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Exit game with private information

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Abstract. The timing of strategic exit is one of the most important but difficult business decisions, especially under competition and uncertainty. Motivated by this problem, we examine a stochastic game of exit in which players are uncertain about their competitor's exit value. We construct an equilibrium for a large class of payoff flows driven by a general one-dimensional diffusion. In the equilibrium, the players employ sophisticated exit strategies involving both the state variable and the posterior belief process. These strategies are specified explicitly in terms of the problem data and a solution to an auxiliary optimal stopping problem. The equilibrium we obtain is further shown to be unique within a wide subclass of symmetric Bayesian equilibria.

Key words: non-zero sum game, exit game, asymmetric information, optimal stopping, Nash equilibrium, declining market

1. Introduction The timing of strategic exit is one of the most important but difficult business decisions. According to anecdotes and empirical studies, many firms in declining industries miss the optimal time to exit and amass substantial financial loss (Horn et al. [26], Elfenbein and Knott [13]). Exit decisions are even more complicated when the firms are uncertain about the future profits such as in the cases of 7 declining industries studied by Harrigan [25]. Furthermore, firms are generally uncertain about their rival firms' exit value from the outside option. Even though these two types of uncertainty pose practical and managerial difficulties, there has been a paucity of attempts to investigate their combined impact on the exit strategy. The goal of this paper is to study an exit game under both types of uncertainty and obtain an equilibrium exit strategy.

In the model that we examine, we incorporate two salient features of an exit game: a stochastic profit stream and private random exit values, both of which are realistic features of an exit game

between competing businesses. Initially, firms operate in a duopoly and earn identical profit streams dependent on a one-dimensional diffusion modelling economic factors. Each firm is allowed to exit at any point in time, but the remaining firm becomes a monopolist and enjoys a monopoly profit flow. The exiting firm obtains an exit value which is its private information unknown to the rival firm. The exit value incorporates the outside option for the firm as well as the cost of shutting down the enterprise. The exit values of both firms have the same distribution and are independent.

We assume that each firm's profit stream is publicly known, as it depends on the revenue and the public demand. This is a common assumption in many game-theoretic models of duopoly exit games [17, 19, 20, 31, 44]. The underlying dynamics of economic factors (the state process) is a general one-dimensional diffusion. A firm's exit value is private information, hidden from the rival firm. This reflects the fact that a firm's exit value depends on many internal factors that are not observable by outsiders, such as alternative business opportunities, salvage values [25], or even managerial behavioural biases [13]. The uncertainty about the rival's exit value is also a standard assumption in many economic models of exit games [17, 30, 33, 41].

The first main result of the paper is to obtain a novel equilibrium. Specifically, we obtain a perfect Bayesian equilibrium that is succinctly characterised by two variables: the underlying state X_t and the belief Y_t . According to the equilibrium strategy, a player of type θ_i exits when (Y_t) falls below θ_i and (X_t, Y_t) is in an explicitly given action region, see (19). The belief process (Y_t) has the meaning of the maximum type of the opponent that has remained in the game, i.e., a player believes that his opponent's type at time t is less than Y_t .

The novel characteristic of the equilibrium lies in the complexity of its strategies not found in the deterministic counterpart. The value of (Y_t) depends on the history of the sample path (X_t) , so it is not a simple function of the current value of (X_t) ; see Eq. (28). Most of the extant models of exit decisions prescribe either a deterministic timeline to exit in the deterministic exit games [17, 20, 33] or a profit-threshold policy in the case of a single-player model. In contrast, our results suggest that a much more elaborate strategy is called for: the players should continuously monitor the evolution of (X_t) and update their beliefs (Y_t) regarding their opponent's type and exit as soon as (Y_t) falls below their own type; see an alternative expression for the strategy in (14).

To our knowledge, this is the first equilibrium solution obtained for a stochastic exit game with a diffusive state variable and a continuously distributed private type. In the deterministic model such as in Fudenberg and Tirole [17], the time variable is a sufficient state variable, so the equilibrium

generating process depends on time alone. In contrast, in the stochastic game, the state of the market evolves separately from the time variable, and hence, the dimensionality of the problem increases.

It is worthy of note that our equilibrium is a natural extension of the known results in the extant literature. Recall that our exit game model incorporates both a diffusive state variable and asymmetric information. Previous studies have obtained an equilibrium of exit games with one of the two features: either a diffusive state variable [19, 44] or private types [17, 30]. In this paper, we bridge these two strands of literature by showing that our result coincides with the extant results when one of the two features is absent; see Section 5.

The striking feature of our equilibrium solution is that it is given explicitly for a large class of underlying diffusion processes and payoff functions, so it cannot be obtained by a guess-and-verify approach. Instead, we only require that an auxiliary optimal stopping problem of exit from a duopoly of one player has a solution of a threshold type and the threshold depends continuously on the exit value; see Section 3.

Our second main result concerns the possibility of other symmetric equilibria. Non-zero sum games typically have multiple equilibria, and identifying them is a formidable task (see Feinstein et al. [15] for recent results in discrete time games). Although uniqueness is rarely studied in the continuous-time literature, we are able to demonstrate that our exit game has a unique symmetric equilibrium in a large class of symmetric equilibria in which the belief process (Y_t) (or, more precisely, the generating process (A_t) in a one-to-one map correspondence with (Y_t)) has a generalised derivative that satisfies certain semi-continuity criteria (Thm. 3).

The proof of uniqueness is a significant mathematical result. There are mathematical difficulties stemming from the continuum of player types and the diffusive dynamics of the underlying state process. The proof requires a combination of probabilistic and analytical methods, and it demonstrates technical complexity involved in establishing such results in a diffusive setting with asymmetric information. Further mathematical details about our approach are summarised in the beginning of Section 6.

In the context of exit games, the uniqueness question is also of game-theoretic interest. A war of attrition under incomplete information is known to have, in general, a continuum of equilibria [33, 41], but there are variants of exit game models that have unique equilibria due to special conditions [17, 37]. This paper adds to this strand of the literature by establishing uniqueness results in an exit game with state diffusive dynamics.

In addition to the two main results, our paper also provides a new framework for stopping games with continuously distributed private types. In our equilibrium solution, each player employs a pure strategy stopping time that depends on the private type. However, from a player's perspective, the opponent's exit times resemble a mixed strategy [44], albeit with the mixing variable which is not uniform on $(0, 1)$ but distributed as a player's type. Mathematically, a mixed stopping strategy can be represented as a randomised stopping time characterised by an increasing process adapted to a player's filtration (a generating process) and a randomisation device which is independent from the underlying randomness in the game (see, e.g., De Angelis et al. [9, Def. 2.2]); the stopping time is defined as the first time that the generating process exceeds the value of the randomisation device. Despite the similarity, there is one fundamental difference: our equilibrium does not introduce a private randomisation device because it is not a mixed strategy equilibrium. The apparent randomness comes from not knowing the opponent's type, i.e., the asymmetric information. Nevertheless, because a player's actions resemble randomised stopping when perceived from the rival's perspective, we can exploit a similar mathematical framework. This observation is key to the reformulation of the problem in terms of best response optimal stopping problems in Lemma 2, where the exit time of the opponent is replaced with a functional of the belief process.

The best response formulation recasts an equilibrium as a solution to a fixed-point problem whereby the strategy of a player's opponent (driven by the belief process) is also the best response for any value of the player's type. We emphasise that the symmetry of the game in which players are identical is crucial for this approach and cannot be naturally relaxed.

Lastly, one of the mathematical challenges of our model is the construction of (Y_t) . Just as in our paper, the introduction of an appropriate belief process is often encountered in papers studying games with asymmetric information (Grün [21], Gensbittel and Grün [18], Ekström et al. [12], De Angelis and Ekström [7]) and is akin to the filter process in problems with partial information. Our model is unique because the belief process (Y_t) is defined as a solution to an ordinary differential equation (ODE) (28) with a discontinuous right-hand side. Solutions to ODEs with discontinuous right-hand side are usually non-unique (Filippov [16, Ch. 2]) and related to differential inclusions. There are several definitions of solutions to such ODEs. In this paper, we adopt Caratheodory's approach in which the solution is a continuous function which satisfies the integral version of the ODE with probability one. Classical results from the ODE theory require that the discontinuities are located on smooth surfaces, the condition that is not satisfied for our ODE in which the discontinuity points are determined by a path of a diffusion process. Instead, we obtain a maximal Caratheodory solution as a monotone limit of upper approximations of the right-hand side.

1.1. Literature review Our paper extends the literature on stochastic stopping games under asymmetric information. The asymmetry of information poses mathematical challenges, but there has been a flurry of recent contributions, particularly, in zero-sum games. Games examined in the literature possess various information structures and sources of uncertainty. One-sided asymmetry, where one player has a strictly larger information flow, were studied by Grün [21], Lempa and Matomäki [29], and De Angelis et al. [8]. Gensbittel and Grün [18] examine a zero-sum game in which each player can only observe a private continuous time finite-state Markov chain while the payoff is a function of both players' processes. A recent paper De Angelis et al. [9] shows the existence of a Nash equilibrium (saddle point) for general payoff processes in the framework with asymmetric and partial information.

In non-zero sum games, players' payoffs do not have to sum up to zero, which results in a much richer set of equilibria even in the case of full information. PDE results are often in the form of a verification theorem for a solution to a system of quasi-variational inequalities (Bensoussan and Friedman [4]). The existence of an appropriate solution of this system is studied in Nagai [32] for symmetric Markov processes and continued in Cattaiaux and Lepeltier [6] for Ray-Markov processes. Superharmonic characterisation of players' value functions for strong Markov processes is provided in Attard [3]. Hamadene and Zhang [24] and Hamadene and Hassani [23] use iterative methods to construct equilibria in games with two or more players which, even in a Markovian setting, are not in the form of hitting times. Sub-game perfect equilibria are examined in Steg [44] and Riedel and Steg [40].

In economics literature, the framework of non-zero sum games has been applied to a game of exit from a declining industry. Murto [31] investigates an exit game with a geometric Brownian motion as the profit flow and characterises Markov perfect equilibria. Steg [44] studies the subgame perfection concept in a class of stochastic exit games and finds a mixed strategy equilibrium analogous to the one in the deterministic war of attrition. Georgiadis et al. [19] investigate an exit game under complete information with a stochastic profit flow and find that the stochasticity combined with asymmetry between the players destabilise the mixed strategy equilibrium.

Closest to this paper are studies of non-zero sum games with asymmetric information. In particular, Fudenberg and Tirole [17] examines a duopoly game of exit with a continuous distribution of private types as in our paper, but it studies a deterministic game unlike our model. Décamps and Mariotti [11] examines a duopoly game of investment in a common project with Poisson signals about its quality. Players have incomplete information about their opponent's investment costs, so

the problem is cast as a stopping game under asymmetric information. In another strand of research, ghost games in which a player does not know if his opponent exists (De Angelis and Ekström [7], Ekström et al. [12]) are solved using a verification approach and result in an equilibrium in randomised stopping times. Pérez et al. [35] consider a game where one player can only stop at random times indicated by a Poisson process. Using a fixed point theorem, the authors show that the game has a Nash equilibrium in threshold strategies, i.e., in pure stopping times. The optimality of this equilibrium is then extended to the class of all stopping times using optimal stopping theory arguments applied to the best response problems. Conceptually, our paper has similarities to both lines of research. The equilibrium strategies we find are akin to randomised stopping times. However, instead of postulating a PDE for value functions, we use probabilistic optimal stopping methods for best response problems to prove that a postulated pair of stopping strategies is a perfect Bayesian equilibrium.

1.2. Summary of results Our model considers an economy in which the underlying economic factors are described by a one-dimensional diffusion

$$dX_t = \mu(X_t)dt + b(X_t)dW_t, \quad t \geq 0,$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. There are two players, each of whom has a private type θ_i , $i = 1, 2$, that describes their exit value. The distribution of both private types, denoted by F , is identical and known to both players, but the player's own exit value is private and unknown to his opponent. Players decide when to exit the market by choosing the stopping times $\tau_1(\theta_1)$ and $\tau_2(\theta_2)$ that depend on their types. The player who remains in the market becomes a monopolist and never exits. The expected payoff to Player i is then given by

$$\int \mathbb{E}_x \left[\int_0^{\tau_i \wedge \tau_j(\theta_j)} e^{-rt} D(X_t) dt + 1_{\tau_i \leq \tau_j(\theta_j)} e^{-r\tau_i} \theta_i + 1_{\tau_i > \tau_j(\theta_j)} \int_{\tau_j(\theta_j)}^{\infty} e^{-rt} M(X_t) dt \right] dF(\theta_j),$$

where x is the initial value of the process (X_t) , D is the duopoly profit flow and M is the monopoly profit flow.

The model is completely symmetric, and this symmetry enables and will be exploited in construction of a symmetric equilibrium. We do not rule out the existence of asymmetric equilibria, but our arguments strongly rely on the symmetry. For this reason, we are also not able to allow for any differences between players be it in the distribution of their private types or in their payoff flows D and M .

We first note that in equilibrium a player of type θ' would want to exit earlier than a player of type θ'' if $\theta' > \theta''$ because a player would exit earlier if his outside option is more attractive. Therefore, it is natural that the player's stopping time $\tau_i(\theta)$ be monotone in the type θ . Based on this monotonicity, we can deduce the existence of a stochastic process (Y_t) that has the interpretation of the highest value of the remaining type for both players. In turn, we hypothesise

$$\tau_i(\theta_i) = \inf\{t \geq 0 : \theta_i > Y_t\}. \quad (1)$$

We show that a symmetric equilibrium ensues when the process (Y_t) solves the differential equation

$$dY_t = -\frac{F(Y_t)}{F'(Y_t)} \frac{rY_t - D(X_t)}{m(X_t) - Y_t} 1_{X_t \leq \alpha(Y_t)},$$

where $\alpha(\theta)$ is the optimal exit threshold for a player with exit value θ whose opponent is committed to never exit the market while m is the expected total discounted monopoly profit: $m(x) = \mathbf{E}_x \int_0^\infty e^{-rt} M(X_t) dt$. The results are stated as Theorems 1 and 2. It turns out that this equilibrium is unique (Theorem 3) in the class of symmetric equilibria of the form (1) for a large class of processes (Y_t) .

1.3. Outline of the paper and terminology The paper is organised as follows. In Section 2 we introduce the framework for the exit game and a sufficient condition for a Nash equilibrium in terms of best response optimal stopping problems. An optimal stopping problem of exit from a duopoly is briefly discussed in Section 3. Its solution plays a pivotal role in the construction of an equilibrium of the exit game in Section 4. Section 4.2 provides a heuristic derivation of the equilibrium in (17). The uniqueness of the equilibrium is demonstrated in Section 6. Extreme cases when the underlying dynamics are deterministic or the distribution of exit value collapses to a point are covered in Section 5. Appendix develops asymptotic bounds for exit times of a diffusion and contains detailed calculations for an example discussed in the text. To get a basic understanding of the motivation, definition, and properties of the equilibrium without having to read technical details of the mathematical framework, we recommend reading Sections 1.2, 4.1, 4.2, 4.6, 4.7, and 5.

As a matter of convention throughout the paper, we write increasing/decreasing for non-strict monotonicity, and strictly increasing/decreasing for strict monotonicity.

2. Model Consider a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. The underlying state of the system is described by a one-dimensional diffusion

$$dX_t = \mu(X_t)dt + b(X_t)dW_t, \quad t \geq 0, \quad (2)$$

where $(W_t)_{t \geq 0}$ is an (\mathcal{F}_t) -Brownian motion. We assume that $\mu(\cdot)$ and $b(\cdot)$ are Lipschitz continuous so that $(X_t)_{t \geq 0}$ is a unique strong solution which is a strong Markov process. We further assume that $b(\cdot) > 0$. We denote by $\mathcal{I} = (x_L, x_U)$ the (potentially infinite) interval in which X_t takes values and assume that its boundaries are not attainable.

The model includes two agents (players), each having a private random variable θ_i , $i = 1, 2$, describing their exit value (type). At the outset, the players do not know the exit value of either player although the probability distribution of the types is public knowledge. At time 0, each player learns his own private type which remains unknown to his opponent throughout the game. At any time $t > 0$, a player is aware of his own type, but he only holds a belief (a probability distribution) about his opponent's type. This flow of information simulates a game of exit among firms that learn of their own exit value upon entering an industry or a new market while remaining uncertain about their opponent's exit value until the end of the game. The exit value represents the reward from exit that may include the salvage value and the value of an alternative business venture. Once a player exits, the opponent player is assumed to hold perpetual monopoly of the industry/market.

Supporting those random variables are two complete probability spaces $(\Omega^{\theta_i}, \mathcal{F}^{\theta_i}, \mathbf{P}^{\theta_i})_{i=1,2}$. Put

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}) = (\Omega, \mathcal{F}, \mathbf{P}) \otimes (\Omega^{\theta_1}, \mathcal{F}^{\theta_1}, \mathbf{P}^{\theta_1}) \otimes (\Omega^{\theta_2}, \mathcal{F}^{\theta_2}, \mathbf{P}^{\theta_2}), \quad (3)$$

and denote by $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ the embedding of the filtration $(\mathcal{F}_t)_{t \geq 0}$ onto $(\tilde{\Omega}, \tilde{\mathcal{F}})$, i.e., $\tilde{\mathcal{F}}_t = \sigma(\{A \times \Omega^{\theta_1} \times \Omega^{\theta_2} : A \in \mathcal{F}_t\})$. With an abuse of notation, we will write $(X_t)_{t \geq 0}$, θ_1 and θ_2 for an embedding of the process $(X_t)_{t \geq 0}$ and the random variables θ_1, θ_2 into $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$. We will denote by \mathbf{E} the expectation with respect to \mathbf{P} and by $\tilde{\mathbf{E}}$ the expectation with respect to $\tilde{\mathbf{P}}$. With the lower index by \mathbf{P}, \mathbf{E} and $\tilde{\mathbf{P}}, \tilde{\mathbf{E}}$ we indicate the initial value X_0 .

The information flow of player i is modelled by filtration $(\mathcal{F}_t^i)_{t \geq 0} := \tilde{\mathcal{F}}_t \vee \sigma(\theta_i)$, i.e., \mathcal{F}_t^i is the smallest σ -algebra containing $\tilde{\mathcal{F}}_t$ and with respect to which θ_i is measurable. His action is given by a (\mathcal{F}_t^i) -stopping time τ_i which we denote by $\tau_i \in \mathcal{T}(\mathcal{F}_t^i)$. The payoff of Player 1 is

$$J_1(x, \tau_1, \tau_2) = \tilde{\mathbf{E}}_x \left[\int_0^{\tau_1 \wedge \tau_2} e^{-rt} D(X_t) dt + 1_{\tau_1 \leq \tau_2} e^{-r\tau_1} \theta_1 + 1_{\tau_1 > \tau_2} e^{-r\tau_2} m(X_{\tau_2}) \right],$$

and, analogously, the payoff of Player 2 is

$$J_2(x, \tau_1, \tau_2) = \tilde{\mathbf{E}}_x \left[\int_0^{\tau_1 \wedge \tau_2} e^{-rt} D(X_t) dt + 1_{\tau_2 \leq \tau_1} e^{-r\tau_2} \theta_2 + 1_{\tau_2 > \tau_1} e^{-r\tau_1} m(X_{\tau_1}) \right],$$

where

$$m(x) = \mathbf{E}_x \left[\int_0^\infty e^{-rs} M(X_s) ds \right], \quad x \in \mathcal{I}.$$

Function $D(x)$ represents the profit flow to a player in a duopoly while $M(x)$ is the profit flow to the remaining player in the monopoly, hence $m(x)$ is the cumulative discounted profit earned by a monopolist given $X_0 = x$.

REMARK 1. When $\tau_1 = \tau_2$ both players exit the market and earn their exit value. This has a clear explanation from a managerial perspective as players make the exit decision independently. From a mathematical perspective, it will never happen as in an equilibrium that we study in this paper the probability of a double exit is zero.

We make the following assumptions.

ASSUMPTION 1. *Random variables θ_i , $i = 1, 2$, have the support $[\theta_L, \theta_U]$.*

ASSUMPTION 2. *Functions $D, M : \mathcal{I} \rightarrow [0, \infty)$ are continuous, increasing and bounded, and $M > D$. Furthermore, $\inf_{x \in \mathcal{I}} m(x) > \theta_U$, and the interest rate $r > 0$.*

ASSUMPTION 3. *Coefficients μ and b are Lipschitz continuous and $b > 0$.*

Assumption 3 means that b is uniformly non-degenerate on any compact subset of \mathcal{I} and $(X_t)_{t \geq 0}$ is a weak Feller process, i.e., its semigroup maps continuous bounded functions into continuous functions. Following from this observation, thanks to Assumption 2, function m defined above and function d given as

$$d(x) = \mathbf{E}_x \left[\int_0^\infty e^{-rs} D(X_s) ds \right], \quad x \in \mathcal{I},$$

are continuous and bounded.

REMARK 2. In our model, $m(x)$ is the cumulative future profit flow for a player who becomes a monopolist when the underlying process is in state x . One might argue that the monopolist should be allowed to exit the market, i.e., in the firm's payoffs, m should be replaced by

$$\hat{m}(x; \theta) = \sup_{\tau} \mathbf{E}_x \left[\int_0^\tau e^{-rs} M(X_s) ds + e^{-r\tau} \theta \right],$$

where $\theta \in [\theta_L, \theta_U]$ is the exit value of the monopolist. Clearly, $\hat{m}(x; \theta) \geq m(x)$. By Assumption 2, we have $m(x) > \theta_U$, so $\hat{m}(x; \theta) > \theta$ for every possible exit value θ . This means that the optimal stopping time in $\hat{m}(x; \theta)$ is $\tau = \infty$, so $\hat{m}(x; \theta) = m(x)$. This simplification has been accounted for in the definition of player payoffs J_1 and J_2 . The case when $\inf_{x \in \mathcal{I}} m(x) < \theta_U$ is significantly more difficult and beyond the scope of this paper. It will certainly lead to a different behaviour of players as it is possible that both players exit the market at a finite time when it is suboptimal to continue as a monopolist even with the lowest exit value θ_L .

We now introduce the notion of a Nash equilibrium in the context of our game.

DEFINITION 1. A strategy profile (a pair of strategies) $(\tau_1^*, \tau_2^*) \in \mathcal{T}(\mathcal{F}_t^1) \times \mathcal{T}(\mathcal{F}_t^2)$ is called a *Nash equilibrium* for x if for any other pair of strategies $(\tau_1, \tau_2) \in \mathcal{T}(\mathcal{F}_t^1) \times \mathcal{T}(\mathcal{F}_t^2)$ we have

$$\begin{aligned} J_2(x, \tau_1^*, \tau_2^*) &\geq J_2(x, \tau_1^*, \tau_2), & \tilde{\mathbf{P}}_x - a.s., \\ J_1(x, \tau_1^*, \tau_2^*) &\geq J_1(x, \tau_1, \tau_2^*), & \tilde{\mathbf{P}}_x - a.s. \end{aligned}$$

To construct a Nash equilibrium, we need to understand the structure of (\mathcal{F}_t^i) -stopping times. The reader is referred to [14, Proposition 3.3] for related results in a more general setting and with a different method of proof.

PROPOSITION 1. *Let Assumption 1 hold. If $\hat{\tau} : (\Omega, \mathcal{F}) \otimes ([\theta_L, \theta_U], \mathcal{B}([\theta_L, \theta_U])) \rightarrow ([0, \infty), \mathcal{B}([0, \infty)))$ is measurable and*

$$\{(\omega_0, z) \in \Omega \times [\theta_L, \theta_U] : \hat{\tau}(\omega_0, z) \leq t\} \in \mathcal{F}_t \otimes \mathcal{B}([\theta_L, \theta_U]) \quad (4)$$

for all $t \geq 0$, then $\tau = \hat{\tau}(\cdot, \theta_i)$ is an (\mathcal{F}_t^i) -stopping time.

Conversely, for every (\mathcal{F}_t^i) -stopping time τ there is a mapping $\hat{\tau}$ satisfying the above conditions such that $\tau = \hat{\tau}(\cdot, \theta_i)$, \mathbf{P} -a.s. Furthermore, $\hat{\tau}(\cdot, \theta) : \Omega \rightarrow [0, \infty)$ is an (\mathcal{F}_t) -stopping time for each $\theta \in [\theta_L, \theta_U]$.

Proof Fix $i \in \{1, 2\}$ and let $\tau = \hat{\tau}(\cdot, \theta_i)$. By a standard fact that a composition of measurable functions is measurable (c.f. [22, Sec. 19, Thm. B]), τ is a random variable on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$. To show that τ is an (\mathcal{F}_t^i) -stopping, we need to prove that

$$\{(\omega_0, \omega_1, \omega_2) \in \tilde{\Omega} : \tau(\omega_0, \omega_1, \omega_2) \leq t\} \in \mathcal{F}_t^i, \quad \text{for all } t \geq 0, \quad (5)$$

where, recalling the definition of $\tilde{\Omega}$ in (3), we write its elements as $(\omega_0, \omega_1, \omega_2) \in \Omega \times \Omega^{\theta_1} \times \Omega^{\theta_2}$. Fix $t \geq 0$. Applying [22, Sec. 19, Thm. B] to the indicator function of the set in (4), we have

$$\{(\omega_0, \omega_i) \in \Omega \times \Omega^{\theta_i} : \hat{\tau}(\omega_0, \theta_i(\omega_i)) \leq t\} \in \mathcal{F}_t \otimes \tilde{\sigma}(\theta_i),$$

where $\tilde{\sigma}(\theta_i)$ is the sub- σ -algebra of \mathcal{F}^{θ_i} generated by θ_i (recall that we denote by θ_i the random variable on $(\Omega^{\theta_i}, \mathcal{F}^{\theta_i}, \mathbf{P}^{\theta_i})$ and its embedding into $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$). Hence,

$$\{(\omega_0, \omega_1, \omega_2) \in \tilde{\Omega} : \hat{\tau}(\omega_0, \theta_i(\omega_i)) \leq t\} \in \tilde{\mathcal{F}}_t \vee \sigma(\theta_i).$$

Since $\mathcal{F}_t^i = \tilde{\mathcal{F}}_t \vee \sigma(\theta_i)$ and $t \geq 0$ is arbitrary, we obtain (5).

The proof of the converse is more involved. Fix $i \in \{1, 2\}$ and let $\mathcal{G} = \hat{\mathcal{F}} \vee \sigma(\theta_i)$, where $\hat{\mathcal{F}} = \sigma\{A \times \Omega^{\theta_1} \times \Omega^{\theta_2} : A \in \mathcal{F}\}$. Consider first a function $\varphi(\omega_0, \omega_1, \omega_2) = \varphi'(\omega_0)1_A(\omega_i)$ for $i \in \{1, 2\}$, $A \in \sigma(\theta_i)$ and φ' an \mathcal{F} -measurable bounded function. By [22, p. 76] there is $B \in \mathcal{B}([\theta_L, \theta_U])$ such that $A = \theta_i^{-1}(B)$. Hence $\varphi(\omega_0, \omega_1, \omega_2) = \hat{\varphi}(\omega_0, \theta_i(\omega_i))$ for $\hat{\varphi}(\omega_0, z) = \varphi'(\omega_0)1_B(z)$. This representation extends to any \mathcal{G} -measurable non-negative function φ using Monotone Class Theorem, i.e., any such function has a representation as $\varphi(\omega_0, \omega_1, \omega_2) = \hat{\varphi}(\omega_0, \theta_i(\omega_i))$ for a measurable $\hat{\varphi} : (\Omega, \mathcal{F}) \otimes ([\theta_L, \theta_U], \mathcal{B}([\theta_L, \theta_U])) \rightarrow ([0, \infty), \mathcal{B}([0, \infty)))$. Applying the above statement to an (\mathcal{F}_t^i) -stopping time τ (which can be viewed as a \mathcal{F} -measurable function) yields the representation with $\hat{\tau}$ as in the statement of the theorem.

From now on, we will use that τ is an (\mathcal{F}_t^i) -stopping time (and not merely an $\hat{\mathcal{F}} \vee \sigma(\theta_i)$ -measurable random variable). To prove (4), fix $t \geq 0$ and let $A = \{\tau \leq t\} \in \mathcal{F}_t^i$. By analogous arguments as above applied to $\mathcal{G} = \mathcal{F}_t^i = \tilde{\mathcal{F}}_t \vee \sigma(\theta_i)$, there is an $\mathcal{F}_t \otimes \mathcal{B}([\theta_L, \theta_U])$ -measurable function \hat{f} such that $1_A(\omega_0, \omega_1, \omega_2) = \hat{f}(\omega_0, \theta_i(\omega_i))$. Hence

$$\{(\omega_0, \omega_1, \omega_2) \in \tilde{\Omega} : \hat{\tau}(\omega_0, \theta_i(\omega_i)) \leq t\} = A = \{(\omega_0, \omega_1, \omega_2) \in \tilde{\Omega} : \hat{f}(\omega_0, \theta_i(\omega_i)) = 1\}.$$

By mapping the above sets through $(\omega_0, \omega_1, \omega_2) \mapsto (\omega_0, \theta_i(\omega_i))$ and since θ_i has support $[\theta_L, \theta_U]$ (see Assumption 1), the above equality implies

$$\{(\omega_0, z) \in \Omega \times [\theta_L, \theta_U] : \hat{\tau}(\omega_0, z) \leq t\} = \{(\omega_0, z) \in \Omega \times [\theta_L, \theta_U] : \hat{f}(\omega_0, z) = 1\}.$$

The set on the right belongs to $\mathcal{F}_t \otimes \mathcal{B}([\theta_L, \theta_U])$ as \hat{f} is measurable with respect to this σ -algebra. This completes the proof of (4).

It remains to show that $\hat{\tau}(\cdot, z)$ is an (\mathcal{F}_t) -stopping time for any $z \in [\theta_L, \theta_U]$. Consider function \hat{f} defined in the previous paragraph. By [22, Sec. 34, Thm. B], for any $z \in [\theta_L, \theta_U]$, the section $A_z := \{\omega_0 \in \Omega : \hat{f}(\omega_0, z) = 1\}$ is \mathcal{F}_t -measurable, so $\{\omega_0 \in \Omega : \hat{\tau}(\omega_0, z) \leq t\} \in \mathcal{F}_t$ upon recalling the definition of \hat{f} . As $t \geq 0$ is arbitrary, we conclude that $\hat{\tau}(\cdot, z)$ is an (\mathcal{F}_t) -stopping time for any $z \in [\theta_L, \theta_U]$. \square

It will be convenient to define a payoff functional for a deterministic exit value: for $\sigma \in \mathcal{T}(\mathcal{F}_t)$, $\gamma \in [\theta_L, \theta_U]$ and a random time τ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$,

$$J(x, \sigma, \tau; \gamma) = \tilde{\mathbb{E}}_x \left[\int_0^{\sigma \wedge \tau} e^{-rt} D(X_t) dt + 1_{\sigma \leq \tau} e^{-r\sigma} \gamma + 1_{\sigma > \tau} e^{-r\tau} m(X_\tau) \right]. \quad (6)$$

The function $J(x, \sigma, \tau; \gamma)$ has the meaning of the expected payoff to player i whose type is γ when player i 's strategy is to stop at σ and player j 's strategy is to stop at τ .

Fundamental to our construction of the Nash equilibrium is the following sufficient condition enabled by the structure of (\mathcal{F}_t^i) -stopping times established in Proposition 1.

COROLLARY 1. *Let Assumption 1 hold and assume that $\hat{\tau}_1, \hat{\tau}_2 : \Omega \times [\theta_L, \theta_U] \rightarrow [0, \infty)$ are as in Proposition 1. Define $\tau_1 = \hat{\tau}_1(\cdot, \theta_1)$ and $\tau_2 = \hat{\tau}_2(\cdot, \theta_2)$. If for each $\theta \in [\theta_L, \theta_U]$ we have*

$$\hat{\tau}_1(\cdot, \theta) \in \arg \max_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_2; \theta) \quad \text{and} \quad \hat{\tau}_2(\cdot, \theta) \in \arg \max_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_1; \theta)$$

then (τ_1, τ_2) is a Nash equilibrium.

Proof Take any (\mathcal{F}_t^1) -stopping time τ'_1 . By Proposition 1, it can be written as $\hat{\tau}'_1(\cdot, \theta_1)$ for a $\mathcal{F} \otimes \mathcal{B}([\theta_L, \theta_U])$ -measurable function $\hat{\tau}'_1$ such that $\hat{\tau}'_1(\cdot, z)$ is an (\mathcal{F}_t) -stopping time for each $z \in [\theta_L, \theta_U]$. By the tower property of conditional expectation and the independence of $\tau_2 = \hat{\tau}_2(\cdot, \theta_2)$ from θ_1 we have

$$\begin{aligned} J_1(x, \tau'_1, \tau_2) &= \tilde{\mathbb{E}}_x \left[\tilde{\mathbb{E}}_x \left[\int_0^{\tau'_1 \wedge \tau_2} e^{-rt} D(X_t) dt + 1_{\tau'_1 \leq \tau_2} e^{-r\tau'_1} \theta_1 + 1_{\tau'_1 > \tau_2} e^{-r\tau_2} m(X_{\tau_2}) \middle| \sigma(\theta_1) \right] \right] \\ &= \int_{\gamma \in [\theta_L, \theta_U]} J(x, \hat{\tau}'_1(\cdot, \gamma), \tau_2; \gamma) dF_{\theta_1}(\gamma) \leq \int_{\gamma \in [\theta_L, \theta_U]} J(x, \hat{\tau}_1(\cdot, \gamma), \tau_2; \gamma) dF_{\theta_1}(\gamma) \\ &= \tilde{\mathbb{E}}_x \left[\tilde{\mathbb{E}}_x \left[\int_0^{\tau_1 \wedge \tau_2} e^{-rt} D(X_t) dt + 1_{\tau_1 \leq \tau_2} e^{-r\tau_1} \theta_1 + 1_{\tau_1 > \tau_2} e^{-r\tau_2} m(X_{\tau_2}) \middle| \sigma(\theta_1) \right] \right] \\ &= J_1(x, \tau_1, \tau_2), \end{aligned}$$

where F_{θ_1} is the cumulative distribution function of θ_1 and for the inequality we used that $\hat{\tau}_1(\cdot, \gamma)$ maximises $J(x, \cdot, \tau_2; \gamma)$. We repeat the same arguments for J_2 . \square

3. Single player problem In this section, we assume that Player 2 never exits. Player 1's decision problem reduces to an optimal stopping problem parametrised by θ with the payoff functional

$$J(x, \tau; \theta) = \mathbb{E}_x \left[\int_0^\tau e^{-rs} D(X_s) ds + e^{-r\tau} \theta \right], \quad (7)$$

where τ is an (\mathcal{F}_t) -stopping time. The solution of this problem will be used in the construction of the equilibrium for the exit game.

Denote the value function corresponding to (7) by

$$u(x; \theta) = \sup_{\tau \in \mathcal{T}(\mathcal{F}_t)} J(x, \tau; \theta). \quad (8)$$

Given that the profit flow D is increasing, the payoff $J(x, \tau; \theta)$ is increasing in x . Therefore, we conclude that the value function $u(x; \theta)$ is increasing in x , so if x is in the stopping set (i.e., it is optimal to stop when $X_t = x$), then $(x_L, x]$ is in the stopping set. This implies that the optimal strategy should be given by the first entry time $\tau = \inf\{t \geq 0 : X_t \leq \alpha\}$ for a threshold α depending on θ . We will impose assumptions sufficient to deduce this result and a characterisation of the threshold α from [1, Thm. 3].

Let $\phi(\cdot)$ denote the decreasing fundamental solution to the ordinary differential equation $(\mathcal{L}_X - r)\phi(x) = 0$, where \mathcal{L}_X is the generator of $(X_t)_{t \geq 0}$ given by

$$\mathcal{L}_X := \frac{1}{2}b^2(x) \frac{\partial^2}{\partial x^2} + \mu(x) \frac{\partial}{\partial x}. \quad (9)$$

ASSUMPTION 4.

(i) For each $\theta \in [\theta_L, \theta_U]$, there exists a critical value $c(\theta) \in \mathcal{I} \cup \{x_U\}$ such that $D(x) \leq r\theta$ for $x < c(\theta)$ and $D(x) > r\theta$ for $x > c(\theta)$.

(ii) For each $\theta \in [\theta_L, \theta_U]$, the function

$$a_\theta(x) := \frac{\theta - d(x)}{\phi(x)} \quad (10)$$

attains a unique global maximum at $\alpha(\theta) \in \mathcal{I}$, is differentiable at $\alpha(\theta)$ and is increasing for $x < \alpha(\theta)$.

(iii) Function $\alpha : [\theta_L, \theta_U] \rightarrow \mathcal{I}$ defined in (ii) is strictly increasing.

REMARK 3. A sufficient condition for Assumption 4(ii)-(iii) is that, for each $\theta \in [\theta_L, \theta_U]$, the function $x \mapsto a_\theta(x)$ is continuously differentiable and there is $\alpha(\theta) \in \mathcal{I}$ such that $a'_\theta(x) > 0$ for $x < \alpha(\theta)$ and $a'_\theta(x) < 0$ for $x > \alpha(\theta)$. These conditions immediately give (ii). For (iii), fix θ and $x^* = \alpha(\theta)$. Take any $\theta' > \theta$ and notice that

$$a'_{\theta'}(x^*) = a'_\theta(x^*) + \frac{(\theta' - \theta)\phi'(x^*)}{\phi^2(x)} < a'_\theta(x^*) = 0,$$

because ϕ is a strictly decreasing function, so $\phi' < 0$. This implies that $x^* < \alpha(\theta')$, hence the required monotonicity.

LEMMA 1. *Under Assumptions 2, 3 and 4, we have:*

(i) for each $\theta \in [\theta_L, \theta_U]$, the optimal policy is to exit at the stopping time

$$\tau_\theta^* = \inf \{t \geq 0 : X_t \leq \alpha(\theta)\}, \quad (11)$$

where $\alpha(\theta)$ is defined in Assumption 4. Furthermore, $u(x; \theta)$ is given by

$$u(x; \theta) = \begin{cases} a_\theta(\alpha(\theta))\phi(x) + d(x) & \text{for } x > \alpha(\theta) \\ \theta & \text{for } x \leq \alpha(\theta) \end{cases}, \quad (12)$$

and $u(x; \theta) > \theta$ for all $x > \alpha(\theta)$.

(ii) $u(x; \theta)$ is continuous in θ .

(iii) Function α is continuous and $\alpha(\theta) \leq c(\theta)$.

Proof Statement (i) follows directly from [1, Thm. 3]. For the second part of (iii), we rewrite (7) as

$$J(x, \tau; \theta) := \theta + \mathbb{E}_x \left[\int_0^\tau e^{-rs} (D(X_s) - r\theta) ds \right],$$

from which it is clear that stopping is not optimal whenever $D(X_s) - r\theta > 0$. The continuity of α is proved by contradiction. Assume that there is a sequence $(\theta_n) \subset [\theta_L, \theta_U]$ converging to θ and $\alpha(\theta_n) \rightarrow \hat{\alpha} \neq \alpha(\theta)$. Since $\alpha(\theta_n)$ is a global maximum of a_{θ_n} , we have $a_{\theta_n}(\alpha(\theta_n)) \geq a_{\theta_n}(\alpha(\theta))$. The mapping $(x, \theta) \mapsto a_\theta(x)$ is continuous, so $a_{\theta_n}(\alpha(\theta)) \rightarrow a_\theta(\alpha(\theta))$ and $a_{\theta_n}(\alpha(\theta_n)) \rightarrow a_\theta(\hat{\alpha})$. This means that $a_\theta(\hat{\alpha}) \geq a_\theta(\alpha(\theta))$. This contradicts that $\alpha(\theta)$ is the unique global maximum of $a_\theta(\cdot)$.

Statement (ii) can be deduced from the explicit formula (12) and the continuity of $\alpha(\cdot)$. \square

We turn the attention to an example on which we will illustrate our theory.

EXAMPLE 1. Consider a geometric Brownian motion (X_t) , i.e., a solution to the SDE given by $dX_t = \mu X_t dt + b X_t dW_t$ for some $\mu < 0$ and $b > 0$. Its generator takes the form

$$\mathcal{L}_X = \frac{1}{2} b^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}.$$

The fundamental solutions to the differential equation $(\mathcal{L}_X - r)\varphi(x) = 0$ are $\psi(x) = x^{\gamma_+}$ and $\phi(x) = x^{\gamma_-}$, where

$$\gamma_\pm = \frac{1}{2} - \frac{\mu}{b^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{b^2}\right)^2 + \frac{2r}{b^2}}.$$

From $\mu < 0$, it is obvious that $\gamma_+ > 1$ and $\gamma_- < 0$, i.e., ψ is increasing while ϕ is decreasing.

We state now all conditions on the coefficients which will be required in this example:

$$\beta \in (0, 1), \quad r > \delta, \quad \beta - 1 > \gamma_- > -1, \quad \beta b^2 |\gamma_-| < 2r, \quad (13)$$

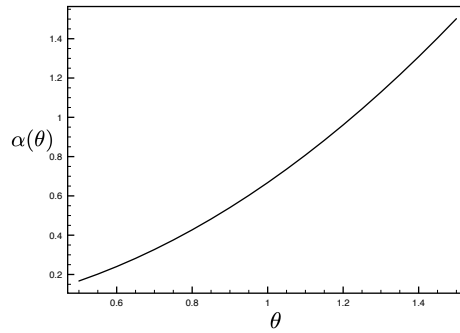


FIGURE 1. Optimal stopping threshold $\alpha(\theta)$ for single player problem with exit value θ .

where $\delta = \beta\mu + b^2\beta(\beta - 1)/2$ and β will be used in the statement of the profit flow D and M below. Notice that the second condition is automatically satisfied on a declining market, $\mu \leq 0$, because $\delta \leq 0$ while r is required to be strictly positive. We keep it for reference.

We examine the case in which the duopoly and monopoly profit flows are given by

$$D(x) = \begin{cases} x^\beta & x \in (0, x_M] \\ x_M^\beta & x > x_M \end{cases}, \quad M(x) = D(x) + M_0,$$

for some fixed (large) x_M to be determined later. We assume that $M_0 > r\theta_U$ so that $m(x) > \theta$ for all $\theta \in [\theta_L, \theta_U]$. We also assume a sufficiently large value of x_M so that $x_M^\beta > r\theta_U$. The function $D(\cdot)$ is strictly increasing for $x < x_M$ and constant for $x \geq x_M$. This form of D is economically realistic because it is impossible to achieve an unboundedly large value of profit stream. Appendix B provides the proof that this example satisfies all the assumptions of the paper and derives the explicit form of $d(\cdot)$ and $m(\cdot)$.

For numerical illustration, we examine the case $\mu = -0.5$, $b = 1$, $r = 1$, $\beta = 0.5$, $\theta_L = 0.5$, $\theta_U = 1.5$, $M_0 = 2$, and $x_M = 1000$. It can be verified that $\gamma_- = -0.732$, so it satisfies $\beta - 1 > \gamma_- > -1$, and $\beta b^2 |\gamma_-| = 0.366 < 2 = 2r$. Furthermore, $r > \delta = -0.375$. Figure 1 shows a graph of $\alpha(\theta)$, the optimal stopping threshold for a single player problem. As expected, it is an increasing function because players with higher exit values exit earlier.

4. Symmetric equilibrium In this section, we construct a Nash equilibrium in the exit game introduced in Section 2. Apart from all assumptions introduced so far in Sections 2 and 3, we make an additional standing assumption:

ASSUMPTION 5. *Random variables θ_i , $i = 1, 2$, have the same cumulative distribution function F which is strictly increasing and continuous on its support $[\theta_L, \theta_U]$, and has a Lipschitz continuous inverse.*

We defer the discussion of the requirement for the symmetry of the game and the reasons behind considering only symmetric strategy profiles to Section 4.2 where we also heuristically derive the form of the equilibrium postulated in this paper.

4.1. The strategy profile We will now introduce a symmetric strategy profile which will be shown to be a Nash equilibrium in the sense of Def. 1 as well as a perfect Bayesian equilibrium. We start from an intuitive derivation of the form of such a strategy profile and then provide a formal mathematical definition. Notice that in equilibrium a player of type θ' would want to exit earlier than a player of type θ'' if $\theta' > \theta''$ because a player would exit earlier if his outside option is more attractive. Therefore, it is natural that the symmetric equilibrium strategy $\hat{\tau}(\cdot, \theta)$ (c.f. Corollary 1) should be monotone in the type θ . Based on the monotone property of $\hat{\tau}$, we can hypothesise the existence of a well-defined stochastic process (Y_t) that has the interpretation of the highest value of the remaining type for both players. Thus, the posterior distribution of the remaining types at any point in time can be succinctly characterised by (Y_t) alone. Furthermore, it would be natural (bar technical difficulties) that (Y_t) defined the strategy $\hat{\tau}$ via its inverse: $\hat{\tau}(\cdot, \theta) = \inf\{t \geq 0 : Y_t < \theta\}$.

The above intuitive arguments motivate the introduction of player i 's strategy of the form

$$\tau_i = \inf\{t \geq 0 : Y_t < \theta_i\}, \quad i = 1, 2, \quad (14)$$

where the process $(Y_t)_{t \geq 0}$ is decreasing, (\mathcal{F}_t) -adapted and right-continuous with values in $[\theta_L, \theta_U]$ such that $Y_{0-} = \theta_U$. With an abuse of notation, we will treat $(Y_t)_{t \geq 0}$ as a process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ when necessary; clearly, it is $(\tilde{\mathcal{F}}_t)$ -adapted then. Parametrisation (14) of the strategy reflects the intuitive meaning of the process (Y_t) introduced above: on $\{Y_t = y\}$ all players of type $\theta > y$ have left the game before or at time t .

From a mathematical perspective the process (Y_t) is inconvenient to work with as it starts from $Y_{0-} = \theta_U$ and decreases to θ_L , i.e, it depends explicitly on the support $[\theta_L, \theta_U]$ and the distribution of types; see the dynamics (28) of (Y_t) in the case when the type θ_i has an absolutely continuous cumulative distribution function F . It turns out that a more convenient parametrisation is given by

$$A_t = \begin{cases} -\log(F(Y_t)), & Y_t > \theta_L, \\ \infty, & Y_t = \theta_L. \end{cases}$$

We postulate a strategy profile defined in terms of (A_t) as

$$dA_t = \lambda(X_t, Y(A_t))dt, \quad (15)$$

where

$$Y(a) = F^{-1}(e^{-a}), \quad a \geq 0, \quad (16)$$

and

$$\lambda(x, y) = \frac{ry - D(x)}{m(x) - y} 1_{x \leq \alpha(y)}. \quad (17)$$

By Assumption 4 and Lemma 1, the numerator of the function λ is non-negative. The denominator is positive as $m(x) > \theta_U$ for any $x \in \mathcal{I}$, see Assumption 2. Intuitively, $\exp(-A_t) = F(Y_t)$ has the interpretation of the proportion of the types that remain in the game at time t . It follows that $\lambda(X_t, Y(A_t))$ has the interpretation of the rate of exit of an opponent at time t .

The heuristic motivation for the above form of the Nash equilibrium will be provided in Section 4.2 with formal mathematical derivation presented in Section 6 along the uniqueness results; arguments used there require properties of the process (A_t) and of the best response stopping problems which we derive Sections 4.3-4.8.

For notational convenience, we introduce the following function:

$$A(y) = -\log(F(y)), \quad y \in (\theta_L, \theta_U], \quad (18)$$

and $A(\theta_L) = \infty$. Notice that $a \mapsto Y(a)$ and $y \mapsto A(y)$ are decreasing and continuous functions on their domains. Furthermore, $A(Y(a)) = a$ and $Y(A(y)) = y$.

We remark that the stopping times τ_i have an equivalent representation (c.f. Lemma 10)

$$\tau_i = \inf\{t \geq 0 : Y_t \leq \theta_i, X_t \leq \alpha(\theta_i)\}, \quad i = 1, 2, \quad (19)$$

which naturally links with the classical form of a solution to the best response optimal stopping problem and furnishes a perfect Bayesian equilibrium (PBE). The condition $Y_t \leq \theta_i$ originates from the notion that Y_t is the maximum type remaining in the game; if Y_t hits θ_i , it signals that it is time for player of type θ_i to exit. The other condition suggests that a player exits only when $X_t \leq \alpha(\theta_i)$; this condition originates from the interpretation of $\alpha(\theta_i)$ as the exit threshold in the single-player problem. If $X_t > \alpha(\theta_i)$, the prospective profit stream is sufficiently large that a player does not have an incentive to exit.

The representation (19) does not allow us to derive an optimisation problem in which the type of the opponent is integrated out and the action of his stopping time replaced by appropriate functional of the process (Y_t) , see Lemma 2. In the following subsections the reader will notice the importance of this detail and how it is overcome in order to establish the PBE. Indeed, in Sections 4.3-4.6 we prove that (τ_1, τ_2) is a Nash equilibrium; this is followed in Section 4.7 by arguments showing that (19) yields a PBE.

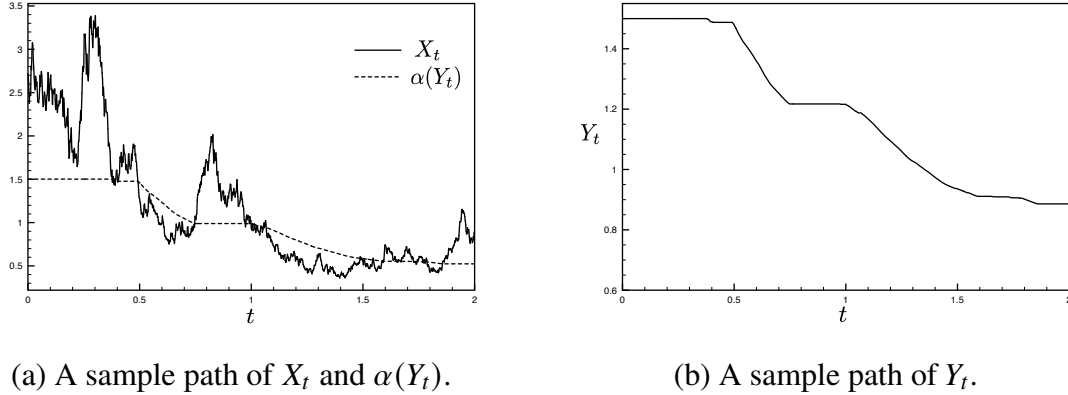
(a) A sample path of X_t and $\alpha(Y_t)$.(b) A sample path of Y_t .

FIGURE 2. Example of the evolution of (X_t) and the corresponding processes (Y_t) and $\alpha(Y_t)$ with initial conditions $X_0 = 2.72$ and $Y_0 = \theta_U$ over time interval $[0, 2]$.

EXAMPLE 1 (CONTINUED). For intuitive understanding, we now return to the numerical example introduced in Section 3 and assume that the type θ_i is uniformly distributed within an interval $[\theta_L, \theta_U]$; recall that $\theta_L = 0.5$ and $\theta_U = 1.5$. A simulated realisation of the game is presented in Figure 2. It illustrates a sample path of (X_t, Y_t) and $\alpha(Y_t)$ with initial conditions $X_0 = 2.72$ and $Y_0 = \theta_U$ over a time interval $[0, 2]$.

The functional form of $\lambda(\cdot, \cdot)$ given in (17) suggests that there are two regions: an exit region $X_t \leq \alpha(Y_t)$ and a no-action region $X_t > \alpha(Y_t)$. According to the strategy profile (19), players may exit only when $X_t \leq \alpha(Y_t)$, which results in a positive value of $\lambda(X_t, Y_t)$. This feature of the strategy profile is illustrated by Figure 2, where (Y_t) decreases only when $X_t \leq \alpha(Y_t)$. For instance, in the time intervals $[0, 0.378]$ and $[0.748, 0.974]$, X_t stays above $\alpha(Y_t)$, so Y_t stays constant within these intervals. In contrast, $X_t \leq \alpha(Y_t)$ in the intervals $[0.494, 0.746]$ and $[1.068, 1.486]$, so (Y_t) declines steadily in time.

Next, we illustrate an individual player's strategy. A player of type $\theta \geq Y_t$ follows a threshold exit strategy: to exit as soon as $X_t \leq \alpha(\theta)$. This is intuitively consistent with the optimal policy for a single-player case. On the other hand, any player of type $\theta' < Y_t$ will wait until Y_t hits θ' , at which point in time he will also adopt a threshold exit strategy, i.e., to exit when $X_t \leq \alpha(\theta)$. In Nash equilibrium, one would never encounter a player of type $\theta > Y_t$; when $\theta = Y_t$ and $X_t \leq \alpha(\theta)$, the dynamics (15) of A_t implies that $Y_s = Y(A_s) < \theta$ for any $s > t$, so infimum of times that the condition $Y_s < \theta$ is satisfied is t which justifies the equivalence of definitions (14) and (19). This equivalence does not hold when $\theta > Y_t$, so only a strategy of the form (19) defines a PBE.

In the example shown in Figure 2, a player of type $\theta = 1.5$ does not exit until X_t hits $\alpha(1.5)$ at $t = 0.378$. Because a player of type 1.5 is supposed to be the first type to exit under the prescribed

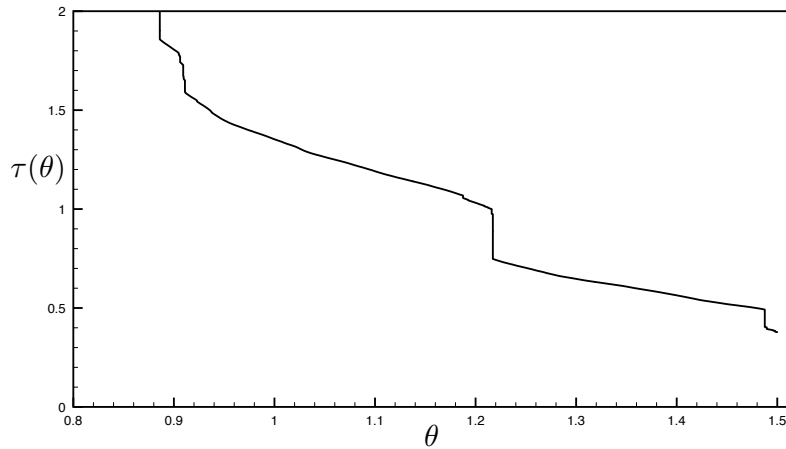


FIGURE 3. Simulated values of $\hat{\tau}(\cdot, \theta)$.

strategy profile ($\theta_U = 1.5$), he exits as soon as $X_t \leq \alpha(1.5) = \alpha(Y_0)$ is satisfied. If the process (X_t) had started out below $\alpha(1.5)$, then the player would have exited right away at time $t = 0$.

On the other hand, any player of type $\theta < Y_0$ has to wait beyond $t = 0.378$ because his prescribed time of exit is $\hat{\tau}(\cdot, \theta) = \inf\{t \geq 0 : Y_t < \theta\}$. It follows that Figure 2(B) can be utilised to determine the time of exit for any given type: we simply have to invert the $t - Y_t$ graph into $Y_t - t$ graph and relabel the horizontal variable as θ and the vertical variable as $\hat{\tau}(\cdot, \theta)$. The result is Figure 3. It illustrates the property of the strategy profile that the exit time is monotonically decreasing in the type.

The PBE strategy (19) of type θ can be succinctly represented as an exit region in the x - y space defined as $\{(x, y) : x \leq \alpha(\theta), y \leq \theta\}$. Figure 4 shows a simulated sample path of (X_t, Y_t) as well as the exit region for type $\theta = 1$ indicated by the shaded rectangle. The player of type $\theta = 1$ exits as soon as (X_t, Y_t) hits the shaded exit region, which takes place at time $t = 1.352$ when $X_t = 0.432$ and $Y_t = 1$.

Next, we illustrate the temporal evolution of a game by analysing the dynamics of (X_t, Y_t) and the prescribed strategy for each player. Suppose that $\theta_1 = 1.4$ and $\theta_2 = 1$. At the beginning of the game, each player knows his own type but not his opponent's. However, they both know the strategy profile and the initial probability distribution of their opponent's type, just as in the standard game-theoretic assumption. According to the strategy profile, player i exits at the earliest time such that $Y_t < \theta_i$. As the game progresses, each player observes whether his opponent exits or not. If both players remain in the game until time t , they update their posterior beliefs about their opponent's type by using the dynamics of (Y_t) given by (15). Finally, Player 1 exits at time $\hat{\tau}(\cdot, 1.4) = 0.564$ (see Figure 3),

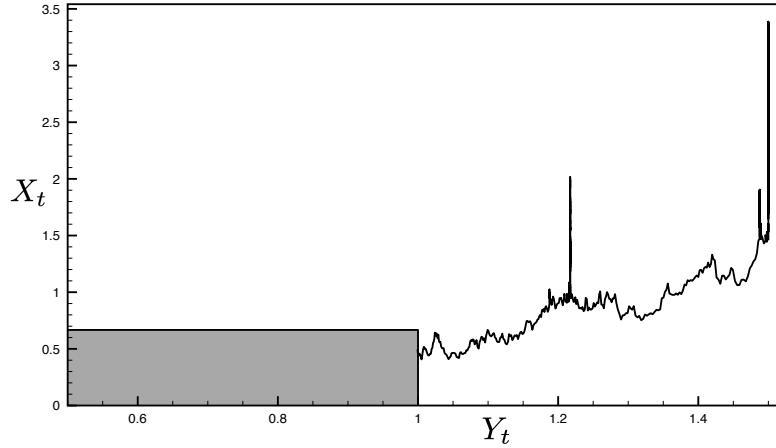


FIGURE 4. Simulated path of (X_t, Y_t) . The exit region for $\theta = 1$ is shaded.

revealing his type publicly. Player 2 never exits under this scenario, and he consequently enjoys the monopoly from time $\hat{\tau}(\cdot, 1.4)$ onwards.

As indicated before (see Corollary 1), the proof that the strategy profile given by (14) with $Y_t = Y(A_t)$, for A_t defined above, is a Nash equilibrium will require examination of the optimal stopping problem with the functional $J(x, \sigma, \tau_i; \gamma)$ defined in (6). Recall that τ_i depends on θ_i which is not observable by the rival firm; formally, θ_i is independent from \mathcal{F} . We will therefore integrate out θ_i in the following lemma. The statement is formulated for a general process $(A_t)_{t \geq 0}$ as this does not lead to any additional complications in the proof.

LEMMA 2. *Let τ_i be of the form (14) with $Y_t = Y(A_t)$ for a process $(A_t)_{t \geq 0}$ which is (\mathcal{F}_t) -adapted, right-continuous and increasing with $A_{0-} = 0$. For any (\mathcal{F}_t) -stopping time σ we have*

$$J(x, \sigma, \tau_i; \gamma) = \mathbf{E}_x \left[\int_0^\sigma e^{-rs - A_s} D(X_s) ds + \gamma e^{-r\sigma - A_{\sigma-}} + \int_{[0, \sigma)} e^{-rs - A_s} m(X_s) dA_s \right]. \quad (20)$$

Proof of this and other technical results are collected in Section 4.8.

4.2. Heuristic derivation and interpretation of $\lambda(x, y)$ Before we delve into the mathematical proof of the equilibrium, we provide a heuristic derivation of the form of $\lambda(x, y)$ given in (17). We first assume a current value of $Y_t = \theta_c$ at time t and consider the regime $X_t > \alpha(Y_t) = \alpha(\theta_c)$. Recall that we have established that the optimal policy for a single-player problem is never to exit for $X_t > \alpha(\theta_c)$ if the player is of type θ_c . Even if his opponent is present, this optimal policy does not change for $X_t > \alpha(\theta_c)$ as the opponent's action can only increase the profit flow compared to duopoly payoff in the single-player problem. Hence, a player of type θ_c should not exit in equilibrium if $X_t > \alpha(\theta_c)$. From the monotone property of $\theta \mapsto \hat{\tau}(\cdot, \theta)$, it follows that any type $\theta' < \theta_c$

should not exit if $X_t > \alpha(\theta_c)$. Since all types of $\theta' > \theta_c$ already exited in the past, no one exits for as long as $X_t > \alpha(Y_t)$. It follows that $\lambda(X_t, Y_t) = 0$ if $X_t > \alpha(Y_t)$.

We now consider the exit region $X_t \leq \alpha(\theta_c)$. By the arguments established above, it is the player of type θ_c that should decide when to exit; the types $\theta < \theta_c$ would first wait until after θ_c exits, and the types $\theta > \theta_c$ should have already exited by now. Thus, we postulate that the best response of the player of type $\theta < \theta_c$ is to wait at least an infinitesimal time.

Next, let $X_t = x \leq \alpha(\theta_c)$ and $A_t = A(\theta_c)$. As argued above, it is suboptimal for Player i of type $\theta < \theta_c$ to exit immediately at t . Assuming that waiting an infinitesimally short time $\delta t > 0$ is strictly better (which we leave without any formal justification) and using (20) and (15), we have

$$\begin{aligned} 0 &< \mathbb{E} \left[\int_0^{t+\delta t} e^{-rs-A_s} D(X_s) ds + \theta e^{-r(t+\delta t)-A_{(t+\delta t)-}} \right. \\ &\quad \left. + \int_{[0,t+\delta t)} e^{-rs-A_s} m(X_s) dA_s \middle| X_t = x, A_t = A(\theta_c) \right] \\ &\quad - \mathbb{E} \left[\int_0^t e^{-rs-A_s} D(X_s) ds + \theta e^{-rt-A_t-} + \int_{[0,t)} e^{-rs-A_s} m(X_s) dA_s \middle| X_t = x, A_t = A(\theta_c) \right] \\ &= e^{-rt-A_t-} \left(\delta t \left[D(x) + \lambda(x, \theta_c) m(x) - \theta(r + \lambda(x, \theta_c)) \right] + \mathcal{O}((\delta t)^2) \right). \end{aligned}$$

From this, the leading-order term of δt must be positive, which yields the inequality

$$\lambda(x, \theta_c) > \frac{r\theta - D(x)}{m(x) - \theta}, \quad \theta < \theta_c. \quad (21)$$

Letting $\theta \uparrow \theta_c$ on the right-hand side, we conclude that

$$\lambda(x, \theta_c) \geq \frac{r\theta_c - D(x)}{m(x) - \theta_c}.$$

In particular, $\lambda(X_t, \theta_c) > 0$ so A_t increases at t ($A_s > A_t$, $s > t$), see (15). Hence, it is optimal for the player of type $\theta_c = Y(A_t)$ to exit immediately at time t . By the arbitrariness of $x \leq \alpha(\theta_c)$, the player of type $\theta_c = Y(A_t)$ exits immediately at time t as long as $X_t \leq \alpha(\theta_c)$. We have also justified above that the player does not exit at t if $X_t > \alpha(\theta_c)$. Summarising, $\lambda(x, \theta_c) > 0$ for $x \leq \alpha(\theta_c)$ and $\lambda(x, \theta_c) = 0$ for $x > \alpha(\theta_c)$, which explains the link between the stopping threshold α of the single player problem and the behaviour of players in the game.

As delaying exit by a small time $\delta t > 0$ may only be suboptimal for the player of type $\theta_c = Y(A_t)$ when $X_t \leq \alpha(\theta_c)$, an analogous argument as above gives

$$\lambda(x, \theta_c) \leq \frac{r\theta_c - D(x)}{m(x) - \theta_c}, \quad x \leq \alpha(\theta_c). \quad (22)$$

Note that both conditions (21) and (22) automatically enforce that $\lambda(x, \theta_c) = (r\theta_c - D(x))/(m(x) - \theta_c)$ for $x \leq \alpha(\theta_c)$. From the arbitrariness of θ_c and $x \leq \alpha(\theta_c)$ and together with the condition that $\lambda(x, y) = 0$ for $x > \alpha(y)$, we finally conclude the form (17) for λ .

The dynamics of (A_t) lends itself to the following interpretation. As argued above, whenever $X_t > \alpha(Y_t)$, the player of any type $\theta_c \leq Y_t$ does not exit. Let us consider $X_t \leq \alpha(Y_t)$. The definition of $\lambda(X_t, Y_t)$ is equivalently written as

$$D(X_t) + \lambda(X_t, Y_t)m(X_t) = Y_t(r + \lambda(X_t, Y_t)). \quad (23)$$

We will look from the point of view of a market observer, i.e., someone who cannot observe either player's private exit value and views their exit strategies as stopping at the rate $\lambda(X_t, Y_t)$. The left-hand side can be interpreted as the profit stream arising from the presence in the market, $D(X_t)$, and from the actions of Player 2 who, from the point of view of the market observer (and Player 1), exits the market with intensity $\lambda(X_t, Y_t)$ and, consequently, leaves Player 1 as a monopolist with the expected cumulative future discounted profit flow $m(X_t)$. Profit flow from exit of Player 1 is captured by the right-hand side of (23). Indeed, Player 1 exits at the rate $r + \lambda(X_t, Y_t)$ at time t with the payoff Y_t . The rate r comes from the (widely used) interpretation of the discounting of the terminal payoff as stopping at an independent exponential time with intensity r while $\lambda(X_t, Y_t)$ is the rate at which Player 1 exits the market. Intuitively, profit streams arising from actions of Player 1 (right-hand side) and Player 2 (left-hand side) must be equal in equilibrium; otherwise, one of the players would be at a disadvantage, for example, Player 2 exiting too early and giving Player 1 an excessive chance of becoming a monopolist. Due to the symmetry of the setup, the roles of the players can be swapped without changing the arguments above.

The interpretation of (23) as the equality between profit streams arising from actions of Players 1 and 2 sheds light on the motivation for considering symmetric equilibria. This equality determines an equilibrium only if the intensity of stopping of both players is identical. Indeed, the intensity λ on the left-hand side pertains to actions of Player 2 while the intensity λ on the right-hand side is related to the exit strategy of Player 1. Notice that these arguments do not rule out asymmetric equilibria but demonstrate the convenience of postulating symmetric equilibria.

4.3. Construction and properties of $(A_t)_{t \geq 0}$ Standard theory of ODEs cannot be applied to obtain existence and uniqueness of solutions to (15) because the function λ is discontinuous for each trajectory of (X_t) . Instead, we construct the process (A_t) that satisfies the integral form of (15). We start with a number of technical results.

We first review the continuity of the process $(X_t^x)_{t \geq 0}$ with respect to the initial value x . [28, Thm. 1, p. 102] implies that for any $T > 0$ and $q \geq 1$ we have for any $x^n \rightarrow x$ in \mathcal{I} ,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{s \in [0, T]} |X_s^{x^n} - X_s^x|^q \right] = 0. \quad (24)$$

The dependence of X_t^x on the initial point x is monotone due to the comparison principle for diffusions. Hence, by the monotone convergence theorem, (24) implies that if the sequence (x_n) is monotone, then for each $T > 0$, there is a \mathbb{P} -negligible subset of Ω outside of which

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |X_s^{x_n} - X_s^x| = 0. \quad (25)$$

The infinite variation of trajectories of the process (X_t) , which follows from the non-degeneracy of the diffusion coefficient σ , yield the following result.

LEMMA 3. *For any càdlàg finite variation (\mathcal{F}_t) -adapted process (φ_t) , we have*

$$\int_0^\infty 1_{X_s = \varphi_s} ds = 0, \quad \mathbb{P}_x - a.s.$$

for any $x \in \mathcal{I}$.

We illustrate now one application of this lemma. Assume that there is a function $\tilde{\lambda}$ which equals to λ everywhere apart from a (countable) number of curves of the form $x = h(y)$ for continuous h . Lemma 3 implies that if a process (A_t) satisfies (15) with the right-hand side λ then it does so with the right-hand side $\tilde{\lambda}$. This will play an important role in the construction of the solution in Proposition 2 as well as in Section 6 in which the uniqueness of equilibrium is established.

LEMMA 4. *For any $x \leq \alpha(\theta_U)$, the mapping*

$$[\theta_L, \theta_U] \ni y \mapsto l(x, y) := \frac{ry - D(x)}{m(x) - y}$$

is strictly increasing with the derivative

$$0 < \frac{rm(x) - D(x)}{(m(x) - y)^2} \leq \frac{r m_{\min}}{(m_{\min} - \theta_U)^2}, \quad (26)$$

where $m_{\min} = \inf_{x \in \mathcal{I}} m(x)$.

LEMMA 5. *The mapping $(x, a) \mapsto \lambda(x, Y(a))$ is decreasing in x and a .*

The above basic properties of the expression defining λ in (17) are key to the construction of a solution to (15) as well as to the study of the best response optimal stopping problems. Notice also that $l(x, y) \geq 0$ for $x \leq \alpha(y)$ thanks to Lemma 1, so the function λ is non-negative.

PROPOSITION 2. *There is a strong Markov process $(X_t, A_t)_{t \geq 0}$ such that (X_t) solves (2) and (A_t) is a continuous process which satisfies*

$$A_t = A_0 + \int_0^t \lambda(X_s, Y(A_s)) ds, \quad t \geq 0, \quad \mathbf{P}_x - a.s. \quad (27)$$

i.e., it is a Carathéodory solution to (15).

The solution of (27) in the above proposition is constructed as a limit of solutions (A_t^ε) of ODEs with the right-hand side λ^ε . Functions λ^ε converge from above to λ and are Lipschitz continuous, so that (A_t^ε) are uniquely determined for each ω . The monotone limit $A^0 = \lim_{\varepsilon \downarrow 0} A^\varepsilon$ is shown to satisfy (27) with Lemma 3 playing an important role.

The process $(A_t^0)_{t \geq 0}$ is a Carathéodory solution to (15): it is a continuous process such that for almost every ω and almost every t the equality (15) holds. The concept of Carathéodory solutions was introduced in the theory of ODEs to make sense of equations with discontinuous right-hand side, i.e., equations whose solutions cannot be continuously differentiable functions. Here, we extended the notion to random ODEs by studying the equation pathwise, for each ω (and the resulting trajectory $t \mapsto X_t(\omega)$) separately. The reason that the equality in (15) is \mathbf{P} -a.s. is due to the use of Lemma 3. We finish with the following important remark.

REMARK 4. There may be many Carathéodory solutions to (27) but it follows from the above discussion that $(A_t^0)_{t \geq 0}$ is the largest of them.

It should be noted, however, that this extremal property of the solution does not play any role in our considerations below. It may well be that there is a unique Carathéodory solution but the theory of such equations with discontinuities driven by infinite variation process (X_t) is not well developed and beyond the scope of this paper.

Notice that when the CDF F of θ is differentiable then $Y_t = Y(A_t)$, where (A_t) satisfies (27), is a Carathéodory solution to

$$dY_t = -\frac{F(Y_t)}{F'(Y_t)} \lambda(X_t, Y_t) dt, \quad Y_0 = Y(A_0). \quad (28)$$

As the mapping Y is decreasing, it is the smallest Carathéodory solution.

We will indicate an initial point $(X_0, A_0) = (x, a)$ of the Markov process $(X_t, A_t)_{t \geq 0}$ either as a subscript in \mathbf{P}_{xa} or in the process itself $X_t^x, A_t^{x,a}$ and mix the notations depending on which is beneficial for the clarity of exposition. Since $(X_t)_{t \geq 0}$ is a strong solution of (2) the process $(X_t, A_t)_{t \geq 0}$ can be considered as a family of processes on the original probability space $(\Omega, \mathcal{F}, \mathbf{P})$ as well as a Markov family.

The final auxiliary result concerns the dependence of the process (A_t) on the initial point a and the initial state x of (X_t) . The Lipschitz continuity with respect to a will be instrumental in proving the continuity of the value function to the best response optimal stopping problem, a key result to prove the equilibrium property of the postulated strategies.

LEMMA 6. *For any $x \in \mathcal{I}$, the mapping $a \mapsto A_t^{x,a} =: A_t^a$ is increasing and for $a, a' \geq 0$*

$$|A_t^a - A_t^{a'}| \leq |a - a'|, \quad t \geq 0.$$

For any $a \geq 0$, the mapping $x \mapsto A_t^{x,a}$ is decreasing and sequentially continuous in the following sense: for every $T > 0$, $x_0 \in \mathcal{I}$ and $x_n \rightarrow x_0$, there is a \mathbf{P} -negligible subset of Ω outside of which $A_t^{x_n,a}$ converges to $A_t^{x_0,a}$ in the supremum norm $\|Z\|_T = \sup_{t \in [0,T]} |Z_t|$.

REMARK 5. It can be deduced from the above lemma and its proof that the mapping $(x, a) \mapsto A_t^{x,a}$ is sequentially continuous at each point $(\bar{x}, \bar{a}) \in \mathcal{I} \times [0, \infty)$ outside of a \mathbf{P} -negligible set which depends on (\bar{x}, \bar{a}) and the sequence. Indeed, the continuity in a is uniform over x , while by applying Lemma 6 for $T = 1, 2, 3, \dots$, we obtain the sequential continuity at \bar{x} for any $t \geq 0$. The aforementioned negligible set arises because of the occupation measure formula applied to a semimartingale $Z_t = X_t^{\bar{x}} - \alpha(Y(A_t^{\bar{x}, \bar{a}}))$ which itself depends on (\bar{x}, \bar{a}) , and because of the convergence of the process (X_t^x) which depends on the sequence $x_n \rightarrow \bar{x}$.

4.4. Best-response value function Equilibrium strategy profile is characterised by the strategy of one player being the best response to the strategy of the opponent. Given the symmetry of the studied strategy profile it is sufficient to fix the strategy of Player 2 (as an opponent) and consider the best response of Player 1 for each possible private exit value. Motivated by Corollary 1, we assume now that Player 2 follows the strategy τ_2^* given by (14) with (A_t) stated in (16). We will study the best response of Player 1 when $\theta_1 = \theta \in [\theta_L, \theta_U]$, i.e., the optimal stopping problem

$$\sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_2^*; \theta).$$

This problem does not have a structure of a Markovian optimal stopping problem, but thanks to the strong Markov property of (X_t, A_t) and the expression (20) for $J(\cdot)$, we will solve it by considering the following Markovian optimal stopping problem on the extended state space (X_t, A_t) :

$$v(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbb{E}_{xa} \left[\int_0^\sigma e^{-rs - A_s} D(X_s) ds + \theta e^{-r\sigma - A_\sigma} + \int_0^\sigma e^{-rs - A_s} m(X_s) \lambda(X_s, Y(A_s)) ds \right]. \quad (29)$$

Notice that (A_t) is absolutely continuous, so $A_{t-} = A_t$. By (15) and Lemma 2, we have

$$v(x, 0; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_2^*; \theta).$$

We can rewrite the middle term of (29) in an integral form

$$\theta e^{-r\sigma - A_\sigma} = \theta e^{-A_0} - \int_0^\sigma e^{-rs - A_s} \theta (r + \lambda(X_s, Y(A_s))) ds. \quad (30)$$

Letting

$$\tilde{v}(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbb{E}_{xa} \left[\int_0^\sigma e^{-rs - A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \quad (31)$$

we have $v(x, a; \theta) = \tilde{v}(x, a; \theta) + \theta e^{-a}$. This equivalent formulation of the optimal stopping problem will be used often in the paper and it underlies the arguments of the following proposition.

PROPOSITION 3. *The value function $v(x, a; \theta)$ is continuous in $(x, a) \in \mathcal{I} \times [0, \infty)$. Furthermore, an optimal stopping time for $v(x, a; \theta)$ is $\sigma^* = \inf\{t \geq 0 : (X_t, A_t) \in \mathcal{S}_v^\theta\}$, where*

$$\mathcal{S}_v^\theta = \{(x, a) \in \mathcal{I} \times [0, \infty] : \tilde{v}(x, a; \theta) = 0\}$$

is a closed set.

The main finding of the above proposition is the continuity of the value function v , or, equivalently, the continuity of \tilde{v} . The form of an optimal stopping time follows then from the standard theory.

The continuity of \tilde{v} in (x, a) does not follow from standard results because the functional is not continuous due to the discontinuity of λ and the process (X_t, A_t) has not been shown to be Feller continuous. Instead, we approximate the value function \tilde{v} from above and from below by value functions \tilde{v}^ε , \tilde{v}_ε corresponding to optimal stopping problems with λ in the functional (31) (but not in the dynamics of (A_t)) replaced by continuous λ^ε from above (as in Proposition 2) and by continuous λ_ε from below (constructed analogously as λ^ε). We prove directly the continuity of \tilde{v}^ε and \tilde{v}_ε using Lemma 6 and Eq. (25). Thanks to the monotonicity of \tilde{v}^ε in ε , which follows immediately from the monotonicity of λ^ε in ε and the fact that $m > \theta_U$, the function $\tilde{v}^0 = \liminf_{\varepsilon \downarrow 0} \tilde{v}^\varepsilon$ is upper semi-continuous. Similarly, $\tilde{v}_0 = \limsup_{\varepsilon \downarrow 0} \tilde{v}_\varepsilon$ is lower semi-continuous. The proof is concluded by showing that $\tilde{v} = \tilde{v}^0 = \tilde{v}_0$.

REMARK 6. It would be tempting to apply a smoothing technique with λ^ε and λ_ε to prove the continuity of $(x, a) \mapsto A_t^{x,a}$. However, the use of λ^ε leads to the largest Carathéodory solution to (15), while, by analogy, we expect λ_ε to yield the smallest Carathéodory solution. This approach would, therefore, require proving the uniqueness of Carathéodory solutions which is known to not be true in general.

4.5. Best response stopping set By Proposition 3, the mapping $(x, a) \mapsto v(x, a; \theta)$ is continuous (as is $\tilde{v}(x, a; \theta) = v(x, a; \theta) - \theta e^{-a}$) and the stopping sets for \tilde{v} and v coincide: $\mathcal{S}_{\tilde{v}}^\theta = \mathcal{S}_v^\theta$. We denote by $C_{\tilde{v}}^\theta$ the continuation set, i.e., $C_{\tilde{v}}^\theta = \mathcal{I} \times [0, \infty) \setminus \mathcal{S}_{\tilde{v}}^\theta$. We also have $u(x; \theta) = \tilde{u}(x; \theta) + \theta$, where u is defined in (12) and

$$\tilde{u}(x; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbb{E}_x \left[\int_0^\sigma e^{-rs} (D(X_s) - r\theta) ds \right]. \quad (32)$$

Recall that the smallest optimal stopping time for $u(x; \theta)$ and $\tilde{u}(x; \theta)$ is

$$\eta = \inf\{t \geq 0 : \tilde{u}(X_t; \theta) = 0\} = \inf\{t \geq 0 : X_t \in \mathcal{S}_{\tilde{u}}\},$$

where $\mathcal{S}_{\tilde{u}} = \{x \in \mathcal{I} : x \leq \alpha(\theta)\}$ is the stopping set and its complement $C_{\tilde{u}} = \mathcal{I} \setminus \mathcal{S}_{\tilde{u}}$ is the continuation set.

We will turn our attention to the study of the stopping and continuation sets for \tilde{v} . In a sequence of 3 lemmas, we will establish the regions of the state space which are subsets of the stopping or continuation sets for \tilde{v} . Each lemma uses different mathematical tools, which guided the split of the material. For the clarity of the presentation, we outline here the steps:

- $\{(x, a) \in \mathcal{I} \times [0, \infty) : x > \alpha(\theta)\} \subset C_{\tilde{v}}^\theta$ (Lemma 7);
- $\{(x, a) \in \mathcal{I} \times [0, \infty) : x < \alpha(Y(a)), a < A(\theta)\} \subset C_{\tilde{v}}^\theta$ (Lemma 8);
- $\{(x, a) \in \mathcal{I} \times [0, \infty) : x \leq \alpha(Y(a)), a \geq A(\theta)\} \subset \mathcal{S}_{\tilde{v}}^\theta$ (Lemma 9).

Corollary 2 combines these properties into a complete characterisation of the stopping set $\mathcal{S}_{\tilde{v}}^\theta$.

LEMMA 7. $\tilde{v}(x, a; \theta) > 0$ for $(x, a) \in \mathcal{I} \times [0, \infty)$ such that $x > \alpha(\theta)$.

Proof Fix $x > \alpha(\theta)$ and $a \geq 0$. Consider a process

$$\begin{cases} dA_t = \lambda(X_t, Y(A_t))dt, \\ A_{0-} = A_0 = a. \end{cases}$$

Define $F_{|Y(a)}(y) = \frac{F(y \wedge Y(a))}{F(Y(a))}$ which is the cumulative distribution function F conditioned on the outcome being smaller than $Y(a)$. Let $\theta_a = F_{|Y(a)}^{-1}(F(\theta_2)) \sim F_{|Y(a)}$, where we recall that θ_2 is the exit value (type) of player 2. We choose θ_a in this way so that we can integrate it out as we did for θ_i using arguments as in Lemma 2 with a different cumulative distribution function.

Let $\bar{A}_t = A_t - a$ and $\tau_a = \inf\{t \geq 0 : F_{|Y(a)}^{-1}(e^{-\bar{A}_t}) < \theta_a\}$. Taking $\sigma = \inf\{t \geq 0 : X_t \leq \alpha(\theta)\}$, we obtain

$$\begin{aligned}
& u(x; \theta) \\
&= \mathbf{E}_x \left[\int_0^\sigma e^{-rs} D(X_s) ds + e^{-r\sigma} \theta \right] \\
&= \tilde{\mathbf{E}}_x \left[\int_0^{\sigma \wedge \tau_a} e^{-rs} D(X_s) ds + 1_{\sigma \leq \tau_a} e^{-r\sigma} \theta + 1_{\sigma > \tau_a} \left(\int_\sigma^{\tau_a} e^{-rs} D(X_s) ds + e^{-r\sigma} \theta \right) \right] \\
&\leq \tilde{\mathbf{E}}_x \left[\int_0^{\sigma \wedge \tau_a} e^{-rs} D(X_s) ds + 1_{\sigma \leq \tau_a} e^{-r\sigma} \theta + 1_{\sigma > \tau_a} \left(\int_\sigma^{\tau_a} e^{-rs} M(X_s) ds + e^{-r\sigma} m(X_\sigma) \right) \right] \\
&= \tilde{\mathbf{E}}_x \left[\int_0^{\sigma \wedge \tau_a} e^{-rs} D(X_s) ds + 1_{\sigma \leq \tau_a} e^{-r\sigma} \theta + 1_{\sigma > \tau_a} e^{-r\tau_a} m(X_{\tau_a}) \right] = J(x, \sigma, \tau_a; \theta),
\end{aligned}$$

where in the inequality we used that $M(x) \geq D(x)$ and $m(x) > \theta_U$. Notice also that we integrate over the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ since τ_a depends on θ_2 via θ_a . We apply analogous arguments as in Lemma 2 and rewrite the middle term as in (30) to get

$$u(x; \theta) \leq \mathbf{E}_x \left[\int_0^\sigma e^{-rs - \bar{A}_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] + \theta.$$

Taking the supremum over $\sigma \in \mathcal{T}(\mathcal{F}_t)$ and subtracting θ from both sides, we see that

$$\begin{aligned}
\tilde{u}(x; \theta) &\leq \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_x \left[\int_0^\sigma e^{-rs - \bar{A}_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\
&= e^a \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_x \left[\int_0^\sigma e^{-rs - A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\
&= e^a \tilde{v}(x, a; \theta),
\end{aligned}$$

where in the first equality we used the relationship between A_t and \bar{A}_t , and the last one is by the definition of \tilde{v} . Since $x > \alpha(\theta)$, it is in the continuation region $C_{\tilde{u}}$ of \tilde{u} , so $\tilde{u}(x; \theta) > 0$. Hence, $\tilde{v}(x, a; \theta) > 0$. \square

LEMMA 8. $\tilde{v}(x, a; \theta) > 0$ for $(x, a) \in \mathcal{I} \times [0, \infty)$ such that $x < \alpha(Y(a))$ and $a < A(\theta)$.

Proof For any $(x, a) \in \mathcal{I} \times [0, \infty)$ satisfying $x < \alpha(Y(a))$ and $a < A(\theta)$, we have

$$\begin{aligned} D(x) - r\theta + \lambda(x, Y(a))(m(x) - \theta) &= D(x) - r\theta + \frac{rY(a) - D(x)}{m(x) - Y(a)}(m(x) - \theta) \\ &> D(x) - r\theta + \frac{r\theta - D(x)}{m(x) - \theta}(m(x) - \theta) = 0, \end{aligned}$$

where the inequality uses $Y(a) > \theta$ and Lemma 4.

The stopping time $\eta := \inf\{t \geq 0 : X_t \geq \alpha(Y(A_t)) \text{ and } A_t < A(\theta)\}$ is \mathbf{P}_{xa} -a.s. strictly positive by the continuity of (X_t, A_t) . On the interval $[0, \eta)$ the above estimate applies. Hence

$$\tilde{v}(x, a; \theta) \geq \mathbf{E}_{xa} \left[\int_0^\eta e^{-rs - A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] > 0.$$

□

LEMMA 9. *We have $\mathcal{S}_v^\theta \supseteq \{(x, a) \in \mathcal{I} \times [0, \infty) : x \leq \alpha(\theta) \text{ and } a \geq A(\theta)\}$.*

Proof Consider an optimal stopping problem $\tilde{v}(x, a; \theta)$ on $(x, a) \in \mathcal{I} \times [A(\theta), \infty) =: \mathcal{O}$. Then $\varphi(x, a) = e^a \tilde{v}(x, a; \theta)$ is the smallest non-negative function satisfying for $(x, a) \in \mathcal{O}$ the supermartingale property (the justification of this fact is relegated to the end of the proof):

$$\begin{aligned} \mathbf{E}_{xa} \left[e^{-rt - \int_0^t \lambda(X_u, Y(A_u)) du} \varphi(X_t, A_t) \right. \\ \left. + \int_0^t e^{-rs - \int_0^s \lambda(X_u, Y(A_u)) du} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \leq \varphi(x, a). \end{aligned} \quad (33)$$

We will show that (33) is satisfied by $\varphi(x, a) = \tilde{u}(x; \theta)$, from which we immediately conclude that $\tilde{u}(x; \theta) \geq e^a \tilde{v}(x, a; \theta)$ for $(x, a) \in \mathcal{O}$ as \tilde{u} is non-negative. Since $\tilde{u}(x; \theta) = 0$ for $x \leq \alpha(\theta)$, the stopping region for \tilde{v} must contain $\mathcal{O} \cap ((x_L, \alpha(\theta)] \times [0, \infty))$ which is the statement of the proposition.

Since $\tilde{u}(\cdot; \theta)$ is $C^1(\mathcal{I})$ and the second derivative lies in L_{loc}^∞ (see (12) and classical results on the smoothness of the value function on the continuation set), Itô-Tanaka formula [39, Ch. VI, Thm. 1.5] yields

$$e^{-rt} \tilde{u}(X_t; \theta) = \tilde{u}(x; \theta) + \int_0^t e^{-rs} (\mathcal{L}_X - r) \tilde{u}(X_s; \theta) ds + \int_0^t e^{-rs} dM_s,$$

where $(M_t)_{t \geq 0}$ is a square integrable martingale and \mathcal{L}_X is the infinitesimal generator of $(X_t)_{t \geq 0}$. Using the supermartingale property of the process $t \mapsto \int_0^t e^{-rs} (D(X_s) - r\theta) ds + e^{-rt} \tilde{u}(X_t; \theta)$, we obtain

$$\mathcal{L}_X \tilde{u} - r\tilde{u} + D - r\theta \leq 0, \quad \text{for } x \in \mathcal{I} \setminus \{\alpha(\theta)\}. \quad (34)$$

Let $\bar{A}_t = A_t - a$. We apply the product rule

$$\begin{aligned} e^{-rt-\bar{A}_t}\tilde{u}(X_t;\theta) &= \tilde{u}(x;\theta) + \int_0^t e^{-rs-\bar{A}_s}(\mathcal{L}_X - r)\tilde{u}(X_s;\theta)ds - \int_0^t e^{-rs-\bar{A}_s}\tilde{u}(X_s;\theta)d\bar{A}_s \\ &\quad + \int_0^t e^{-rs-\bar{A}_s}dM_s \end{aligned}$$

and take expectations on both sides to arrive at

$$\mathbf{E}_{xa} \left[e^{-rt-\bar{A}_t}\tilde{u}(X_t;\theta) + \int_0^t e^{-rs-\bar{A}_s}(-(\mathcal{L}_X - r) + \lambda(X_s, Y(A_s)))\tilde{u}(X_s;\theta)ds \right] = \tilde{u}(x;\theta). \quad (35)$$

Recall that $\tilde{u}(x';\theta) = 0$ for $x' \leq \alpha(\theta)$ and that $\alpha(\theta) \geq \alpha(Y(A_s))$ since $A_s \geq A(\theta)$. Hence

$$\begin{aligned} &\int_0^t e^{-rs-\bar{A}_s}(-(\mathcal{L}_X - r) + \lambda(X_s, Y(A_s)))\tilde{u}(X_s;\theta)ds \\ &= \int_0^t 1_{X_s > \alpha(Y(A_s))} e^{-rs-\bar{A}_s} \left(-(\mathcal{L}_X - r)\tilde{u}(X_s;\theta) + \lambda(X_s, Y(A_s))\tilde{u}(X_s;\theta) \right) ds \\ &\geq \int_0^t 1_{X_s > \alpha(Y(A_s))} e^{-rs-\bar{A}_s} \left(D(X_s) - r\theta + \lambda(X_s, Y(A_s))\tilde{u}(X_s;\theta) \right) ds \\ &= \int_0^t 1_{X_s > \alpha(Y(A_s))} e^{-rs-\bar{A}_s} \left(D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta) \right) ds, \end{aligned}$$

where the last equality uses that $\lambda(x', Y(a')) = 0$ for $x' > \alpha(Y(a'))$. On $x' \leq \alpha(Y(a'))$ we have

$$D(x') - r\theta + \lambda(x', Y(a'))(m(x') - \theta) \leq D(x') - r\theta + \lambda(x', \theta)(m(x') - \theta) = 0,$$

where the inequality is by Lemma 4 and $Y(a') \leq \theta$. Therefore,

$$0 \geq \int_0^t 1_{X_s \leq \alpha(Y(A_s))} e^{-rs-\bar{A}_s} \left(D(X_s) - r\theta + \lambda(X_s, Y(A_s))\tilde{u}(X_s;\theta) \right) ds.$$

Inserting the above two estimates into (35) we get

$$\mathbf{E}_{xa} \left[e^{-rt-\bar{A}_t}\tilde{u}(X_t;\theta) + \int_0^t e^{-rs-\bar{A}_s} \left(D(X_s) - r\theta + \lambda(X_s, Y(A_s))\tilde{u}(X_s;\theta) \right) ds \right] \leq \tilde{u}(x;\theta),$$

which completes the proof that $\varphi(x, a) = \tilde{u}(x;\theta)$ satisfies (33).

Derivation of (33): Define

$$F(x, a) = \mathbf{E}_{xa} \left[\int_0^\infty e^{-rs-A_s} \left(D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta) \right) ds \right].$$

Similar arguments as in the proof of Proposition 3 show that F is continuous and bounded. Furthermore, by the strong Markov property of (X_t, Y_t) , we have

$$\tilde{v}(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_{xa} \left[F(x, a) - e^{-r\sigma} F(X_\sigma, A_\sigma) \right].$$

Recall that a function ψ is called r -excessive if $\psi(x, a) \geq \mathbf{E}_{xa} [e^{-rt} \psi(X_t, A_t)]$, $t \geq 0$. Define $\eta(x, a) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_{xa} [-e^{-r\sigma} F(X_\sigma, A_\sigma)]$. By [43, Sec. 3.3, Thm. 1], η is the smallest r -excessive function dominating $-F$. Using that $\tilde{v}(x, a; \theta) = \eta(x, a) + F(x, a)$, we obtain that \tilde{v} is the smallest non-negative function satisfying, for $t \geq 0$,

$$\tilde{v}(x, a; \theta) \geq \mathbf{E}_{xa} \left[e^{-rt} \tilde{v}(X_t, A_t; \theta) + \int_0^t e^{-rs-A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right].$$

Inserting $\tilde{v}(x, a; \theta) = e^{-a} \varphi(x, a)$ above and noticing that $A_s - a = \int_0^s \lambda(X_u, Y(A_u)) du$ yields (33). \square

Having established properties of the continuation and stopping sets for \tilde{v} and a given θ , we combine them into a complete characterisation of the stopping set.

COROLLARY 2. *Stopping region for \tilde{v} (and v) is*

$$\mathcal{S}_{\tilde{v}}^\theta = \{(x, a) \in \mathcal{I} \times [0, \infty) : x \leq \alpha(\theta) \text{ and } a \geq A(\theta)\}.$$

Proof Denote by $\tilde{\mathcal{S}}^\theta$ the right-hand side of the equality in the statement of the corollary. From Lemma 9, we have $\tilde{\mathcal{S}}^\theta \subset \mathcal{S}_{\tilde{v}}^\theta$. The proof is completed when we show that $(\tilde{\mathcal{S}}^\theta)^c \subset C_{\tilde{v}}^\theta$. Lemma 7 implies that

$$\{(x, a) \in \mathcal{I} \times [0, \infty) : x > \alpha(\theta)\} \subset C_{\tilde{v}}^\theta. \quad (36)$$

Lemma 8 shows that

$$\{(x, a) \in \mathcal{I} \times [0, \infty) : x < \alpha(Y(a)) \text{ and } a < A(\theta)\} \subset C_{\tilde{v}}^\theta. \quad (37)$$

Take $a < A(\theta)$. From the first inclusion, $(\alpha(\theta), x_U) \times \{a\} \subset C_{\tilde{v}}^\theta$, where we recall that $\mathcal{I} = (x_L, x_U)$. From the second inclusion, $(x_L, \alpha(Y(a))) \times \{a\} \subset C_{\tilde{v}}^\theta$, but $a < A(\theta)$ means that $Y(a) > \theta$, so $\alpha(Y(a)) > \alpha(\theta)$. Hence $\mathcal{I} \times \{a\} \subset C_{\tilde{v}}^\theta$. This, together with inclusions (36) and (37), implies

$$C_{\tilde{v}}^\theta \supseteq \{(x, a) \in \mathcal{I} \times [0, \infty) : x > \alpha(\theta) \text{ or } a < A(\theta)\} = (\tilde{\mathcal{S}}^\theta)^c$$

and the proof is completed. \square

4.6. Nash equilibrium The description of the stopping set in Corollary 2 is natural in the framework of optimal stopping of two-dimensional dynamics. The resulting optimal stopping time, however, is not of the form (14). It turns out that due to the specific form of the stopping region and of the dynamics of (X_t, A_t) this optimal stopping time can be equivalently described as the first time that A_t exceeds $A(\theta)$, hence, in the form of (14) when one recalls that $A_t = A(Y_t)$. Leaving technical complications aside, this can be seen as follows. Denoting by σ the first entry time of (X_t, A_t) to \mathcal{S}_v^θ and by τ the first time that $A_t > A(\theta)$, it is clear that $\tau \geq \sigma$. Obviously, if $A_\sigma > A(\theta)$ then $\tau = \sigma$. Assume $A_\sigma = A(\theta)$. When $X_\sigma < \alpha(\theta)$, the process A_t is strictly increasing at σ as $\lambda(X_\sigma, \theta) > 0$, so again $\tau = \sigma$. A more delicate argument is needed when $X_\sigma = \alpha(\theta)$, but then the regularity of the point $\alpha(\theta)$ for the process (X_t) implies that (X_t) enters the open interval $(x_L, \alpha(\theta))$ immediately, so a similar argument as before can be used.

LEMMA 10. For $(x, a) \in \mathcal{I} \times [0, A(\theta))$, we have

$$\inf\{t \geq 0 : A_t > A(\theta)\} = \inf\{t \geq 0 : (X_t, A_t) \in \mathcal{S}_v^\theta\}, \quad \mathbf{P}_{xa} - a.s.$$

Proof Define $\tau = \inf\{t \geq 0 : A_t > A(\theta)\}$ and

$$\sigma = \inf\{t \geq 0 : (X_t, A_t) \in \mathcal{S}_v^\theta\} = \inf\{t \geq 0 : A_t \geq A(\theta) \text{ and } X_t \leq \alpha(\theta)\}.$$

Recall that \mathcal{S}_v^θ is closed, so σ is a stopping time. Fix (x, a) as in the statement of the lemma. We will argue omega by omega. Fix $\omega \in \Omega$ and take any $t \geq 0$ such that $A_t(\omega) > A(\theta)$. By the assumption that $a < A(\theta)$ we have $t > 0$. Due to the dynamics of (A_t) there is $s \leq t$ such that $X_s(\omega) \leq \alpha(\theta)$ and $A_s(\omega) > A(\theta)$. This implies that $s \in \{u \geq 0 : A_u(\omega) \geq A(\theta) \text{ and } X_u(\omega) \leq \alpha(\theta)\}$ and shows that $\tau(\omega) \geq \sigma(\omega)$. From the arbitrariness of ω we conclude that $\tau \geq \sigma$.

Since $\tau \geq \sigma$, \mathbf{P}_{xa} -a.s., by the strong Markov property of (X_t, A_t) and the continuity of the trajectories we have $\mathbf{P}_{xa}(\tau > \sigma) = \mathbf{E}_{xa}[\mathbf{P}_{X_\sigma A_\sigma}(\tau > 0)]$ and $X_\sigma \leq \alpha(\theta)$, $A_\sigma = A(\theta)$. In order to show that $\mathbf{P}_{xa}(\tau > \sigma) = 0$ it suffices to demonstrate that

$$\mathbf{P}_{xa}(\tau > 0) = 0 \quad \text{for } (x, a) \in \mathcal{O}_\theta := \{(x, a) \in \mathcal{I} \times [0, \infty) : x \leq \alpha(\theta), a = A(\theta)\}. \quad (38)$$

Take $(x, a) \in \mathcal{O}_\theta$. If $x < \alpha(\theta)$ then $\eta = \inf\{t \geq 0 : X_t > \alpha(Y(A_t))\} > 0$ \mathbf{P}_{xa} -a.s. and (A_t) is strictly increasing on $[0, \eta)$. Hence, $\mathbf{P}_{xa}(\tau > 0) = 0$. Consider now $x = \alpha(\theta)$ and $a = A(\theta)$. By the non-degeneracy of the diffusion around x , we have that $\eta^\circ = \inf\{t \geq 0 : X_t < \alpha(\theta)\} = 0$ \mathbf{P}_{xa} -a.s. Let $B = \{\tau > 0 \text{ and } \eta^\circ = 0\}$. For $\omega \in B$, $A_t(\omega) = A_0(\omega) = A(\theta)$ for $t \in [0, \tau(\omega))$. Since $\eta^\circ(\omega) = 0$, there

is $t(\omega) < \tau(\omega)$ such that $X_{t(\omega)} < \alpha(\theta)$. By the continuity of (X_t) , the Lebesgue measure of the set $\{s \in [0, t(\omega)] : X_s(\omega) < \alpha(\theta)\}$ is greater than zero which contradicts that $\lambda(X_s(\omega), A_s(\omega)) = 0$ for $s \in [0, \tau(\omega))$ (recall the dynamics (15) of (A_t)). This contradiction shows that the set B is empty. Since $\mathbf{P}_{xa}(\eta^\circ = 0) = 1$, we conclude that $\mathbf{P}_{xa}(\tau > 0) = 0$. \square

We are now in a position to state the main result of this section.

THEOREM 1. *Let τ_1, τ_2 be given by (14) with (A_t) stated in (16). Then (τ_1, τ_2) is a Nash equilibrium in the sense of Definition 1.*

Proof We apply Corollary 1. Take $\hat{\tau}_1(\cdot, \theta) = \hat{\tau}_2(\cdot, \theta) = \inf\{t \geq 0 : Y(A_t) < \theta\}$. We now prove (4). The monotonicity of (A_t) implies that for any $t > 0$

$$\begin{aligned} & \{(\omega_0, \theta) \in \Omega \times [\theta_L, \theta_U] : \hat{\tau}_1(\omega_0, \theta) < t\} \\ &= \bigcup_{q \in [0, t) \cap \mathbb{Q}} \{(\omega_0, \theta) \in \Omega \times [\theta_L, \theta_U] : Y(A_q(\omega_0)) - \theta < 0\} \in \mathcal{F}_t \otimes \mathcal{B}([\theta_L, \theta_U]). \end{aligned}$$

We also note that for any $t \geq 0$

$$\begin{aligned} & \{(\omega_0, \theta) \in \Omega \times [\theta_L, \theta_U] : \hat{\tau}_1(\omega_0, \theta) \leq t\} \\ &= \bigcap_{n \in \mathbb{N}} \{(\omega_0, \theta) \in \Omega \times [\theta_L, \theta_U] : \hat{\tau}_1(\omega_0, \theta) < t + 1/n\} \in \mathcal{F}_{t+} \otimes \mathcal{B}([\theta_L, \theta_U]), \end{aligned}$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathcal{F}_{t+} = \bigcap_{s > t} \mathcal{F}_s$. This yields (4) upon recalling that the filtration (\mathcal{F}_t) is assumed to be right-continuous, i.e., $\mathcal{F}_{t+} = \mathcal{F}_t$.

Due to the symmetry of the problem, it is sufficient to show that for every $\theta \in [\theta_L, \theta_U]$ and $x \in \mathcal{I}$, the stopping time $\sigma^* = \hat{\tau}_1(\cdot, \theta)$ solves the optimal stopping problem $\sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_2; \theta)$, where $\tau_2 = \hat{\tau}_2(\cdot, \theta_2)$ is an \mathcal{F}_t^2 -stopping time. Recall that by Lemma 2 and (29) we have

$$v(x, 0; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} J(x, \sigma, \tau_2; \theta).$$

We show in Corollary 2 that the optimal stopping time for v is given by

$$\begin{aligned}\sigma^* &= \inf\{t \geq 0 : X_t \leq \alpha(\theta) \text{ and } A_t \geq A(\theta)\} \\ &= \inf\{t \geq 0 : A_t > A(\theta)\} \\ &= \inf\{t \geq 0 : Y(A_t) < \theta\},\end{aligned}$$

where the second equality follows from Lemma 10 and the last equality is because $Y = A^{-1}$. \square

4.7. Perfect Bayesian Equilibrium The equilibrium established above is not only a Nash equilibrium, but it also furnishes a *perfect Bayesian equilibrium* via (19). A perfect Bayesian equilibrium requires two conditions [34]: sequential rationality and the correct dynamics (Bayesian updating) of the beliefs of the players. The correct belief dynamics is automatically taken care of through the dynamics of Y_t that unambiguously determines the posterior probability distribution of θ_1 and θ_2 . The sequential rationality stipulates that each player's strategy is a best response at any time t with the knowledge of his own type at any value of X_t and any belief represented by Y_t . In particular, a perfect Bayesian equilibrium has to take into account the best response even in case of deviations. In the context of the strategy profile we defined, a deviation happens only if a player of type θ fails to exit even though $Y_t < \theta$. If deviation takes place, the dynamics of (Y_t) is unaltered because his opponent can never find out if deviation occurred or not. Indeed, the only case in which the opponent would find out about the deviation is when $Y_t = \theta_L$, i.e., all types should have exited by t . However, due to the boundedness of λ in (15), the process A_t cannot reach the value ∞ in finite time. This implies that $Y_t > \theta_L$ for all $t \geq 0$.

To formally define a perfect Bayesian equilibrium in our setting with asymmetric information, we adapt the notions of extended strategy and time-consistent extended strategy from Riedel and Steg [40]. For the sake of simplicity, we present the framework for equilibria in pure strategies which can be written as hitting times of a measurable set Υ by the process (X_t, A_t, θ_i) , where the last component is the type of the player; the strategy (19) is of this form. The stopping time employed by the player i of type θ_i , $i = 1, 2$, for the game starting at time $t \geq 0$ is given by

$$\tau(t; \theta_i) = \inf\{s \geq t : (X_s, A_s, \theta_i) \in \Upsilon\}.$$

Notice that such a family of strategies for player i and a fixed value of θ_i satisfies the conditions of time-consistent extended strategy in [40, Def. 2.13], i.e., it is an extension of that concept to games with private information.

To this end, for any $t \geq 0$, we define the posterior distribution $F_{|Y(A_t)}$ of θ_i , $i = 1, 2$, given the value of the belief process A_t , as $F_{|y}(z) = F(z \wedge y)/F(y)$. Therefore, the random variables denoting the remaining types of player i conditional that the game has not ended by time t , are constructed as

$$\hat{\theta}_i^t = F_{|Y(A_t)}^{-1}(F(\theta_i)) \sim F_{|Y(A_t)}, \quad i = 1, 2. \quad (39)$$

Define a functional \hat{J}_t by

$$\hat{J}_t(x, \tau_1, \tau_2; \theta) = \tilde{\mathbb{E}}_x \left[\int_t^{\tau_1 \wedge \tau_2} e^{-r(s-t)} D(X_s) ds + 1_{\tau_1 \leq \tau_2} e^{-r(\tau_1-t)} \theta + 1_{\tau_1 > \tau_2} e^{-r(\tau_2-t)} m(X_{\tau_2}) \middle| \tilde{\mathcal{F}}_t \right]. \quad (40)$$

DEFINITION 2. The symmetric strategy profile $\tau(t; \theta)$ given by Υ and the posterior probability distribution (39) constitute a perfect Bayesian equilibrium if

$$\hat{J}_t(x, \sigma, \tau(t; \hat{\theta}_i^t); \theta) \leq \hat{J}_t(x, \tau(t; \theta), \tau(t; \hat{\theta}_i^t); \theta), \quad \mathbf{P}_x\text{-a.s.},$$

for any $i = 1, 2$, $x \in \mathcal{I}$, $\theta \in [\theta_L, \theta_U]$, $t \geq 0$, and $\sigma \in \mathcal{T}(\tilde{\mathcal{F}}_t)$, $\sigma \geq t$.

A formal treatment of the conditional random distribution $F_{|A(Y_t)}$ is beyond the scope of this paper but the functional \hat{J}_t evaluated at $\tau(t, \hat{\theta}_i^t)$ can be given the formal meaning:

$$\hat{J}_t(x, \sigma, \tau(t; \hat{\theta}_i^t); \theta) = \mathbb{E}_x \left[\int_{\theta_L}^{Y(A_t)} \left(\int_t^{\sigma \wedge \tau(t; \gamma)} e^{-r(s-t)} D(X_s) ds + 1_{\sigma \leq \tau(t; \gamma)} e^{-r(\sigma-t)} \theta + 1_{\sigma > \tau(t; \gamma)} e^{-r(\tau(t; \gamma)-t)} m(X_{\tau(t; \gamma)}) \right) \frac{dF(\gamma)}{F(Y(A_t))} \middle| \mathcal{F}_t \right].$$

For technical reasons and mathematical convenience, we define

$$J_t(x, \tau_1, \tau_2; \theta) = \tilde{\mathbb{E}}_x \left[1_{\tau_2 > t} \left(\int_t^{\tau_1 \wedge \tau_2} e^{-r(s-t)} D(X_s) ds + 1_{\tau_1 \leq \tau_2} e^{-r(\tau_1-t)} \theta + 1_{\tau_1 > \tau_2} e^{-r(\tau_2-t)} m(X_{\tau_2}) \right) \middle| \tilde{\mathcal{F}}_t \right] \quad (41)$$

and notice that

$$J_t(x, \sigma, \tau(0; \theta_i); \theta) = e^{-A_t} \hat{J}_t(x, \sigma, \tau(t; \hat{\theta}_i^t); \theta), \quad (42)$$

where we used $F(Y(A_t)) = e^{-A_t}$. The symmetry of the condition in Definition 2, the identical distribution of θ_i , and equality (42) imply that $\tau(t; \theta)$ furnishes a perfect Bayesian equilibrium if

$$J_t(x, \sigma, \tau(0; \theta_2); \theta) \leq J_t(x, \tau(t; \theta), \tau(0; \theta_2); \theta), \quad \mathbf{P}_x\text{-a.s.}, \quad (43)$$

for any $x \in \mathcal{I}$, $\theta \in [\theta_L, \theta_U]$, $t \geq 0$ and $\sigma \in \mathcal{T}(\tilde{\mathcal{F}}_t)$ with $\sigma \geq t$.

Consider now Υ corresponding to the Nash equilibrium (19):

$$\Upsilon^* = \{(x, a, \theta) \in \mathcal{I} \times [0, \infty) \times [\theta_L, \theta_U] : x \leq \alpha(\theta), a \geq A(\theta)\}. \quad (44)$$

In order to verify (43), we apply arguments similar as in the proof of Lemma 2 to integrate out θ_2 in J_t :

$$\begin{aligned} & J_t(x, \sigma, \tau(0; \theta_2); \theta) \\ &= \mathbf{E}_x \left[\int_t^\sigma e^{-r(s-t)-A_s} D(X_s) ds + \theta e^{-r(\tau-t)-A_\tau} + \int_{[t, \tau)} e^{-r(s-t)-A_s} m(X_s) dA_s \middle| \mathcal{F}_t \right], \end{aligned}$$

where we used that $\sigma \geq t$ and $\sigma \in \mathcal{T}(\mathcal{F}_t)$. To show that Υ^* defines a perfect Bayesian equilibrium, it is enough to prove that for any $t \geq 0$ and $\theta \in [\theta_L, \theta_U]$, the stopping time $\sigma^* = \tau(t; \theta)$ solves the optimal stopping problem

$$\text{ess sup}_{\sigma \geq t} \mathbf{E}_x \left[\int_t^\sigma e^{-r(s-t)-A_s} D(X_s) ds + \theta e^{-r(\sigma-t)-A_\sigma} + \int_{[t, \sigma)} e^{-r(s-t)-A_s} m(X_s) dA_s \middle| \mathcal{F}_t \right]. \quad (45)$$

Due to the Markov property of (X_t, A_t) and the boundedness and continuity of D and m , the classical theory of optimal stopping yields the optimal stopping time of the form $\sigma^* = \inf\{s \geq t : U(X_s, A_s; \theta) = \theta e^{-A_s}\}$, where

$$U(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_{xa} \left[\int_0^\sigma e^{-rs-A_s} D(X_s) ds + \theta e^{-r\sigma-A_\sigma} + \int_{[0, \sigma)} e^{-rs-A_s} m(X_s) dA_s \right].$$

Recalling the form of A_t in (15), notice that $U(x, a; \theta) = v(x, a; \theta)$, where v is defined in (29). Corollary 2 implies that the solution of the stopping problem (45) is indeed given by $\sigma^* = \tau(t; \theta)$. This completes the proof that the strategy profile derived in previous sections gives rise to a perfect Bayesian equilibrium.

THEOREM 2. *Symmetric strategy profile given by Υ^* is a perfect Bayesian equilibrium in the sense of Definition 2.*

We emphasise again the difference between the definition of the strategy profile (14) that forms a Nash equilibrium and the strategy profile (19) that corresponds to Υ^* and forms the perfect Bayesian equilibrium. These strategies coincide along the equilibrium path for the game started at time 0.

The former definition is fundamental for the reformulation of the best response problem where the type of the opponent is integrated out (Lemma 2). We further exploit it above where we rewrite the functional \hat{J}_t as J_t in (42) with the opponent following the equilibrium path $\tau(0; \theta_i)$. The reader can further notice the interaction between these definitions in the following remark.

REMARK 7. The equality (42) enables a mathematically equivalent formulation of Definition 2: a symmetric strategy profile $\tau(t; \theta)$ given by Υ is a perfect Bayesian equilibrium if

$$\{\tau(0; \theta_i) < t\} = \{Y(A_t) < \theta_i\}, \quad \mathbb{P}_x\text{-a.s.}$$

and

$$J_t(x, \sigma, \tau(0; \theta_i); \theta) \leq J_t(x, \tau(t; \theta), \tau(0; \theta_i); \theta), \quad \mathbb{P}_x\text{-a.s.},$$

for any $i = 1, 2$, $x \in \mathcal{I}$, $\theta \in [\theta_L, \theta_U]$, $t \geq 0$, and $\sigma \in \mathcal{T}(\tilde{\mathcal{F}}_t)$, $\sigma \geq t$.

4.8. Remaining proofs

Proof of Lemma 2 Since $(A_t)_{t \geq 0}$ is (\mathcal{F}_t) -adapted, right-continuous and increasing, the process $(Y_t)_{t \geq 0}$ retains the same adaptivity and right-continuity but is decreasing. From $A_{0-} = 0$ and $F(\theta_U) = 1$, we obtain $Y_{0-} = \theta_U$. We also have $Y_t \in (\theta_L, \theta_U]$, $t \geq 0$.

The rest of the proof follows similar arguments as in [9, Section 4]. For simplicity of notation, we omit the index i in τ_i and θ_i . Here, we treat $(Y_t)_{t \geq 0}$ and $(A_t)_{t \geq 0}$ as stochastic processes on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ in an obvious way due to the product form of the probability space. Let $\tilde{\mathcal{F}}_\infty = \bigvee_{t \geq 0} \tilde{\mathcal{F}}_t$. We first note that

$$\{t > \tau\} \subseteq \{Y_t < \theta\} \subseteq \{t \geq \tau\}.$$

From the first inclusion, we get that

$$\tilde{\mathbb{E}}[1_{\sigma > \tau} | \tilde{\mathcal{F}}_\infty] = \lim_{\varepsilon \downarrow 0} \tilde{\mathbb{E}}[1_{\sigma - \varepsilon > \tau} | \tilde{\mathcal{F}}_\infty] \leq \lim_{\varepsilon \downarrow 0} (1 - F(Y_{\sigma - \varepsilon})) = 1 - F(Y_{\sigma -}),$$

where we used that (Y_t) is monotone, so the limits exist. The second inclusion gives the opposite estimate:

$$\tilde{\mathbb{E}}[1_{\sigma > \tau} | \tilde{\mathcal{F}}_\infty] = \lim_{\varepsilon \downarrow 0} \tilde{\mathbb{E}}[1_{\sigma - \varepsilon \geq \tau} | \tilde{\mathcal{F}}_\infty] \geq \lim_{\varepsilon \downarrow 0} (1 - F(Y_{\sigma - \varepsilon})) = 1 - F(Y_{\sigma -}).$$

By (18), we conclude that

$$\tilde{\mathbb{E}}[1_{\sigma > \tau} | \tilde{\mathcal{F}}_\infty] = 1 - e^{-A_{\sigma -}} \quad \text{and} \quad \tilde{\mathbb{E}}[1_{\sigma \leq \tau} | \tilde{\mathcal{F}}_\infty] = e^{-A_{\sigma -}}. \quad (46)$$

Let $\varphi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable bounded function. We shall show that

$$\tilde{\mathbb{E}}_x [1_{\sigma > \tau} \varphi(\tau, X_\tau) | \tilde{\mathcal{F}}_\infty] = \int_{[0, \sigma)} e^{-A_s} \varphi(s, X_s) dA_s. \quad (47)$$

Define $\hat{\tau}(u) = \inf\{t \geq 0 : Y_t < u\}$ so that $\tau = \hat{\tau}(\theta)$; notice that this is the decomposition of τ from Proposition 1 in which we suppress in the notation the dependence on ω . To shorten notation, define an $(\tilde{\mathcal{F}}_t)$ -adapted process

$$Z_t = \varphi(t, X_t) 1_{\sigma > t}, \quad t \geq 0.$$

Using the independence of θ from $\tilde{\mathcal{F}}_\infty$, we have

$$\tilde{\mathbb{E}}_x [1_{\sigma > \tau} \varphi(\tau, X_\tau) | \tilde{\mathcal{F}}_\infty] = \tilde{\mathbb{E}}_x [Z_\tau | \tilde{\mathcal{F}}_\infty] = \int_{\theta_L}^{\theta_U} Z_{\hat{\tau}(v)} dF(v) = \int_0^1 Z_{\hat{\tau}(F^{-1}(u))} du,$$

where $F^{-1}(\cdot)$ is the inverse of F (which exists by Assumption 5) and in the last equality we used [39, Ch. 0, Prop. 4.9]. We rewrite $\hat{\tau}(F^{-1}(u))$ as follows:

$$\begin{aligned} \hat{\tau}(F^{-1}(u)) &= \inf\{t \geq 0 : Y_t < F^{-1}(u)\} = \inf\{t \geq 0 : F(Y_t) < u\} = \inf\{t \geq 0 : e^{-A_t} < u\} \\ &= \inf\{t \geq 0 : 1 - e^{-A_t} > 1 - u\} =: \tilde{\tau}(1 - u). \end{aligned}$$

This allows us to write

$$\int_0^1 Z_{\hat{\tau}(F^{-1}(u))} du = \int_0^1 Z_{\tilde{\tau}(1-u)} du = \int_0^1 Z_{\tilde{\tau}(u)} du = \int_0^\infty Z_s d(1 - e^{-A_s}) = \int_0^\infty e^{-A_s} Z_s dA_s,$$

where we apply [39, Ch. 0, Prop. 4.9] in the third equality. Recalling the definition of $(Z_t)_{t \geq 0}$ completes the derivation of (47).

Using (46)-(47), the functional J takes an equivalent form

$$J(x, \sigma, \tau; \gamma) = \tilde{\mathbb{E}}_x \left[\int_0^\sigma e^{-rs - A_s} D(X_s) ds + \gamma e^{-r\sigma - A_\sigma} + \int_{[0, \sigma)} e^{-rs - A_s} m(X_s) dA_s \right].$$

Since (A_t) is increasing, it has only a countable number of jumps, so

$$\int_0^\sigma e^{-rs - A_s} D(X_s) ds = \int_0^\sigma e^{-rs - A_s} D(X_s) ds$$

and (20) is proved. \square

Proof of Lemma 3 Define a semimartingale $Z_t = X_t - \varphi_t$. The occupation times formula [38, Cor. 1, p. 216] shows that for any $\varepsilon > 0$ and $t > 0$, \mathbf{P}_x -a.s.,

$$\int_{-\varepsilon}^{\varepsilon} L_t^u du = \int_0^t 1_{Z_{s-} \in [-\varepsilon, \varepsilon]} d[Z, Z]_s^c = \int_0^t 1_{\{Z_{s-} \in [-\varepsilon, \varepsilon]\}} b^2(X_s) ds, \quad (48)$$

where L_t^u is the local time of $(Z_t)_{t \geq 0}$ at the level u and $[Z, Z]^c$ is the path-by-path continuous part of the quadratic variation $[Z, Z]$ and equals to the quadratic variation $[Z^c, Z^c]$ of the continuous local martingale part of Z (see [38, p. 70]). Clearly, the continuous local martingale part Z^c of Z equals to the continuous local martingale part of X^c and, using [38, Thm. 29, p. 75], we have $[Z^c, Z^c]_t = [X^c, X^c]_t = \int_0^t b^2(X_s) ds$. The equality (48) holds outside of \mathbf{P}_x -negligible set common for every $t \geq 0$; indeed, it is sufficient to apply the above formula for a sequence $t_n \rightarrow \infty$.

Taking the limit in (48) as $\varepsilon \downarrow 0$, the dominated convergence theorem implies that

$$0 = \lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\varepsilon} L_t^u du = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{(X_{s-} - \varphi_{s-}) \in [-\varepsilon, \varepsilon]\}} b^2(X_s) ds = \int_0^t 1_{X_s = \varphi_s} b^2(X_s) ds,$$

where in the last equality we used the continuity of (X_t) . To conclude, we recall that $b(\cdot) > 0$. \square

Proof of Lemma 4 Take any $x \leq \alpha(\theta_U)$. The formula (26) for the derivative of the mapping from the statement of the lemma follows by straightforward differentiation. Denote it by $g(y; x)$. By assumptions, we have $m(x) \geq m_{\min} > \theta_U$ and $D(x) \leq r\theta_U$ since $x \leq \alpha(\theta_U)$. Hence $g(y; x) > 0$. We also note that

$$\frac{rm(x) - D(x)}{(m(x) - y)^2} \leq \frac{rm(x)}{(m(x) - \theta_U)^2} \leq \frac{r m_{\min}}{(m_{\min} - \theta_U)^2},$$

since the function $z \mapsto z/(z - \theta_U)^2$ is decreasing for $z > \theta_U$. \square

Proof of Lemma 5 Recalling that $a \mapsto Y(a)$ is decreasing, Lemma 4 shows that $a \mapsto \lambda(x, Y(a))$ is decreasing on $\{a \geq 0 : x \leq \alpha(Y(a))\}$ which is either a closed interval $[0, a^*(x)]$ or an empty set (in which case we set $a^*(x) = 0$), since $a \mapsto \alpha(Y(a))$ is continuous and decreasing. As $\lambda \geq 0$ and $\lambda(x, \cdot) \equiv 0$ on $(a^*(x), \infty)$, we conclude that $a \mapsto \lambda(x, a)$ is decreasing.

Fix now $a \geq 0$ and notice that

$$x \mapsto \frac{rY(a) - D(x)}{m(x) - Y(a)}$$

is decreasing on $x \leq \alpha(Y(a))$. Indeed, $rY(a) - D(x) > 0$ for such x and decreasing and the numerator is increasing in x . We also have $\lambda(x, a) = 0$ for $x > \alpha(Y(a))$ and $\lambda \geq 0$, so a potential jump at $x = \alpha(Y(a))$ is downward. This completes the proof of monotonicity. \square

Proof of Proposition 2

Construction of (A_t) : Function α is strictly increasing and continuous (Lemma 1), so its inverse α^{-1} is well defined, continuous and strictly increasing. Hence, $x \leq \alpha(y)$ can be equivalently written as $y \geq \alpha^{-1}(x)$. Define

$$\lambda^\varepsilon(x, y) = l(x, y)1_{y \geq \alpha^{-1}(x)} + l(x, \alpha^{-1}(x))1_{y < \alpha^{-1}(x)} \frac{1}{\varepsilon} (y - \alpha^{-1}(x) + \varepsilon)^+, \quad (49)$$

where $l(x, y)$ is defined in Lemma 4. Using this lemma and the above definition, the mapping $y \mapsto \lambda^\varepsilon(x, y)$ is Lipschitz with the constant independent of x . Due to the continuity of α and its inverse, λ^ε is continuous. Hence, for any $\omega \in \Omega$, using the continuity of trajectories of (X_t) and the Lipschitz continuity of $Y(\cdot)$, there is a unique solution of the ODE

$$dA_t^\varepsilon = \lambda^\varepsilon(X_t, Y(A_t^\varepsilon))dt, \quad A_0^\varepsilon = a \geq 0, \quad (50)$$

and it depends continuously on a . Since λ^ε is increasing in ε and non-negative, by the comparison principle for ODEs, the solution A_t^ε is increasing in ε and non-negative. Hence the limit $A_t^0 := \lim_{\varepsilon \downarrow 0} A_t^\varepsilon$ exists, is increasing and right-continuous. We will show that it satisfies (27) P_x -a.s. It suffices to show that

$$\lim_{\varepsilon \downarrow 0} \int_0^t \lambda^\varepsilon(X_s, Y(A_s^\varepsilon))ds = \int_0^t \lambda(X_s, Y(A_s^0))ds, \quad P_x - a.s. \quad (51)$$

From (49), we have the lower bound $\lambda^\varepsilon(x, y) \geq l(x, y)1_{y > \alpha^{-1}(x)}$. As l is non-negative, Fatou's lemma implies

$$\liminf_{\varepsilon \downarrow 0} \int_0^t \lambda^\varepsilon(X_s, Y(A_s^\varepsilon))ds \geq \int_0^t l(X_s, Y(A_s^0))1_{Y(A_s^0) > \alpha^{-1}(X_s)}ds$$

using that l is continuous and $\varepsilon \mapsto Y(A_s^\varepsilon)$ is continuous and increasing for each $s \geq 0$. For the upper bound, we write

$$\lambda^\varepsilon(x, y) \leq l(x, y)1_{y > \alpha^{-1}(x)} + l(x, \alpha^{-1}(x))1_{\alpha^{-1}(x) - \varepsilon < y \leq \alpha^{-1}(x)}.$$

This and the fact that $\limsup_{\varepsilon \downarrow 0} \{\alpha^{-1}(X_s) - \varepsilon < Y(A_s^\varepsilon) \leq \alpha^{-1}(X_s)\} \subset \{Y(A_s^0) = \alpha^{-1}(X_s)\}$ yield

$$\limsup_{\varepsilon \downarrow 0} \lambda^\varepsilon(X_s, Y(A_s^\varepsilon)) \leq l(X_s, Y(A_s^0))1_{Y(A_s^0) > \alpha^{-1}(X_s)} + l(X_s, \alpha^{-1}(X_s))1_{Y(A_s^0) = \alpha^{-1}(X_s)}.$$

Recall that l is bounded, so we can apply reverse Fatou's lemma

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} \int_0^t \lambda^\varepsilon(X_s, Y(A_s^\varepsilon))ds \\ & \leq \int_0^t l(X_s, Y(A_s^0))1_{Y(A_s^0) > \alpha^{-1}(X_s)} + l(X_s, \alpha^{-1}(X_s))1_{Y(A_s^0) = \alpha^{-1}(X_s)}ds. \end{aligned}$$

The process $s \mapsto A_s^0$ is increasing, hence of finite variation, and right-continuous. Functions Y and α are continuous and $\{Y(A_s^0) = \alpha^{-1}(X_s)\} = \{\alpha(Y(A_s^0)) = X_s\}$. Since function l is bounded, Lemma 3 implies that the integral $\int_0^\infty l(X_s, \alpha^{-1}(X_s)) 1_{Y(A_s^0) = \alpha^{-1}(X_s)} ds = 0$ \mathbb{P}_x -a.s. We can therefore conclude that the following limit exists \mathbb{P}_x -a.s. (with the measure zero set independent of t)

$$\lim_{\varepsilon \downarrow 0} \int_0^t \lambda^\varepsilon(X_s, Y(A_s^\varepsilon)) ds = \int_0^t l(X_s, Y(A_s^0)) 1_{Y(A_s^0) \geq \alpha^{-1}(X_s)} ds = \int_0^t \lambda(X_s, Y(A_s^0)) ds.$$

Hence $(A_t^0)_{t \geq 0}$ satisfies (27) and we will use it as a definition of the process $(A_t)_{t \geq 0}$ from the statement of the proposition. From (27) we deduce that (A_t) is continuous \mathbb{P}_x -a.s.

Markov property: As the process (A_t) is constructed ω by ω , it is sufficient to show that for any $u \geq 0$ and $s \geq 0$ we have $A_{u+s}^0 = \bar{A}_s^0$, where $\bar{A}_t^0 = \lim_{\varepsilon \downarrow 0} \bar{A}_t^\varepsilon$ with (\bar{A}_t^ε) being the unique solution of

$$d\bar{A}_t^\varepsilon = \lambda^\varepsilon(X_{u+t}, Y(\bar{A}_t^\varepsilon)) dt, \quad \bar{A}_0^\varepsilon = A_u^0.$$

This is not immediate as it is possible that $A_u^\varepsilon > A_u^0$ for all $\varepsilon > 0$ which implies $\bar{A}_t^\varepsilon < A_{u+t}^\varepsilon$, at least for $t \leq T(\omega)$ for some $T(\omega) > 0$. To overcome this problem, define $(\bar{A}_t^{\varepsilon, \delta})$ as a solution to (50) with the initial condition $\bar{A}_0^{\varepsilon, \delta} = A_u^0 + \delta$, for $\delta > 0$. By the continuous dependence of the solution to (50) on the initial condition and the comparison principle for ODEs, we have $\bar{A}_t^\varepsilon = \inf_{\delta > 0} \bar{A}_t^{\varepsilon, \delta}$. The mapping $\varepsilon \mapsto \bar{A}_t^\varepsilon$ is increasing, hence, $\bar{A}_t^0 = \inf_{\varepsilon, \delta > 0} \bar{A}_t^{\varepsilon, \delta}$.

Fix $\omega \in \Omega$ (we omit it in the notation for the clarity of exposition). For any $\delta > 0$ there is $\varepsilon > 0$ such that $A_u^\varepsilon \leq A_u^0 + \delta$. By the uniqueness of solutions to (50) and the comparison principle, we have $A_{u+s}^\varepsilon \leq \bar{A}_s^{\varepsilon, \delta}$, $s \geq 0$, so $A_{u+s}^0 \leq \bar{A}_s^{\varepsilon, \delta}$. This implies that $A_{u+s}^0 \leq \bar{A}_s^0$. The opposite inequality follows from $\bar{A}_0^\varepsilon \leq A_u^\varepsilon$ and analogous arguments as above.

Strong Markov property: We apply [5, Ch. I, Prop. 8.2]. Condition (S.R.) for the process $(X_t, A_t)_{t \geq 0}$ is immediate. Indeed, fix a stopping time σ . Then X_σ is \mathcal{F}_σ -measurable since (X_t) is strong Markov. Due to the boundedness of λ , the process (A_t) does not explode at a finite time. Furthermore, for any $\varepsilon > 0$, $A_\sigma^\varepsilon(\omega)$ is defined based on the trajectory $(X_t(\omega))_{t \leq \sigma(\omega)}$, so A_σ^ε is \mathcal{F}_σ -measurable. The random variable A_σ^0 is also \mathcal{F}_σ -measurable as an \mathbb{P}_x -a.s. limit of a sequence of \mathcal{F}_σ -measurable random variables; recall that the probability zero set from Lemma 3 is universal for all σ .

Condition (S.M.)' of [5, Ch. I, Prop. 8.2] is proved analogously as the Markov property but with u replaced by $\sigma(\omega)$. \square

Proof of Lemma 6 The monotonicity of the mapping $a \mapsto A_t^a$ follows from the comparison principle for ODEs applied to (50). To prove the second statement, we take $a > a'$ and write

$$A_t^a - A_t^{a'} = A_0^a - A_0^{a'} + \int_0^t [\lambda(X_s, Y(A_s^a)) - \lambda(X_s, Y(A_s^{a'}))] ds \leq A_0^a - A_0^{a'} = a - a',$$

where we used that the integrand is non-positive because $A_s^a \geq A_s^{a'}$ and the mapping $a \mapsto \lambda(x, Y(a))$ is decreasing (by Lemma 5). The difference $A_t^a - A_t^{a'}$ is bounded from below by 0 from the first part of the statement.

The monotonicity in x follows from the observation that $X_t^x \geq X_t^{x'}$ for $x > x'$ and $t \geq 0$ by the comparison principle for SDEs, so, using Lemma 5,

$$\lambda(X_t^x, y) \leq \lambda(X_t^{x'}, y) \quad \text{for all } y \in [\theta_L, \theta_U].$$

For the continuity, fix $a \geq 0$ and take $x_n \uparrow x_0$. Then $A_t^{x_n} \geq A_t^{x_0}$ and so $\alpha(Y(A_t^{x_n})) \leq \alpha(Y(A_t^{x_0}))$. Notice that $\lambda(x, a)$ is bounded from above by $r\theta_U/(m_{\min} - \theta_U) < \infty$, where $m_{\min} = \inf_{x \in \mathcal{I}} m(x)$. By Fatou's lemma

$$\begin{aligned} A_t^\infty := \limsup_{n \rightarrow \infty} A_t^{x_n} &\leq a + \int_0^t \limsup_{n \rightarrow \infty} \lambda(X_s^{x_n}, Y(A_s^{x_n})) ds \leq a + \int_0^t \limsup_{n \rightarrow \infty} \lambda(X_s^{x_n}, Y(A_s^{x_0})) ds \\ &= a + \int_0^t \lambda(X_s^{x_0}, Y(A_s^{x_0})) ds = A_t^{x_0}, \end{aligned}$$

where the second inequality is based on the monotonicity of $a \mapsto \lambda(x, Y(a))$ (Lemma 5) and the penultimate equality is because $\lambda(z_n, y) \downarrow \lambda(z, y)$ for $z_n \uparrow z$, and $X_s^{x_n} \uparrow X_s^{x_0}$ by (25) (the convergence is for ω outside of \mathbf{P} -negligible set independent from s). Combined with the opposite inequality as $A_t^{x_n} \geq A_t^{x_0}$, we obtain $A_t^x = A_t^\infty$. By the arbitrariness of t , $A_t^{x_n} \downarrow A_t^{x_0}$ for all $t \geq 0$. Dini's theorem implies that the convergence is uniform on compact sets.

We now turn our attention to the case of $x_n \downarrow x_0$. Then $A_t^{x_n} \leq A_t^{x_0}$ and so $\alpha(Y(A_t^{x_n})) \geq \alpha(Y(A_t^{x_0}))$. By Fatou's lemma

$$\begin{aligned} A_t^\infty := \liminf_{n \rightarrow \infty} A_t^{x_n} &\geq a + \int_0^t \liminf_{n \rightarrow \infty} \lambda(X_s^{x_n}, Y(A_s^{x_n})) ds \geq a + \int_0^t \liminf_{n \rightarrow \infty} \lambda(X_s^{x_n}, Y(A_s^{x_0})) ds \\ &\geq a + \int_0^t \frac{rY(A_s^{x_0}) - D(X_s^{x_0})}{m(X_s^{x_0}) - Y(A_s^{x_0})} \mathbf{1}_{\{X_s^{x_0} < \alpha(Y(A_s^{x_0}))\}} ds, \end{aligned}$$

where in the last inequality, we use $X_s^{x_n} \downarrow X_s^{x_0}$ by (25) to argue the convergence of the fraction due to its continuity, and the convergence of the indicator functions because of $\liminf_{n \rightarrow \infty} \{X_s^{x_n} \leq \alpha(Y(A_s^{x_0}))\} \supset \{X_s^{x_0} < \alpha(Y(A_s^{x_0}))\}$. Denote by \bar{A}_t the right-hand side of the above estimate. When we recall that $A_t^\infty \leq A_t^{x_0}$, we obtain $A_t^{x_0} \geq \bar{A}_t$ and both processes are continuous. For any $t \geq 0$, we compute

$$\begin{aligned} A_t^{x_0} - \bar{A}_t &= \int_0^t \frac{rY(A_s^{x_0}) - D(X_s^{x_0})}{m(X_s^{x_0}) - Y(A_s^{x_0})} (1_{\{X_s^{x_0} \leq \alpha(Y(A_s^{x_0}))\}} - 1_{\{X_s^{x_0} < \alpha(Y(A_s^{x_0}))\}}) ds \\ &\leq \lambda_{\max} \int_0^t 1_{\{X_s^{x_0} = \alpha(Y(A_s^{x_0}))\}} ds, \end{aligned}$$

where λ_{\max} is the upper bound for λ . By Lemma 3, the right-hand side is \mathbf{P} -a.s. zero and the null set can be taken independent of t . Hence $A_t^{x_0}$ and \bar{A}_t are \mathbf{P} -indistinguishable. Since $A_t^{x_0} \geq A_t^{x_n}$ and $A_t^\infty = \liminf_{n \rightarrow \infty} A_t^{x_n} \geq \bar{A}_t$, we have that $A_t^{x_n}$ converges to $A_t^{x_0}$ for all t outside of \mathbf{P} -negligible set. By Dini's theorem, the convergence is uniform for t on compact sets. Due to the monotonicity of $x \mapsto A_t^x$ the convergence over monotone sequences x_n extends to general sequences. \square

Proof of Proposition 3 There are continuous bounded functions $\lambda^\varepsilon, \lambda_\varepsilon$ such that $\lambda^\varepsilon \geq \lambda \geq \lambda_\varepsilon$, and $\lambda^\varepsilon \downarrow \lambda$ and $\lambda_\varepsilon \uparrow \lambda$ pointwise as $\varepsilon \downarrow 0$; see (49) for an explicit definition of λ^ε . This is because the discontinuity of λ is on a continuous curve $\{(x, a) : x = \alpha(Y(a))\}$. Consider

$$\tilde{v}^\varepsilon(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} \mathbf{E}_{xa} \left[\int_0^\sigma e^{-rs - A_s} (D(X_s) - r\theta + \lambda^\varepsilon(X_s, Y(A_s))(m(X_s) - \theta)) ds \right],$$

and analogously \tilde{v}_ε ; notice that the dynamics of the process (A_t) does not depend on ε . Since $m(x) > \theta_U$ for all $x \in \mathcal{I}$, we have $\tilde{v}^\varepsilon \geq \tilde{v} \geq \tilde{v}_\varepsilon$.

Assume that \tilde{v}^ε and \tilde{v}_ε are continuous; the proof will come shortly. For any δ -optimal stopping time σ_δ for \tilde{v} , we have

$$\begin{aligned} \tilde{v}(x, a; \theta) - \delta &\leq \mathbf{E}_{xa} \left[\int_0^{\sigma_\delta} e^{-rs - A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbf{E}_{xa} \left[\int_0^{\sigma_\delta} e^{-rs - A_s} (D(X_s) - r\theta + \lambda_\varepsilon(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\ &\leq \liminf_{\varepsilon \downarrow 0} \tilde{v}_\varepsilon(x, a; \theta), \end{aligned}$$

where the equality is by the dominated convergence theorem and the last inequality is because the expectation in the second line is dominated by $\tilde{v}_\varepsilon(x, a; \theta)$. Combining this for arbitrary $\delta > 0$ with $\tilde{v} \geq \tilde{v}_\varepsilon$, we obtain that \tilde{v}_ε converges pointwise from below to \tilde{v} , hence \tilde{v} is lower semicontinuous.

The mapping $\varepsilon \mapsto \tilde{v}^\varepsilon(x, a; \theta)$ is increasing for each fixed x, a, θ hence the limit $\tilde{v}^0(x, a; \theta) := \lim_{\varepsilon \downarrow 0} \tilde{v}^\varepsilon(x, a; \theta)$ exists. Using the continuity of \tilde{v}^ε , by the general theory of optimal stopping (see, e.g., [27, 36]), the stopping time $\sigma^\varepsilon = \inf\{t \geq 0 : \tilde{v}^\varepsilon(X_t, A_t; \theta) = 0\}$ is optimal for \tilde{v}^ε . It is also increasing in ε due to the monotonicity of \tilde{v}^ε in ε . Hence

$$\begin{aligned} \tilde{v}^0(x, a; \theta) &= \lim_{\varepsilon \downarrow 0} \mathbf{E}_{xa} \left[\int_0^\infty 1_{t \leq \sigma^\varepsilon} e^{-rs - A_s} (D(X_s) - r\theta + \lambda^\varepsilon(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\ &\leq \mathbf{E}_{xa} \left[\int_0^\infty 1_{t \leq \sigma^0} e^{-rs - A_s} (D(X_s) - r\theta + \lambda(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \\ &\leq \tilde{v}(x, a; \theta), \end{aligned}$$

where in the first inequality we used the reverse Fatou's lemma (as the integrand is bounded from above) and $\sigma^0 = \lim_{\varepsilon \downarrow 0} \sigma^\varepsilon = \inf_{\varepsilon > 0} \sigma^\varepsilon$ is a stopping time as an infimum of stopping times. The inequality $\tilde{v}^0 \geq \tilde{v}$ is immediate from $\tilde{v}^\varepsilon \geq \tilde{v}$. Hence \tilde{v}^ε converges to \tilde{v} pointwise from above, which implies that \tilde{v} is upper semicontinuous.

Combining the above two semicontinuity results shows the continuity of \tilde{v} . By the general optimal stopping theory, see [27, 36], the optimal stopping time is given by the formula in the statement of the proposition.

It remains to show the continuity of \tilde{v}^ε . The proof for \tilde{v}_ε is analogous. Take (x_n, a_n) converging to (x, a) and such that (x_n) is monotone (which can be assumed without loss of generality). For any

$T > 0$, we have the following estimate

$$\begin{aligned}
& \left| \tilde{v}^\varepsilon(x_n, a_n; \theta) - \tilde{v}^\varepsilon(x, a; \theta) \right| \\
& \leq \sup_{\sigma \in \mathcal{F}(\mathcal{F}_T), \sigma \leq T} \left\{ \mathbb{E} \left[\int_0^\sigma e^{-rs} \left| e^{-A_s^{x_n, a_n}} (D(X_s^{x_n}) - \theta) - e^{-A_s^{x, a}} (D(X_s^x) - \theta) \right| ds \right] \right. \\
& \quad \left. + \mathbb{E} \left[\int_0^\sigma e^{-rs} \left| e^{-A_s^{x_n, a_n}} \lambda^\varepsilon(X_s^{x_n}, Y(A_s^{x_n, a_n})) (m(X_s^{x_n}) - \theta) \right. \right. \right. \\
& \quad \quad \left. \left. \left. - e^{-A_s^{x, a}} \lambda^\varepsilon(X_s^x, Y(A_s^{x, a})) (m(X_s^x) - \theta) \right| ds \right] \right\} + 2e^{-rT} \frac{1}{r} C \\
& \leq \mathbb{E} \left[\int_0^T e^{-rs} \left| e^{-A_s^{x_n, a_n}} (D(X_s^{x_n}) - \theta) - e^{-A_s^{x, a}} (D(X_s^x) - \theta) \right| ds \right] \\
& \quad + \mathbb{E} \left[\int_0^T e^{-rs} \left| e^{-A_s^{x_n, a_n}} \lambda^\varepsilon(X_s^{x_n}, Y(A_s^{x_n, a_n})) (m(X_s^{x_n}) - \theta) \right. \right. \\
& \quad \quad \left. \left. - e^{-A_s^{x, a}} \lambda^\varepsilon(X_s^x, Y(A_s^{x, a})) (m(X_s^x) - \theta) \right| ds \right] + 2e^{-rT} \frac{1}{r} C,
\end{aligned}$$

where

$$C := \sup_{(x, a) \in \mathcal{I} \times [0, \infty)} (D(x) - r\theta + \lambda^\varepsilon(x, Y(a))(m(x) - \theta)) < \infty.$$

By Lemma 6 (c.f. Remark 5), $A_s^{x_n, a_n}$ converges pointwise to $A_s^{x, a}$ for all $s \in [0, T]$ and ω outside of a \mathbf{P} -negligible set. We also have convergence of $(X_s^{x_n})_{s \in [0, T]}$ to $(X_s^x)_{s \in [0, T]}$ outside of a \mathbf{P} -negligible set. We can therefore conclude, by the dominated convergence theorem, that

$$\lim_{n \rightarrow \infty} \left| \tilde{v}^\varepsilon(x_n, a_n; \theta) - \tilde{v}^\varepsilon(x, a; \theta) \right| \leq 2e^{-rT} \frac{1}{r} C.$$

Since T is arbitrary, this shows the continuity of \tilde{v}^ε . \square

5. Two special cases In this section, we explore how our solution behaves when we remove either the stochastic state variable or the private information from our model. These are two special cases that have been studied in the previous literature on exit games. By comparing our results with known results from the literature, we establish the robustness and generality of our solution.

First, we remove the dynamics of the stochastic state variable from the solution by setting (formally) $\mu(\cdot) = b(\cdot) = 0$ with an initial value of the state variable set to x . We assume that $D(x)/r < \theta_L$ so that all types of players have an incentive to exit. In this case, $\alpha(\theta) = \infty$ for all $\theta \in [\theta_L, \theta_U]$, so the rate of exit reduces to

$$\lambda(x, y) = \frac{ry - D(x)}{m(x) - y}.$$

One striking difference from the case of a dynamic state variable is that $\lambda(x, y)$ is always strictly positive, i.e., there is a type θ that exists at any given time t . Furthermore, from (14), the exit strategy of a player of type θ is given by

$$\hat{\tau}(\theta) = \inf\{t \geq 0 : Y_t < \theta\}.$$

Because Y_t possesses a deterministic dynamics, $\hat{\tau}(\theta)$ is also deterministic. Thus, each type of a player chooses a deterministic time to exit at the outset of the game. This feature is consistent with [17, 30], who examined deterministic exit games with private types. In contrast, the random dynamics of the underlying process (X_t) introduces the randomness into player's behaviour. Not only are there times that the player of no type exits but also the time of exit of player of type θ is not deterministic.

Next, we keep the stochastic state variable but remove the uncertainty in the exit value. More precisely, we take the limits $\theta_L \rightarrow \theta$ and $\theta_U \rightarrow \theta$ and study the behaviour of the equilibrium strategy profile. This requires care as in the limit the distribution of a player's type degenerates to a deterministic quantity θ . To get around this difficulty, we re-express (14) in terms of the process A_t as follows:

$$\tau_i = \inf\{t \geq 0 : e^{-A_t} < F(\theta_i)\}.$$

Recall that $F(\theta_i)$ is uniformly distributed within the interval $[0, 1]$, and hence, we can reformulate the condition $e^{-A_t} < F(\theta_i)$ as $e^{-A_t} < \hat{\epsilon}_i$, where $\hat{\epsilon}_i$ is a random variable uniformly distributed on $[0, 1]$. We now take the limits $\theta_L \rightarrow \theta$ and $\theta_U \rightarrow \theta$:

$$\tau_i = \inf\{t \geq 0 : e^{-A_t} < \hat{\epsilon}_i\}. \quad (52)$$

In the limit the dynamics of (A_t) takes the form

$$dA_t = \frac{r\theta - D(X_t)}{m(X_t) - \theta} 1_{X_t \leq \alpha(\theta)} =: \tilde{\lambda}(X_t).$$

We can identify (52) with a mixed strategy equilibrium for stochastic exit games with known exit values. Indeed, in accordance with the standard definition of a mixed strategy, each player can be viewed as having a randomisation device uniformly distributed on $[0, 1]$ which generates a random value in the beginning of the game; we can identify this randomisation device with $\hat{\epsilon}_i$. The player then exits at the first instance that e^{-A_t} falls below $\hat{\epsilon}_i$. Because the exit rate $\tilde{\lambda}(X_t)$ is positive only when $X_t \leq \alpha(\theta)$, the strategy (52) coincides with the mixed strategy found by [19, 44]. We conclude

that our equilibrium converges to the established results as θ_L and θ_U approach θ . Notice also the similarities between the dynamics of A_t above and in (17), and the key role played by the threshold $\alpha(\theta)$ of the single player problem.

We emphasise the key difference between equilibria with fixed exit value θ for both players and the setting of the present paper. The former equilibrium is in mixed strategies in order to resolve the war of attrition conundrum. In our paper, the player of each type plays a pure strategy (a stopping time). However, the randomness and privacy of exit value of each player means that actions of the opponent are perceived as a mixed strategy. This argument is apparent from the derivation above.

6. Uniqueness of absolutely continuous symmetric Bayesian equilibria In the previous section, we have constructed a symmetric equilibrium. Here we show that this is the unique equilibrium in a certain subclass of symmetric equilibria in which individual strategies are of the form (14).

Difficulties in the proof of uniqueness of symmetric equilibria driven by a belief process stem from the continuum of player types and the diffusive dynamics of the underlying state process. The first technical result shows that the player's strategy is a solution to a Markovian optimal stopping problem for almost every value of player type, i.e., the best response optimal stopping problem. The Markovian structure is provided by the state process and the belief process that defines the equilibrium. Classically, the solution of a stopping problem is given by the hitting time of a stopping set on which the value function coincides with the payoff. This is the smallest optimal stopping time, however it may not be the only one, so we cannot assume that the player's strategy determines the stopping set for the best response problem. Instead, we work with an action set which collects all points in which the equilibrium strategy prescribes to stop immediately and describe it uniquely in terms of the function $\alpha(\cdot)$ from Section 3. Hence, every symmetric equilibrium is described by the same action set in the 2-dimensional space comprising the state process and the belief, but there may be potentially many equilibria driven by different belief processes. The uniqueness of the latter is the second technical result of this section. The optimality of the equilibrium strategy for the best response optimal stopping problem yields that the equilibrium strategy established in the paper provides an upper bound for infinitesimal changes of the equilibrium generating process (A_t) . The lower bound for infinitesimal changes of (A_t) arises from examining deviations in equilibrium strategies for players with nearly identical types.

As in the previous section, for the convenience of presentation, we rewrite strategies are of the form (14) in terms of an increasing right-continuous process $(A_t)_{t \geq 0}$ with $A_0 = 0$:

$$\tau_i = \inf\{t \geq 0 : A_t > A(\theta_i)\}, \quad (53)$$

where $A(\cdot)$ is defined in (18). We also impose Assumption 5. In this section, we restrict our attention to absolutely continuous equilibria. The reader is referred to [10, Section 3.2] for a complete characterisation of stopping times of Markovian type of the form (53).

DEFINITION 3. A strategy profile (τ_1, τ_2) with τ_i given by (53) equipped with the prior distribution F is called an *absolutely continuous symmetric Bayesian Nash equilibrium* if

- (i) the process $(X_t, A_t)_{t \geq 0}$ is a strong Markov process;
- (ii) for $x \in \mathcal{I}$ and $a \geq 0$, the process $(A_t)_{t \geq 0}$ is absolutely continuous with respect to the Lebesgue measure and satisfies for $t \geq 0$

$$A_t = a + \int_0^t \varphi(X_s, Y(A_s)) ds, \quad \mathbf{P}_x - a.s. \quad (54)$$

for a measurable function $\varphi : \mathcal{I} \times [\theta_L, \theta_U] \rightarrow [0, \infty)$ called a *generator*.

- (iii) For any $a \geq 0$ and $x \in \mathcal{I}$, stopping times τ_1, τ_2 given by

$$\tau_i = \inf\{t \geq 0 : A_t > A(\hat{\theta}_i)\}, \quad i = 1, 2, \quad (55)$$

with A_t satisfying (54), form a Nash equilibrium for x in the sense of Def. 1 with $\hat{\theta}_i = F_{|Y(a)}^{-1}(F(\theta_i)) \sim F_{|Y(a)}$, $i = 1, 2$, where $F_{|y}(z) = F(z \wedge y)/F(y)$.

DEFINITION 4. An absolutely continuous symmetric Bayesian Nash equilibrium is called *lower semi-continuous* if the generator φ is a lower semi-continuous function, and *upper semi-continuous* if the generator φ is an upper semi-continuous function.

The above definition of an absolutely continuous symmetric Bayesian Nash equilibrium is weaker than a PBE adopted in Subsection 4.7 as it does not require the strategy profile to be an equilibrium at any time t even in the case of a deviation of one of a player. This weaker notion is however sufficient to prove uniqueness.

REMARK 8. Since F is continuous and strictly increasing, $F(\theta_i) \sim U(0, 1)$, so $\hat{\theta}_i$ as defined above has the distribution $F_{|Y(a)}$. The cumulative distribution function $F_{|y}$ is strictly increasing on its domain, hence $\sigma(\theta_i) = \sigma(\hat{\theta}_i)$ (here $\sigma(Z)$ is the σ -algebra generated by Z), so the information of a player observing θ_i is identical to the information obtained from observing $\hat{\theta}_i$.

REMARK 9. Notice that φ on the right-hand side of (54) can be discontinuous, so the solution does not exist in the classical sense. It may also not be unique as the discontinuity may be crossed infinitely many times in any time interval (due to the infinite variation of (X_t)) and φ is not assumed to be Lipschitz. This lack of uniqueness does not pose any mathematical difficulties for the analysis as we only need that (X_t, A_t) is a strong Markov process and (A_t) is increasing and absolutely continuous with a lower semi-continuous weak derivative. An example of such an absolutely continuous symmetric Bayesian Nash equilibrium can be found in Section 4.

REMARK 10. Notice that for any $a \geq 0$ and $x \in \mathcal{I}$, τ_i given by (53) or (55) with $A_0 = a$ is an (\mathcal{F}_t^i) -stopping time, $i = 1, 2$, because the filtration $(\mathcal{F}_t)_{t \geq 0}$ is complete with respect to $\mathbf{P}_x, x \in \mathcal{I}$, see [5, Chapter I, Theorem 10.7].

Before formulating main results of this section, we need to introduce the following notation. We define an upper semi-continuous envelope φ^* of φ by

$$\varphi^*(x, y) = \limsup_{(x', y') \rightarrow (x, y)} \varphi(x', y')$$

and a lower semi-continuous envelope φ_* of φ by

$$\varphi_*(x, y) = \liminf_{(x', y') \rightarrow (x, y)} \varphi(x', y').$$

THEOREM 3. *For an absolutely continuous Bayesian Nash equilibrium with a lower semi-continuous generator φ , its upper semi-continuous envelope φ^* coincides with λ , i.e., we have $\varphi^* = \lambda$.*

Before proving this theorem, we discuss its consequences. Theorem 3 establishes the uniqueness of the absolutely continuous symmetric Bayesian Nash equilibrium with a lower semi-continuous generator. This does not cover the case of the equilibrium defined by λ in the previous section which is upper semi-continuous, but we will be able to strengthen this result.

DEFINITION 5. Assume that a measurable function φ is a generator of an absolutely continuous Bayesian Nash equilibrium. If the set $\Delta\varphi := \{(x, y) \in \mathcal{I} \times [\theta_L, \theta_U] : \varphi^*(x, y) \neq \varphi_*(x, y)\}$ is a countable union of graphs of the form $x = h(y)$ for a function $h : [\theta_L, \theta_U] \rightarrow \mathcal{I}$ of finite variation, we will call the generator *semi-continuously bounded*.

Notice that the generator λ from the previous section is semi-continuously bounded as the set $\Delta\lambda$ from the above definition consists of a graph of function α , the stopping boundary of the single-player problem.

LEMMA 11. *Let φ be a semi-continuously bounded generator and (X_t, A_t) the pair of processes from Definition 3. The process $(A_t)_{t \geq 0}$ satisfies (54) with the upper semi-continuous envelope φ^* and with the lower semi-continuous envelope φ_* of φ .*

Proof The result is an immediate consequence of Lemma 3. \square

We recall, see Remark 4, that the process (A_t) may not be uniquely determined. However, the process constructed in Section 4 is the maximal solution, so it yields the smallest stopping times.

We conclude with the main uniqueness result of the paper.

COROLLARY 3. *Generator λ determines the unique absolutely continuous Bayesian Nash equilibrium in the family of equilibria with semi-continuously bounded generators.*

Proof Let φ be a semi-continuously bounded generator of an absolutely continuous Bayesian Nash equilibrium. By applying Theorem 3 to φ_* , we have $\varphi^* = \lambda$. Furthermore, Lemma 11 implies that the process (A_t) satisfies (54) with λ as well. Noting that (A_t) itself determines player's strategies, we obtain uniqueness. \square

The above corollary is the main uniqueness result of the paper. We argue that the assumption of semi-continuous boundedness of a generator is quite natural. As remarked, the function λ generating the equilibrium of the previous section is semi-continuously bounded with the jump set $\Delta\lambda$ consisting of only the graph of the stopping boundary of a single player problem. As the generator φ from (54) has the interpretation of the intensity of exiting of an opponent, it is unlikely that one can explicitly construct φ with a more complex jump set $\Delta\varphi$ than stipulated in Definition 5, so our result may be viewed as a guarantee that there will be no other *explicitly* constructed absolutely continuous symmetric Bayesian Nash equilibrium in the problem.

The remaining of this section is divided into two parts. In the first part, we define the action sets for the best response problem and establish their properties. The second part is devoted to the derivation of an upper and lower bounds for φ and its upper semi-continuous envelope and the proof of Theorem 3.

6.1. Properties of action sets for best response problems Before proceeding further, we recall properties of lower semi-continuous functions.

LEMMA 12. [42, p. 51] *The following hold:*

1. *Function φ is lower semi-continuous iff $\liminf_{(x', y') \rightarrow (x, y)} \varphi(x', y') \geq \varphi(x, y)$ for any $(x, y) \in \mathcal{I} \times [\theta_L, \theta_U]$;*

2. Function φ is lower semi-continuous iff the set $\{(x, y) \in \mathcal{I} \times [\theta_L, \theta_U] : \varphi(x, y) > z\}$ is open in $\mathcal{I} \times [\theta_L, \theta_U]$ (i.e., its complement is closed) for any $z \in \mathbb{R}$.

3. Function φ is upper semi-continuous iff $(-\varphi)$ is lower semi-continuous.

Throughout the remaining of this section, we assume that $(A_t)_{t \geq 0}$ given by (54) is the process that characterises a *lower semi-continuous* absolutely continuous symmetric Bayesian Nash equilibrium (τ_1, τ_2) . For $\theta \in [\theta_L, \theta_U]$ and $a \leq A(\theta)$, define $v(x, a; \theta)$ by (29) with λ replaced by φ . In the remainder of this section, we will refer to equations from Section 4 without further mentioning that λ is to be replaced by φ . The first key step is proving a converse of Corollary 1.

PROPOSITION 4. *The stopping time*

$$\tau_\theta = \inf\{t \geq 0 : A_t > A(\theta)\}, \quad (56)$$

with $A_0 = a$ and the dynamics (54), is an optimal stopping time for $v(x, a; \theta)$ for $\theta \in (\theta_L, Y(a)]$.

Proof Fix $a \geq 0$ and consider an equilibrium (τ_1, τ_2) in Definition 3(iii). The decomposition of the stopping time τ_i from Proposition 1 is $\hat{\tau}_i(\omega, \theta) = \tau_\theta(\omega)$ with τ_θ from (56). We further have

$$J_1(x, \tau_1, \tau_2) = \int_{\theta_L}^{Y(a)} J(x, \tau_\theta, \tau_2; \theta) F_{|Y(a)}(d\theta),$$

where $J(x, \sigma, \tau_2; \theta)$ is defined in (6). Using representation (20) of J and taking into account that $A_0 = a \geq 0$ while Lemma 2 assumed $a = 0$, we have

$$J(x, \sigma, \tau_2; \theta) = e^a \mathbf{E}_x \left[\int_0^\sigma e^{-rs - As} D(X_s) ds + \theta e^{-r\sigma - A\sigma} + \int_{[0, \sigma)} e^{-rs - As} m(X_s) dA_s \right]. \quad (57)$$

Hence, from (29)

$$v(x, a; \theta) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} e^{-a} J(x, \sigma, \tau_2; \theta), \quad \theta \in [\theta_L, Y(a)].$$

The proof will now follow by contradiction. Assume that there is $\hat{\theta} \in (\theta_L, Y(a)]$ such that

$$J(x, \tau_{\hat{\theta}}, \tau_2; \hat{\theta}) \leq e^a v(x, a; \hat{\theta}) - \varepsilon \quad (58)$$

for some $\varepsilon > 0$. Notice that the mapping $(t, \theta) \mapsto J(x, t, \tau_2; \theta)$ is continuous due to the continuity of A_t , see (57). We also have that the mapping $\theta \mapsto \tau_\theta$ is left-continuous. Indeed, take a sequence $\theta_n \uparrow \theta$ and fix $\omega \in \Omega$; we will argue pointwise. Fix any $t > \tau_\theta(\omega)$. We have $A_t(\omega) > A(\theta)$. Using that $A(\theta_n) \downarrow A(\theta)$, there is k such that $A_t(\omega) > A(\theta_k)$, which implies $t > \tau_{\theta_k}(\omega) \geq \inf_n \tau_{\theta_n}(\omega) =$

$\lim_{n \rightarrow \infty} \tau_{\theta_n}(\omega)$, where the last equality follows from the fact that the sequence $\tau_{\theta_n}(\omega)$ is decreasing in n . From the arbitrariness of t , we obtain that $\tau_\theta(\omega) \geq \lim_{n \rightarrow \infty} \tau_{\theta_n}(\omega)$. The opposite inequality is obvious as $A(\theta_n) > A(\theta)$.

The above two observations as well as the boundedness of the terms under the integrals in (57) imply that

$$\theta \mapsto J(x, \tau_\theta, \tau_2; \theta)$$

is left-continuous. Hence, there is $\delta > 0$ such that $\hat{\theta} - \delta \geq \theta_L$ and $J(x, \tau_\theta, \tau_2; \theta) \leq e^a v(x, a; \theta) - \varepsilon/2$ for $\theta \in [\hat{\theta} - \delta, \hat{\theta}]$. This will allow us to improve the equilibrium strategy τ_1 defined in (55) and lead to a contradiction. Take $\sigma^* \in \mathcal{T}(\mathcal{F}_t)$ such that $J(x, \sigma^*, \tau_2; \hat{\theta}) > e^a v(x, a; \hat{\theta}) - \varepsilon/4$. By the continuity of $\theta \mapsto J(x, \sigma^*, \tau_2; \theta)$, there is $\delta' \in (0, \delta)$ such that

$$J(x, \sigma^*, \tau_2; \theta) > J(x, \tau_\theta, \tau_2; \theta), \quad \theta \in [\hat{\theta} - \delta', \hat{\theta}].$$

Hence the strategy $\tau'_1 = \hat{\tau}(\cdot, \theta_1)$ (c.f. Proposition 1) with

$$\hat{\tau}(\cdot, \theta) = \begin{cases} \sigma^*, & \theta \in [\hat{\theta} - \delta', \hat{\theta}], \\ \tau_\theta, & \text{otherwise,} \end{cases}$$

is a strictly better response to τ_2 than $\tau_1 = \tau_{\theta_1}$, which contradicts that (τ_1, τ_2) given by (55) is an absolutely continuous Bayesian Nash equilibrium, contradicting Definition 3(iii). \square

For $\theta \in [\theta_L, \theta_U]$, denote $S_\theta = \{x \in \mathcal{I} : \mathbf{P}_{xA(\theta)}(\tau_\theta = 0) = 1\}$ and $O_\theta = \{x \in \mathcal{I} : \varphi(x, \theta) > 0\}$, which is an open set by Lemma 12. Recall that by the 0-1 law, $\mathbf{P}_{xA(\theta)}(\tau_\theta = 0) \in \{0, 1\}$, so on the complement of S_θ we have $\mathbf{P}_{xA(\theta)}(\tau_\theta > 0) = 1$. We will call S_θ *the action set* for reasons explained in the remark below.

REMARK 11. We cannot assume that the stopping time defined in (53) coincides with the first hitting time of the stopping set on which the value function $v(x, a; \theta)$ coincides with the payoff as there may be many optimal stopping times; the aforementioned hitting time is the smallest of them. Therefore, the optimality of the stopping rule (53) does not determine the stopping set. This motivates our less direct approach and the introduction of the set S_θ of those values of x, a, θ for which the stopping rule (53) stops immediately with probability one.

LEMMA 13. *We have $O_\theta \subset S_\theta$.*

Proof If $\varphi(x, \theta) > 0$, then the first inclusion follows from the fact that $\varphi > 0$ in an open neighbourhood of (x, θ) by Lemma 12. \square

LEMMA 14. *Set S_θ is closed in \mathcal{I} .*

Proof Assume that S_θ is not closed. There is $x \in \mathcal{I} \setminus S_\theta$ such that $B_\varepsilon(x) \cap S_\theta \neq \emptyset$ for all $\varepsilon > 0$, where $B_\varepsilon(x) = \{x' \in \mathcal{I} : |x - x'| < \varepsilon\}$. We can find a monotone sequence $(x_n) \subset S_\theta$ converging to x . We will assume that the sequence is increasing; arguments for a decreasing sequence are analogous. By the regularity of (X_t) , we have $\mathbf{P}_x(\sigma_{(x_L, x)} = 0) = 1$, where we write $\sigma_B = \inf\{t \geq 0 : X_t \in B\}$ for a Borel set $B \subset \mathcal{I}$ and σ_z when $B = \{z\}$. For any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \mathbf{E}_{xA(\theta)}[\tau_\theta \wedge 1] &= \mathbf{E}_{xA(\theta)}[1_{\sigma_{x_n} \leq \varepsilon}(\tau_\theta \wedge 1) + 1_{\sigma_{x_n} > \varepsilon}(\tau_\theta \wedge 1)] \\ &\leq \mathbf{E}_{xA(\theta)}[1_{\sigma_{x_n} \leq \varepsilon}(\tau_\theta \wedge 1) + 1_{\sigma_{x_n} > \varepsilon}] \\ &\leq \mathbf{E}_{xA(\theta)}[1_{\sigma_{x_n} \leq \varepsilon}(1_{\tau_\theta < \sigma_{x_n}} \sigma_{x_n} + 1_{\tau_\theta \geq \sigma_{x_n}}(\tau_\theta \wedge 1))] + \mathbf{P}_{xA(\theta)}(\sigma_{x_n} > \varepsilon) \\ &\leq \mathbf{E}_{xA(\theta)}[1_{\sigma_{x_n} \leq \varepsilon}(1_{\tau_\theta < \sigma_{x_n}} \varepsilon + 1_{\tau_\theta \geq \sigma_{x_n}}(\sigma_{x_n} + \mathbf{E}_{x_n A_{\sigma_{x_n}}}[\tau_\theta]))] + \mathbf{P}_{xA(\theta)}(\sigma_{x_n} > \varepsilon) \\ &\leq \varepsilon \mathbf{P}_{xA(\theta)}(\sigma_{x_n} \leq \varepsilon) + \mathbf{P}_{xA(\theta)}(\sigma_{x_n} > \varepsilon), \end{aligned}$$

where we used the strong Markov property of (X_t, A_t) and $\mathbf{E}_{x_n A_{\sigma_{x_n}}}[\tau_\theta] = 0$ since $x_n \in S_\theta$ and $A_{\sigma_{x_n}} \geq A(\theta)$. By the dominated convergence theorem and the regularity of (X_t) , we have

$$\lim_{n \rightarrow \infty} \mathbf{P}_{xA(\theta)}(\sigma_{x_n} \leq \varepsilon) = 1, \quad \lim_{n \rightarrow \infty} \mathbf{P}_{xA(\theta)}(\sigma_{x_n} > \varepsilon) = 0.$$

Inserting this into the above estimates gives $\mathbf{E}_{xA(\theta)}[\tau_\theta \wedge 1] \leq \varepsilon$. This means that $\mathbf{E}_{xA(\theta)}[\tau_\theta \wedge 1] = 0$ as $\varepsilon \in (0, 1)$ was arbitrary. We therefore conclude that $\mathbf{P}_{xA(\theta)}(\tau_\theta = 0) = 1$, which contradicts the assumption that $x \notin S_\theta$. \square

LEMMA 15. *We have $\mathbf{P}_{xA(\theta)}(X_{\tau_\theta} \in S_\theta \text{ or } \tau_\theta = \infty) = 1$.*

Proof Recall that the filtration (\mathcal{F}_t) is right continuous. Define $\sigma_\theta = \tau_\theta + 1_{\tau_\theta < \infty} \tau_\theta \circ \theta_{\tau_\theta}$, where θ_t is the shift operator for the Markov process $(X_t, A_t)_{t \geq 0}$. This is a stopping time thanks to [5, Chapter I, Thm. 8.7]. We have $A_{\tau_\theta} = A(\theta)$ because of the continuity of (A_t) and the definition of τ_θ . Combining this with the strong Markov property of (X_t, A_t) we can write

$$\begin{aligned} \mathbf{E}_{xA(\theta)}[A_{\sigma_\theta}] &= \mathbf{E}_{xA(\theta)}[1_{\tau_\theta = \infty} A(\theta) + 1_{\tau_\theta < \infty} \mathbf{E}_{X_{\tau_\theta} A_{\tau_\theta}}[A_{\tau_\theta}]] \\ &= \mathbf{E}_{xA(\theta)}[1_{\tau_\theta = \infty} A(\theta) + 1_{\tau_\theta < \infty} \mathbf{E}_{X_{\tau_\theta} A(\theta)}[A_{\tau_\theta}]] \\ &= \mathbf{E}_{xA(\theta)}[1_{\tau_\theta = \infty} A(\theta) + 1_{\tau_\theta < \infty} \mathbf{E}_{X_{\tau_\theta} A(\theta)}[A(\theta)]] = A(\theta). \end{aligned}$$

Hence, recalling the definition of τ_θ , this implies $\mathbf{P}_{xA(\theta)}(\sigma_\theta > \tau_\theta \text{ and } \tau_\theta < \infty) = 0$ and, consequently, $\mathbf{P}_{xA(\theta)}(\tau_\theta \circ \theta_{\tau_\theta} = 0 \mid \tau_\theta < \infty) = 1$. Notice now that $\{X_{\tau_\theta} \notin S_\theta, \tau_\theta < \infty\} = \{\tau_\theta \circ \theta_{\tau_\theta} > 0, \tau_\theta < \infty\}$. As the latter event has probability zero, we conclude that $\mathbf{P}_{xA(\theta)}(X_{\tau_\theta} \notin S_\theta \mid \tau_\theta < \infty) = 0$. \square

LEMMA 16. *For any $\theta \in (\theta_L, \theta_U]$, we have $S_\theta = (x_L, \alpha(\theta)]$.*

Proof We first show that $S_\theta \cap (\alpha(\theta), x_U) = \emptyset$. Take $x > \alpha(\theta)$. By assumption τ_θ is optimal for $v(x, A(\theta); \theta)$, so also for $\tilde{v}(x, A(\theta); \theta)$ defined in (31). Following arguments of Lemma 7 with λ replaced by φ we obtain $\tilde{v}(x, A(\theta); \theta) > 0$. Hence, stopping immediately is suboptimal, so $\tau_\theta > 0$ $\mathbf{P}_{xA(\theta)}$ -a.s. and $x \notin S_\theta$.

Assume that there are $b, c \in \mathcal{I}$, $b < c$, such that $[b, c] \cap S_\theta = \{b, c\}$. Take any $x \in (b, c)$. Thanks to Lemma 15, we have $\sigma_{b,c} \leq \tau_\theta$, where $\sigma_{b,c}$ is the first entry time to the set $\{b, c\}$. By the definition of S_θ we further have that $\sigma_{b,c} = \tau_\theta$. By the optimality of $\sigma_{b,c}$ for $\tilde{v}(x, A(\theta); \theta)$ we have

$$\tilde{v}(x, A(\theta); \theta) = \mathbf{E}_x \left[\int_0^{\sigma_{b,c}} e^{-rs-A(\theta)} (D(X_s) - r\theta) ds \right] < 0, \quad (59)$$

where the last inequality is because $X_s \leq \alpha(\theta) \leq c(\theta)$ (Lemma 1) and hence $D(X_s) - r\theta < 0$ (Assumption 4). This contradicts the lower bound $\tilde{v} \geq 0$ which can be obtained by stopping immediately. This means that S_θ is either an empty set or an interval.

The set S_θ cannot be empty as by Lemma 15 that would mean $\tau_\theta = \infty$, $\mathbf{P}_{xA(\theta)}$ -a.s. But clearly the best response to an opponent who never stops is the optimal stopping time from Section 3 which is not infinite \mathbf{P}_x -a.s. Hence, in equilibrium $S_\theta \neq \emptyset$.

We shall prove that x_L is the left endpoint of S_θ . Assume that $\inf S_\theta =: b > x_L$. Take any $x \in (x_L, b)$. As above, the stopping time τ_b is optimal for $\tilde{v}(x, A(\theta); \theta)$. The estimate (59) with $\sigma_{b,c}$ replaced by $\sigma_b := \inf\{t \geq 0 : X_t = b\}$ holds true and contradicts $\tilde{v}(x, A(\theta); \theta) \geq 0$.

It remains to show that $\sup S_\theta =: c = \alpha(\theta)$. Assume, by contradiction, that $c < \alpha(\theta)$ and take any $x \in (c, \alpha(\theta))$. Then σ_c is optimal for $\tilde{v}(x, A(\theta); \theta)$ and yields the payoff

$$\begin{aligned} e^{-A(\theta)} \mathbf{E}_x \left[\int_0^{\sigma_c} e^{-rs} (D(X_s) - r\theta) ds \right] &\leq \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t)} e^{-A(\theta)} \mathbf{E}_x \left[\int_0^\sigma e^{-rs} (D(X_s) - r\theta) ds \right] \\ &= e^{-A(\theta)} \tilde{u}(x; \theta), \end{aligned}$$

where $\tilde{u}(x; \theta)$ is defined in (32). Since $x < \alpha(\theta)$ then $\tilde{u}(x; \theta) = 0$. We will show that the inequality above is strict, i.e., σ_c is suboptimal for $\tilde{u}(x; \theta)$. Assume optimality of σ_c and select z_1, z_2 so that $c < z_1 < x < z_2 < \alpha(\theta)$. From the dynamic programming principle for \tilde{u} we obtain

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{\sigma_c} e^{-rs} (D(X_s) - r\theta) ds \right] &\leq \mathbb{E}_x \left[\int_0^{\sigma_{z_1, z_2}} e^{-rs} (D(X_s) - r\theta) ds + e^{-r\sigma_{z_1, z_2}} \tilde{u}(X_{\sigma_{z_1, z_2}}; \theta) \right] \\ &< 0, \end{aligned}$$

where we used that $\tilde{u}(X_{\sigma_{z_1, z_2}}; \theta) = 0$ and the integrand is strictly negative for $X_s \leq \alpha(\theta)$. This contradicts that $\tilde{u} \geq 0$. \square

6.2. Properties of φ We start from an immediate corollary which follows by combining Lemma 13 and Lemma 16.

COROLLARY 4. $\varphi(x, y) = 0$ for $y \in (\theta_L, \theta_U]$ and $x > \alpha(y)$.

We turn attention to upper and lower bounds for φ .

LEMMA 17. We have $\varphi(x, y) \leq \lambda(x, y)$ for $(x, y) \in \mathcal{I} \times [\theta_L, \theta_U]$.

Proof Fix any (x, θ) such that $\varphi(x, \theta) > 0$ and set $a = A(\theta)$. We must have $x \leq \alpha(\theta)$ thanks to Corollary 4. From Lemma 13, we have that the optimal stopping time τ_θ for $\tilde{v}(x, a; \theta)$ satisfies $\tau_\theta = 0$ \mathbb{P}_{xa} -a.s. Then, for any $t > 0$, we have

$$\mathbb{E}_{xa} \left[\int_0^t e^{-rs - A_s} (D(X_s) - r\theta + \varphi(X_s, Y(A_s))(m(X_s) - \theta)) ds \right] \leq \tilde{v}(x, a; \theta) = 0.$$

We divide both sides by t and change the variable of integration to $z = s/t$:

$$\mathbb{E}_{xa} \left[\int_0^1 e^{-rtz - A_{tz}} (D(X_{tz}) - r\theta + \varphi(X_{tz}, Y(A_{tz}))(m(X_{tz}) - \theta)) dz \right] \leq 0.$$

Since D is bounded from below, $\varphi \geq 0$ and $m > \theta$, we can apply Fatou's lemma

$$\begin{aligned} 0 &\geq \liminf_{t \rightarrow 0} \mathbb{E}_{xa} \left[\int_0^1 e^{-rtz - A_{tz}} (D(X_{tz}) - r\theta + \varphi(X_{tz}, Y(A_{tz}))(m(X_{tz}) - \theta)) dz \right] \\ &\geq D(x) - r\theta + \varphi(x, Y(a))(m(x) - \theta), \end{aligned}$$

where in the last inequality follows from the continuity of trajectories (X_t, A_t) , the continuity of functions D , m , Y , and the lower semi-continuity of φ . The above inequality is equivalent to $\varphi(x, \theta) \leq \lambda(x, \theta)$, where we also used that $x \leq \alpha(\theta)$. The proof is concluded when we notice that $\lambda \geq 0$, so $\varphi(x, a) \leq \lambda(x, a)$ when $\varphi(x, a) = 0$. \square

LEMMA 18. *We have $\varphi^*(x, y) \geq \lambda(x, y)$ for $(x, y) \in \mathcal{I} \times [\theta_L, \theta_U]$.*

Proof Denote by \tilde{v} the value function of the problem (31) with (A_t) given by (54). First notice that there exists $(x, \theta) \in \mathcal{I} \times (\theta_L, \theta_U]$ such that $\varphi(x, \theta) > 0$. Otherwise, we would have a contradiction with $S_\theta = (x_L, \alpha(\theta)]$ asserted in Lemma 16.

Fix (x, θ) such that $\varphi(x, \theta) > 0$. Denote $a = A(\theta)$ and

$$\Gamma(x', a') = e^{-a'} (D(x') - r\theta + \varphi^*(x', Y(a'))(m(x') - \theta)).$$

Take $\varepsilon > 0$. By the lower semi-continuity of φ and the upper semi-continuity of Γ , using Lemma 12, there is $\delta > 0$ such that $U := [x - \delta, x + \delta] \times [\theta - \delta, \theta] \subset \mathcal{I} \times [\theta_L, \theta_U]$,

$$\inf_{(x', y') \in U} \varphi(x', y') > \varphi(x, \theta)/2, \quad \text{and} \quad \sup_{(x', y') \in U} \Gamma(x', A(y')) \leq \Gamma(x, a) + \varepsilon. \quad (60)$$

Recall the definition (55) of an optimal stopping time τ_γ for $\tilde{v}(x, a; \gamma)$ for $\theta - \delta \leq \gamma \leq \theta$. Let $\sigma_\delta = \inf\{t \geq 0 : X_t \notin (x - \delta, x + \delta)\}$. By the optimality of τ_γ we have

$$\begin{aligned} \tilde{v}(x, a; \gamma) &= \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma} e^{-rs - A_s} (D(X_s) - r\gamma + \varphi(X_s, Y(A_s))(m(X_s) - \gamma)) ds \right] \\ &= \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma \wedge \sigma_\delta} e^{-rs - A_s} (D(X_s) - r\gamma + \varphi(X_s, Y(A_s))(m(X_s) - \gamma)) ds \right. \\ &\quad \left. + 1_{\sigma_\delta < \tau_\gamma} \int_{\tau_\gamma \wedge \sigma_\delta}^{\tau_\gamma} e^{-rs - A_s} (D(X_s) - r\gamma + \varphi(X_s, Y(A_s))(m(X_s) - \gamma)) ds \right] \\ &\leq \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma \wedge \sigma_\delta} e^{-rs - A_s} (D(X_s) - r\gamma + \varphi(X_s, Y(A_s))(m(X_s) - \gamma)) ds \right. \\ &\quad \left. + 1_{\sigma_\delta < \tau_\gamma} e^{-r\sigma_\delta} \tilde{v}(X_{\sigma_\delta}, A_{\sigma_\delta}; \gamma) \right] \\ &\leq \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma \wedge \sigma_\delta} e^{-rs - A_s} (D(X_s) - r\gamma + \varphi^*(X_s, Y(A_s))(m(X_s) - \gamma)) ds \right. \\ &\quad \left. + 1_{\sigma_\delta < \tau_\gamma} e^{-r\sigma_\delta} \tilde{v}(X_{\sigma_\delta}, A_{\sigma_\delta}; \gamma) \right], \end{aligned}$$

where the first inequality follows from the strong Markov property and the definition of $\tilde{v}(X_{\sigma_\delta}, A_{\sigma_\delta}; \gamma)$, and the second inequality uses $\varphi^* \geq \varphi$.

We transform the final expression above and use $\tilde{v} \geq 0$ to obtain

$$\begin{aligned} 0 &\leq \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma \wedge \sigma_\delta} e^{-rs} \Gamma(X_s, A_s) ds \right] \\ &+ \mathbf{E}_{xa} \left[\int_0^{\tau_\gamma \wedge \sigma_\delta} e^{-rs - A_s} (r + \varphi^*(X_s, Y(A_s)))(\theta - \gamma) ds \right] \\ &+ \mathbf{E}_{xa} \left[1_{\sigma_\delta < \tau_\gamma} e^{-r\sigma_\delta} \tilde{v}(X_{\sigma_\delta}, A_{\sigma_\delta}; \gamma) \right] = (I) + (II) + (III). \end{aligned} \quad (61)$$

We divide both sides by $\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]$; this can be done as $\mathbf{P}_{xa}(\tau_\gamma \wedge \sigma_\delta > 0) = 1$ due to the 0-1 law. Using the bounds (60) the first terms yields

$$\frac{(I)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]} \leq \Gamma(x, a) + \varepsilon.$$

Recall that λ is bounded from above, while Lemma 17 shows that $\varphi \leq \lambda$. Since $\varphi^* \leq \sup_{(x,y)} \varphi(x, y)$, we conclude that $r + \varphi^*$ is bounded above by some constant C_1 and

$$\frac{(II)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]} \leq \frac{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta] C_1 (\theta - \gamma)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]} = C_1 (\theta - \gamma).$$

To estimate the last term, notice that $\tilde{v}(x', a'; \gamma) = v(x', a'; \gamma) - e^{-a'} \gamma \leq m(x') - e^{-a'} \gamma \leq C_2$ for some $C_2 > 0$ since the function m is bounded. Then

$$\frac{(III)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]} \leq C_2 \frac{\mathbf{P}_{xa}(\sigma_\delta < \tau_\gamma)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]}.$$

We apply the above estimates to the right-hand side of (61) and take limit as $\gamma \uparrow \theta$:

$$0 \leq \Gamma(x, a) + \varepsilon + 0 + C_2 \lim_{\gamma \uparrow \theta} \frac{\mathbf{P}_{xa}(\sigma_\delta < \tau_\gamma)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]}. \quad (62)$$

Let $\underline{\varphi} = \inf_{(x', y') \in U} \varphi(x', y') \geq \frac{1}{2} \varphi(x, \theta)$ by (60) and $\bar{\varphi} = \sup_{(x', y') \in U} \varphi(x', y') < \infty$ by $\varphi \leq \lambda$. Those bounds on φ allow us to bound the numerator and denominator under the limit:

$$\{\sigma_\delta < \tau_\gamma\} = \{A_{\sigma_\delta} \leq A(\gamma)\} \subseteq \{a + \underline{\varphi} \sigma_\delta \leq A(\gamma)\} = \left\{ \sigma_\delta \leq \frac{A(\gamma) - a}{\underline{\varphi}} \right\},$$

and

$$\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta] \geq \mathbf{E}_{xa}[1_{\sigma_\delta > \tau_\gamma} \tau_\gamma] \geq \mathbf{E}_{xa} \left[1_{\sigma_\delta > \tau_\gamma} \frac{A(\gamma) - a}{\bar{\varphi}} \right] = \frac{A(\gamma) - a}{\bar{\varphi}} \mathbf{P}_{xa}(\sigma_\delta > \tau_\gamma).$$

Combining these estimates yields

$$\lim_{\gamma \uparrow \theta} \frac{\mathbf{P}_{xa}(\sigma_\delta < \tau_\gamma)}{\mathbf{E}_{xa}[\tau_\gamma \wedge \sigma_\delta]} \leq \lim_{\gamma \uparrow \theta} \frac{\mathbf{P}_{xa} \left(\sigma_\delta \leq \frac{A(\gamma) - a}{\underline{\varphi}} \right)}{\frac{A(\gamma) - a}{\bar{\varphi}} \mathbf{P}_{xa}(\sigma_\delta > \tau_\gamma)} = \frac{\underline{\varphi}}{\bar{\varphi}} \lim_{\gamma \uparrow \theta} \frac{\mathbf{P}_{xa} \left(\sigma_\delta \leq \frac{A(\gamma) - a}{\underline{\varphi}} \right)}{\frac{A(\gamma) - a}{\underline{\varphi}}},$$

where the last equality follows from $\lim_{\gamma \uparrow \theta} \mathbf{P}_{xa}(\sigma_\delta > \tau_\gamma) = 1$ (as $a = A(\theta)$). It remains to study the asymptotic behaviour of σ_δ near 0, i.e., the limit $\lim_{t \downarrow 0} \mathbf{P}_x(\sigma_\delta < t)/t$. It is well known that $\mathbf{P}_x(\sigma_\delta < t)$ decreases exponentially fast as it does for the Brownian motion, but we could not find a direct reference for this fact. For completeness, we provide a derivation of a bound for this probability in the appendix. From (63), we have

$$\mathbf{P}_{xa}(\sigma_\delta < t) \leq C_3 e^{-C_4/t^2},$$

for some constants $C_3, C_4 > 0$. It is now immediate to see that

$$\lim_{t \downarrow 0} \mathbf{P}_{xa}(\sigma_\delta < t)/t \leq C_3 \lim_{t \downarrow 0} e^{-C_4/t^2}/t = 0.$$

Returning to (62), we have shown that $\Gamma(x, a) + \varepsilon \geq 0$ for any $\varepsilon > 0$, i.e., $\Gamma(x, a) \geq 0$. This implies that

$$D(x) - r\theta + \varphi^*(x, Y(a))(m(x) - \theta) \geq 0,$$

which is equivalent to $\varphi^*(x, \theta) \geq \lambda(x, \theta)$ upon recollection that $a = A(\theta)$.

We have therefore demonstrated that $\varphi^*(x, \theta) \geq \lambda(x, \theta)$ for (x, θ) such that $\varphi(x, \theta) > 0$. Define $O_+ = \{(x, \theta) \in \mathcal{I} \times [\theta_L, \theta_U] : \varphi(x, \theta) > 0\}$ and $O_\alpha = \{(x, \theta) \in \mathcal{I} \times [\theta_L, \theta_U] : x \leq \alpha(\theta)\}$. Due to the upper bound $\varphi \leq \lambda$ (see Lemma 17) and the fact that $\lambda \equiv 0$ on the complement of O_α , we have $O_+ \subset O_\alpha$. By the upper semi-continuity of φ^* and the continuity of λ on O_α , we further have that $\varphi^* \geq \lambda$ on $\text{cl}(O_+)$, where $\text{cl}(\cdot)$ denotes the closure. Let $U_0 = O_\alpha \setminus \text{cl}(O_+)$. This is a relatively open set in O_α . Furthermore, $\varphi \equiv 0$ on U_0 . Assume that U_0 is non-empty. Due to the closedness of O_α , it has a non-empty interior. Take any (x, θ) in the interior of U_0 . We immediately have that $\mathbf{P}_{xA(\theta)}(\tau_\theta > 0) = 1$. However, $(x, \theta) \in O_\alpha$, so $x \leq \alpha(\theta)$ and, by Lemma 16, $x \in S_\theta$. This means that $\mathbf{P}_{xA(\theta)}(\tau_\theta = 0) = 1$, a contradiction. This completes the proof that $\varphi^*(x, \theta) \geq \lambda(x, \theta)$ for $(x, \theta) \in O_\alpha$. Recall that $\lambda \equiv 0$ on the complement of O_α . Since φ^* is non-negative, it trivially dominates λ on the complement of O_α , which finishes the proof. \square

Combining Lemma 17 and 18 yields the proof of the main result of this section.

Proof of Theorem 3 Notice that λ is upper semi-continuous and it majorises φ by Lemma 17. Hence, it also majorises φ^* which is the smallest upper semi-continuous function dominating φ . However, λ also bounds φ^* from below, which completes the proof. \square

Appendix A: Asymptotics of σ_δ near 0 We provide a sketch of an asymptotic bound for the behaviour of $\mathbf{P}_x(\sigma_\delta < u)$ as $u \downarrow 0$, where $\sigma_\delta = \inf\{t \geq 0 : X_t \notin (x - \delta, x + \delta)\}$. Notice that the probability is identical when the coefficients of X_t are replaced with

$$\tilde{\mu}(y) = \mu(y \wedge (x + \delta) \vee (x - \delta)), \quad \tilde{b}(y) = b(y \wedge (x + \delta) \vee (x - \delta)),$$

i.e., we can assume that μ and b in (2) are bounded, continuous and b is uniformly bounded away from 0. Consider the change of measure given by

$$\frac{d\tilde{\mathbf{P}}_x}{d\mathbf{P}_x} = \eta_1,$$

where

$$\eta_t = \exp\left(-\int_0^t \mu(X_s)/b(X_s)dW_s - \int_0^t \mu^2(X_s)/b^2(X_s)ds\right).$$

Then for $u \leq 1$, we have

$$\mathbf{P}_x(\sigma_\delta < u) = \tilde{\mathbf{E}}_x(1_{\sigma_\delta < u}\eta_1^{-1}) \leq (\tilde{\mathbf{P}}_x(\sigma_\delta < u))^{1/2} \|\eta_1^{-1}\|_{L^2} = c_1(\tilde{\mathbf{P}}_x(\sigma_\delta < u))^{1/2}$$

for some constant $c_1 > 0$ and

$$dX_t = b(X_t)d\tilde{W}_t, \quad X_0 = x$$

for $\tilde{\mathbf{P}}_x$ -Brownian motion \tilde{W}_t . Consider $\Lambda_t = \int_0^t b^2(X_s)ds$ and the time change T_t being the inverse of Λ_t which exists since b is separated from 0. Then $Y_t = X_{T_t}$ is a Brownian motion. We have the sequence of inclusions:

$$\{\sigma_\delta < t\} = \left\{ \sup_{u \in [0, t]} |X_u - x| \geq \delta \right\} \subset \left\{ \sup_{u \in [0, \text{ess sup}_{\omega \in \Omega} \Lambda_t(\omega)]} |Y_u - x| \geq \delta \right\},$$

where we used that $X_t = Y_{\Lambda_t}$. Let $\bar{b} = \sup_{y \in [x - \delta, x + \delta]} b(y) > 0$. Then $\Lambda_t \leq \bar{b}t$ and

$$\left\{ \sup_{u \in [0, \text{ess sup}_{\omega \in \Omega} \Lambda_t(\omega)]} |Y_u - x| \geq \delta \right\} \subset \left\{ \sup_{u \in [0, t\bar{b}]} |Y_u - x| \geq \delta \right\}.$$

This gives us the following estimate

$$\begin{aligned} \tilde{\mathbf{P}}_x(\sigma_\delta < t) &\leq \tilde{\mathbf{P}}_x\left(\sup_{u \in [0, t\bar{b}]} |Y_u - x| \geq \delta\right) = \tilde{\mathbf{P}}_0\left(\sup_{u \in [0, t\bar{b}]} |Y_u| \geq \delta\right) \\ &\leq 2\tilde{\mathbf{P}}_0\left(\sup_{u \in [0, t\bar{b}]} Y_u \geq \delta\right) \leq 2\sqrt{\frac{2}{\pi}} \int_{\delta/(t\bar{b})}^{\infty} e^{-z^2/2} dz \leq c_2 e^{-c_3/t^2}, \end{aligned}$$

where the penultimate inequality follows from [39, Proposition 3.7, Chapter III] and the last inequality holds for sufficiently small t (precisely, t such that $\delta/(t\bar{b}) \geq 1$). Combining together the above estimates yields

$$\mathbb{P}_x(\sigma_\delta < u) \leq c_1 \sqrt{c_2} e^{-0.5c_3/u^2} \quad (63)$$

for sufficiently small u .

Appendix B: Example from the text Below we provide the proof that the model in Example 1 satisfies all the assumptions of the paper. First, we obtain the explicit form of $d(x)$ and $m(x)$. Clearly, $m(x) = d(x) + \frac{M_0}{r}$. To find d , we utilise the fact that $(\mathcal{L}_X - r)d(x) + D(x) = 0$ for $x \in (0, x_M) \cup (x_M, \infty)$ and that $d(x)$ is a continuously differentiable bounded function. It can be verified that (see [2])

$$d(x) = \begin{cases} \frac{x^\beta}{r-\delta} + c_1(x_M)\psi(x), & x \in (0, x_M], \\ \frac{x_M^\beta}{r} + c_2(x_M)\phi(x), & x > x_M, \end{cases}$$

where δ was defined after (13). Because $d(\cdot)$ must be a bounded function, the fundamental solutions $\phi(\cdot)$ and $\psi(\cdot)$ do not show respectively in the general form of the solution $d(x)$ in the intervals $(0, x_M]$ and (x_M, ∞) . The coefficients $c_1(x_M)$ and $c_2(x_M)$ are determined by the continuity and the differentiability of $d(\cdot)$ at x_M :

$$\begin{aligned} \frac{x_M^\beta}{r-\delta} + c_1(x_M)\psi(x_M) &= \frac{x_M^\beta}{r} + c_2(x_M)\phi(x_M) \\ \frac{\beta x_M^{\beta-1}}{r-\delta} + c_1(x_M)\psi'(x_M) &= c_2(x_M)\phi'(x_M), \end{aligned}$$

from which we obtain

$$\begin{aligned} c_1(x_M) &= \frac{1}{\xi(x_M)} \left[-\phi'(x_M)x_M^\beta \left(\frac{1}{r} - \frac{1}{r-\delta} \right) - \beta\phi(x_M) \frac{x_M^{\beta-1}}{r-\delta} \right], \\ c_2(x_M) &= \frac{1}{\xi(x_M)} \left[-\psi'(x_M)x_M^\beta \left(\frac{1}{r} - \frac{1}{r-\delta} \right) - \beta\psi(x_M) \frac{x_M^{\beta-1}}{r-\delta} \right], \end{aligned}$$

with $\xi(x) = \phi(x)\psi'(x) - \psi(x)\phi'(x) = (\gamma_+ - \gamma_-)x^{\gamma_+ + \gamma_- - 1}$. Using the definition of $\phi(\cdot)$, we simplify expression for $c_1(x_M)$ as

$$c_1(x_M) = \frac{\gamma_- \delta - \beta r}{(\gamma_+ - \gamma_-)r(r - \delta)} x_M^{\beta - \gamma_+}.$$

Its numerator can be further rewritten as follows:

$$\begin{aligned} \gamma_- \delta - \beta r &= \beta \left[\frac{b^2}{2} \gamma_- (\beta - 1) + \mu \gamma_- - r \right] \\ &= \beta \left[\frac{b^2}{2} \gamma_- (\gamma_- - 1) + \mu \gamma_- - r \right] + \beta \frac{b^2}{2} \gamma_- (\beta - \gamma_-) = \beta \frac{b^2}{2} \gamma_- (\beta - \gamma_-), \end{aligned} \quad (64)$$

where we used the equality $\frac{b^2}{2} \gamma_- (\gamma_- - 1) + \mu \gamma_- - r = 0$ satisfied by γ_- . Since $\gamma_- < 0$ and $\beta - \gamma_- > 0$, it follows that $c_1(x_M) < 0$. Furthermore, using that $\gamma_+ > 1 > \beta$, we have $\lim_{x_M \rightarrow \infty} c_1(x_M) = 0$.

We now show that $D(\cdot)$ and $M(\cdot)$ satisfy all the assumptions given in the paper if x_M is taken sufficiently large. First, Assumptions 1–3 are trivially satisfied. Our remaining task is to show that Assumption 4 is satisfied. Assumption 4(i) holds because $D(\cdot)$ is increasing from zero to $x_M^\beta > r\theta_U$. By Remark 3, Assumption 4(ii) and (iii) are satisfied if there is $x^* = \alpha(\theta)$ such that $a'_\theta(x) > 0$ for $x < x^*$ and $a'_\theta(x) < 0$ for $x > x^*$. Hence, we examine the derivative of $a_\theta(x) = (\theta - d(x))/\phi(x)$ given by

$$a'_\theta(x) = \begin{cases} -\gamma_- \theta x^{-\gamma_- - 1} - (\beta - \gamma_-) \frac{x^{\beta - \gamma_- - 1}}{r - \delta} - (\gamma_+ - \gamma_-) c_1(x_M) x^{\gamma_+ - \gamma_- - 1}, & x \leq x_M, \\ -\gamma_- (\theta - \frac{x_M^\beta}{r}) x^{-\gamma_- - 1}, & x > x_M. \end{cases} \quad (65)$$

From (65), $a'_\theta(x) < 0$ for $x > x_M$, because $\theta - x_M^\beta/r \leq \theta_U - x_M^\beta/r < 0$ by the assumption that $x_M^\beta > r\theta_U$. We also note that $a'_\theta(x) > 0$ for sufficiently small values of x because of (13) and $\gamma_+ - \gamma_- > 0$. We shall show that $a'_\theta(x)$ is decreasing on $(0, x_M)$ for sufficiently large x_M , which, together with the above observations about $a'_\theta(x)$ for x close to 0 and $x > x_M$ allows us to conclude that there is $x^* = \alpha(\theta)$ such that $a'_\theta(x) > 0$ for $x < x^*$ and $a'_\theta(x) < 0$ for $x > x^*$.

For convenience, we express $a'_\theta(x) = A(x) - B(x) + C(x)$ for $x \in (0, x_M)$, where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are positive functions given by

$$A(x) = -\gamma_- \theta x^{-\gamma_- - 1}; \quad B(x) = (\beta - \gamma_-) \frac{x^{\beta - \gamma_- - 1}}{r - \delta}; \quad C(x) = -(\gamma_+ - \gamma_-) c_1(x_M) x^{\gamma_+ - \gamma_- - 1}.$$

Note that $A(\cdot)$ is decreasing while $B(\cdot)$ and $C(\cdot)$ are increasing. Furthermore, it can be easily checked that $C(x)/B(x)$ increases in x because $\gamma_+ > 1 > \beta$ and $c_1(x_M) < 0$. Hence, within the interval $(0, x_M]$, $C(x)/B(x)$ takes the maximum value at x_M , which is given by

$$g := \frac{C(x_M)}{B(x_M)} = \frac{(r - \delta)}{(\beta - \gamma_-)} \cdot (\gamma_+ - \gamma_-) \frac{-(\gamma_- \delta - \beta r)}{(\gamma_+ - \gamma_-)r(r - \delta)} = \frac{\beta b^2 |\gamma_-|}{2r},$$

where we used the alternative expression of $\gamma_- \delta - \beta r$ in (64). Here we have $g < 1$ from the assumption that $1 > \beta b^2 |\gamma_-| / (2r)$. Thus, we obtain for $x \leq x_M$

$$B(x) - C(x) = B(x) \left(1 - \frac{C(x)}{B(x)}\right) \geq B(x) \left(1 - \frac{C(x_M)}{B(x_M)}\right) > 0. \quad (66)$$

Fix $\epsilon > 0$. There is $y_\epsilon > 0$ such that $A(y_\epsilon) < \epsilon$ and $B(y_\epsilon)(1 - g) > 2\epsilon$. We compute the derivative of $B(x) - C(x)$ to judge its monotonicity on $(0, y_\epsilon)$:

$$B'(x) - C'(x) = x^{\beta - \gamma_- - 2} \left[\frac{(\beta - \gamma_-)(\beta - \gamma_- - 1)}{r - \delta} + (\gamma_+ - \gamma_-)(\gamma_+ - \gamma_- - 1)C_1(x_M)x^{\gamma_+ - \beta} \right].$$

By (13), we have $\beta - \gamma_- - 1 > 0$, $\gamma_+ - \gamma_- - 1 > 0$ and $\gamma_+ - \beta > 0$. Recalling that $C_1(x_M) < 0$ and converges to 0 as $x_M \rightarrow \infty$, there is $x_M > y_\epsilon$ that satisfies assumptions stated previously and such that $B(x) - C(x)$ is increasing on $(0, y_\epsilon)$. Since $A(x)$ is decreasing, this implies that $a'_\theta(x)$ is decreasing for $x \in (0, y_\epsilon)$. Using (66) and the fact that B is increasing, we have $B(x) - C(x) > B(x)(1 - g) > B(y_\epsilon)(1 - g) > 2\epsilon$ for $x \in (y_\epsilon, x_M]$. Thus, $A(x) - B(x) + C(x) < \epsilon - 2\epsilon < 0$ for $x \in (y_\epsilon, x_M]$, i.e., a'_θ is decreasing. We conclude that there is a unique value of $x = \alpha(\theta)$ that satisfies $a'_\theta(x) = 0$ within the interval $(0, x_M]$. We further recall that $a'_\theta(x) < 0$ for $x > x_M$. This establishes that $a'_\theta(x) > 0$ for $x < \alpha(\theta)$ and $a'_\theta(x) < 0$ for $x > \alpha(\theta)$.

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References

- [1] Alvarez LHR (2001) Reward functionals, salvage values and optimal stopping. *Mathematical Methods of Operations Research* 54(2):315–337.
- [2] Alvarez LHR, Lempa J (2008) On the optimal stochastic impulse control of linear diffusions. *SIAM Journal on Control and Optimization* 47(2):703–732, URL <http://dx.doi.org/10.1137/060659375>.
- [3] Attard N (2018) Nonzero-sum games of optimal stopping for Markov processes. *Applied Mathematics & Optimization* 77:567–597.

- [4] Bensoussan A, Friedman A (1977) Nonzero-sum stochastic differential games with stopping times and free boundary value problem. *Transactions of American Mathematical Society* 213:275–327.
- [5] Blumenthal R, Gettoor R (1968) *Markov Processes and Potential Theory* (Academic Press, New York).
- [6] Cattiaux P, Lepeltier JP (1990) Existence of a quasi-markov nash equilibrium for non-zero sum Markov stopping games. *Stochastics and Stochastic Reports* 30(2):85–103.
- [7] De Angelis T, Ekström E (2020) Playing with ghosts in a Dynkin game. *Stochastic Processes and their Applications* 130(10):6133–6156.
- [8] De Angelis T, Ekström E, Glover K (2022) Dynkin games with incomplete and asymmetric information. *Mathematics of Operations Research* 47:560–586.
- [9] De Angelis T, Merkulov N, Palczewski J (2022) On the value of non-Markovian Dynkin games with partial and asymmetric information. *Annals of Applied Probability* 32(3):1774–1813.
- [10] Décamps JP, Gensbittel F, Mariotti T (2022) Mixed-Strategy Equilibria in the War of Attrition under Uncertainty. *arXiv:2210.08848* URL <http://dx.doi.org/10.48550/arXiv.2210.08848>.
- [11] Décamps JP, Mariotti T (2004) Investment timing and learning externalities. *Journal of Economic Theory* 118:80–102.
- [12] Ekström E, Lindensjö K, Olofsson M (2022) How to detect a salami slicer: A stochastic controller-and-stopper game with unknown competition. *SIAM Journal on Control and Optimization* 60(1):545–574.
- [13] Elfenbein DW, Knott AM (2015) Time to exit: Rational, behavioral, and organizational delays. *Strategic Management Journal* 36(7):957–975, ISSN 1097-0266, URL <http://dx.doi.org/10.1002/smj.2262>.
- [14] Esmaeeli N, Imkeller P (2018) American options with asymmetric information and reflected BSDE. *Bernoulli* 24(2):1394–1426.
- [15] Feinstein Z, Rudloff B, Zhang J (2022) Dynamic set values for nonzero-sum games with multiple equilibriums. *Mathematics of Operations Research* 47(1):616–642.
- [16] Filippov A (1988) *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Mathematics and its Applications (Springer Netherlands).
- [17] Fudenberg D, Tirole J (1986) A theory of exit in duopoly. *Econometrica* 54(4):943–960, ISSN 00129682, 14680262, URL <http://www.jstor.org/stable/1912845>.
- [18] Gensbittel F, Grün C (2019) Zero-sum stopping games with asymmetric information. *Mathematics of Operations Research* 44(1):277–302, URL <http://dx.doi.org/10.1287/moor.2017.0924>.
- [19] Georgiadis G, Kim Y, Kwon HD (2022) The absence of attrition in a war of attrition under complete information. *Games and Economic Behavior* 131:171–185, URL <http://dx.doi.org/10.1016/j.geb.2021.11.004>.
- [20] Ghemawat P, Nalebuff B (1985) Exit. *Rand Journal of Economics* 16(2):184–194.

- [21] Grün C (2013) On Dynkin games with incomplete information. *SIAM Journal on Control and Optimization* 51(5):4039–4065, URL <http://dx.doi.org/10.1137/120891800>.
- [22] Halmos PR (1974) *Measure theory* (Springer-Verlag New York).
- [23] Hamadène S, Hassani M (2014) The multiplayer nonzero-sum Dynkin game in continuous time. *SIAM Journal on Control and Optimization* 52(2):821–835.
- [24] Hamadène S, Zhang J (2010) The continuous time nonzero-sum Dynkin game problem and application in game options. *SIAM Journal on Control and Optimization* 48(5):3659–3669.
- [25] Harrigan KR (2003) *Declining demand, divestiture, and corporate strategy* (Beard Books, Washington, D.C.).
- [26] Horn JT, Lovallo DP, Viguerie SP (2006) Learning to let go: Making better exit decisions. *McKinsey Quarterly* 2:64, URL <https://www.mckinsey.com/business-functions/strategy-and-corporate-finance/our-insights/learning-to-let-go-making-better-exit-decisions>.
- [27] Karoui NE (1981) Les aspects probabilistes du contrôle stochastique. *École d'été de Probabilités de Saint-Flour IX-1979*, 73–238 (Springer).
- [28] Krylov N (1980) *Controlled Diffusion Processes* (Springer).
- [29] Lempa J, Matomäki P (2013) A Dynkin game with asymmetric information. *Stochastics* 85(5):763–788.
- [30] Milgrom PR, Weber RJ (1985) Distributional strategies for games with incomplete information. *Mathematics of Operations Research* 10(4):619–632, URL <http://dx.doi.org/10.1287/moor.10.4.619>.
- [31] Murto P (2004) Exit in duopoly under uncertainty. *RAND Journal of Economics* 35(1):111–127, URL <http://dx.doi.org/10.2307/1593732>.
- [32] Nagai H (1987) Non zero-sum stopping games of symmetric markov processes. *Probability Theory and Related Fields* 75:487–497.
- [33] Nalebuff B, Riley J (1985) Asymmetric equilibria in the war of attrition. *Journal of Theoretical Biology* 113(3):517–527, URL [http://dx.doi.org/10.1016/S0022-5193\(85\)80036-9](http://dx.doi.org/10.1016/S0022-5193(85)80036-9).
- [34] Osborne MJ, Rubinstein A (1994) *A course in game theory* (Cambridge, MA: MIT press).
- [35] Pérez JL, Rodosthenous N, Yamazaki K (2021) Non-zero-sum optimal stopping game with continuous versus periodic observations. *arXiv:2107.08243*.
- [36] Peskir G, Shiryaev A (2006) *Optimal stopping and free-boundary problems* (Springer).
- [37] Ponsati C, Sákovics J (1995) The war of attrition with incomplete information. *Mathematical Social Sciences* 29(3):239–254, ISSN 0165-4896, URL [http://dx.doi.org/10.1016/0165-4896\(94\)00771-y](http://dx.doi.org/10.1016/0165-4896(94)00771-y).
- [38] Protter PE (2005) *Stochastic integration and differential equations* (Springer), 2nd edition.
- [39] Revuz D, Yor M (1999) *Continuous Martingales and Brownian Motion* (Springer-Verlag Berlin Heidelberg).

- [40] Riedel F, Steg JH (2017) Subgame-perfect equilibria in stochastic timing games. *Journal of Mathematical Economics* 72:36–50.
- [41] Riley JG (1980) Strong evolutionary equilibrium and the war of attrition. *Journal of Theoretical Biology* 82(3):383 – 400, ISSN 0022-5193, URL [http://dx.doi.org/10.1016/0022-5193\(80\)90244-1](http://dx.doi.org/10.1016/0022-5193(80)90244-1).
- [42] Royden HL (1988) *Real analysis* (New York: Macmillan), 3rd edition.
- [43] Shiryaev AN (2008) *Optimal stopping rules* (Springer-Verlag, Berlin).
- [44] Steg JH (2015) Symmetric equilibria in stochastic timing games. *arXiv:1507.04797* .