j} c \circ_{i-1} d &= \delta^{-}_{i-1} s_{n-1} \dots s_j \delta^{\alpha}_{j} c, \\ \delta^{\alpha}_{j} c \circ_i e &= \delta^{-}_{i} \delta^{\alpha}_{j} c, \end{aligned}$$

  where we write $e = \widetilde{s}_j \dots \widetilde{s}_{n-1} d$ for short. Similarly we can show that $e \circ_i \delta^{\alpha}_{j} c = \delta^{+}_{i} \delta^{\alpha}_{j} c$, $\Delta_i(\delta^{\alpha}_{j} c, e)$ and $\Delta_i(e, \delta^{\alpha}_{j} c)$. Hence e is the r_i-inverse of $\delta^{\alpha}_{j} c$.

- Alternatively, the hypothesis implies that $\partial^{\alpha}_{n,j} c$ has an $R_{n-1,i}$-inverse, d say, for each $i < j \leqslant n$. So

$$\begin{aligned} \partial^{\alpha}_{n,j} c \star_{n-1,i} d &= \varepsilon_{n-1,i} \partial^{-}_{n-1,i} \partial^{\alpha}_{n,j} c, \\ s_{n-1} \dots s_j \delta^{\alpha}_{j} c \circ_i d &= \delta^{-}_{i} s_{n-1} \dots s_j \delta^{\alpha}_{j} c, \\ \delta^{\alpha}_{j} c \circ_i e &= \delta^{-}_{i} \delta^{\alpha}_{j} c, \end{aligned}$$

  where we abbreviate $e = \widetilde{s}_j \dots \widetilde{s}_{n-1} d$. Again we can prove that $e \circ_i \delta^{\alpha}_{j} c = \delta^{+}_{i} \delta^{\alpha}_{j} c$, $\Delta_i(\delta^{\alpha}_{j} c, e)$ and $\Delta_i(e, \delta^{\alpha}_{j} c)$, and e is the r_i-inverse of $\delta^{\alpha}_{j} c$.

This shows that c has an r_i-invertible $(n-1)$-shell and hence the r_i-inverse $r_i c \in \mathcal{S}^{>n}$ by Lemma 2.4.7. It satisfies $\Delta_i(c, r_i c)$, $\Delta_i(r_i c, c)$, $c \circ_i r_i c = \delta_i^- c$ and $r_i c \circ_i c = \delta_i^+ c$. It thus follows that $r_i c \in C_n$, and that

$$c \star_{n,i} r_i c = c \circ_i r_i c = \delta_i^- c = \widetilde{s}_i \dots \widetilde{s}_{n-1} s_{n-1} \dots s_i \delta_i^- c = \varepsilon_{n,i} \partial_{n,i}^- c,$$
$$r_i c \star_{n,i} c = r_i c \circ_i c = \delta_i^+ c = \widetilde{s}_i \dots \widetilde{s}_{n-1} s_{n-1} \dots s_i \delta_i^+ c = \varepsilon_{n,i} \partial_{n,i}^+ c.$$

Therefore, c is $R_{n,i}$-invertible in C and C is a cubical (ω, p)-category.

Second, suppose C is a classical (ω, p)-category. Let $\mathcal{S} = C^{s\gamma}$ and, for all $n > p$ and $i \geqslant 1$, pick a cell s in $\mathcal{S}^{>n}$ with an r_i-invertible $(n-1)$-shell. For each $j \geqslant 1$ with $i \neq j$, $\delta_j^\alpha s$ then has an r_i-inverse $t = r_i \delta_j^\alpha s$. Pick representatives c and d in C_n of s and t, respectively, which is possible because $s \in \mathcal{S}^{>n}$. Then

$$\delta_j^\alpha s \circ_i t = \delta_i^- \delta_j^\alpha s = \delta_i^- \delta_j^\alpha s \quad \text{and} \quad \varepsilon_{n,j} \partial_{n,j}^\alpha c \star_{n,i} d = \varepsilon_{n,j} \partial_{n,j}^\alpha \varepsilon_{n,i} \partial_{n,i}^- c.$$

- If $i < j$, then $\partial_{n,j}^\alpha c \star_{n-1,i} e = \partial_{n,j}^\alpha \varepsilon_{n,i} \partial_{n,i}^- c = \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,j}^\alpha c$, where we have abbreviated $e = \partial_{n,j}^\alpha d$. Similarly we can show that $e \star_{n-1,i} \partial_{n,j}^\alpha c = \varepsilon_{n-1,i} \partial_{n-1,i}^+ \partial_{n,j}^\alpha c$. Thus e is the $R_{n-1,i}$-inverse of $\partial_{n,j}^\alpha c$.
- Alternatively, if $i > j$, then $\partial_{n,j}^\alpha c \star_{n-1,i-1} e = \partial_{n,j}^\alpha \varepsilon_{n,i} \partial_{n,i}^- c = \varepsilon_{n-1,i-1} \partial_{n-1,i-1}^- \partial_{n,j}^\alpha c$, where we abbreviate $e = \partial_{n,j}^\alpha d$. Similarly we can show that $e \star_{n-1,i-1} \partial_{n,j}^\alpha c = \varepsilon_{n-1,i-1} \partial_{n-1,i-1}^+ \partial_{n,j}^\alpha c$. Hence once again e is the $R_{n-1,i-1}$-inverse of $\partial_{n,j}^\alpha c$.

It follows from the definition that c has an $R_{n-1,i}$-invertible shell and hence an $R_{n,i}$-inverse $R_{n,i} c \in C_n$. It satisfies $c \star_{n,i} R_{n,i} c = \varepsilon_{n,i} \partial_{n,i}^- c$ and $R_{n,i} c \star_{n,i} c = \varepsilon_{n,i} \partial_{n,i}^+ c$. Suppose $u = [R_{n,i} c]_\sim \in \mathcal{S}$. The previous equations then imply that $s \circ_i u = \delta_i^- s$ and $u \circ_i s = \delta_i^+ s$. Hence u is the r_i-inverse of s. This shows that $\mathcal{S}$ is a single-set (ω, p)-category. □

4. Formalisation with Isabelle/HOL

The Isabelle/HOL proof assistant [53] has been indispensable for developing our axiomatisation of single-set cubical ω-categories in Section 2. In this section we describe our Isabelle components, report on our particular usage of its support for proof automation in taming these axioms and give an example of a non-trivial proof (of Proposition 2.4.8) using our formalisation.

4.1. Isabelle/HOL in a nutshell

Isabelle/HOL is based on a simply-typed classical higher-order logic, which in practice often gives the impression of working in typed set theory. Among similar proof assistants such as Coq [7] or Lean [1], it stands out due to its support for proof automation. On one hand, Isabelle employs internal simplification and proof procedures as well as external proof search tools for first-order logic, which can be invoked using the *Sledgehammer* tactic. On the other, it integrates SAT solvers for counterexample search using the *Nitpick* tactic.

Yet these strengths come at a price: Isabelle's type classes, one of its two main mechanisms for modelling and working with algebraic hierarchies, allow only one single type parameter, which essentially imposes a single-set approach to categories when using type classes. Consequently, the numbers n or ω

do not feature as type parameters in the formalisation of n- or ω-categories and cannot be instantiated easily to fixed finite dimensions. Further, Isabelle's type system does not support dependent types, which may sometimes be desirable in mathematical specifications.

To overcome the first limitation, Isabelle offers locales as a more set-based specification mechanism. This allows more than one type parameter and hence the standard approach to categories with objects and arrows. Yet a locale-based approach to formalising mathematics [5], which has been developed over many years, had recently to be revised [6] for modelling more advanced mathematical concepts such as Grothendieck schemes [11], at the expense of losing many benefits of types and type checking as well as a weaker coupling with Isabelle's main libraries. It might therefore be desirable to use proof assistants with more expressive type systems, such as Coq or Lean, for formalising more advanced features of higher categories. However, more work is needed to evaluate the strengths and weaknesses of different proof assistants in this regard. The formalisation of higher categories would certainly provide excellent test cases.

4.2. Formalising single-set categories

As previously for globular ω-categories [20], the basic Isabelle type class for our formalisation of cubical ω-categories is that of a *catoid*, a structure mentioned en passant in Remark 2.1.2. We start with recalling the basic features of the formalisation of catoids and single-set categories with Isabelle from the Archive of Formal proofs [60].

class *multimagma* =
 fixes *mcomp* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$ (**infixl** $\odot$ *70*)

class *multisemigroup* = *multimagma* +
 assumes *assoc*: $(\bigcup v \in y \odot z.\ x \odot v) = (\bigcup v \in x \odot y.\ v \odot z)$

class *st-op* =
 fixes *src* :: $'a \Rightarrow 'a$ (σ)
 and *tgt* :: $'a \Rightarrow 'a$ (τ)

class *st-multimagma* = *multimagma* + *st-op* +
 assumes *Dst*: $x \odot y \neq \{\} \Longrightarrow \tau\ x = \sigma\ y$
 and *s-absorb* [*simp*]: $\sigma\ x \odot x = \{x\}$
 and *t-absorb* [*simp*]: $x \odot \tau\ x = \{x\}$

class *catoid* = *st-multimagma* + *multisemigroup*

The type classes *multimagma* and *st-op* introduce a multioperation and source and target maps, together with notation $\odot$, σ and τ. The classes *multisemigroup*, *st-op* and *catoid* add structure, extending the classes previously defined. The catoid class, for instance, extends the st-multimagma and multisemigroup classes, while the multisemigroup class extends the class defining its multiplication.

Catoids are extended to single-set categories by imposing the locality and functionality axioms from Section 2.1.1.

class *single-set-category* = *functional-catoid* + *local-catoid*

Each of the above type class features one single type parameter $'a$ (spelled α) and is polymorphic in this parameter. It can therefore be instantiated to more concrete types. In the class *multisemigroup*, for instance, α could be instantiated to the type of strings and $\odot$ to the shuffle operation on strings.

Note also that multisemigroups or catoids have been specified without carrier sets. While such sets can be added easily, it often suffices to regard the type α roughly as a set.

The standard function type in proof assistants such as Coq, Isabelle or Lean models total functions. Partiality is usually modelled using a monad or by adjoining a zero. Here instead we take the multioperation $\odot$ as a basis, mapping to the empty set when two elements cannot be composed. We define the resulting domain of definition as

abbreviation $\Delta\ x\ y \equiv (x \odot y \neq \{\})$

and the partial operation $\otimes$ (denoted $\circ$ in previous sections), using the definite description operator *THE* as

definition *pcomp* :: $'a \Rightarrow\ 'a \Rightarrow\ 'a$ (**infixl** $\otimes$ *70*) **where**
$x \otimes y = (\mathit{THE}\ z.\ z \in x \odot y)$

If $x \odot y$ is empty, Isabelle maps $x \otimes y$ to a value about which nothing particular can be proved. Using $\otimes$ in place of $\odot$ with single-set categories allows us to be precise about definedness conditions in higher categories, which may be subtle, while avoiding clumsy specifications with many set braces and a proliferation of cases in proofs due to undefinedness. In proof assistants with dependent types, the partiality of composition in categories can alternatively be formalised at type level, defining arrow composition on homsets. This approach might be harder do integrate with tools like Sledgehammer.

4.3. Formalising single-set cubical ω-categories

To formalise ω-categories, we have first created indexed variants of the classes leading to *single-set-category* above, that is, classes based on $\odot_i$, $\otimes_i$ for the compositions and $\partial\ i\ \alpha$ for face maps. Unlike in previous sections, our indices start with 0 and we write ∂ instead of δ for single-set face maps. With Isabelle, we can formally link these indexed classes with the non-indexed ones (using so-called sublocale statements between classes) so that all theorems about single-set categories are in scope in the indexed variants. In particular, we have introduced the definedness conditions *DD i* and linked them formally with Δ.

Using the class *icategory* for indexed single-set categories, we have first defined an auxiliary class for ω-categories without symmetries.

class *semi-cubical-omega-category* = *icategory* +
assumes *face-comm*: $i \neq j \Longrightarrow \partial\ i\ \alpha \circ \partial\ j\ \beta = \partial\ j\ \beta \circ \partial\ i\ \alpha$
and *face-func*: $i \neq j \Longrightarrow DD\ j\ x\ y \Longrightarrow \partial\ i\ \alpha\ (x \otimes_j y) = \partial\ i\ \alpha\ x \otimes_j \partial\ i\ \alpha\ y$
and *interchange*: $i \neq j \Longrightarrow DD\ i\ w\ x \Longrightarrow DD\ i\ y\ z \Longrightarrow DD\ j\ w\ y \Longrightarrow DD\ j\ x\ z$
$\Longrightarrow (w \otimes_i x) \otimes_j (y \otimes_i z) = (w \otimes_j y) \otimes_i (x \otimes_j z)$
and *fin-fix*: $\exists\, k.\ \forall\, i.\ k \leq i \longrightarrow \mathit{fFx}\ i\ x$

In the last axiom, $fFx\ i\ x \equiv (\partial\ i\ ff\ x = x)$, which we use in place of the predicate $x \in \mathcal{S}^i$ from Section 2.2, has been defined in the context of a class on which *icategory* is based.

We have further extended this class to one for ω-categories with the remaining axioms for symmetries and reverse symmetries from Definition 2.2.1, after introducing a separate class for these two maps.

class *symmetry-ops* =
  fixes *symmetry* :: $nat \Rightarrow 'a \Rightarrow 'a$ (σ)
  and *inv-symmetry* :: $nat \Rightarrow 'a \Rightarrow 'a$ (ϑ)

class *cubical-omega-category* = *semi-cubical-omega-category* + *symmetry-ops* +
  assumes *sym-type*: $\sigma\sigma\ i\ (\mathit{face\text{-}fix}\ i) \subseteq \mathit{face\text{-}fix}\ (i + 1)$
  and *inv-sym-type*: $\vartheta\vartheta\ i\ (\mathit{face\text{-}fix}\ (i + 1)) \subseteq \mathit{face\text{-}fix}\ i$
  and *sym-inv-sym*: $fFx\ (i + 1)\ x \Longrightarrow \sigma\ i\ (\vartheta\ i\ x) = x$
  and *inv-sym-sym*: $fFx\ i\ x \Longrightarrow \vartheta\ i\ (\sigma\ i\ x) = x$
  and *sym-face1*: $fFx\ i\ x \Longrightarrow \partial\ i\ \alpha\ (\sigma\ i\ x) = \sigma\ i\ (\partial\ (i + 1)\ \alpha\ x)$
  and *sym-face2*: $i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow fFx\ j\ x \Longrightarrow \partial\ i\ \alpha\ (\sigma\ j\ x) = \sigma\ j\ (\partial\ i\ \alpha\ x)$
  and *sym-func*: $i \neq j \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ j\ x\ y \Longrightarrow$
    $\sigma\ i\ (x \otimes_j y) = (\mathit{if}\ j = i + 1\ \mathit{then}\ \sigma\ i\ x \otimes_i \sigma\ i\ y\ \mathit{else}\ \sigma\ i\ x \otimes_j \sigma\ i\ y)$
  and *sym-fix*: $fFx\ i\ x \Longrightarrow fFx\ (i + 1)\ x \Longrightarrow \sigma\ i\ x = x$
  and *sym-sym-braid*: $\mathit{diffSup}\ i\ j\ 2 \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ j\ x \Longrightarrow \sigma\ i\ (\sigma\ j\ x) = \sigma\ j\ (\sigma\ i\ x)$

In Axioms *sym-type* and *inv-sym-type*, the functions $\sigma\sigma$ and $\vartheta\vartheta$ are the image maps corresponding to symmetries and reverse symmetries. Further *face-fix i* denotes the set $\mathcal{S}^i$ of fixed points of the lower face map in direction i. It is defined as $\mathit{face\text{-}fix}\ i \equiv \mathit{Fix}\ (\partial\ i\ ff)$ as in Section 2.2. Other axioms in the class use the predicate $\mathit{diffSup}\ i\ j\ k \equiv (i - j \geq k \vee j - i \geq k)$.

Though our formalisation is a so-called shallow embedding of categories in Isabelle, as it uses Isabelle's built-in types for functions, sets and numbers to axiomatise ω-categories, it has nevertheless some deep features, as we do not define a (sub)type for each $fFx\ i\ x \equiv (\partial\ i\ ff\ x = x)$ and we capture the partiality of cell composition in terms of the predicate DD, but not at type level. Such typing or composition conditions must therefore be declared explicitly in axioms and lemmas, and they need to be checked explicitly in proofs.

Finally, we have provided a class for connections and defined a class for ω-categories with connections.

class *cubical-omega-category-connections* = *cubical-omega-category* + *connection-ops* +
  assumes *conn-face1*: $fFx\ j\ x \Longrightarrow \partial\ j\ \alpha\ (\Gamma\ j\ \alpha\ x) = x$
  and *conn-face2*: $fFx\ j\ x \Longrightarrow \partial\ (j + 1)\ \alpha\ (\Gamma\ j\ \alpha\ x) = \sigma\ j\ x$
  and *conn-face3*: $i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow fFx\ j\ x \Longrightarrow \partial\ i\ \alpha\ (\Gamma\ j\ \beta\ x) = \Gamma\ j\ \beta\ (\partial\ i\ \alpha\ x)$
  and *conn-corner1*: $fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ (i + 1)\ x\ y \Longrightarrow$
    $\Gamma\ i\ tt\ (x \otimes_{(i+1)} y) = (\Gamma\ i\ tt\ x \otimes_{(i+1)} \sigma\ i\ x) \otimes_i (x \otimes_{(i+1)} \Gamma\ i\ tt\ y)$
  and *conn-corner2*: $fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ (i + 1)\ x\ y \Longrightarrow$
    $\Gamma\ i\ ff\ (x \otimes_{(i+1)} y) = (\Gamma\ i\ ff\ x \otimes_{(i+1)} y) \otimes_i (\sigma\ i\ y \otimes_{(i+1)} \Gamma\ i\ ff\ y)$
  and *conn-corner3*: $j \neq i \wedge j \neq i + 1 \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ j\ x\ y \Longrightarrow \Gamma\ i\ \alpha\ (x \otimes_j y) = \Gamma\ i\ \alpha\ x \otimes_j \Gamma\ i\ \alpha\ y$
  and *conn-fix*: $fFx\ i\ x \Longrightarrow fFx\ (i + 1)\ x \Longrightarrow \Gamma\ i\ \alpha\ x = x$
  and *conn-zigzag1*: $fFx\ i\ x \Longrightarrow \Gamma\ i\ tt\ x \otimes_{(i+1)} \Gamma\ i\ ff\ x = x$
  and *conn-zigzag2*: $fFx\ i\ x \Longrightarrow \Gamma\ i\ tt\ x \otimes_i \Gamma\ i\ ff\ x = \sigma\ i\ x$

and *conn-conn-braid*: *diffSup i j 2* $\Longrightarrow$ *fFx j x* $\Longrightarrow$ *fFx i x* $\Longrightarrow$ $\Gamma\ i\ \alpha\ (\Gamma\ j\ \beta\ x) = \Gamma\ j\ \beta\ (\Gamma\ i\ \alpha\ x)$
and *conn-shift*: *fFx i x* $\Longrightarrow$ *fFx* $(i + 1)\ x \Longrightarrow \sigma\ (i + 1)\ (\sigma\ i\ (\Gamma\ (i + 1)\ \alpha\ x)) = \Gamma\ i\ \alpha\ (\sigma\ (i + 1)\ x)$

4.4. Example proofs

We present two Isabelle proofs as examples. The first one shows a proof of Lemma 2.2.10(ii) by automated proof search.

lemma *sym-func1*:
assumes *fFx i x*
and *fFx i y*
and *DD i x y*
shows $\sigma\ i\ (x \otimes_i y) = \sigma\ i\ x \otimes_{(i + 1)} \sigma\ i\ y$
by (*metis assms icid.ts-compat local.iDst local.icat.sscatml.l0-absorb sym-type-var1*)

Isabelle's Sledgehammer tactic has returned the proof shown. Sledgehammer invoques external proof-search tools for first-order logic, which are internally reconstructed by Isabelle's *metis* tool, which itself has been verified using Isabelle to increase trustworthiness. The proof statement lists the lemmas used. All of them are part of our Isabelle component for cubical ω-categories and the components on which it builds.

The second proof shows how a proof by hand can be typed into Isabelle line by line, and each line then be verified automatically using Isabelle's proof tactics – here the third case in Lemma 2.2.11(i).

lemma *inv-sym-face*:
assumes $i \neq j$
and $i \neq j + 1$
and *fFx* $(j + 1)\ x$
shows $\partial\ i\ \alpha\ (\vartheta\ j\ x) = \vartheta\ j\ (\partial\ i\ \alpha\ x)$
proof–
have $\sigma\ j\ (\partial\ i\ \alpha\ (\vartheta\ j\ x)) = \sigma\ j\ (\partial\ i\ \alpha\ (\partial\ j\ \mathit{ff}\ (\vartheta\ j\ x)))$
using *assms*(*3*) *inv-sym-type-var* **by** *simp*
also have $\ldots = \partial\ i\ \alpha\ (\sigma\ j\ (\partial\ j\ \alpha\ (\vartheta\ j\ x)))$
by (*metis assms*(*1*) *assms*(*2*) *assms*(*3*) *inv-sym-type-var local.fFx-prop sym-face-var1*)
also have $\ldots = \partial\ i\ \alpha\ (\sigma\ j\ (\vartheta\ j\ x))$
using *assms*(*1*) *assms*(*2*) *assms*(*3*) *calculation inv-sym-type-var local.sym-face* **by** *presburger*
also have $\ldots = \partial\ i\ \alpha\ (\partial\ (j + 1)\ \alpha\ x)$
by (*metis assms*(*3*) *local.face-compat-var sym-inv-sym-var1*)
finally have $\sigma\ j\ (\partial\ i\ \alpha\ (\vartheta\ j\ x)) = \partial\ i\ \alpha\ (\partial\ (j + 1)\ \alpha\ x)$**.**
thus *?thesis*
by (*smt* (*z3*) *assms*(*3*) *icid.st-eq1 inv-sym-type-var local.face-comm-var local.inv-sym-sym*)
qed

Beyond fully automatic proofs found using Sledgehammer and interactive proofs using Isabelle's proof scripting language Isar, as here, Isabelle also offers so-called apply-style proofs, in which simplification steps, application of rules or substitutions of particular formulas can be combined step-wise with Sledgehammer proofs. Examples can be found in our Isabelle component.

4.5. Taming ω-categories with Isabelle

We have already outlined in the introduction how Isabelle has helped developing the single-set axioms for ω-categories. Here we provide more details. Recall that we have justified these axioms via the equivalence proofs in Section 3. Their selection was driven by the construction of the functors $(-)^c$ and $(-)^s$ and their extensions, which relate classical and single-set concepts. We aimed for a small set of structurally meaningful axioms to make the equivalence proofs smooth and simple.

We started with translating the axioms for $\partial^\alpha_{n,i} : C_n \to C_{n-1}$ into those for $\delta^\alpha_i : \mathcal{S} \to \mathcal{S}$. This was straightforward, for instance, for (i), (ii) or (iii) in (2.2.1), but others required encoding the index-shift in $C_n \to C_{n-1}$ of face maps in the single-set axioms, where graded sets C_n are not immediately available. Instead we used the fixed point sets $\mathcal{S}^i$ or the face maps δ^α_i as guards in the single-set axioms; Isabelle helped us to bring them into convenient form. While this obviated degeneracies, we had to introduce symmetries to relate fixed points at the same dimension but in different directions, and to model the rotations of degenerated cubes through the interactions of symmetries with face maps and compositions. Starting from lattices like that in Subsection 2.2.3, the translations between symmetries and degeneracies in $(-)^c$ and $(-)^s$ and geometric intuitions, we used Isabelle, in particular Sledgehammer in combination with other proof tactics, in an iterative process to adapt or simplify candidate axioms, to analyse their dependencies, and to add axioms in light of the equivalence proof. Beyond symmetries, the equivalence proof for ω-categories led us to experiment with dimensionality axioms using Isabelle. This resulted in Axiom (x), and enabled the colimit and filtration constructions for $(-)^s$.

During this process, we compressed the single-set axiomatisation for ω-categories by a factor >2 to a size similar to the classical one. A significant part of the process was automatic. Most redundant candidate axioms now feature in Lemmas 2.2.10 and 2.2.11. Our work flow for ω-categories with connections has been similar and resulted in a similar compression. Redundant laws are shown in Lemma 2.3.4. Interestingly, we found Axiom (vi) quite late through the equivalence proof.

Our insights in the strengths and weaknesses of Isabelle's proof automation might be valuable for mathematicians working with higher categories, where proofs tend to be highly combinatorial, axiomatisations often fill pages and there can be a big formalisation gap with respect to geometric or (string) diagrammatical intuition. In our work, we were sometimes surprised when Sledgehammer managed to derive seemingly independent conjectures, such as the Yang-Baxter identity in Lemma 2.2.10 or the face identities in Lemma 2.2.11(1). But we also spent hours feeding paper-and-pencil proofs into Isabelle and hard-coding rule applications, including the proof in the following subsection. Overall, interactive proofs with higher categories at the granularity of paper and pencil proofs seem nowadays feasible – and highly beneficial for activities like the one described in this article. Yet a main source of disappointment was that, unlike in previous work, we could not use Isabelle's Nitpick tool for verifying the irredundancy of our axiomatisation: it seems that the underlying SAT solver cannot cope with the arithmetic constraints in our axioms, though that should certainly be possible for SMT solvers.

4.6. A non-trivial proof

At the end of this section, we show our formalisation at work, presenting a proof of Proposition 2.4.8. This example shows that Isabelle's proof automation smoothly supports interactive proofs in higher category theory. For this we have formalised $(\omega, 0)$-categories with Isabelle. A formalisation of (ω, p)-categories based on type classes seems impossible as it would require more than one type parameter.

Defining a type class for $(\omega, 0)$-categories needs some preliminary definitions. First we have defined compositions of sequences of symmetries and reverse symmetries.

primrec *symcomp* :: $nat \Rightarrow nat \Rightarrow 'a \Rightarrow 'a$ (Σ) **where**
  $\Sigma\ i\ 0\ x = x$
| $\Sigma\ i\ (Suc\ j)\ x = \sigma\ (i + j)\ (\Sigma\ i\ j\ x)$

primrec *inv-symcomp* :: $nat \Rightarrow nat \Rightarrow 'a \Rightarrow 'a$ (Θ) **where**
  $\Theta\ i\ 0\ x = x$
| $\Theta\ i\ (Suc\ j)\ x = \Theta\ i\ j\ (\vartheta\ (i + j)\ x)$

Then we have defined r_i-invertibility and shell r_i-invertibility, following Definition 2.4.1.

abbreviation (**in** *cubical-omega-category-connections*)
  ri-inv $i\ x\ y \equiv (DD\ i\ x\ y \wedge DD\ i\ y\ x \wedge x \otimes_i y = \partial\ i\ ff\ x \wedge y \otimes_i x = \partial\ i\ tt\ x)$

abbreviation (**in** *cubical-omega-category-connections*)
  ri-inv-shell $k\ i\ x \equiv (\forall\ j\ \alpha.\ j + 1 \leq k \wedge j \neq i \longrightarrow (\exists\ y.\ ri\text{-}inv\ i\ (\partial\ j\ \alpha\ x)\ y))$

This allowed us to specify a class for $(\omega, 0)$-categories following Definition 2.4.1.

class *cubical-omega-zero-category-connections* = *cubical-omega-category-connections* +
 assumes *ri-inv*: $k \geq 1 \Longrightarrow i \leq k - 1 \Longrightarrow dim\text{-}bound\ k\ x \Longrightarrow ri\text{-}inv\text{-}shell\ k\ i\ x \Longrightarrow \exists\ y.\ ri\text{-}inv\ i\ x\ y$

In the axiom *ri-inv*, the predicate $dim\text{-}bound\ k\ x \equiv (\forall\ i.\ k \leq i \longrightarrow fFx\ i\ x)$, which we use in place of the set $\mathcal{S}^{>k}$ from Section 2.2, has been defined in the context of the class *icategory*.

We have shown uniqueness of r_i-inverses and used this property, together with Isabelle's definite description operator *THE*, to define an inversion map.

lemma *ri-unique*: $(\exists\ y.\ ri\text{-}inv\ i\ x\ y) = (\exists!y.\ ri\text{-}inv\ i\ x\ y)$
 ⟨*proof*⟩

definition $ri\ i\ x = (THE\ y.\ ri\text{-}inv\ i\ x\ y)$

Our proof of Proposition 2.4.8 with Isabelle follows that in Subsection 2.4 quite directly. We do not show the verifications of individual proof steps by Isabelle's proof tools. Details can be found in our Isabelle component [52]. The main part of the proof is captured by a technical lemma that proceeds by induction on the dimension k of the cell x.

lemma *every-dim-k-ri-inv*:
 assumes *dim-bound k x*
 shows $\forall\ i.\ \exists\ y.\ ri\text{-}inv\ i\ x\ y$ **using** *dim-bound k x*
proof (*induction k arbitrary*: *x*)
 case *0*
 thus *?case*
  ⟨*proof*⟩
next
 case (*Suc k*)

{fix *i*
have $\exists\, y.$ *ri-inv i x y*

Here we start a proof by cases for $i \geqslant k+1$ as in Section 2.4. As in the proof by hand, the first case is trivial, and automatic with Isabelle.

proof (*cases Suc k* $\geq$ *i*)
case *True*
thus *?thesis*
⟨*proof*⟩
next
case *False*
{fix *j* α
assume *h*: $j \leq k \wedge j \neq i$

While the proof of $\delta_j^\alpha x \in \mathcal{S}^{j,k+1,k+2,\ldots}$ with Isabelle is automatic, we need to check $s_{k-1}\ldots s_j \delta_j^\alpha x \in \mathcal{S}^{>k}$.

hence *a*: *has-dim-bound k* (Σ *j* (*k* − *j*) (∂ *j* α *x*))
⟨*proof*⟩
have $\exists\, y.$ *ri-inv i* (∂ *j* α *x*) *y*

To construct an r_i-inverse of $\delta_j^\alpha x$, we perform a proof by cases on j.

proof (*cases j* < *i*)
case *True*

For $j < i$, we introduce y as the r_{i-1}-inverse of $s_{k-1}\ldots s_j \delta_j^\alpha x$ using the induction hypothesis.

obtain *y* **where** *b*: *ri-inv* (*i* − *1*) (Σ *j* (*k* − *j*) (∂ *j* α *x*)) *y*
⟨*proof*⟩

We check that $\widetilde{s}_j \ldots \widetilde{s}_{k-1} y$ and $\delta_j^\alpha x$ are composable and then show that these expressions are inverses.

have *c*: *dim-bound k y*
⟨*proof*⟩
hence *d*: *DD i* (∂ *j* α *x*) (Θ *j* (*k* − *j*) *y*)
⟨*proof*⟩
hence *e*: *DD i* (Θ *j* (*k* − *j*) *y*) (∂ *j* α *x*)
⟨*proof*⟩
have *f*: (∂ *j* α *x*) $\otimes_i$ (Θ *j* (*k* − *j*) *y*) = Θ *j* (*k* − *j*) ((Σ *j* (*k* − *j*) (∂ *j* α *x*)) $\otimes_{(i-1)}$ *y*)
⟨*proof*⟩
have (Θ *j* (*k* − *j*) *y*) $\otimes_i$ (∂ *j* α *x*) = Θ *j* (*k* − *j*) (*y* $\otimes_{(i-1)}$ (Σ *j* (*k* − *j*) (∂ *j* α *x*)))
⟨*proof*⟩
thus *?thesis*
⟨*proof*⟩

We proceed similarly in the case of $j > i$.

```
next
  case False
  obtain y where b: ri-inv i (Σ j (k − j) (∂ j α x)) y
    ⟨proof⟩
  have c: dim-bound k y
    ⟨proof⟩
  hence d: DD i (∂ j α x) (Θ j (k − j) y)
    ⟨proof⟩
  hence e: DD i (Θ j (k − j) y) (∂ j α x)
    ⟨proof⟩
  have f: (∂ j α x) ⊗i (Θ j (k − j) y) = Θ j (k − j) ((Σ j (k − j) (∂ j α x)) ⊗i y)
    ⟨proof⟩
  have (Θ j (k − j) y) ⊗i (∂ j α x) = Θ j (k − j) (y ⊗i (Σ j (k − j) (∂ j α x)))
    ⟨proof⟩
  thus ?thesis
    ⟨proof⟩
qed}
```

This shows that x is r_i-invertible. We can now conclude that x is r_i-invertible in each direction i.

```
    thus ?thesis
      ⟨proof⟩
  qed}
  thus ?case
    ⟨proof⟩
qed
```

Every cell in a single-set cubical $(\omega, 0)$-category has finite dimension. Lemma *every-dim-k-ri-inv* therefore shows that every cell is r_i-invertible in every direction i, which is Proposition 2.4.8.

```
lemma every-ri-inv: ∃ y. ri-inv i x y
  using every-dim-k-ri-inv local.fin-fix by blast
```

Relative to the proof by hand, we had to prove several fixed points and definedness conditions for compositions due the deep features of our embedding. These are usually left implicit in proofs by hand or with proper shallow embeddings, where they are discharged by type inference. Except for such proof steps, the granularity of this formal proof is similar to that of a proof by hand owing to Isabelle's proof automation. Nevertheless, in some longer formulas we had to tell Isabelle in detail how assumptions had to be matched with proof goals. These details are visible in our proof document.

As already mentioned, formalising (ω, p)-categories with Isabelle for a finite p is at least complicated with Isabelle's type classes. Particular instances of p can be given, but an arbitrary p would require more than one type parameter, and therefore locales.

5. Conclusion

We have introduced single-set axiomatisations of cubical categories with additional structure such as connections and inverses. We have justified their adequacy through equivalence proofs relative to their classical counterparts. The Isabelle/HOL proof assistant, with its powerful support for proof automation and counterexample search, has been instrumental in this development. We might not have undertaken this research without it. Cubical set and categories have a broad range of applications in mathematics and computer science: from homotopy theory and algebraic topology to homotopy type theory, concurrency theory and rewriting theory. Our formalisation might therefore support innovative applications of proof assistants in these fields. In this regard, our results allow us to outline several lines of research that we hope to explore in the future.

First, our formalisation work yields an initial step towards formal tools and methods that tame the combinatorial complexity of proofs in higher categories. These are meant to support users in reasoning formally with higher categories, based on geometric intuitions if available, and with a high degree of automation. For this, a single-set approach seems relevant because of its algebraic simplicity. As our axiomatisation is essentially single-sorted first-order, we expect it to work well with SMT solvers and similar tools. Yet further experiments, and in particular comparisons with formalisations using locales in Isabelle or proof assistants such as Coq [7] and Lean [1], are needed to identify the most suitable approach. Dependent types, as supported by Coq and Lean, might offer advantages in specifying and reasoning with higher categories orthogonal to the proof automation supplied by Isabelle.

Second, this work is part of a programme on proof support for higher-dimensional rewriting. Higher globular and cubical categories and higher globular and cubical polygraphs, which correspond essentially to higher path categories organised in globes or cubes, are particularly appropriate for this [4]. Cubical categories, for instance, allow natural explicit formulations of confluence results such as the Church-Rosser theorem, Newman's lemma and their higher-dimensional extensions. Higher algebras such as globular Kleene algebras and quantales have already been developed for reasoning about such properties [18, 19] and the initial steps of the globular approach have already been formalised with Isabelle [20]. The construction of similar Kleene algebras and quantales related to cubical categories and their formalisation along similar lines seem equally desirable.

Third, one can construct polygraphic resolutions related to rewriting properties in globular or cubical categories [45, 46]. In a companion article [49] we are using our single-set approach to formalise normalisation strategies for higher abstract rewriting systems, which provide a constructive approach to their polygraphic resolutions in the cubical setting. Based on this, we aim to formalise polygraphic constructions of higher-dimensional rewriting on categories [39, 40] and higher algebras [38, 50] using proof assistants and the mathematical components created for them.

A fourth line of work might concern the categorical constructions from Section 3. Theorem 3.2.1, in particular shows that the category SCub_ω of single-set cubical ω-categories is equivalent to its classical counterpart Cub_ω. This equivalence of categories lives in the 2-category of categories, where the natural isomorphisms 3.2.6 and 3.2.8 are 2-cells. One may wonder how to formulate such an equivalence in a single-set globular 2-category. This would constitute a formalisation of the proof of justification of the single-set axiomatisation within a single-set approach. The question arises in the same way for the equivalences of Theorems 3.3.1 and 3.4.1.

Acknowledgements. The authors would like to thank Uli Fahrenberg for a preliminary discussion about single-set cubical 2-categories. The third author would like to thank the CNRS for an invited professorship at the *Institut Camille Jordan* at Lyon, during which part of this work has been conducted.

References

[1] Lean. https://lean-lang.org.

[2] Fahd Ali Al-Agl, Ronald Brown, and Richard Steiner. Multiple categories: the equivalence of a globular and a cubical approach. *Adv. Math.*, 170(1):71–118, 2002.

[3] Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Robert Harper, Kuen-Bang Hou (Favonia), and Daniel R. Licata. Syntax and models of cartesian cubical type theory. *Math. Struct. Comput. Sci.*, 31(4):424–468, 2021.

[4] Dimitri Ara, Albert Burroni, Yves Guiraud, Philippe Malbos, François Métayer, and Samuel Mimram. Polygraphs: from rewriting to higher categories. London Mathematical Society Lecture Note Series, 666pp, arXiv:2312.00429, to appear, 2024.

[5] Clemens Ballarin. Locales: A module system for mathematical theories. *J. Autom. Reason.*, 52(2):123–153, 2014.

[6] Clemens Ballarin. Exploring the structure of an algebra text with locales. *J. Autom. Reason.*, 64(6):1093–1121, 2020.

[7] Yves Bertot and Pierre Castéran. *Interactive theorem proving and program development. Coq'Art: the calculus of inductive constructions. Foreword by Gérard Huet and Christine Paulin-Mohring.* Texts Theor. Comput. Sci., EATCS Ser. Berlin: Springer, 2004.

[8] Marc Bezem and Thierry Coquand. A Kripke model for simplicial sets. *Theoret. Comput. Sci.*, 574:86–91, 2015.

[9] Marc Bezem, Thierry Coquand, and Simon Huber. A model of type theory in cubical sets. In *19th International Conference on Types for Proofs and Programs*, volume 26 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 107–128. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2014.

[10] Marc Bezem, Thierry Coquand, and Simon Huber. The univalence axiom in cubical sets. *J. Automat. Reason.*, 63(2):159–171, 2019.

[11] Anthony Bordg, Lawrence C. Paulson, and Wenda Li. Grothendieck's schemes in algebraic geometry. *Arch. Formal Proofs*, 2021, 2021.

[12] Ronald Brown and Philip J. Higgins. Colimit theorems for relative homotopy groups. *J. Pure Appl. Algebra*, 22(1):11–41, 1981.

[13] Ronald. Brown and Philip J. Higgins. The equivalence of ∞-groupoids and crossed complexes. *Cahiers de topologie et géométrie différentielle catégoriques*, 22(4):371–383, 1981.

[14] Ronald Brown and Philip J. Higgins. On the algebra of cubes. *J. Pure Appl. Algebra*, 21(3):233–260, 1981.

[15] Ronald Brown, Philip J. Higgins, and Rafael Sivera. *Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev*, volume 15 of *EMS Tracts Math.* Zürich: European Mathematical Society (EMS), 2011.

[16] Ronald Brown and Jean-Louis Loday. Van Kampen theorems for diagrams of spaces. *Topology*, 26(3):311–335, 1987.

[17] Ronald Brown and Christopher B. Spencer. Double groupoids and crossed modules. *Cah. Topologie Géom. Différ. Catégoriques*, 17:343–362, 1976.

[18] Cameron Calk, Eric Goubault, Philippe Malbos, and Georg Struth. Algebraic coherent confluence and higher globular Kleene algebras. *Logical Methods in Computer Science*, Volume 18, Issue 4, November 2022.

[19] Cameron Calk, Philippe Malbos, Damien Pous, and Georg Struth. Higher catoids, higher quantales and their correspondences, 2023. arXiv 2307.09253.

[20] Cameron Calk and Georg Struth. Higher globular catoids and quantales. *Archive of Formal Proofs*, January 2024. `https://isa-afp.org/entries/OmegaCatoidsQuantales.html`, Formal proof development.

[21] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017.

[22] James Cranch, Simon Doherty, and Georg Struth. Relational semigroups and object-free categories. *CoRR*, abs/2001.11895, 2020.

[23] Jules Desharnais and Georg Struth. Internal axioms for domain semirings. *Sci. Comput. Program.*, 76(3):181–203, 2011.

[24] Uli Fahrenberg, Christian Johansen, Georg Struth, and Krzysztof Ziemiański. Kleene theorem for higher-dimensional automata, 2023. arXiv 2202.03791, 2023.

[25] Uli Fahrenberg, Christoph Johansen, Georg Struth, and Krzysztof Ziemiański. Catoids and modal convolution algebras. *Algebra Universalis*, 84:10, 2023.

[26] Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, and Martin Raussen. *Directed algebraic topology and concurrency.* Springer, 2016.

[27] Lisbeth Fajstrup, Eric Goubault, and Martin Raußen. Detecting deadlocks in concurrent systems. In Davide Sangiorgi and Robert de Simone, editors, *CONCUR '98*, volume 1466 of *LNCS*, pages 332–347. Springer, 1998.

[28] Lisbeth Fajstrup, Martin Raußen, and Eric Goubault. Algebraic topology and concurrency. *Theoret. Comput. Sci.*, 357(1-3):241–278, 2006.

[29] Peter Freyd. *Abelian categories. An introduction to the theory of functors.* Harper's Series in Modern Mathematics. Harper & Row, Publishers, New York, 1964.

[30] Peter J. Freyd and Andre Scedrov. *Categories, allegories*, volume 39 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1990.

[31] Hitoshi Furusawa and Georg Struth. Concurrent dynamic algebra. *ACM Trans. Comput. Log.*, 16(4):30:1–30:38, 2015.

REFERENCES

[32] Philippe Gaucher. Homotopy invariants of higher dimensional categories and concurrency in computer science. volume 10, pages 481–524. 2000.

[33] Philippe Gaucher. About the globular homology of higher dimensional automata. *Cah. Topol. Géom. Différ. Catég.*, 43(2):107–156, 2002.

[34] Eric Goubault and Thomas P. Jensen. Homology of higher-dimensional automata. In *CONCUR '92*, volume 630 of *LNCS*, pages 254–268. Springer, Berlin, 1992.

[35] Marco Grandis. Higher cospans and weak cubical categories (cospans in algebraic topology, i). *Theory and Applications of Categories*, 18:321–347, 07 2007.

[36] Marco Grandis. *Directed algebraic topology*, volume 13 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2009.

[37] Marco Grandis and Luca Mauri. Cubical sets and their site. *Theory Appl. Categ.*, 11:No. 8, 185–211, 2003.

[38] Yves Guiraud, Eric Hoffbeck, and Philippe Malbos. Convergent presentations and polygraphic resolutions of associative algebras. *Math. Z.*, 293(1-2):113–179, 2019.

[39] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.

[40] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Math. Structures Comput. Sci.*, 28(2):155–201, 2018.

[41] Thomas Kahl. Labeled homology of higher-dimensional automata. *Journal of Applied and Computational Topology*, 2(3-4):271–300, 2018.

[42] Daniel M. Kan. Abstract homotopy. I. *Proc. Natl. Acad. Sci. USA*, 41:1092–1096, 1955.

[43] Chris Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after voevodsky), 2018. arXiv 1211.2851, 2018.

[44] Jean-Louis Loday. Spaces with finitely many nontrivial homotopy groups. *J. Pure Appl. Algebra*, 24(2):179–202, 1982.

[45] Maxime Lucas. *Cubical categories for homotopy and rewriting*. Theses, Université Paris 7, Sorbonne Paris Cité, December 2017.

[46] Maxime Lucas. A cubical squier's theorem. *Mathematical Structures in Computer Science*, 30(2):159–172, 2020.

[47] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[48] Saunders MacLane. Groups, categories and duality. *PNAS*, 34(6):263–267, 1948.

[49] Philippe Malbos, Tanguy Massacrier, and Georg Struth. Higher dimensional cubical equations from confluence. 2024. (not yet published).

[50] Philippe Malbos and Isaac Ren. Shuffle polygraphic resolutions for operads. *Journal of the London Mathematical Society*, 107(1):61–122, 2023.

[51] Georges Maltsiniotis. La catégorie cubique avec connexions est une catégorie test stricte. *Homology Homotopy Appl.*, 11(2):309–326, 2009.

[52] Tanguy Massacrier and Georg Struth. Cubical categories. *Archive of Formal Proofs*, January 2024. `https://isa-afp.org/entries/CubicalCategories.html`, Formal proof development.

[53] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL. A proof assistant for higher-order logic*, volume 2283 of *Lect. Notes Comput. Sci.* Berlin: Springer, 2002.

[54] Vaughn Pratt. Modeling concurrency with geometry. In *Proceedings of the 18th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '91, pages 311–322, New York, NY, USA, 1991. Association for Computing Machinery.

[55] Jean-Pierre Serre. Homologie singulière des espaces fibrés. Applications. *Ann. of Math. (2)*, 54:425–505, 1951.

[56] Eugene W. Stark. Category theory with adjunctions and limits. *Archive of Formal Proofs*, June 2016. `https://isa-afp.org/entries/Category3.html`, Formal proof development.

[57] Eugene W. Stark. Monoidal categories. *Archive of Formal Proofs*, May 2017. `https://isa-afp.org/entries/MonoidalCategory.html`, Formal proof development.

[58] Eugene W. Stark. Bicategories. *Archive of Formal Proofs*, January 2020. `https://isa-afp.org/entries/Bicategory.html`, Formal proof development.

[59] Richard Steiner. Omega-categories and chain complexes. *Homology, Homotopy and Applications*, 6(1):175–200, 2004.

[60] Georg Struth. Catoids, categories, groupoids. *Archive of Formal Proofs*, August 2023. `https://isa-afp.org/entries/Catoids.html`, Formal proof development.

[61] Terese. *Term Rewriting Systems.* Cambridge University Press, 2003.

[62] A. P. Tonks. Cubical groups which are Kan. *J. Pure Appl. Algebra*, 81(1):83–87, 1992.

[63] R. J. van Glabbeek. On the expressiveness of higher dimensional automata. *Theoret. Comput. Sci.*, 356(3):265–290, 2006.

A. Appendices

A.1. End of the proof of Lemma 3.2.3

To show that $\mathcal{S}^c$ is a cubical ω-category, we derive the remaining axioms:

(iv) if x, y are $\star_{k,j}$-composable, by Axioms 2.2.1(ii), (vii) and other ones,

– if $i < j$ then

$$\begin{aligned}
\partial^\alpha_{k,i}(x \star_{k,j} y) &= s_{k-1} \dots s_i \delta^\alpha_i (x \circ_j y) \\
&= s_{k-1} \dots s_{j-1}(s_{j-2} \dots s_i \delta^\alpha_i x \circ_j s_{j-2} \dots s_i \delta^\alpha_i y) \\
&= s_{k-1} \dots s_j (s_{j-1} \dots s_i \delta^\alpha_i x \circ_{j-1} s_{j-1} \dots s_i \delta^\alpha_i y) \\
&= s_{k-1} \dots s_i \delta^\alpha_i x \circ_{j-1} s_{k-1} \dots s_i \delta^\alpha_i y \\
&= \partial^\alpha_{k,i} x \star_{k,j-1} \partial^\alpha_{k,i} y,
\end{aligned}$$

– if $i > j$ then

$$\partial^\alpha_{k,i}(x \star_{k,j} y) = s_{k-1} \dots s_i \delta^\alpha_i (x \circ_j y) = s_{k-1} \dots s_i \delta^\alpha_i x \circ_j s_{k-1} \dots s_i \delta^\alpha_i y = \partial^\alpha_{k,i} x \star_{k,j} \partial^\alpha_{k,i} y,$$

– if $i = j$ then

$$\partial^-_{k,i}(x \star_{k,i} y) = s_{k-1} \dots s_i \delta^-_i (x \circ_i y) = s_{k-1} \dots s_i \delta^-_i x = \partial^-_{k,i} x,$$

$$\partial^+_{k,i}(x \star_{k,i} y) = s_{k-1} \dots s_i \delta^+_i (x \circ_i y) = s_{k-1} \dots s_i \delta^+_i y = \partial^+_{k,i} y,$$

(v) if a, b are $\star_{k,i}$-composable, c, d are $\star_{k,i}$-composable, a, c are $\star_{k,j}$-composable, b, d are $\star_{k,j}$-composable, and if $i \neq j$, then $\Delta_i(a, b)$, $\Delta_i(c, d)$, $\Delta_j(a, c)$ and $\Delta_j(b, d)$, so by Axiom 2.2.1(iii) and other ones

$$\begin{aligned}(a \star_{k,i} b) \star_{k,j} (c \star_{k,i} d) &= (a \circ_i b) \circ_j (c \circ_i d)\\ &= (a \circ_j c) \circ_i (b \circ_j d)\\ &= (a \star_{k,j} c) \star_{k,i} (b \star_{k,j} d),\end{aligned}$$

(vi) – if $i < j$ then by Axioms 2.2.1(iv), (v), (vi), (viii) and other ones

$$\begin{aligned}\partial^\alpha_{k,i} \varepsilon_{k,j} &= s_{k-1} \dots s_i \delta^\alpha_i \widetilde{s}_j \dots \widetilde{s}_{k-1}\\ &= s_{k-1} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta^\alpha_i\\ &= s_{k-1} \dots s_j s_{j-2} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta^\alpha_i\\ &= s_{j-2} \dots s_i s_{k-1} \dots s_j \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta^\alpha_i\\ &= s_{j-2} \dots s_i \delta^\alpha_i\\ &= \varepsilon_{k-1,j-1} \partial^\alpha_{k-1,i},\end{aligned}$$

– if $i > j$ then similarly $\partial^\alpha_{k,i} \varepsilon_{k,j} = s_{k-1} \dots s_i \delta^\alpha_i \widetilde{s}_j \dots \widetilde{s}_{k-1} = s_{j-1} \dots s_{i-1} \delta^\alpha_{i-1} = \varepsilon_{k-1,j} \partial^\alpha_{k-1,i-1}$,

– if $i = j$ then $\partial^\alpha_{k,i} \varepsilon_{k,i} = s_{k-1} \dots s_i \delta^\alpha_i \widetilde{s}_i \dots \widetilde{s}_{k-1} = s_{k-1} \dots s_i \widetilde{s}_i \dots \widetilde{s}_{k-1} = id$,

(vii) if a, b are $\star_{k,j}$-composable,

– if $i \leqslant j$ then

$$\begin{aligned}\varepsilon_{k+1,i}(a \star_{k,j} b) &= \widetilde{s}_i \dots \widetilde{s}_k (a \circ_j b)\\ &= \widetilde{s}_i \dots \widetilde{s}_j (\widetilde{s}_{j+1} \dots \widetilde{s}_k a \circ_j \widetilde{s}_{j+1} \dots \widetilde{s}_k b)\\ &= \widetilde{s}_i \dots \widetilde{s}_{j-1} (\widetilde{s}_j \dots \widetilde{s}_k a \circ_{j+1} \widetilde{s}_j \dots \widetilde{s}_k b)\\ &= \widetilde{s}_i \dots \widetilde{s}_k a \circ_{j+1} \widetilde{s}_i \dots \widetilde{s}_k b\\ &= \varepsilon_{k+1,i} a \star_{k+1,j+1} \varepsilon_{k+1,i} b,\end{aligned}$$

– if $i > j$ then

$$\varepsilon_{k+1,i}(a \star_{k,j} b) = \widetilde{s}_i \dots \widetilde{s}_k (a \circ_j b) = \widetilde{s}_i \dots \widetilde{s}_k a \circ_j \widetilde{s}_i \dots \widetilde{s}_k b = \varepsilon_{k+1,i} a \star_{k+1,j} \varepsilon_{k+1,i} b,$$

(viii) if $i \leqslant j$ then by Axioms 2.2.1(viii), (ix) and other ones

$$\begin{aligned}\varepsilon_{k+1,j+1} \varepsilon_{k,i} &= \widetilde{s}_{j+1} \dots \widetilde{s}_k \widetilde{s}_i \dots \widetilde{s}_{k-1}\\ &= \widetilde{s}_i \dots \widetilde{s}_{j-1} \widetilde{s}_{j+1} \dots \widetilde{s}_k \widetilde{s}_j \dots \widetilde{s}_{k-1}\\ &= \widetilde{s}_i \dots \widetilde{s}_k \widetilde{s}_j \dots \widetilde{s}_{k-1}\\ &= \varepsilon_{k+1,i} \varepsilon_{k,j}.\end{aligned}$$

A.2. End of the proof of Lemma 3.2.5

To show that C^s is a single-set cubical ω-category, we derive the remaining axioms:

(iv) if $\Delta_j(x,y)$,

- if $i<j$ then

$$\begin{aligned}\delta_i^\alpha(x\circ_j y) &= \varepsilon_{n,i}\partial_{n,i}^\alpha(x\star_{n,j} y)\\ &= \varepsilon_{n,i}(\partial_{n,i}^\alpha x\star_{n-1,j-1}\partial_{n,i}^\alpha y)\\ &= \varepsilon_{n,i}\partial_{n,i}^\alpha x\star_{n,j}\varepsilon_{n,i}\partial_{n,i}^\alpha y)\\ &= \delta_i^\alpha x\circ_j\delta_i^\alpha y,\end{aligned}$$

- if $i>j$ then

$$\begin{aligned}\delta_i^\alpha(x\circ_j y) &= \varepsilon_{n,i}\partial_{n,i}^\alpha(x\star_{n,j} y)\\ &= \varepsilon_{n,i}(\partial_{n,i}^\alpha x\star_{n-1,j}\partial_{n,i}^\alpha y)\\ &= \varepsilon_{n,i}\partial_{n,i}^\alpha x\star_{n,j}\varepsilon_{n,i}\partial_{n,i}^\alpha y)\\ &= \delta_i^\alpha x\circ_j\delta_i^\alpha y,\end{aligned}$$

(v) exchange law: if $i\neq j$, $\Delta_i(w,x)$, $\Delta_i(y,z)$, $\Delta_j(w,y)$ and $\Delta_j(x,z)$ then

$$(w\circ_i x)\circ_j(y\circ_i d)=(w\star_{n,i}x)\star_{n,j}(y\star_{n,i}z)=(w\star_{n,j}y)\star_{n,i}(x\star_{n,j}z)=(w\circ_j y)\circ_i(x\circ_j z),$$

(vi)
- if $x\in\mathcal{S}^i$ then $\delta_{i+1}^-s_ix=\varepsilon_{n,i+1}\partial_{n,i+1}^-\varepsilon_{n,i+1}\partial_{n,i}^-x=s_ix$, so $s_ix\in\mathcal{S}^{i+1}$,
- if $y\in\mathcal{S}^{i+1}$ then $\delta_i^-\widetilde{s}_iy=\varepsilon_{n,i}\partial_{n,i}^-\varepsilon_{n,i}\partial_{n,i+1}^-y=\widetilde{s}_iy$, so $\widetilde{s}_iy\in\mathcal{S}^i$,

(vii)
- if $x\in\mathcal{S}^i$ then $\widetilde{s}_is_ix=\varepsilon_{n,i}\partial_{n,i+1}^-\varepsilon_{n,i+1}\partial_{n,i}^-x=\delta_i^-x=x$,
- if $y\in\mathcal{S}^{i+1}$ then $s_i\widetilde{s}_iy=\varepsilon_{n,i+1}\partial_{n,i}^-\varepsilon_{n,i}\partial_{n,i+1}^-y=\delta_{i+1}^-y=y$,

(viii) if $x\in\mathcal{S}^j$,

- then

$$\delta_j^\alpha s_jx=\varepsilon_{n,j}\partial_{n,j}^\alpha\varepsilon_{n,j+1}\partial_{n,j}^-x=\varepsilon_{n,j+1}\varepsilon_{n-1,j}\partial_{n-1,j}^\alpha\partial_{n,j+1}^-x=\varepsilon_{n,j}\varepsilon_{n-1,j}\partial_{n-1,j}^\alpha\partial_{n,j}^-x=s_j\delta_{j+1}^\alpha x,$$

- if $i<j$ then

$$\delta_i^\alpha s_jx=\varepsilon_{n,i}\partial_{n,i}^\alpha\varepsilon_{n,j+1}\partial_{n,j}^-x=\varepsilon_{n,i}\varepsilon_{n-1,j}\partial_{n-1,i}^\alpha\partial_{n,j}^\beta x=\varepsilon_{n,j+1}\varepsilon_{n-1,i}\partial_{n-1,j-1}^\alpha\partial_{n,i}^\beta x=s_j\delta_i^\alpha x,$$

- $i>j+1$ then

$$\delta_i^\alpha s_jx=\varepsilon_{n,i}\partial_{n,i}^\alpha\varepsilon_{n,j+1}\partial_{n,j}^-x=\varepsilon_{n,i}\varepsilon_{n-1,j+1}\partial_{n-1,i-1}^\alpha\partial_{n,j}^\beta x=\varepsilon_{n,j+1}\varepsilon_{n-1,i-1}\partial_{n-1,j}^\alpha\partial_{n,i}^\beta x=s_j\delta_i^\alpha x,$$

(ix) if $x,y\in\mathcal{S}^i$ and $\Delta_j(x,y)$,

– if $j = i+1$ then

$$\begin{aligned} s_i(x \circ_{i+1} y) &= \varepsilon_{n,i+1}\partial^-_{n,i}(x \star_{n,i+1} y) \\ &= \varepsilon_{n,i+1}(\partial^-_{n,i}x \star_{n,i} \partial^-_{n,i}y) \\ &= \varepsilon_{n,i+1}\partial^-_{n,i}x \star_{n,i} \varepsilon_{n,i+1}\partial^-_{n,i}y \\ &= s_i x \circ_i s_i y \end{aligned}$$

– if $j < i$ then

$$\begin{aligned} s_i(x \circ_j y) &= \varepsilon_{n,i+1}\partial^-_{n,i}(x \star_{n,j} y) \\ &= \varepsilon_{n,i+1}(\partial^-_{n,i}x \star_{n,j} \partial^-_{n,i}y) \\ &= \varepsilon_{n,i+1}\partial^-_{n,i}x \star_{n,j} \varepsilon_{n,i+1}\partial^-_{n,i}y \\ &= s_i x \circ_j s_i y \end{aligned}$$

– if $j > i+1$ then

$$\begin{aligned} s_i(x \circ_j y) &= \varepsilon_{n,i+1}\partial^-_{n,i}(x \star_{n,j} y) \\ &= \varepsilon_{n,i+1}(\partial^-_{n,i}x \star_{n,j-1} \partial^-_{n,i}y) \\ &= \varepsilon_{n,i+1}\partial^-_{n,i}x \star_{n,j} \varepsilon_{n,i+1}\partial^-_{n,i}y \\ &= s_i x \circ_j s_i y \end{aligned}$$

(x) if $x \in \mathcal{S}^{i,i+1}$ then

$$s_i x = s_i \delta^-_i \delta^-_{i+1} x = \varepsilon_{n,i+1}\partial^-_{n,i}\varepsilon_{n,i+1}\partial^-_{n,i+1}x = \varepsilon_{n,i+1}\partial^-_{n,i+1}\varepsilon_{n,i}\partial^-_{n,i}x = \delta^-_{i+1}\delta^-_i x = x,$$

(xi) if $x \in \mathcal{S}^{i,j}$,

– if $i < j-1$ then

$$s_i s_j x = \varepsilon_{n,i+1}\partial^-_{n,i}\varepsilon_{n,j+1}\partial^-_{n,j}x = \varepsilon_{n,i+1}\varepsilon_{n,j}\partial^-_{n,i}\partial^-_{n,j}x = \varepsilon_{n,j+1}\varepsilon_{n-1,i+1}\partial^-_{n-1,j-1}\partial^-_{n,i}x = s_j s_i x,$$

– if $i > j+1$ then

$$s_i s_j x = \varepsilon_{n,i+1}\partial^-_{n,i}\varepsilon_{n,j+1}\partial^-_{n,j}x = \varepsilon_{n,i+1}\varepsilon_{n,j+1}\partial^-_{n,i-1}\partial^-_{n,j}x = \varepsilon_{n,j+1}\varepsilon_{n-1,i}\partial^-_{n-1,j}\partial^-_{n,i}x = s_j s_i x,$$

(xii) each $x \in \mathcal{S}$ has a representative $a \in C_n$ for some $n \in \mathbb{N}$, so let $i \geqslant n+1$, then $a' = \varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a \in C_i$ represents x, so by definition

$$\delta^-_i x = cl_\sim(\varepsilon_{i,i}\partial^-_{i,i}a') = cl_\sim(\varepsilon_{i,i}\partial^-_{i,i}\varepsilon_{i,i} \dots \varepsilon_{n+1,n+1}a) = cl_\sim(\varepsilon_{i,i} \dots \varepsilon_{n+1,n+1}a) = x,$$

so $x \in \mathcal{S}^{>n}$.

A.3. End of the proof of Lemma 3.3.3

To show that $\mathcal{S}^{c\Gamma}$ is a cubical ω-category with connections, we prove the remaining axioms:

(i)
- $\partial^{\alpha}_{k,i+1}\Gamma^{\alpha}_{k,i} = s_{k-1}\ldots s_{i+1}\delta^{\alpha}_{i+1}\gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = id$,
- $\partial^{\alpha}_{k,i}\Gamma^{-\alpha}_{k,i} = s_{k-1}\ldots s_i\delta^{\alpha}_{i+1}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \widetilde{s}_i\ldots\widetilde{s}_{k-2}s_{k-2}\ldots s_i\delta^{\alpha}_{i} = \varepsilon_{k-1,i}\partial^{\alpha}_{k-1,i}$,
- $\partial^{\alpha}_{k,i+1}\Gamma^{-\alpha}_{k,i} = s_{k-1}\ldots s_{i+1}\delta^{\alpha}_{i+1}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \widetilde{s}_i\ldots\widetilde{s}_{k-2}s_{k-2}\ldots s_i\delta^{\alpha}_{i} = \varepsilon_{k-1,i}\partial^{\alpha}_{k-1,i}$,
- if $i > j+1$ then

$$\partial^{\alpha}_{k,i}\Gamma^{\beta}_{k,j} = \gamma^{\beta}_{j}s_{k-1}\ldots s_i\widetilde{s}_j\ldots\delta^{\alpha}_{i}\widetilde{s}_{i-1}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \gamma^{\beta}_{j}\widetilde{s}_j\ldots\widetilde{s}_{i-2}\delta^{\alpha}_{i-1} = \Gamma^{\beta}_{k-1,j}\partial^{\alpha}_{k-1,i-1},$$

(ii) if a, b are $\star_{k,j}$-composable,

- $i = j$ then

$$\begin{aligned}\Gamma^{+}_{k+1,i}(a\star_{k,i}b) &= \gamma^{+}_{i}(\widetilde{s}_i\ldots\widetilde{s}_k a\circ_{i+1}\widetilde{s}_i\ldots\widetilde{s}_k b)\\ &= (\gamma^{+}_{i}\widetilde{s}_i\ldots\widetilde{s}_k a\circ_i\widetilde{s}_i\ldots\widetilde{s}_k a)\circ_{i+1}(s_i\widetilde{s}_i\ldots\widetilde{s}_k a\circ_i\gamma^{+}_{i}\widetilde{s}_i\ldots\widetilde{s}_k b)\\ &= (\Gamma^{+}_{k+1,i}a\star_{k+1,i}\varepsilon_{k+1,i}a)\star_{k+1,i+1}(\varepsilon_{k+1,i+1}a\star_{k+1,i}\Gamma^{+}_{k+1,i}b),\end{aligned}$$

- if $i > j$ then $\Gamma^{\alpha}_{k+1,i}(a\star_{k,j}b) = \gamma^{\alpha}_{i}(\widetilde{s}_i\ldots\widetilde{s}_k a\circ_j\widetilde{s}_i\ldots\widetilde{s}_k b) = \Gamma^{\alpha}_{k+1,i}a\star_{k+1,j}\Gamma^{\alpha}_{k+1,i}b$,

(iii)
- $\Gamma^{+}_{k,i}a\star_{k,i}\Gamma^{-}_{k,i}a = \gamma^{+}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1}a\circ_i\gamma^{-}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1}a = \varepsilon_{k,i+1}a$ by Axiom 2.3.1(iv) and other ones,
- $\Gamma^{+}_{k,i}a\star_{k,i+1}\Gamma^{-}_{k,i}a = \gamma^{+}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1}a\circ_{i+1}\gamma^{-}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1}a = \varepsilon_{k,i}a$ by Axiom 2.3.1(iv) and other ones,

(iv)
- by Axiom 2.3.1(iii) and other ones $\Gamma^{\alpha}_{k+1,i}\varepsilon_{k,i} = \gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_k\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \varepsilon_{k+1,i}\varepsilon_{k,i}$,
- if $i > j$ then

$$\Gamma^{\alpha}_{k+1,i}\varepsilon_{k,j} = \widetilde{s}_j\ldots\widetilde{s}_{i-2}\gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_k\widetilde{s}_{i-1}\ldots\widetilde{s}_{k-1} = \widetilde{s}_j\ldots\widetilde{s}_i\gamma^{\alpha}_{i-1}\widetilde{s}_{i+1}\ldots\widetilde{s}_k\widetilde{s}_{i-1}\ldots\widetilde{s}_{k-1} = \varepsilon_{k+1,j}\Gamma^{\alpha}_{k,i-1},$$

(v)
- if $i < j$ then using Axioms 2.3.1(v), (vi) and other ones

$$\begin{aligned}\Gamma^{\alpha}_{k+1,i}\Gamma^{\beta}_{k,j} &= \gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_j\widetilde{s}_{j+1}\gamma^{\beta}_{j}\widetilde{s}_{j+2}\ldots\widetilde{s}_k\widetilde{s}_j\ldots\widetilde{s}_{k-1}\\ &= \gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_{j-1}\gamma^{\beta}_{j+1}\widetilde{s}_{j+1}\ldots\widetilde{s}_k\widetilde{s}_j\ldots\widetilde{s}_{k-1}\\ &= \Gamma^{\beta}_{k+1,j+1}\Gamma^{\alpha}_{k,i},\end{aligned}$$

- $\Gamma^{\alpha}_{k+1,i}\Gamma^{\alpha}_{k,i} = \widetilde{s}_i\widetilde{s}_{i+1}\gamma^{\alpha}_{i}\widetilde{s}_{i+2}\ldots\widetilde{s}_k\gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \gamma^{\alpha}_{i+1}\widetilde{s}_{i+1}\ldots\widetilde{s}_k\gamma^{\alpha}_{i}\widetilde{s}_i\ldots\widetilde{s}_{k-1} = \Gamma^{\alpha}_{k+1,i+1}\Gamma^{\alpha}_{k,i}$.

A.4. End of the proof of Lemma 3.3.5

To show that $C^{s\gamma}$ is a single-set cubical ω-category with connections, we prove the remaining axioms:

(i) if $i \neq j, j+1$ and $x \in \mathcal{S}^j$, then $\partial^{-}_{n,j}x = \partial^{+}_{n,j}x$ so

- $\delta^\alpha_{j+1}\gamma^\alpha_j x = \varepsilon_{n,j+1}\partial^\alpha_{n,j+1}\Gamma^\alpha_{n,j}\partial^\alpha_{n,j}x = \varepsilon_{n,j+1}\partial^\alpha_{n,j}x = s_j x$,
- if $i < j$ then $\delta^\alpha_i\gamma^\beta_j x = \varepsilon_{n,i}\Gamma^\beta_{n-1,j-1}\partial^\alpha_{n-1,i}\partial^\beta_{n,j}x = \Gamma^\beta_{n,j}\varepsilon_{n-1,i}\partial^\beta_{n-1,j-1}\partial^\alpha_{n,i}x = \gamma^\beta_j\delta^\alpha_i x$,
- if $i > j+1$ then $\delta^\alpha_i\gamma^\beta_j x = \varepsilon_{n,i}\Gamma^\beta_{n-1,j}\partial^\alpha_{n-1,i-1}\partial^\beta_{n,j}x = \Gamma^\beta_{n,j}\varepsilon_{n-1,i-1}\partial^\beta_{n-1,j}\partial^\alpha_{n,i}x = \gamma^\beta_j\delta^\alpha_i x$,

(ii) if $j \neq i, i+1$, $x, y \in \mathcal{S}^i$,

- if $\Delta_{i+1}(x,y)$ then

$$\begin{aligned}\gamma^-_i(x \circ_{i+1} y) &= \Gamma^-_{n,i}(\partial^-_{n,i}x \star_{n,i} \partial^-_{n,i}y)\\ &= (\Gamma^-_{n,i}\partial^-_{n,i}x \star_{n+1,i+1} \varepsilon_{n,i}\partial^-_{n,i}y) \star_{n+1,i} (\varepsilon_{n,i+1}\partial^-_{n,i}y \star_{n+1,i+1} \Gamma^-_{n,i}\partial^-_{n,i}y)\\ &= (\gamma^-_i x \circ_{i+1} y) \circ_i (s_i y \circ_{i+1} \gamma^-_i y),\end{aligned}$$

- if $j < i$ and $\Delta_j(x,y)$ then $\gamma^\alpha_i(x \circ_j y) = \Gamma^\alpha_{n,i}(\partial^\alpha_{n,i}x \star_{n-1,j} \partial^\alpha_{n,i}y) = \gamma^\alpha_i x \circ_j \gamma^\alpha_i y$,
- if $j > i+1$ and $\Delta_j(x,y)$ then $\gamma^\alpha_i(x \circ_j y) = \Gamma^\alpha_{n,i}(\partial^\alpha_{n,i}x \star_{n-1,j-1} \partial^\alpha_{n,i}y) = \gamma^\alpha_i x \circ_j \gamma^\alpha_i y$,

(iii) if $x \in \mathcal{S}^{i,i+1}$ then $x = \varepsilon_{n,i}\varepsilon_{n-1,i}\partial^-_{n-1,i}\partial^-_{n,i+1}x$ so

$$\gamma^\alpha_i x = \Gamma^\alpha_{n,i}\varepsilon_{n-1,i}\partial^-_{n-1,i}\partial^-_{n,i+1}x = \varepsilon_{n,i}\varepsilon_{n-1,i}\partial^-_{n-1,i}\partial^-_{n,i+1}x = x,$$

(iv) if $x \in \mathcal{S}^i$, then $\partial^+_{n,i}x = \partial^-_{n,i}x$ so

- $\gamma^+_i x \circ_{i+1} \gamma^-_i x = \Gamma^+_{n,i}\partial^+_{n,i}x \star_{n,i+1} \Gamma^-_{n,i}\partial^-_{n,i}x = \varepsilon_{n,i}\partial^+_{n,i}x = x$,
- $\gamma^+_i x \circ_i \gamma^-_i x = \Gamma^+_{n,i}\partial^+_{n,i}x \star_{n,i} \Gamma^-_{n,i}\partial^-_{n,i}x = \varepsilon_{n,i+1}\partial^+_{n,i}x = s_i x$,

(v) if $x \in \mathcal{S}^{i,j}$ and $i > j+1$ then

$$\gamma^\alpha_i\gamma^\beta_j x = \Gamma^\alpha_{n,i}\Gamma^\beta_{n-1,j}\partial^\alpha_{n-1,i-1}\partial^\beta_{n,j}x = \Gamma^\beta_{n,j}\Gamma^\alpha_{n-1,i-1}\partial^\beta_{n-1,j}\partial^\alpha_{n,i}x = \gamma^\beta_j\gamma^\alpha_i x.$$

Philippe Malbos
malbos@math.univ-lyon1.fr
Université Claude Bernard Lyon 1
ICJ UMR5208, CNRS
F-69622 Villeurbanne cedex, France

Tanguy Massacrier
massacrier@math.univ-lyon1.fr
Université Claude Bernard Lyon 1
ICJ UMR5208, CNRS
69622 Villeurbanne cedex, France

Georg Struth
g.struth@sheffield.ac.uk
University of Sheffield
Department of Computer Science
Regent Court, 211 Portobello
Sheffield S1 4DP, United Kingdom