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On the solution uniqueness in portfolio optimization and risk analysis

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We consider the issue of solution uniqueness of the mean-deviation portfolio optimization problem and its inverse for asset returns distributed over a finite number of scenarios. Due to the asymmetry of returns, the risk is assessed by a general deviation measure introduced by [Rockafellar et al., Mathematical Programming, Ser. B, 108 (2006), pp. 515–540] instead of the standard deviation as in the classical Markowitz optimization problem. We demonstrate that, in general, one cannot expect the uniqueness of Pareto-optimal profit sharing in cooperative investment and the uniqueness of solutions in the mean-deviation Black-Litterman asset allocation model. For a large class of deviation measures, we provide a resolution of the above non-uniqueness issues based on the principle of law-invariance. We provide several examples illustrating the non-uniqueness and the law-invariant solution.

Keywords: Portfolio optimization; Cooperative investment; Black-Litterman model.

1. Introduction

In the realm of portfolio analysis we consider a market with a riskless asset and n risky assets. Portfolios are represented as combinations $x_1 R^{(1)} + \cdots + x_n R^{(n)}$, where the vector random variable $R = (R^{(1)}, \ldots, R^{(n)})^T$ denotes excess returns of risky assets. The objective is to find a portfolio allocation (fractions of wealth invested in the risky assets) $x = (x_1, \ldots, x_n)^T$ that solves the following optimization problem:

$$\min_x \mathcal{D}(R^T x) \quad \text{subject to } \mu^T x \geq \Delta, \tag{1.1}$$

where $\mathcal{D}$ is a deviation measure Rockafellar *et al.* (2006a) which measures the portfolio risk, Δ is the target excess return and

$$\mu = (\mu_1, \ldots, \mu_n)^T = (E[R^{(1)}], \ldots, E[R^{(n)}])^T.$$

This problem generalizes the classical Markowitz portfolio optimization problem of minimizing the standard deviation of porfolio return subject to a constraint on the expected return. Deviation measures, which include the standard deviation as a special case, need not be symmetric and offer more flexibility in the assessment of various aspects of portfolio risk. The key feature of those measures is that, as the standard deviation, they are location invariant so that the measurements of the "risk" and of the "expected return" are explicitly separated. The reader is referred to Rockafellar *et al.* (2006a) for an extended discussion of (1.1).

The suggestion to use lower semideviation $\sigma^-(X)$ instead of standard deviation in (1.1) goes back to the original work of Markowitz (1959). The study of (1.1) with mean absolute deviation $\mathrm{MAD}(X) = E[|X - E[X]|]$ was initiated in Konno & Yamazaki (1991). Rockafellar *et al.* (2000) suggest to use the so-called conditional value-at-risk $\mathrm{CVaR}_\alpha(X)$ in various optimization problems including portfolio optimization. Recent works, Moresco *et al.* (2023) and Zabarankin *et al.* (2024), introduce new important families of deviation measures called Minkowski deviation measures and benchmark-based deviation measures, respectively. Since $\sigma^-(X)$, $\mathrm{MAD}(X)$, and $E[X] + \mathrm{CVaR}_\alpha(X)$ are deviation measures, optimization problem (1.1) generalises the portfolio problems studied in these and many other papers. We also remark that deviation measures are also extensively used in many applications other than portfolio optimization, e.g., in entropy maximization Grechuk *et al.* (2009) and insurance Boonen & Han (2024).

In this paper we study two uniqueness problems. The first one, called in the sequel the *forward problem*, concerns the uniqueness of solution x, representing portfolio weights, to the portfolio optimization problem (1.1). The second one pertains to the following *inverse portfolio problem:* given a vector x^*, the information on the distribution of R sufficient to compute $\mathcal{D}(R^T x)$ for any x, and $\Delta > 0$ find a vector of mean returns μ such that x^* is a solution to problem (1.1) for that μ.

The motivation for analyzing the uniqueness of forward and inverse optimization problems stated above comes from the cooperative investment, c.f. Grechuk *et al.* (2013), and from the Black-Litterman asset allocation model, cf. Black & Litterman (1992) where the model is formulated and Litterman *et al.* (2004) for a more detailed presentation. In the cooperative investment, agents pool resources, invest together and share the rewards. We show that the uniqueness of the forward problem is linked to the uniqueness of the fair sharing scheme. In the classical Black-Litterman model, where the risk is modeled by the variance, the inverse problem is used to infer the so-called equilibrium mean return vector from the market portfolio and it has a unique solution. The variance, however, is a poor measure of risk for non-Gaussian distributions. Rockafellar *et al.* (2006a) promotes deviations measures which are rooted in coherent risk measures but are indifferent to the location parameter of the distribution (as is the variance). The optimization problem (1.1) retains its convexity in x, but the uniqueness of solutions to the forward and inverse problems has not been studied. A general theory of convex optimization implies that it depends on the interplay between the distribution of R and the deviation measure $\mathcal{D}$.

In the context of asset management, many papers assume a finite (but possibly large) number of scenarios for the future excess return R (for example a historical time series of asset returns) and this is the case that we research in this paper. The reader is referred to, e.g., Krokhmal *et al.* (2002), Fabozzi *et al.* (2010), Lim *et al.* (2011), Grechuk & Zabarankin (2018) for theoretical and finance-centred contributions and Gaivoronski & Pflug (2005), Lim *et al.* (2010), Lwin *et al.* (2017) for numerical methods; applications outside of finance can be found in the monograph Conejo *et al.* (2010) and references therein. Although the question of existence of optimal solutions has been solved, the problem of uniqueness for a finite number of scenarios has not been analyzed carefully enough. We perform detailed analysis of that problem for arbitrary discrete scenarios and a class of deviation measures that we call "finitely generated risk measures" which includes Conditional Value-at-Risk (CVaR), mixed CVaR and mean absolute deviation. In our approach we use the characterization of deviation measures by their risk envelopes introduced in Rockafellar *et al.* (2006b).

We mention that several authors investigated other formulations of inverse portfolio problems. For example, Bertsimas *et al.* (2012) considers an inverse optimization in a robust optimization framework with the portfolio mean as the objective function and risk accounted for in constraints. The problem of uniqueness is not addressed in that paper, particularly because under their assumption of normality of asset returns the forward problem always has a unique solution. Due to the number of degrees of freedom (in the mean-variance case it is both the mean, variance and the target return Δ that are to be inferred from the optimal portfolio), the inverse problem inherently has many solutions. Grechuk & Zabarankin (2014, 2016) attempt to infer risk preferences of an investor: assuming a complete knowledge of the distribution of R and portfolio x^*, they look for a deviation measure $\mathcal{D}$ for which x^* is an optimal solution to (1.1).

This paper has four main contributions. They are based on a new link between the uniqueness of an optimal portfolio x^* in (1.1) and the number of risk identifiers that uniquely characterize the deviation measure $\rho(R^T x^*)$. The first contribution concerns the forward optimization problem. We show that the solution to (1.1) is unique for any $\mu \in \mathbb{R}^n$ that does not belong to a finite number of hyperplanes; therefore, when μ is estimated from the data the uniqueness can be safely assumed.

Our second contribution demonstrates that this uniqueness has negative consequences for cooperative investment. We prove that a unique optimal portfolio x^* for the cooperative investment problem corresponds to many risk identifiers and that this implies that there are many Pareto-optimal profit sharing arrangements in the cooperative investment problem. It is surprising as in Grechuk & Zabarankin (2017), this possibility was only inferred from the general convexity theory and treated as an unlikely and inconvenient case that was not of prime importance.

Our third contribution is related to the extension of the Black-Litterman model (Black & Litterman 1992) to arbitrary distributions and deviation measures (Meucci 2005, Palczewski & Palczewski 2019). A key step of the classical model as well as in the extended model is the solution of the inverse portfolio problem described above in which x^* is a market or benchmark portfolio. We demonstrate that if x^* is a unique optimal solution for a particular μ then the inverse problem has multiple solutions for a large class of deviation measures ρ. As a consequence, the final investment recommendation coming out of the Black-Litterman methodology is not unique.

The fourth contribution pertains to the resolution of the non-uniqueness in the cooperative investment and inverse optimization. We introduce the concept of a law-invariant selector and prove its uniqueness for certain deviation measures such as CVaR or mixed-CVaR. The law-invariance is a natural concept as it is intuitively related to the solution being the same for any realization of the distributions of returns on a probability space. The law-invariance in the framework of convex risk measures has been widely studied in the literature, see, e.g., Kusuoka (2001), Frittelli & Gianin (2005), Jouini *et al.* (2006), due to its attractive mathematical and financial properties.

The rest of the paper is organized as follows. Section 2 formulates the portfolio optimization problem in the framework of deviation measures, defines portfolio risk generators and discusses the portfolio uniqueness problem in terms of portfolio risk generators. Section 3 formulates the cooperative investment problem and discusses the issue of non-uniqueness of its solution. Section 4 discusses the dichotomy between uniqueness of solutions of the forward and inverse optimization problems. Section 5 presents a solution to the non-uniqueness problem, which works for many important examples of deviation measures. Section 6 considers consequences of non-uniqueness for the Black-Litterman model for non-Gaussian distributions. Section 7 concludes the work.

2. Mean-deviation portfolio optimization

2.1. *Finitely generated deviation measures*

We assume that the probability space Ω is finite with $N = |\Omega|$ and $\mathbb{P}(\omega) > 0$ for any $\omega \in \Omega$. A finite probability space Ω will be called *uniform*, if $\mathbb{P}(\omega_1) = \cdots = \mathbb{P}(\omega_N) = \frac{1}{N}$.

Let $R^{(i)}$, $i = 1, \dots, n$, be random variables denoting the rates of return of financial instruments. We assume that there exists also a risk-free instrument with a constant rate of return $R^{(0)} =: r_0$. Following Rockafellar *et al.* (2006b), we also assume that

(M) any portfolio $X = \sum_{i=1}^n x_i R^{(i)}$ is a non-constant random variable for any non-zero $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Rockafellar *et al.* (2006b) formulated portfolio optimization problem as follows

$$\min_{(x_0,x_1,\dots,x_n)} \mathcal{D}\left(\sum_{i=0}^n x_i R^{(i)}\right), \quad \text{s.t.} \sum_{i=0}^n x_i = 1, \quad \sum_{i=0}^n x_i E[R^{(i)}] \geq r_0 + \Delta, \tag{2.1}$$

where $\Delta > 0$ and $\mathcal{D}$ is a general deviation measure, that is, a functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0;\infty]$ satisfying:

(D1) $\mathcal{D}(X) = 0$ for constant X, but $\mathcal{D}(X) > 0$ otherwise *(non-negativity)*,
(D2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$ *(positive homogeneity)*,
(D3) $\mathcal{D}(X+Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y *(subadditivity)*,
(D4) set $\{X \in \mathcal{L}^2(\Omega) \big| \mathcal{D}(X) \leq C\}$ is closed for all $C < \infty$ *(lower semicontinuity)*.

With centered rates of return $\hat{R}^{(i)} = R^{(i)} - E[R^{(i)}]$, $i = 1, \dots, n$, and $\mu_i = E[R^{(i)}] - r_0$, $i = 1, \dots, n$, problem (2.1) can be reformulated as

$$\min_{x \in \mathbb{R}^n} \mathcal{D}(\hat{R}^T x), \qquad \text{s.t. } \mu^T x \geq \Delta, \tag{2.2}$$

where $\hat{R} = (\hat{R}^{(1)}, \dots, \hat{R}^{(n)})^T$, $x = (x_1, \dots, x_n)^T$, and $\mu = (\mu_1, \dots, \mu_n)^T$.

By Rockafellar *et al.* (2006a, Theorem 1), every deviation measure $\mathcal{D}$ can be represented in the form

$$\mathcal{D}(X) = EX + \sup_{Q \in \mathcal{Q}} E[-XQ], \tag{2.3}$$

where $\mathcal{Q} \subset \mathcal{L}^2(\Omega)$ is called the *risk envelope* and can be recovered from $\mathcal{D}$ by

$$\mathcal{Q} = \left\{ Q \in \mathcal{L}^2(\Omega) \;\middle|\; E[X(1-Q)] \leq \mathcal{D}(X) \;\; \forall X \in \mathcal{L}^2(\Omega) \right\}. \tag{2.4}$$

Moreover, the set $\mathcal{Q}$ is closed and convex in $\mathcal{L}^2(\Omega)$. Elements $Q \in \mathcal{Q}$ for which supremum in (2.3) is attained are called *risk identifiers* of X. The set of all risk identifiers of X is denoted by $\mathcal{Q}(X)$.

A deviation measure $\mathcal{D}$ is *finite*, that is, $\mathcal{D}(X) < \infty, \forall X$ if and only if the corresponding $\mathcal{Q}$ is bounded. In this case, $\mathcal{Q}(X)$ is non-empty for every $X \in \mathcal{L}^2(\Omega)$, and, due to closeness and convexity of $\mathcal{Q}$ and linearity of $Q \mapsto E[-XQ]$, every set $\mathcal{Q}(X)$ must contain at least one extreme point of $\mathcal{Q}$. Therefore, $\sup_{Q \in \mathcal{Q}} E[-XQ] = \max_{Q \in \mathcal{Q}^e} E[-XQ]$, where $\mathcal{Q}^e$ is the set of all extreme points of $\mathcal{Q}$. In fact, a bounded closed convex $\mathcal{Q}$ is the closed convex hull of $\mathcal{Q}^e$, see Theorem 2 in Phelps (1974)[a]. Of particular importance to this paper will be the set of such risk measures for which the set $\mathcal{Q}^e$ is finite:

[a]Because $\mathcal{L}^2(\Omega)$ is a reflexive Banach space, it has the Radon-Nikodym property, and Theorem 2 in Phelps (1974) applies.

Definition 2.1. A finite deviation measure $\mathcal{D}$ is called *finitely generated* if the set $\mathcal{Q}^e$ of all extreme points of $\mathcal{Q}$ is finite. We will call elements of this set *extreme risk generators.*

In other words, $\mathcal{D}$ is finitely generated if and only if $\mathcal{Q}$ is a convex hull of a finite number of points.

Example 2.1. For standard deviation, $\sigma(X) = ||X - E[X]||_2$, the risk envelope is given by Rockafellar *et al.* (2006b, Example 1)

$$\mathcal{Q} = \left\{ Q \;\middle|\; E[Q] = 1, \;\; \sigma(Q) \leq 1 \right\},$$

and, for $N > 2$, has infinitely many extreme points, hence σ is *not* finitely generated.

Example 2.2. For mean absolute deviation, $\text{MAD}(X) = E[|X - E[X]|]$, the risk envelope is given by Rockafellar *et al.* (2006b, Example 2)

$$\mathcal{Q} = \left\{ Q \;\middle|\; E[Q] = 1, \;\; \sup Q - \inf Q \leq 2 \right\},$$

which is a convex polytope in $\mathbb{R}^N$ with a finite number of vertices. Hence MAD is finitely generated. In fact, extreme points $\mathcal{Q}^e$ can be explicitly written as

$$\mathcal{Q}^e = \left\{ Q = 1 + E[Z] - Z \;\middle|\; \exists S \subset \{1, 2, \ldots, N\} : \, Z_i = 1, \; i \in S; \; Z_i = -1, \; i \notin S \right\},$$

which for a uniform probability on Ω simplifies to

$$\mathcal{Q}^e = \left\{ x \in \mathbb{R}^N \;\middle|\; \exists S \subset \{1, 2, \ldots, N\} : \, x_i = \frac{2|S|}{N} - 1, \; i \in S; \; x_i = \frac{2|S|}{N} + 1, \; i \notin S \right\},$$

where the subset S is taken non-empty and proper. Hence, $|\mathcal{Q}^e| = 2^N - 2$.

Example 2.3. For CVaR-deviation

$$\text{CVaR}^{\Delta}_{\alpha}(X) \equiv E[X] - \frac{1}{\alpha} \int_0^{\alpha} q_X(\beta) \, d\beta, \tag{2.5}$$

the risk envelope is given by Rockafellar *et al.* (2006b, Example 4)

$$\mathcal{Q}_{\alpha} = \left\{ Q \;\middle|\; E[Q] = 1, \;\; 0 \leq Q \leq \alpha^{-1} \right\}.$$

The linearity of constraints imply that $\text{CVaR}^{\Delta}_{\alpha}$ is finitely generated. In particular, if the probability is uniform over Ω and $\alpha = \frac{k}{N}$ for some integer $1 \leq k < N$, extreme points $\mathcal{Q}^e$ are

$$\mathcal{Q}^e = \left\{ x \in \mathbb{R}^N \;\middle|\; \exists S \subset \{1, 2, \ldots, N\} : \, |S| = k, \;\; x_i = \frac{N}{k}, \; i \in S; \; x_i = 0, \; i \notin S \right\}.$$

This implies that $|\mathcal{Q}^e| = \frac{N!}{k!(N-k)!}$.

Lemma 2.1. *Let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ be finitely generated deviation measures. Then functionals*

(a) $\mathcal{D}(X) = \sum_{i=1}^m \lambda_i \mathcal{D}_i(X)$, *with* $\lambda_i > 0$, $i = 1 \ldots, m$;
(b) $\mathcal{D}(X) = \max\{\mathcal{D}_1(X), \ldots, \mathcal{D}_m(X)\}$

are also finitely generated deviation measures.

Proof. Proof follows from Rockafellar *et al.* (2006a, Proposition 4), and from the fact that if sets $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m$ are all convex hulls of a finite number of points, then so are the sets: $\lambda_1\mathcal{Q}_1 + \cdots + \lambda_m\mathcal{Q}_m$; the convex hull of $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_m$; and $\{Q \,|\, Q = (1-\lambda) + \lambda Q_i \text{ for some } Q_i \in \mathcal{Q}_i\}$, $\lambda > 0$, $i = 1, \ldots, m$. □

Example 2.4. Mixed CVaR-deviation

$$\mathrm{CVaR}^{\Delta}_{\lambda}(X) = \int_0^1 \mathrm{CVaR}^{\Delta}_{\alpha}(X)\, \lambda(d\alpha), \tag{2.6}$$

where λ is a probability measure on $(0, 1)$, is also finitely generated. Indeed, because the probability space is finite, mixed CVaR-deviation (2.6) can be written as a finite mixture of CVaR-deviations

$$\mathrm{CVaR}^{\Delta}_{\lambda}(X) = \sum_{i=1}^{m} \lambda_i \mathrm{CVaR}^{\Delta}_{\alpha_i}(X),$$

where $\alpha_i \in (0, 1)$, $\lambda_i > 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \lambda_i = 1$, which is a finitely generated deviation measure due to Example 2.3 and Lemma 2.1(a).

2.2. *Optimal portfolios and active portfolio risk generators*

We make the following standing assumptions:

(A) The deviation measure $\mathcal{D}$ is finitely generated.
(B) $\Delta > 0$ and $\mu \neq \mathbf{0}$.

The latter assumption implies the following properties of the optimal solution to (2.2).

Lemma 2.2. *The optimal objective value in* (2.2) *is positive and in optimum the constraint is binding:* $\mu^T x = \Delta$.

Proof. By Theorem 1 in Rockafellar *et al.* (2006b), there is an optimal solution x^*. By assumption (B), $x = \mathbf{0}$ does not satisfy the constraint on the expected return, and so $x^* \neq \mathbf{0}$. Due to assumption (M), we conclude that $\hat{R}^T x^*$ is random and hence $\mathcal{D}(\hat{R}^T x^*) > 0$. For the second part of the statement, assume that $\mu^T x^* > \Delta$. Therefore, there is $\eta < 1$ such that $\mu^T(\eta x^*) \geq \Delta$ and we have $\mathcal{D}(\hat{R}^T(\eta x^*)) = \eta\mathcal{D}(\hat{R}^T x^*) < \mathcal{D}(\hat{R}^T x^*)$, a contradiction. □

Since $\mathcal{D}$ is finitely generated, the deviation measure of a centered return of portfolio $x \in \mathbb{R}^n$ can be expressed as a maximum of a finite number of terms:

$$\mathcal{D}(\hat{R}^T x) = \max_{Q \in \mathcal{Q}^e} E[-\hat{R}^T x\, Q]. \tag{2.7}$$

As the number of extreme risk generators for $\mathcal{D}$ is finite, they can be enumerated: $\mathcal{Q}^e = \{Q_1, \ldots, Q_{M'}\}$. Define $\tilde{D}_i = E[-\hat{R}Q_i]$, $i = 1, \ldots, M'$. It follows from (2.7) that the set of $\tilde{D}_i$'s is sufficient to evaluate $\mathcal{D}(\hat{R}^T x)$ for a portfolio x:

$$\mathcal{D}(\hat{R}^T x) = \max_{i=1,\ldots,M'} \tilde{D}_i^T x. \tag{2.8}$$

It may happen that $\tilde{D}_i = \tilde{D}_j$ for some $i \neq j$; for example, $\hat{R}$ may be constant on a number of elementary events in Ω. It may also happen that not all $\tilde{D}^i$ are necessary to define $\mathcal{D}$. Indeed, by the linearity of the mapping $D \mapsto D^T x$ for a fixed portfolio x, we have

$$\max_{i=1,\ldots,M'} \tilde{D}_i^T x = \sup_{D \in \mathrm{conv}\{\tilde{D}_1,\ldots,\tilde{D}_{M'}\}} D^T x, \tag{2.9}$$

where $\mathrm{conv}\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\}$ is the convex envelope of points $\tilde{D}_1, \ldots, \tilde{D}_{M'}$, i.e., the smallest convex set containing those points, or, in the terminology of (Rockafellar 1970, Section 19) the polyhedral set generated by $\tilde{D}_1, \ldots, \tilde{D}_{M'}$. The convex envelope is bounded and closed so that linear mapping $D \mapsto D^T x$ attains its supremum in an extreme point of $\mathrm{conv}\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\}$. By (Rockafellar 1970, Corollary IV.18.3.1), the extreme points form a subset of $\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\}$, so it is sufficient to restrict optimization on the right-hand side of (2.7) to those extreme points. This will prove convenient in the future arguments motivating the following definition.

Definition 2.2. Extreme points of $\mathrm{conv}\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\}$ are denoted by D_i, $i = 1, \ldots, M$, and called *portfolio risk generators*.

Remark 2.1. Portfolio risk generators are generators (in the sense of Rockafellar (1970, Section 19)) of the polyhedral set $\{E[-\hat{R}Q] \,|\, Q \in \mathcal{Q}\}$, see the proof of Theorem 19.3 in Rockafellar (1970).

By Definition 2.2 and the discussion following (2.8), we have for any portfolio x:

$$\mathcal{D}(\hat{R}^T x) = \max_{i=1,\ldots,M} D_i^T x. \tag{2.10}$$

Definition 2.3. Those D_i that realize the maximum in (2.10) are called *active portfolio risk generators* for the portfolio x.

The following lemma shows that the set of portfolio risk generators is sufficiently rich to span the whole space $\mathbb{R}^n$.

Lemma 2.3. *We have* $\mathrm{lin}(D_1, \ldots, D_M) = \mathbb{R}^n$.

Proof. Assume the opposite and take any non-zero vector x in the orthogonal complement of $\mathrm{lin}(D_1, \ldots, D_M)$. Then $\mathcal{D}(\hat{R}^T x) = 0$. However, $\hat{R}^T x$ is non-constant by assumption (M), so its deviation measure should be strictly positive by (D1). A contradiction. □

The representation (2.10) of the deviation measure of a portfolio x enables an equivalent formulation of optimization problem (2.2) as a linear program:

$$\begin{aligned} &\text{minimize } A, \\ &\text{subject to: } A \geq D_i^T x, \quad i = 1, \dots, M, \\ &\qquad\qquad \mu^T x \geq \Delta, \\ &\qquad\qquad (A, x) \in \mathbb{R} \times \mathbb{R}^n. \end{aligned} \tag{2.11}$$

Indeed, let (A^*, x^*) be the solution. Then A^* is the smallest real number greater or equal to $D_i^T x^*$ for all $i = 1, \dots, M$, so $A^* = \max_{i=1,\dots,M} D_T^i x^* = \mathcal{D}(\hat{R}^T x^*)$. As the objective function of (2.11) equals A, the solution x^* minimizes $x \mapsto \mathcal{D}(\hat{R}^T x)$ over $x \in \mathbb{R}^n$ satisfying $\mu^T x \geq \Delta$ proving the equivalence of the above linear programme to (2.10).

Theorem 2.1. *The linear program* (2.11) *as well as the optimization problem* (2.2) *have the following properties:*

(1) The set of optimal portfolios $\mathbb{X}^*$ is a bounded polyhedral subset of $\mathbb{R}^n$. The set of solutions to (2.11) *is of the form $\{A^*\} \times \mathbb{X}^*$ for some $A^* > 0$.*
(2) If the solution is not unique then μ is a linear combination of at most $n - 1$ portfolio risk generators.
(3) If the solution is unique, then the set of active portfolio risk generators spans the whole space $\mathbb{R}^n$, i.e., there are n linearly independent active portfolio risk generators.

Proof. (2.11) is a linear program, so the set of solutions is polyhedral. The mapping $x \mapsto \mathcal{D}(\hat{R}^T x)$ is convex, hence also continuous (Rudin 1976, Exercise 23, p. 101). Denote by d its minimum on the sphere $\{x \in \mathbb{R}^n \,|\, \|x\| = 1\}$. This minimum is strictly positive due to assumptions (M) and (D1). Employing further assumption (D2) gives that $\{x \in \mathbb{R}^n \,|\, \mathcal{D}(\hat{R}^T x) \leq A\}$ is bounded for any $A > 0$; indeed, it is contained in the ball with radius A/d. Hence, the set of solutions $\mathbb{X}'$ to (2.11) is a bounded polyhedral set. It is expressed by convex combinations of its extreme points at which the objective function is optimal. In each such extreme point the coordinate A is identical, so $\mathbb{X}' = \{A^*\} \times \mathbb{X}^*$ for some $A^* > 0$; the positivity of A^* follows from Lemma 2.2.

If $\mathbb{X}'$ is a single point, then it is an extreme point of the feasible set. Since the constraint $\mu^T x \geq \Delta$ is active (see Lemma 2.2), (Bertsimas & Tsitsiklis 1997, Theorem 2.2) implies that there are n indices $i_1, \dots, i_n$ such that $A = D_{i_j}^T x$, $j = 1, \dots, n$, and vectors $(D_{i_j})_{j=1}^n$ are linearly independent, hence generate $\mathbb{R}^n$.

The proof of assertion 2 uses the dual of problem (2.11):

$$\begin{aligned} &\text{maximize } q\Delta, \\ &\text{subject to: } \sum_{i=1}^{M} p_i D_i - q\mu = 0, \qquad \sum_{i=1}^{M} p_i = 1, \\ &\qquad\qquad q \geq 0,\ p_i \geq 0, \quad i = 1, \ldots, M. \end{aligned} \tag{2.12}$$

By the strong duality, $q\Delta = A^*$ and we know $A^* > 0$, hence $q > 0$. Bertsimas & Tsitsiklis (1997, Theorem 4.5) implies that the dual variables corresponding to inactive constraints are zero. Denote by $i_1, \ldots, i_k$ the active constraints involving portfolio risk generators. Then the first constraint in the above dual problem (2.12) reads:

$$\mu = \frac{1}{q} \sum_{j=1}^{k} p_{i_j} D_{i_j}. \tag{2.13}$$

Assume now that the solution is not unique, i.e., $\mathbb{X}'$ contains at least two extreme points and therefore a line connecting them. Fix an internal point of that line (A^*, x^*). Since (A^*, x^*) is not an extreme point of $\mathbb{X}'$, the linear space spanned by active portfolio risk generators D_{i_j}, $j = 1, \ldots, k$, has dimension not larger than $n - 1$ (there is at least one portfolio risk generator which is active at an extreme point of $\mathbb{X}'$ and does not belong to $\text{lin}\{D_{i_1}, \ldots, D_{i_k}\}$). This proves assertion 2 of the theorem. □

Corollary 2.1. *There is a finite number of hyperplanes (of dimensions from* 1 *to* $n-1$*) such that:* μ *belongs to one of them if and only if a solution to* (2.2) *is not unique. Therefore, the set of* μ *for which the portfolio optimization problem has a unique solution has a full Lebesgue measure.*

Proof. By Theorem 2.1, non-uniqueness of solutions coincides with μ being a linear combination of at most $n-1$ portfolio risk generators, i.e., belongs to a linear space spanned by at most $n-1$ vectors in $\mathbb{R}^n$. This is a hyperplane of dimension at most $n-1$, so it has a Lebesgue measure 0. There is a finite number of ways to choose up to $n-1$ vectors from the set of M vectors, so the number of such hyperplanes is finite. A finite sum of sets of Lebesgue measure zero has the measure zero. Its complement has therefore a full measure. □

A practical consequence of the above theorem and corollary is that there is a unique optimal portfolio in (2.2) unless μ is specially chosen to match the distribution of returns $\hat{R}$ and the risk measure. In the following section we will show that the uniqueness of solution, which implies multiple active portfolio risk generators, leads to issues in optimal cooperative investment. We will also show that there are natural settings when μ happens to be on one of the hyperplanes mentioned in the corollary.

Consider now a portfolio optimization problem with no shortsales of risky assets. This corresponds to the linear program (2.11) with additional constraints $x_i \geq 0$, $i = 1, \ldots, n$. The following lemma shows that the non-uniqueness of portfolio risk generators holds here as well.

Lemma 2.4. *If the solution x^* to the portfolio optimization problem with no short-sales of risky assets is unique, then there are at least k active portfolio risk generators, where k is the number of non-zero coordinates of x^*.*

Proof. As in Lemma 2.2, we show that $\mu^T x^* = \Delta$. The assertion follows from the fact that x^* is an extreme point of the feasible set. □

3. Cooperative investment

3.1. *Theoretical framework*

The general problem of cooperative investment can be formulated as follows, see Grechuk & Zabarankin (2017). Let $\mathcal{F} \subset \mathcal{L}^2(\Omega)$ be a feasible set, representing rates of return from feasible investment opportunities on the market without a riskless asset:

$$\mathcal{F} = \left\{ X \,\middle|\, X = \sum_{i=1}^{n} R^{(i)} x_i, \;\; \sum_{i=1}^{n} x_i = 1 \right\}.$$

An individual portfolio optimization problem for agent i, $i = 1, \ldots, m$, is

$$\max_{X \in \mathcal{F}} U_i(X), \tag{3.1}$$

where $U_i : \mathcal{L}^2(\Omega) \to [-\infty, \infty)$ is the utility function of agent i. If the unit of capital is invested, then rate of return $X \in \mathcal{F}$ can also be interpreted as a monetary profit from the investment. Instead of investing individually, m agents can invest their m units of capital to buy a joint portfolio $X \in m\mathcal{F} := \{mX \,|X \in \mathcal{F}\}$ and distribute it so that agent i receives a share Y_i with $\sum Y_i = X$. An allocation $\mathbb{Y} = (Y_1, \ldots Y_m)$ is called *feasible* if $\sum Y_i \in m\mathcal{F}$, and Pareto optimal if there is no feasible allocation $\mathbb{Z} = (Z_1, \ldots Z_m)$ such that $U_i(Y_i) \leq U_i(Z_i)$ with at least one inequality being strict.

A utility function U is called *cash-invariant* if $U(X + C) = U(X) + C$ for all $X \in \mathcal{L}^2(\Omega)$ and $C \in \mathbb{R}$. Proposition 2 in Grechuk & Zabarankin (2017) implies that if all U_i, $i = 1, \ldots, m$, are cash-invariant, and $\mathbb{Y} = (Y_1, \ldots Y_m)$ is Pareto optimal, then $X^* = \sum Y_i$ solves the optimization problem

$$\sup_{X \in m\mathcal{F}} U^*(X), \tag{3.2}$$

where

$$U^*(X) \equiv \sup_{\mathbb{Z} \in \mathcal{A}(X)} \sum_{i=1}^{m} U_i(Z_i) \tag{3.3}$$

with $\mathcal{A}(X) = \{\mathbb{Z} = (Z_1, \ldots, Z_m) : \sum_{i=1}^m Z_i = X, \ Z_i \in \mathcal{L}^2(\Omega)\}$. Furthermore, if $\mathbb{Y} = (Y_1, \ldots Y_m)$ is *any* Pareto optimal allocation, then *all* Pareto optimal allocations are given by

$$(Y_1 + C_1, \ldots, Y_m + C_m), \tag{3.4}$$

where $C_1, \ldots C_m$ are constants with $\sum_{i=1}^m C_i = 0$. Hence, the coalition should:

(1) solve the portfolio optimization problem (3.2) to find an optimal portfolio X^* for the whole group;
(2) find *any* Pareto optimal way $\mathbb{Y} = (Y_1, \ldots Y_m)$ to distribute X among group members;
(3) agree on constants $C_1, \ldots C_m$ in (3.4) to select a *specific* Pareto-optimal allocation among the ones available.

We consider investors employing the following utility functions:

$$U_i(X) = E[X] - \mathcal{D}_i(X), \tag{3.5}$$

for some deviation measures $\mathcal{D}_i$, $i = 1, \ldots, m$. These utility functions are cash-invariant and the above theory applies. U^* in (3.3) is given by $U^*(X) = E[X] - \mathcal{D}^*(X)$, where

$$\mathcal{D}^*(X) \equiv \inf_{\mathbb{Z} \in \mathcal{A}(X)} \sum_{i=1}^m \mathcal{D}_i(Z_i). \tag{3.6}$$

Remark 3.1. Commonly, an investor's optimization criterion is given by

$$U_i(X) = E[X] - \gamma_i \mathcal{D}_i(X),$$

where $\gamma_i > 0$ is the investor's risk aversion. However, $\gamma_i \mathcal{D}_i$ is a deviation measure whenever $\mathcal{D}_i$ is, so the expression (3.5) covers this example.

In this model, a possible approach to point (iii) above is to select constants C_i in (3.4) such that

$$E[Q^*(Y_1 + C_1)] = \cdots = E[Q^*(Y_m + C_m)] \tag{3.7}$$

where Q^* is the extreme risk identifier in portfolio optimization problem (3.2) with $U^*(X) = E[X] - \mathcal{D}^*(X)$. The intuition is that elements Q of risk envelope represents probability scenarios, Q^* represents the "critical" worst-case scenario for the coalition, and (3.7) states that the investors should receive the same profit under the critical scenario. See Grechuk & Zabarankin (2017, Section 3) for further justification of (3.7) in the model with risk-free asset. Because a concave function is differentiable almost everywhere, one may expect that $\partial U^*(X^*)$ is "typically" a singleton, in which case the extreme risk identifier Q^* is unique, and this approach leads to the unique selection of a "fair" Pareto optimal allocation in (3.4). Below we show, however, that this intuition may be wrong.

Lemma 3.1. *Let $\mathcal{D}_i$ be deviation measures with risk envelopes $\mathcal{Q}_i$, $i = 1, \ldots, m$. Then $\mathcal{D}^*$ is a deviation measure with the risk envelope $\mathcal{Q}^* = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_m$. In particular, if all $\mathcal{D}_i$ are finitely generated, then so is $\mathcal{D}^*$.*

Proof. Proposition 3 in Rockafellar *et al.* (2006a) implies that $\mathcal{Q}_1, \ldots, \mathcal{Q}_m$ are closed, convex subsets of the closed hyperplane $H = \{Q \,|\, EQ = 1\}$ in $\mathcal{L}^2(\Omega)$ such that constant 1 is in their quasi-interior relative to H. Because $\mathcal{Q}_1, \ldots, \mathcal{Q}_m$ have a common point in their relative interiors, Rockafellar (1970, Corollary 16.4.1) implies that $\mathcal{D}^*$ can be represented in the form (2.3) with $\mathcal{Q}^* = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_m$. Because $\mathcal{Q}^*$ is also closed, convex subset of H with constant 1 in quasi-interior relative to H, this implies that $\mathcal{D}^*$ is a deviation measure. Because intersection of polygons is a polygon, $\mathcal{D}^*$ is finitely generated provided that all $\mathcal{D}_i$ are. □

Theorem 3.1. *Assume that investors' utility functions are of the form $U_i(X) = E[X] - \mathcal{D}_i(X)$ with deviation measures $\mathcal{D}_i$ finitely generated and none of the portfolio risk generators D_i^* for $\mathcal{D}^*$ is such that $(D_i^* - E[R])$ is parallel to $\mathbf{1} := (1, \ldots, 1)^T$ or zero. Then any solution $X^* = R^T x^*$ to (3.2) has at least two extreme risk identifiers.*

Proof. We follow ideas from the proof of Theorem 2.1. Denote by $(D_i^*)_{i=1}^M$ the portfolio risk generators for the deviation measure $\mathcal{D}^*$ and let $\hat{D}_i^* = D_i^* - E[R]$. Then, following similar arguments as those used for (2.11), we obtain that (3.2) is equivalent to the following linear problem

$$\begin{aligned} &\text{minimize } A, \\ &\text{subject to: } A \geq x^T \hat{D}_i^*, \quad i = 1, \ldots, M, \\ &\qquad\qquad\quad x^T \mathbf{1} = 1, \qquad (A, x) \in \mathbb{R} \times \mathbb{R}^n. \end{aligned} \tag{3.8}$$

Since x^* is a solution to this program (not necessarily unique), its dual also has a solution (Bertsimas & Tsitsiklis 1997, Theorem 4.4):

$$\begin{aligned} &\text{maximize } q, \\ &\text{subject to: } \sum_{k=1}^M p_k \hat{D}_k^* - q\mathbf{1} = 0, \qquad \sum_{k=1}^M p_k = 1, \\ &\qquad\qquad\quad p_k \geq 0,\ k = 1, \ldots, M, \quad q \in \mathbb{R}. \end{aligned} \tag{3.9}$$

If the optimal solution $q \neq 0$, then the middle equation together with the assumption that none of $\hat{D}_j^*$'s is parallel to $\mathbf{1}$ implies that there must be at least two p_k's strictly positive. Bertsimas & Tsitsiklis (1997, Theorem 4.5) states that the corresponding constraints in the primal problem are active, i.e., their respective portfolio risk generators are active for X^*. When $q = 0$, the assumption that none of $\hat{D}_j^*$'s is zero imply again that at least two p_k's must be non-zero. □

Theorem 3.1 implies that there are at least two linearly independent active portfolio risk generators and there are multiple fair Pareto-optimal solutions to the cooperative investment problem. In Section 5, we will offer a method for selecting a unique portfolio risk generator based on the principle of law-invariance. An alternative approach for resolving the non-uniqueness problem in the risk sharing and cooperative investment based on Steiner point method is discussed in Grechuk (2023).

Remark 3.2. The portfolio optimization problem (3.2) with utility functions (3.5) may not have a solution, i.e., an optimal value may be attained asymptotically on a diverging sequence of portfolios. This happens, for example, when there is x such that $x^T\mathbf{1} = 0$ and $x^T E[R] - \mathcal{D}^*(x^T R) > 0$, i.e., when the projection of $E[R]$ on $\mathbf{1}^\perp := \{y \in \mathbb{R}^d \,|\, y^T\mathbf{1} = 0\}$ is not contained in the convex envelope of projections of portfolio risk generators $(D_i^*)_{i=1}^M$ on $\mathbf{1}^\perp$.

Remark 3.3. The issue with non-existence of solution to the optimization problem of this section

$$\sup_{x:\ x^T\mathbf{1}=1} x^T\mu - \mathcal{D}(x^T R), \tag{3.10}$$

where $\mu = E[R]$, extends to optimization with the risk measured by a coherent risk measure ρ (another criterion popular in the literature)

$$\sup_{x:\ x^T\mathbf{1}=1} x^T\mu - \gamma\rho(x^T R)$$

with the risk aversion $\gamma > 0$. Indeed, using that $\mathcal{D}(X) = \rho(X) + E[X]$ is a deviation measure, the above problem is equivalent to

$$\sup_{x:\ x^T\mathbf{1}=1} x^T\mu - \gamma^*\mathcal{D}(x^T R)$$

with $\gamma^* = \gamma/(1+\gamma)$, which, by Remark 3.1, is of the form (3.10).

3.2. *Explicit example*

Cash-or-nothing binary option O returns some fixed amount of cash $C(O)$ if it expires in-the-money but nothing otherwise. Assume that there are two such options A and B which expire in-the-money if $P > C_1$ and $P > C_2$, respectively, where P is the (random) price of (the same) underlying asset, and $C_1 < C_2$ are constants. Assume that options are offered for the same price p with $C(A) = 2p$ and $C(B) = 8p$. Each agent can invest a unit of capital into A and B, precisely $1-t$ into A and t into B, to get profit $-(1-t) - t = -1$; $(1-t) - t = 1-2t$; or $(1-t) + 7t = 1 + 6t$ depending on the relation of the price P with respect to C_1 and C_2. We assume that two agents think that these three opportunities are equally probable.

For agent 1 with $U_1(X) = E[X] - CVaR^{\Delta}_{\frac{2}{3}}(X) = -CVaR_{\frac{2}{3}}(X)$, an optimal individual investment can be found from the linear program

$$\max_{a_1,t} a_1, \quad \text{s.t. } X = (-1, 1-2t, 1+6t), \quad E[QX] \geq a_1, \forall\, Q \in \mathcal{Q}^1,$$

where $\mathcal{Q}^1 = \left\{\left(\frac{3}{2},\frac{3}{2},0\right),\left(\frac{3}{2},0,\frac{3}{2}\right),\left(0,\frac{3}{2},\frac{3}{2}\right)\right\} = \left\{\text{Perm}\left(\frac{3}{2},\frac{3}{2},0\right)\right\}$, resulting in the optimum $t=0$, $X=(-1,1,1)$, and the optimal value $u_1^*=0$.

Similarly, for agent 2 with $U_2(X) = E[X] - \frac{1}{2}MAD(X)$, the linear program

$$\max_{a_2,t} a_2, \quad \text{s.t. } X = (-1, 1-2t, 1+6t), \quad E[QX] \geq a_2, \forall\, Q \in \mathcal{Q}^2,$$

where $\mathcal{Q}^2 = \left\{\text{Perm}\left(\frac{5}{3},\frac{2}{3},\frac{2}{3}\right), \text{Perm}\left(\frac{4}{3},\frac{4}{3},\frac{1}{3}\right)\right\}$, returns $t=\frac{1}{5}$, with the optimal value $u_2^* = \frac{1}{15}$.

The cooperative investment corresponds to the linear program

$$\begin{aligned}\max_{a_1,a_2,Y_1,Y_2,t} a_1 + a_2, \quad \text{s.t. } & Y_1 + Y_2 = 2(-1, 1-2t, 1+6t),\\ & E[QY_1] \geq a_1, \forall\, Q \in \mathcal{Q}^1,\\ & E[QY_2] \geq a_2, \forall\, Q \in \mathcal{Q}^2,\end{aligned}$$

that is, we are simultaneously looking for optimal portfolio (t), and an optimal way to share it (Y_1, Y_2) to maximize the sum of agents utilities. The optimal t is $t=\frac{1}{5}$, with $Y_1+Y_2 = \left(-2,\frac{6}{5},\frac{22}{5}\right)$, and optimal value is $u^* = \frac{2}{15} > u_1^* + u_2^*$. The simplex method returns a solution $Y_1 = (\frac{2}{15},\frac{2}{15},\frac{2}{15})$, $Y_2 = (-\frac{32}{15},\frac{16}{15},\frac{64}{15})$, with $u_1(Y_1) = \frac{2}{15}$ and $u_2(Y_2)=0$, which is obviously unfair. Because the utilities are cash invariant, any solution in the form $Y_1' = Y_1 + C$, $Y_2' = Y_2 - C$ is Pareto-optimal, and the question is how to select a "fair" C.

To this end, we compute the utility of a coalition $U^*(X)$ as

$$U^*(X) = \min_{Q\in\mathcal{Q}^*} E[QX], \tag{3.11}$$

where $\mathcal{Q}^*$ can be found as (the vertices of) intersection of convex hulls of $\mathcal{Q}^1$ and $\mathcal{Q}^2$. In our case, $\mathcal{Q}^* = \left\{\text{Perm}\left(\frac{3}{2},1,\frac{1}{2}\right), \text{Perm}\left(\frac{4}{3},\frac{4}{3},\frac{1}{3}\right)\right\}$. The optimal portfolio $X^* = \left(-2,\frac{6}{5},\frac{22}{5}\right)$ is a solution to the optimization problem

$$\max U^*(X), \quad \text{s.t. } X = 2(-1, 1-2t, 1+6t). \tag{3.12}$$

Now, let Q^* be the minimizer in (3.11) for X^*. Then, according to (3.7), the fair C should be selected such that

$$E[Q^*(Y_1 + C)] = E[Q^*(Y_2 - C)]. \tag{3.13}$$

An intuition is that the investors should get the same profit under the critical scenario Q^*. The problem is that, for $X^* = \left(-2,\frac{6}{5},\frac{22}{5}\right)$, the minimizer Q^* in (3.11) in not unique! Indeed, $E[QX^*] = \frac{2}{15}$ for $Q = \left(\frac{3}{2},1,\frac{1}{2}\right)$, and also for $Q = \left(\frac{4}{3},\frac{4}{3},\frac{1}{3}\right)$. This is not a coincidence as we have shown in Theorem 3.1. While set of random variables X with non-unique risk identifier has measure 0, the optimal portfolio in (3.12) is guaranteed to belong to this set. Consequently, the cooperative investment does not have a unique solution in the case of finitely generated deviation measures.

In our example, the set of minimizers in (3.11) is the whole line segment with endpoints $\left(\frac{3}{2},1,\frac{1}{2}\right)$ and $\left(\frac{4}{3},\frac{4}{3},\frac{1}{3}\right)$. Consequently, there are infinitely many "fair" choices of C.

4. Inverse portfolio problem

Following Palczewski & Palczewski (2019), let us formulate a problem inverse to (2.2) as follows. Assume that we are given a portfolio $x^M = (x_1^M, \dots, x_n^M) \neq \mathbf{0}$, the distribution of centered rates of return $\hat{R}$, the deviation measure $\mathcal{D}$, and $\Delta_M > 0$ (the expected excess return of the portfolio x^M). The inverse problem concerns finding $\mu = (\mu_1, \dots, \mu_n)^T$ such that x^M is the solution to the optimization problem (2.2). Does such μ exist for any $x^M \neq \mathbf{0}$? Is the vector μ always determined uniquely? We will give a positive answer to the first question and discuss a dichotomy faced by the second: if the solution of the inverse problem is unique then the forward problem with the computed μ has multiple solutions, while if the forward problem has a unique solution then there are many μ's solving the inverse problem.

Assume that $\Delta_M > 0$. Necessarily, $x^M \neq \mathbf{0}$. Theorem 4 in Rockafellar *et al.* (2006b) states that the portfolio x^M is a solution to (2.2) *if and only if* there is a risk identifier Q^* for the random variable $\hat{R}^T x^M$ such that

$$\mu = \frac{\Delta_M}{\mathcal{D}(\hat{R}^T x^M)} E[-\hat{R}Q^*] = \frac{\Delta_M}{(x^M)^T E[-\hat{R}Q^*]} E[-\hat{R}Q^*]. \tag{4.1}$$

This follows since every finite deviation measure on a discrete probability space is continuous, c.f. Rockafellar *et al.* (2006b, page 518).

Let D_{i_j}, $j = 1, \dots, k$, be the set of active portfolio risk generators for x^M. Then (4.1) amounts to the existence of weights $\beta_1, \dots, \beta_k \geq 0$, such that $\sum_{j=1}^k \beta_j = 1$ and

$$\mu = \frac{\Delta_M}{\sum_{j=1}^k \beta_j D_{i_j}^T x^M} \sum_{j=1}^k \beta_j D_{i_j}. \tag{4.2}$$

From the above formula we immediately get the following characterization of vectors μ for which x^M is a solution to (2.2).

Lemma 4.1. *The set of solutions $\mathcal{M}$ to the inverse optimization problem is convex and spanned by points δD_{i_j}, where D_{i_j}, $j = 1, \dots, k$, are active portfolio risk generators for x^M and $\delta = \Delta_M / \mathcal{D}(\hat{R}^T x^M)$:*

$$\mathcal{M} = \Big\{ \delta \sum_{j=1}^k \beta_j D_{i_j} \,\Big|\, \beta \in [0,1]^k \text{ and } \sum_{j=1}^k \beta_j = 1 \Big\}.$$

Proof. Since D_{i_j}, $j = 1, \dots, k$ are active portfolio risk generators, we have $\mathcal{D}(\hat{R}^T x^M) = D_{i_j}^T x^M$. Hence, from (4.2)

$$\mu = \frac{\Delta_M}{\sum_{j=1}^k \beta_j \mathcal{D}(\hat{R}^T x^M)} \sum_{j=1}^k \beta_j D_{i_j} = \frac{\Delta_M}{\mathcal{D}(\hat{R}^T x^M)} \sum_{j=1}^k \beta_j D_{i_j} = \delta \sum_{j=1}^k \beta_j D_{i_j},$$

where in the second equality we used that $\sum_{j=1}^k \beta_j = 1$.

□

Remark 4.1. The conclusions of the above lemma can be immediately deduced from the dual representation (2.12) of the portfolio optimization problem. Indeed, multiplying both sides of (2.13) by x^M yields $q = 1/\delta$.

Equipped with this characterization of the set $\mathcal{M}$ we demonstrate the link between the set of solutions of the inverse and forward optimization problems.

Theorem 4.1.

(1) If x^M is a unique solution to (2.2) *for some μ, then the set of all solutions $\mathcal{M}$ to the inverse optimization problem has at least $n+1$ extreme points. Moreover, all extreme points are of the form δD_{i_j}, where $\delta > 0$ and D_{i_j} is an active portfolio risk generator for x^M.*

(2) If there is a unique active portfolio risk generator for x^M, then the inverse optimization problem has a unique solution μ^* (the set $\mathcal{M}$ consists of one point). However, the optimization problem (2.2) *with $\Delta = \Delta_M$ and $\mu = \mu^*$ has multiple solutions: the set of solutions $\mathbb{X}^*$ is a polyhedron of dimension $n-1$ and has at least n extreme points*[b].

The proof of the above theorem requires the following simple technical result.

Lemma 4.2. *Given $v_i \in \mathbb{R}^n$, $i = 1, \dots, k$, let $\hat{n} = \mathrm{rank}(v_i,\ i = 1, \dots, k) = \mathrm{dim}(\mathrm{lin}(v_1, \dots, v_k))$. Then $\mathcal{N} = \mathrm{conv}(v_1, \dots, v_k)$ has at least $\hat{n}+1$ extreme points and all extreme points are from the set $\{v_1, \dots, v_k\}$.*

Proof. It follows from Rockafellar (1970, Corollary 18.3.1) that all extreme points of $\mathcal{N}$ are in $\{v_1, \dots, v_k\}$. It remains to prove that there are at least $\hat{n}+1$ extreme points. Assume the opposite: there are only $n' < \hat{n}+1$ extreme points $v_{i_1}, \dots, v_{i_{n'}}$ of $\mathcal{N}$. Then $\mathcal{N} \subset A := \mathrm{lin}(v_{i_1}, \dots, v_{i_{n'}})$ and $\mathrm{dim}(A) \le n'+1$. However, A is a linear space containing all points $v_1, \dots, v_k$ so it also contains $\mathrm{lin}(v_1, \dots, v_k)$. The latter space has dimension $\hat{n}+1$ by assumption, hence a contradiction. □

Proof of Theorem 4.1. From Theorem 2.1, the uniqueness of solutions to (2.2) implies that the set of active portfolio risk generators $D_{i_1}, \dots, D_{i_k}$ spans the whole space $\mathbb{R}^n$, i.e., the dimension of a linear space generated by those vectors is n. The conclusions follow from Lemma 4.2.

Assume now that there is a unique active portfolio risk generator. The uniqueness of solution to the inverse optimization problem is clear from formula (4.2). Consider the equivalent form (2.11) for the forward optimization problem. Recall that the set of all solutions to such a linear problem is a convex bounded polyhedral set, a face of a polyhedral set generated by the constraints. The portfolio x_M is a

[b]The dimension of a polyhedron P is the maximum number of affinely independent points contained in P minus 1.

solution for which there are exactly two active constraints: one with the unique portfolio risk generator and one encoding the minimum expected return. This implies that the set of solutions is a polyhedron of dimension $n-1$. By Lemma 4.2 it must have at least n extreme points. □

Corollary 4.1. *In the case 1 of Theorem 4.1, if $\mu \in \operatorname{ri}\mathcal{M}$ (μ is in the relative interior of $\mathcal{M}$), then the forward optimization problem* (2.2) *has a unique solution for $\Delta = \Delta_M$.*

Proof. The implication is equivalent to: solution to (2.2) is not unique $\implies \mu \notin \operatorname{ri}\mathcal{M}$. This follows immediately from assertion 2 of Theorem 2.1 and Rockafellar (1970, Theorem 6.4). □

The inverse problem with no shortsales constraint admits more solutions as shown in the following lemma.

Lemma 4.3. *Let x^M be a solution to the portfolio optimization problem* (2.2) *with additional constraint of no shortsales of risky assets. The set of solutions to the inverse optimization problem is given by*

$$\mathcal{M}' = \big\{\mu \in \mathbb{R}^n \,\big|\, \exists\, m \in \mathcal{M} \text{ s.t. } \mu_i \le m_i, \quad m_i \mathbb{1}_{\{x_i^M \neq 0\}} = \mu_i \mathbb{1}_{\{x_i^M \neq 0\}},\ i = 1, \dots, n\big\}.$$

In particular, $\mathcal{M} \subset \mathcal{M}'$.

Proof. The dual to portfolio optimization problem (2.11) with non-negativity constraints on portfolio weights of risky assets is given by

$$\begin{aligned}
&\text{maximize } q\Delta_M,\\
&\text{subject to: } \sum_{j=1}^{k} p_j D_{i_j} - q\mu \ge 0, \qquad \sum_{j=1}^{k} p_j = 1, \\
&\qquad\qquad q \ge 0,\ p_j \ge 0, \quad j = 1, \dots, k,
\end{aligned} \tag{4.3}$$

where D_{i_j}, $j = 1, \dots, k$, are the active portfolio risk identifiers. By the strong duality, $q\Delta_M = \mathcal{D}(\hat{R}^T x^M) > 0$, hence $q = \mathcal{D}(\hat{R}^T x^M)/\Delta_M > 0$. By complementary slackness conditions, Bertsimas & Tsitsiklis (1997, Theorem 4.5), the inequality in the first constraint above becomes equality for those coordinates for which x^M is non-zero. Using the fact that $\mu^T x^M = \Delta_M$ yields the form of $\mathcal{M}'$. Strong duality implies that for any $\mu \in \mathcal{M}'$, the portfolio x^M is optimal. □

It is well known that optimal portfolios with shortsales constraints are often poorly diversified, i.e., have many null portfolio weights. It then transpires from the definition of the set $\mathcal{M}'$ that the coordinates of μ corresponding to those zero weights are unbounded from below.

Remark 4.2. If the portfolio with no shortsales x^M is fully diversified, i.e., all coordinates are strictly positive, then assertions of Theorem 4.1 apply to the problem with no shortsale constraints for risky assets.

We have seen in Section 3 that cooperative investment often leads to the solution of the coalition optimal portfolio problem (3.2) with multiple active portfolio risk generators. Consequently, there are multiple fair Pareto-optimal solutions to the cooperative investment problem. Non-uniqueness is also observed in the inverse optimization problem. If x^* is a solution to a portfolio optimization problem (2.2) with the deviation measure $\mathcal{D}$ and excess returns $\hat{R}$ and the excess return mean vector μ, Corollary 2.1 shows that this solution is unique unless μ lies on one of a finite number of hyperplanes (belongs to a set of Lebesgue measure zero). This uniqueness implies, in turn, that the inverse optimization problem has multiple solutions (Theorem 4.1). In view of (4.2) as long as there is more than one active portfolio risk generator, there are multiple solutions μ to the inverse problem. Or, in other words, there are multiple risk identifiers $\mathcal{Q}^*$ that determine μ through (4.1).

How to choose a unique point from the set of solutions to the inverse optimization problem or a unique Pareto-optimal solution to the cooperative investment? In view of the above discussion, this is equivalent to the choice of a unique risk identifier Q^* or rather a map $f_{\mathcal{D}} : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ that, for a deviation measure $\mathcal{D}$, assigns to a random variable $X \in \mathcal{L}^2(\Omega)$ one of its risk identifiers. We will call such a map $f_{\mathcal{D}}$ a *selector* corresponding to the deviation measure $\mathcal{D}$. The following section will suggest a natural construction of such a map. An alternative based on Steiner point method is presented in Grechuk (2023).

5. Law invariant selector

5.1. *Theoretical analysis*

This section presents a partial solution to the problem of finding a unique selector, which is based on the principle of law-invariance. Although a law-invariant selector is, in general, not unique, it is natural from the financial and probabilistic point of view and is unique for some important deviation measures such as CVaR and mixed-CVaR.

Definition 5.1. A selector $f_{\mathcal{D}} : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ is called *law-invariant* if $E[Y_1 f_{\mathcal{D}}(X)] = E[Y_2 f_{\mathcal{D}}(X)]$ whenever pairs of r.v.s $(Y_1, X), (Y_2, X) \in \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)$ have the same joint laws.

A deviation measure $\mathcal{D}$ is called *law-invariant* if $\mathcal{D}(X) = \mathcal{D}(Y)$ whenever r.v.s X and Y have the same distribution. For example, $\mathrm{CVaR}_\alpha^\Delta$ (CVaR-deviation) is law invariant for every $\alpha \in (0, 1)$. Notice that not every deviation measure is law-invariant: a simple example of a non-law-invariant deviation measure can be constructed on $\Omega = \{\omega_1, \omega_2\}$, with $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 0.5$, and

$$\mathcal{D}(X) := \max\big\{X(\omega_1) - X(\omega_2), 2(X(\omega_2) - X(\omega_1))\big\}. \tag{5.1}$$

In the framework of uniform probability spaces, we prove below the existence, but not uniqueness, of a law-invariant selector.

Theorem 5.1. *If Ω is uniform, then there exists a law-invariant selector $f_{\mathcal{D}}$ for every law-invariant deviation measure $\mathcal{D}$.*

Proof. It follows from Lemmas 5.1, 5.2, and 5.3 below. □

For non-uniform finite probability spaces, the notion of law-invariance as defined above is of little use for defining a unique selector, because, for example, on $\Omega = \{\omega_1, \omega_2\}$ with $\mathbb{P}(\omega_1) \neq 0.5$, r.v.s X and Y have the same distribution if and only if $X = Y$, and, by definition, *every* deviation measure, including (5.1), is law-invariant. For similar reasons, *every* selector $f_{\mathcal{D}}$ on such probability space is law-invariant. An appropriate extension of the notion of law-invariance to non-uniform probability spaces follows from results below.

A r.v. X dominates a r.v. Y in second order stochastic dominance, denoted $X \succeq_2 Y$, if

$$\int_{-\infty}^{t} F_X(x)dx \leq \int_{-\infty}^{t} F_Y(x)dx, \quad \forall t \in \mathbb{R}.$$

An r.v. X dominates r.v. Y in concave order, denoted $X \succeq_c Y$, if $E[X] = E[Y]$ and $X \succeq_2 Y$. A deviation measure $\mathcal{D}$ is called consistent with concave order if $\mathcal{D}(X) \leq \mathcal{D}(Y)$ whenever $X \succeq_c Y$.

Lemma 5.1. *If a deviation measure $\mathcal{D}$ is consistent with the concave order, it is law-invariant. If Ω is uniform, the converse statement also holds.*

Proof. The first statement is trivial, and the second one is well-known, but the proof is usually presented for atomless probability space, see Dana (2005, Theorem 4.1). For a discrete uniform Ω, let r.v.s X and Y take values $x_1 \leq \cdots \leq x_N$ and $y_1 \leq \cdots \leq y_N$, respectively. Then $X \succeq_c Y$ is equivalent to

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \quad k = 1, \ldots, N, \tag{5.2}$$

with equality for $k = N$. Let us prove that in this case Y can be obtained from X by a finite sequence of operations

$$(z_1, z_2, \ldots, z_N) \to (z_1, \ldots, z_{i-1}, z_i - d, z_{i+1}, \ldots, z_{j-1}, z_j + d, z_{j+1}, \ldots, z_N), \\ d > 0, \ 1 \leq i < j \leq N. \tag{5.3}$$

The statement is trivial for $N = 2$, and the case $N > 2$ can be proved by induction. If $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$ for some $k < N$, we can apply induction hypothesis to pair of r.v.s $X_1 = (x_1, \ldots, x_k)$ and $Y_1 = (y_1, \ldots, y_k)$, and separately to pair $X_2 = (x_{k+1}, \ldots, x_N)$ and $Y_2 = (y_{k+1}, \ldots, y_N)$, to conclude that there exists a sequence of operations (5.3) transforming X_1 to Y_1 and X_2 to Y_2, and hence X to Y. Otherwise, apply operation (5.3) to X with $i = 1$, $j = N$, and $d = \min_k \sum_{i=1}^{k}(x_i - y_i) > 0$, to get $X = (x_1, x_2, \dots, x_N) \to (x_1 - d, x_2, \dots, x_N + d) = (z_1, \dots, z_N) = Z$. Then condition (5.2) holds for $z_1, z_2, \dots, z_N$ in place of $x_1, x_2, \dots, x_N$, with equality for some $k < N$, hence Z can be transformed to Y by the argument above.

Because operation (5.3) can only increase a law-invariant deviation measure $\mathcal{D}$, $\mathcal{D}(X) \leq \mathcal{D}(Y)$ follows. □

Lemma 5.2. *If for any r.v. $X \in \mathcal{L}^2(\Omega)$ the selector $f_\mathcal{D}$ satisfies the condition*

$$Q(\omega_i) = Q(\omega_j) \quad \textit{whenever} \quad X(\omega_i) = X(\omega_j), \tag{5.4}$$

where $Q = f_\mathcal{D}(X)$, then it is law-invariant. If Ω is uniform, the converse statement also holds.

Proof. Condition (5.4) implies that $Q = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$. Then $E[Y_1 Q] = E[Y_1 g(X)] = E[Y_2 g(X)] = E[Y_2 Q]$ whenever pairs of r.v.s (Y_1, X) and (Y_2, X) have the same joint law.

Conversely, let Ω be uniform and $X(\omega_i) = X(\omega_j)$. Then pairs of r.v.s (I_i, X) and (I_j, X) have the same joint law, where I_i and I_j are indicator functions for ω_i and ω_j, respectively. If $f_\mathcal{D}$ is law-invariant, this implies $Q(\omega_i) = N \cdot E[I_i Q] = N \cdot E[I_j Q] = Q(\omega_j)$, where $N = |\Omega|$, and (5.4) follows. □

Lemmas 5.1 and 5.2 imply that consistency with the concave ordering and (5.4) are appropriate extensions of the notion of law-invariance to non-uniform probability spaces for deviation measures and selectors, respectively.

Lemma 5.3. *For every deviation measure $\mathcal{D}$, consistent with concave ordering, there exists a selector $f_\mathcal{D}$ satisfying* (5.4).

Proof. Fix a r.v. X, select *any* risk identifier Q for X, and let $f_\mathcal{D}(X) := E[Q|X]$. Then for all $Y \in \mathcal{L}^2(\Omega)$,

$$E[(1 - f_\mathcal{D}(X))Y] = E[(1 - E[Q|X])Y] = E[(1 - Q)(E[Y|X])] \leq \mathcal{D}(E[Y|X]) \leq \mathcal{D}(Y),$$

where the first inequality follows from $Q \in \mathcal{Q}$ and (2.3), while the second one follows from consistency of $\mathcal{D}$ with concave ordering and the fact that $E[Y|X] \succeq_c Y$, see Föllmer & Schied (2011, Corollary 2.61). Hence, $f_\mathcal{D}(X) \in \mathcal{Q}$ by (2.4). Because also $E[(1 - f_\mathcal{D}(X))X] = E[(1 - E[Q|X])X] = E[(1 - Q)X] = \mathcal{D}(X)$, $f_\mathcal{D}(X)$ is in fact a risk identifier of X, and condition (5.4) trivially holds. □

Example 5.1. For CVaR-deviation $\mathcal{D} = \mathrm{CVaR}_\alpha^\Delta$, there exists a *unique* selector satisfying (5.4), and it is given by (see Cherny (2006))

$$f_\mathcal{D}(X) = Q_\alpha = \begin{cases} 0, & X > -VaR_\alpha(X), \\ c_X, & X = -VaR_\alpha(X), \\ 1/\alpha, & X < -VaR_\alpha(X), \end{cases} \tag{5.5}$$

where constant $c_X \in [0, 1/\alpha]$ is such that $E[Q] = 1$.

Example 5.2. For mixed CVaR-deviation (2.6), there exists a *unique* selector satisfying (5.4), and it is of the form $f_{\mathcal{D}}(X) = Q_\mu = \int_0^1 Q_\alpha \, \mu(d\alpha)$, where Q_α is given by (5.5) (see Cherny (2006)).

5.2. *Explicit example*

Let $\Omega = \{\omega_1, \dots, \omega_N\}$ with $\mathbb{P}(\omega_j) = w_j$, $j = 1, \dots, N$, and $\hat{R}_j = \hat{R}(\omega_j)$. Consider a given portfolio x^M and denote $X^* = \hat{R}^T x^M$ and $x_j^* = X^*(\omega_j)$. Without loss of generality, we assume that $\{\omega_1, \dots, \omega_N\}$ are ordered in such a way that $x_1^* \le x_2^* \le \dots \le x_N^*$. Since $E[\hat{R}] = 0$, we have $E[X^*] = 0$ and either $x_1 = \dots = x_N = 0$ or $x_1 < 0 < x_N$. The former case is impossible for a non-zero portfolio x^M under the assumption (M), therefore, we will concentrate on the non-trivial latter case of non-zero return X^*. We will examine the inverse portfolio problem for risk measured by deviation CVaR.

Let k be the maximal index such that $x_k^* < -\mathrm{VaR}_\alpha(X^*)$ (set $k = 0$ is no such index exists) and m be the maximal index such that $x_m^* \le -\mathrm{VaR}_\alpha(X^*)$. Then any risk identifier $Q^* = (q_1, \dots, q_N)$ of X^* satisfies, c.f. Rockafellar *et al.* (2006b),

$$\begin{aligned} &\sum_{j=1}^N w_j q_j = 1, \quad 0 \le q_j \le 1/\alpha, \\ &q_1 = q_2 = \dots = q_k = 1/\alpha, \quad q_{m+1} = \dots = q_N = 0. \end{aligned} \tag{5.6}$$

Hence,

$$\mu = \frac{\Delta_M}{\mathrm{CVaR}_\alpha^\Delta(X^*)} \left(\frac{1}{\alpha} \sum_{j=1}^k w_j (-\hat{R}_j) + \sum_{j=k+1}^m w_j q_j (-\hat{R}_j) \right), \tag{5.7}$$

where $q_{k+1}, \dots, q_m$ are arbitrary numbers satisfying linear constraints

$$\sum_{j=k+1}^m w_j q_j = 1 - \frac{1}{\alpha} \sum_{j=1}^k w_j, \quad \text{and} \quad 0 \le q_j \le 1/\alpha, \quad j = k+1, \dots, m.$$

If $m = k + 1$, the risk identifier in (5.6) and μ in (5.7) are uniquely defined. For $m > k + 1$, i.e., $x_{k+1}^* = \dots = x_m^* = -\mathrm{VaR}_\alpha(X^*)$, the inverse problem has infinitely many solutions. The robust selector corresponds to $q_{k+1} = \dots = q_m$, that is, $\mu = \frac{\Delta_M}{\mathrm{CVaR}_\alpha^\Delta(X^*)} \big(\frac{1}{\alpha} \sum_{j=1}^k w_j (-\hat{R}_j) + q \sum_{j=k+1}^m w_j (-\hat{R}_j) \big)$, where $q = \big(1 - \frac{1}{\alpha} \sum_{j=1}^k w_j \big) / \big(\sum_{j=k+1}^m w_j \big)$.

Example 5.3. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with uniform probability $\mathbb{P}(\omega_j) = 1/3$. There are two risky assets with centered returns $\hat{R}_1 = (-1, 0)^T$, $\hat{R}_2 = (0, -1)^T$, $\hat{R}_3 = (1, 1)^T$. Fix $\alpha = 0.05$. The solution to the forward portfolio optimization problem with $\mu = (1/3, 2/3)$ and $\Delta_M = 0.5$ is $x_M = (0.5, 0.5)$. Then $X^* = (-0.5, -0.5, 1)$, $\mathrm{VaR}_\alpha(X^*) = 0.5$, and $k = 0$, $m = 2$. The set of risk identifiers of X^* comprises $Q = (q_1, q_2, 0)$, where $0 \le q_1, q_2 \le 20$ and $q_1 + q_2 = 3$. Parameterizing $q_1 = q$ and

$q_2 = 3 - q$ for $q \in [0, 3]$, we obtain

$$\mu = \frac{0.5}{1}\left(\frac{1}{3}q(-\hat{R}_1) + \frac{1}{3}(3-q)(-\hat{R}_2)\right) = \begin{pmatrix} q/3 \\ (3-q)/3 \end{pmatrix}, \qquad q \in [0, 3].$$

The law invariant selector is given by $q = 1.5$, resulting in $\mu_1 = \mu_2 = 0.5$.

6. An application to Black-Litterman portfolio framework

In this section, we apply findings of Section 4 to an extension of the Black-Litterman model of portfolio optimization on markets with discrete distributions of returns. Asset return distributions are commonly approximated with a finite number of scenarios in practical financial applications, see, e.g., Krokhmal *et al.* (2002), Gaivoronski & Pflug (2005), Lim *et al.* (2010), Lwin *et al.* (2017). We start with a short presentation of the extension of market-based Black-Litterman model of Meucci (2005) to general discrete distributions and deviation measures.[c] We demonstrate that the non-uniqueness of solutions to the inverse optimization problem (Section 2.2) is commonly observed in this theory and means that the posterior distribution of returns is not unique. The principle of law invariance brings back the well-definiteness of this portfolio theory.

The underlying assumption of the original Black-Litterman model (Black & Litterman 1992) is that the market is in equilibrium in which the mutual fund theorem holds, i.e., all investors hold risky assets in the same proportions. In the general setting of deviation measures, Rockafellar *et al.* (2007) develops an analogous theory and calls the common portfolio of risky assets a master fund. It can be recovered by solving (2.2) for a particular choice of $\Delta = \Delta_M$. We assume, as in the original framework, that the market is in equilibrium, so the master fund corresponds to relative market capitalizations of stocks: we will call it a *market portfolio* x^M. Further, acting in the spirit of Black & Litterman (1992) we assume that the centered equilibrium distribution is known, for example, it is equal to the centered empirical distribution of asset returns. The only parameter of the distribution which is unknown is its location. To recover the latter, we solve an inverse optimization problem: knowing the solution x^M to problem (2.2) we find the mean excess return vector μ_{eq} for a given expected market return $\Delta = \Delta_M$. The distribution $\mu_{eq} + \hat{R}$ is then called *equilibrium distribution* or prior distribution.

Investor's views are represented by a $m \times n$ 'pick matrix' P and a vector $v \in \mathbb{R}^m$. Each row of P specifies combinations of assets and the corresponding entry in v provides a forecasted excess return. The uncertainty (the lack of confidence) in the forecasts is represented by a zero-mean random variable ε with a continuous distribution with full support on $\mathbb{R}^m$, for example, a normal distribution $N(0, Q)$.

[c]The reader is referred to Palczewski & Palczewski (2019) for a detailed discussion of a parallel extension for continuous distributions.

The resulting Bayesian model is

$$\begin{aligned} &\text{prior:} \quad R \sim \mu_{eq} + \hat{R}, \\ &\text{observation:} \quad V|[R = r] \sim Pr + \varepsilon. \end{aligned}$$

The posterior distribution of future returns R given $V = v$ is concentrated on the same points as the prior distribution but with different probabilities. It can be described by a new probability measure $\mathbb{Q}$ on Ω, i.e., the posterior distribution of asset excess returns is that of $\mu_{eq} + \hat{R}$ under $\mathbb{Q}$[d]. Following Bayes formula, we set the unnormalized "density" of the posterior distribution:

$$X(\omega) = f_\varepsilon\big(v - P\mu_{eq} - P\hat{R}(\omega)\big),$$

where f_ε is the density of ε. Then $\mathbb{Q}(\omega)/\mathbb{P}(\omega) = X(\omega)/E_{\mathbb{P}}[X]$. The posterior distribution of asset returns is then fed into the optimization problem (2.2).

Assume now that the deviation measure $\mathcal{D}$ is finitely generated. By Corollary 2.1 it should be expected that the market portfolio is a unique solution to (2.2). Consequently, the inverse optimization problem that determines the equilibrium distribution has many solutions (Theorem 4.1 and Example 5.3) resulting in multitude of posterior distributions and, in effect, multitude of Black-Litterman optimal portfolios. This is obviously unacceptable in a financial context. This non-uniqueness is caused by the existence of many active portfolio risk generators (active risk identifiers for the deviation measure), Lemma 4.1. Selecting the law-invariant active risk identifier, see Section 5 and Example 5.3, brings back uniqueness of the solution to the inverse optimization problem and, consequently, the uniqueness of solution to the complete portfolio optimization workflow.

In practice, an investor commonly infers the market portfolio from the market capitalization of assets. Such a portfolio is unlikely to have more than one active portfolio risk generator since optimal portfolios with at least two active portfolio risk generators lie on a finite number of hyperplanes in $\mathbb{R}^n$ (their Lebesgue measure is zero). Hence, the market portfolio solves an *unlikely* portfolio optimization problem for which the set of solutions has dimension $n-1$, see Theorem 4.1. The inverse optimization problem has, however, a unique solution.

Example 6.1. Consider the setting of Example 5.3. Extreme risk identifiers for $\mathrm{CVaR}^{\Delta}_{5\%}$ are $\mathcal{Q}^e = \{\mathrm{Perm}(3,0,0)\}$. The set of portfolio risk generators consists of 3 vectors:

$$D_1 = (1,0)^T, \quad D_2 = (0,1)^T, \quad D_3 = (-1,-1)^T.$$

Fix a market portfolio $x^M = (0.2, 0.8)^T$ and its return $\Delta_M = 0.4$. The only active portfolio risk generator for x^M is D_2. From Lemma 4.1, the inverse optimization

[d]The location vector of the posterior distribution is rarely equal to μ_{eq} due to the reweighing of probabilities in $\mathbb{Q}$ relative to $\mathbb{P}$

problem has a unique solution $\mu^* = (0, 0.5)$. Consider now the forward optimization problem with expected excess return Δ_M and mean excess return μ^*:

$$\min_{x_1, x_2} \max\big(x_1; x_2; -x_1 - x_2\big), \qquad \text{s.t. } 0.5x_2 \geq 0.4.$$

The set of solutions is $\mathbb{X}^* = \big\{(x_1, 0.8) \,\big|\, x_1 \in [-1.6, 0.8]\big\}$. Each solution in $\mathbb{X}^*$ has $\text{CVaR}^{\Delta}_{5\%}$ equal to 0.8 and the expected excess return of Δ_M.

7. Conclusions

We have analyzed in depth forward and inverse portfolio optimization problems when asset returns follow a finite number of scenarios and deviation measure is finitely generated (covering popular deviation measures: CVaR, mixed CVaR and MAD). We discovered a dichotomy in the uniqueness of solutions for both problems: the forward and inverse problems cannot be simultaneously uniquely solved (for the same data). Nevertheless, the set of parameters for which the non-uniqueness holds is of measure zero. Although it may seem that the uniqueness problem is practically negligible, we have demonstrated that this is not true in many applications, like capital allocation, cooperative investment, and the generalized Black-Litterman model. In cooperative investment, the non-uniqueness affects a "fair" way of distributing profit of joint investment between participating investors: for investors with preferences described by utility functions derived from finitely generated deviation measures, when the coalition's forward optimization problem has a unique solution (which happens on the set of model parameters of full measure), there are many risk identifiers for the optimal wealth which prevents a unique "fair" allocation of wealth between investors. For the generalized Black-Litterman model, the inverse optimization problem has multiple solutions resulting in multiple posterior distributions and optimal portfolios. This result is in contrast with the classical Black-Litterman model where the uniqueness holds for both forward and inverse problems.

In forward optimization problems, if the solution is not unique, we can optimize amongst those solutions according to a secondary objective. For example, if there are many optimal portfolios, we can choose the one which is the "closest" to our current portfolio to minimize rebalancing, c.f. Palczewski (2018). In the inverse optimization, we advocate selecting a law-invariant solution which is intuitive from the perspective of applications and shown to be unique in many cases of practical importance. This idea reintroduces uniqueness of solution to both the cooperative investment and generalized Black-Litterman model.

While the considered applications are specific in their assumptions, e.g., short selling is allowed, specific risk measures involved, specific constraint structure, etc., the theory itself is very general. Theorems 2.1 and 4.1 can be extended to a broad class of (parametrized) linear programs, and explain why non-uniqueness issue is "common" in direct and inverse linear optimization.

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