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ON SUSTAINABLE EQUILIBRIA

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Abstract. Following the ideas laid out in Myerson [23], Hofbauer [15] defined a Nash equilibrium of a finite game as sustainable if it can be made the unique Nash equilibrium of a game obtained by deleting/adding a subset of the strategies that are inferior replies to it. This paper prove a result about sustainable equilibria and uses it to provide a refinement as well. Our result concerns the Hofbauer-Myerson conjecture about the relationship between the sustainability of an equilibrium and its index: for a generic class of games, an equilibrium is sustainable iff its index is +1. Von Schemde and von Stengel [34] proved this conjecture for bimatrix games; we show that the conjecture is true for all finite games. More precisely, we prove that an equilibrium is isolated and has index +1 if and only if it can be made unique in a larger game obtained by adding finitely many strategies that are inferior replies to that equilibrium. It follows in a straightforward way from our result that sustainable equilibria fail the Decomposition Axiom for games as formulated by Mertens [20]. In order to rectify this problem we propose a refinement, called strongly sustainable equilibria, which is shown to exist for all regular games.

1. Introduction

Myerson [23] proposes a refinement of Nash equilibria of finite games, which he calls *sustainable equilibria*, based on the hypothesis that most games, even if they are one-shot affairs, should be analyzed not as if they are played in isolation, but rather as particular instances of many plays of such games. Myerson argues that when, say, two members of a society play a Battle-of-Sexes game, if the game has a history in this society, it becomes a "culturally familiar" game for these two players, and the past history of plays, by other members of the society, should inform play in this encounter. An equilibrium is then a

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cultural norm, an institution, in this society, and the game is typically played according to this norm. Any Nash equilibrium of the underlying game that can emerge as a norm in some society is sustainable. From this perspective, Myerson claims, the two pure-strategy equilibria in the Battle-of-Sexes game are sustainable while the mixed equilibrium is not. This claim relies on a combination of the intuitive appeal of the pure strategy equilibria and Schelling's concept of a focal point [29].

In his search for a formal definition of a sustainable equilibrium, Myerson considers, and then dismisses, on axiomatic grounds, existing refinements that yield the same prediction in the Battle-of-Sexes game as his heuristic argument does: for example, persistent equilibria [16] fail invariance [1]; and evolutionary stability fails existence. Myerson concludes his paper with a conjecture that the index of an equilibrium is a determinant of its sustainability.[1]

Hofbauer [15] distills the ideas in Myerson's paper to provide a definition of sustainable equilibria in regular games.[2] Hofbauer posits that a minimum requirement of sustainability should be that if an equilibrium of a game is sustainable, it should remain sustainable in the game obtained by restricting players' strategies to the set of best replies to the equilibrium.[3] If one also accepts that an equilibrium that is unique is sustainable, then one is lead to the following definition. Say that a game-equilibrium pair is *equivalent* to another such pair if the restrictions of the two games to the set of best replies to their respective equilibria are the same game (modulo a relabelling of the players and their strategies) and the two equilibria coincide (under the same identification). An equilibrium of a game is *sustainable* if it has an equivalent pair where the equilibrium is unique.

An implicit assumption in the definition of sustainable equilibria is that to judge the reasonableness of an equilibrium, it is sufficient to consider the game obtained by deleting all strategies that are inferior replies to the equilibrium. This assumption is a version of the IIA (Independence of Irrelevant Alternatives) Axiom, if we believe that inferior replies are indeed irrelevant. The IIA axiom has been invoked by other papers—explicitly by Kohlberg and Mertens [17], but also implicitly by, a.o., Myerson and Weibull [22]. Kohlberg and Mertens argue by examples that there is a fundamental asymmetry between adding strategies and deleting strategies; in particular, in their view, IIA cannot be invoked to say that solutions should remain solutions if dominated strategies are added. The games they construct are

[1] Interestingly, Myerson speculates that one could perhaps develop a theory of index of equilibria based on fixed-point theory, seemingly unaware, as Hofbauer [15] observes, of an extant theory in the literature (see Gül, Pearce and Stacchetti [9] and Ritzberger [27]).

[2] Call an equilibrium *regular* if locally the equilibrium is a differentiable function of the payoffs (see section 2.1).

[3] For regular games, this is equivalent to restricting players' strategies to the support of the equilibrium—see Section 4 for a discussion of this point.

nongeneric and we do not believe that their criticism applies to the class of regular games, for which sustainability is defined. (Shortly, we will offer a different critique of IIA.)

Once one accepts the IIA assumption, it seems to us incontrovertible to say that: (1) the reasonableness of an equilibrium is a property of its equivalence class; and (2) the presence of a unique equilibrium in an equivalence class makes this class reasonable. The definition of sustainability captures these twin ideas. However, this interpretation of sustainability says nothing directly about the unreasonableness of equilibria that are not sustainable. To rule out such equilibria requires an assumption that, somehow, a refinement has true predictive power only when it obtains uniqueness. Absent such an assumption, sustainability, or any variant such as our proposed refinement, may not be the final word on equilibrium selection, but it definitely provides a criterion for refining equilibria even in generic games.

The main result of the paper is a proof of the Hofbauer-Myerson conjecture: a regular equilibrium is sustainable iff its index is $+1$.[4] Von Schemde and von Stengel [34] proved this conjecture for bimatrix games. In this paper, we show that the conjecture holds for all N-person games. In fact we prove a slightly stronger version of the conjecture: an equilibrium is sustainable iff it is isolated and has index $+1$.

The validity of the Hofbauer-Myerson conjecture offers a criticism of sustainability, one that is based on the the multiplication property of index of fixed points. The fact that sustainable equilibria are those with index $+1$ means that if we compose two equilibria of two different games, each with index -1, then the result is a sustainable equilibrium. Thus, sustainable equilibria do not satisfy the Decomposition Axiom, which was formulated by Mertens [20]: the solutions of games obtained as the composition of two different games are products of the solutions of the two games.

The violation of the Decomposition Axiom is a result of the fact that the notion of equivalence in sustainability is too broad, i.e., the formulation of the IIA property strips away too many details of the game as being irrelevant. We therefore tighten the requirements for equivalence as follows. If under a game-equilibrium pair, the strategic interaction among a subset of players is independent of the strategies of the others, then it must be so in any equivalent game-equilibrium pair. This stronger equivalence notion results in smaller equivalence classes than those under sustainability, yielding a refinement we call strongly sustainable equilibria.

Our proof of the Hofbauer-Myerson conjecture shows that an equilibrium σ of a finite game G is isolated and has index $+1$ if and only if one can add finitely many new strategies,

[4]The "only if" direction was already observed by Hofbauer [15], and it is a straightforward consequence of the fact that the index of equilibria is invariant under the addition/deletion of strictly worse replies to the equilibrium and the fact that the indices of equilibria must sum to $+1$.

all inferior replies to σ, and obtain a larger game $\hat{G}$ in which σ is the unique equilibrium. To illustrate the statement, consider the battle of the sexes game $G_{1.1}$ below. It has three Nash equilibria: two are strict (t, l) and (b, r) (with index $+1$), and one is mixed (with index -1).

$$G_{1.1} = \begin{array}{c|c|c|} & l & r \\ \hline t & (3,2) & (0,0) \\ \hline b & (0,0) & (2,3) \\ \hline \end{array}$$

By adding one strategy to each player, x for player 1, and y for player 2, we obtain a game $G_{1.2}$ (see below) where b and r are strictly dominated. Removing them yields a game where x and y are strictly dominated, making the strict equilibrium (t, l) of G the unique equilibrium of $G_{1.2}$.

$$G_{1.2} = \begin{array}{c|c|c|c|} & l & r & y \\ \hline t & (3,2) & (0,0) & (0,1) \\ \hline b & (0,0) & (2,3) & (-2,4) \\ \hline x & (1,0) & (4,-2) & (-1,-1) \\ \hline \end{array}$$

This construction easily extends to all finite games: any strict equilibrium (necessarily pure, regular, and with index $+1$, see Ritzberger [27]) can be made the unique equilibrium in a larger game obtained by adding finitely many strategies that are inferior replies to the equilibrium.[5] This paper not only extends this property to isolated mixed equilibria with index $+1$ but shows that they are the only equilibria having that property, that is, if a Nash equilibrium (isolated or not) can be made unique by adding finitely many inferior replies to it, that equilibrium must be isolated and must have index $+1$.

What are the key properties of the index of equilibria that drive this equivalence? To answer this question, let us see a sketch of our proof. In one direction, suppose that an equilibrium σ of a game G is sustainable, and that (G, σ) is equivalent to a pair $(\bar{G}, \sigma)$ where σ is the unique equilibrium of $\bar{G}$. Let G^* be the game obtained from G by deleting strategies that are inferior replies to σ. It follows from a property of the index that the index of σ in G can be computed as the index of σ in G^*. The game G^* is also the game obtained from $\bar{G}$ by deleting inferior replies there. Therefore, the index of σ in G^* can also be computed as the index of σ in $\bar{G}$. As σ is the unique equilibrium of $\bar{G}$, its index is $+1$, which then gives us the result.

Going the other way, if we have a $+1$ index equilibrium σ of a game G, the sum of the indices of the other equilibria is zero, as the sum of indices over all equilibria is $+1$. Now, we can take a map whose fixed points are the Nash equilibria of G and alter it outside a

[5] To make a strict equilibrium $s = (s_1, ..., s_N)$ of a game G unique in a larger game $\hat{G}$, it is enough to add one strategy x_n per player n as in the Battle-of-Sexes, where for each player n, x_n strictly dominates all its pure strategies $t_n \neq s_n$; once the strategies $t_n \neq s_n$ are eliminated, x_n becomes strictly dominated by s_n.

neighbourhood of σ so that the new map has no fixed points other than σ.[6] By a careful addition of strategies and specification of payoffs for these strategies, we obtain a game $\bar{G}$ where any equilibrium must translate to a fixed point of the modified map of G, making σ the unique equilibrium in $\bar{G}$.

A word about our methodology is in order. In a bimatrix game, a player's payoff function is linear in his opponent's strategy and the index can be computed easily using the Shapley formula [31]. Von Schemde and von Stengel [34] were able to exploit those features and use tools from the theory of polytopes (von Schemde [35]) to prove the conjecture. In the general case, their technique is inapplicable. What we do, instead, is start with a construction involving a fixed-point map and then convert it into a game-theoretically meaningful one. In this respect, our approach is similar in spirit to, but different in details from, that in Govindan and Wilson [7].

The idea of equilibria evolving to be cultural norms—and thus favoring, for example, the pure-strategy equilibria the Battle-of-Sexes Game—has a (dynamic) learning aspect to it. This connection is not surprising, since equilibria with index $+1$ are also distinguished from their counterparts with index -1 in terms of their dynamic stability. It is well-known, both in general equilibrium and in game theory, that equilibria with index -1 are asymptotically unstable under any reasonable learning or adjustment process—cf. McLennan [19];[7] in other words, for an equilibrium to be asymptotically stable it is necessary that it have index $+1$. Even computational dynamics like those generated by homotopy algorithms (Lemke-Howson, linear-tracing procedure [10], etc.) converge to a $+1$ index equilibrium (Herings-Peeters [13]). While these results might be seen as eliminating -1 equilibria, they do not come down conclusively in favor of all $+1$ equilibria. The main reason is that it is still an open question as to whether every regular game has at least one equilibrium (necessarily of index $+1$) that is asymptotically stable with respect to some natural dynamical system. Hofbauer observed that some $+1$ index equilibria are indeed unstable for all natural dynamics, as the following potential game $G_{2.1}$ shows.[8]

$$G_{2.1} = \begin{array}{c|c|c|c|} & l & m & r \\ \hline t & (10,10) & (0,0) & (0,0) \\ \hline m & (0,0) & (10,10) & (0,0) \\ \hline b & (0,0) & (0,0) & (10,10) \\ \hline \end{array}$$

[6]The possibility of such a construction follows from a deep result in algebraic topology, the Hopf Extension Theorem (Corollary 8.1.18, [32]).

[7]In fact, this observation leads McLennan to articulate his *index* $+1$ *principle*, which selects $+1$ equilibria.

[8]Note however that in $G_{2.1}$, the three pure equilibria are strict, have index $+1$ and can be shown to be asymptotically stable for all natural dynamics.

The profile where players mix uniformly is an isolated equilibrium with index +1 and so it can be made unique in a larger game, for example by adding the three strategies x, y, and z as in $G_{2.2}$ below.[9]

$G_{2.2} =$

	l	m	r	x	y	z
t	$(10,10)$	$(0,0)$	$(0,0)$	$(0,11)$	$(10,5)$	$(0,-10)$
m	$(0,0)$	$(10,10)$	$(0,0)$	$(0,-10)$	$(0,11)$	$(10,5)$
b	$(0,0)$	$(0,0)$	$(10,10)$	$(10,5)$	$(0,-10)$	$(0,11)$

The fact is that all natural dynamics increase the potential of G_1; since the completely-mixed equilibrium minimizes that potential, it is unstable for all natural dynamics—cf. Hofbauer [15] for more about his second conjecture, which states that any regular game has at least one +1 index equilibrium that is asymptotically stable w.r.t. some natural dynamics.[10]

The rest of the paper is organized as follows. Section 2 sets up the problem and states our main theorem, which is about the Hofbauer-Myerson conjecture. It also gives an informal summary of the theory of index of equilibria. Section 3 is devoted to proving the theorem. Section 4 provides a discussion of the role of unused best replies in the definition of sustainable equilibria. Section 5 shows that sustainability fails the Decomposition Axiom, which leads to the concept of strongly sustainable equilibria in Section 6. Section 7 concludes the main body of the paper. An appendix reviews a construct from the theory of triangulations.

2. Definitions and statement of the main theorem

A finite game in normal form is a triple $(\mathcal{N}, (S_n)_{n\in\mathcal{N}}, G)$ where: $\mathcal{N} = \{1, \ldots, N\}$ is the set of players; for each $n \in \mathcal{N}$, S_n is a finite set of pure strategies; and, letting $S \equiv \prod_{n\in\mathcal{N}} S_n$ be the set of pure strategy profiles, $G : S \to \mathbb{R}^N$ is the payoff function. By a slight abuse of terminology, we will refer to a game by its payoff function G.

Given a game G, for each n, let Σ_n be the set of n's mixed strategies and let $\Sigma \equiv \prod_{n\in\mathcal{N}} \Sigma_n$. Also, for each n, $S_{-n} \equiv \prod_{m\neq n} S_n$, and $\Sigma_{-n} \equiv \prod_{m\neq n} \Sigma_n$. The payoff function G extends to Σ in the usual way and we will denote this extension by G as well.

Define an equivalence relation on game-equilibrium pairs as follows. For $i = 1, 2$, let $((\mathcal{N}^i, (S^i_n)_{n\in\mathcal{N}^i}, G^i), \sigma^i)$ be a game-equilibrium pair, i.e., σ^i is an equilibrium of G^i. Say that $(G^1, \sigma^1) \sim (G^2, \sigma^2)$ if, *up to a relabelling of players and strategies*, the restriction of G^1 to

[9] We refer the interested reader to von Schemde [35], p. 89-91, to understand how the strategies x, y and z are geometrically constructed.

[10] It follows from Demichelis and Ritzberger [5] that Hofbauer's second conjecture is false if we include all games: a necessary condition for a component of Nash equilibria to be asymptotically stable is that its index agree with its Euler characteristic; yet, there are examples (e.g. [12] Fig. 8 or [28] p. 325) where all components of equilibria are convex (and so have Euler characteristic +1), but no component has index +1.

the set of best replies to σ^1 is the same game as the restriction of G^2 to the set of best replies to σ^2, and the equilibria coincide under this identification. It is easily checked that $\sim$ is an equivalence relation. Observe that under this notion of equivalence, if there are best replies to, say, σ^1 in G^1 that are unused, then G^2 must include those strategies as well. In Section 4, we examine the issue of deleting unused best replies.

Definition 2.1. An equilibrium σ of a game G is *sustainable* if $(G, \sigma) \sim (\bar{G}, \bar{\sigma})$ for a game $\bar{G}$ where $\bar{\sigma}$ is the unique equilibrium.

Sustainability is a property of equivalence classes and we could say that the canonical representation of a sustainable equilibrium is the game-equilibrium pair where there are no inferior replies to the equilibrium. A concept closely related to sustainability, and which will be of particular importance for our arguments, is that of an embedding. One game embeds another if the latter is obtained from the former by deleting strategies (without a need for relabeling of players or strategies). Formally:

Definition 2.2. A game $(\mathcal{N}, \bar{S}, \bar{G})$ *embeds* $(\mathcal{N}, S, G)$ if: (1) for each n: $S_n \subseteq \bar{S}_n$; and (2) the restriction of $\bar{G}$ to S equals G.

For notational convenience, we will talk of a game $\bar{G}$ embedding G. When $\bar{G}$ embeds G, we view the set Σ of mixed strategies of G as a subset of the set $\bar{\Sigma}$ of mixed strategies in $\bar{G}$. Obviously, if $\bar{G}$ embeds G and σ is an equilibrium of $\bar{G}$ where for each n, the strategies that are not in S_n are inferior replies, then $(G, \sigma) \sim (\bar{G}, \sigma)$.

2.1. **Index and degree of equilibria.** In this section we recall the properties of index and degree of equilibria that are required for Theorem 2.3. Both the index and the degree of equilibria are integer measures of the robustness of components of equilibria to perturbations. They differ in the space of perturbations they consider (perturbations of fixed-point maps vs payoffs perturbations) but assign the same integer to components of equilibria. We start with the degree of equilibria. First we give a definition of degree only for regular equilibria and then we extend the definition to components of equilibria. This approach allows to bypass the use of algebraic topology. For the general definition, see for e.g., Govindan and Wilson [7].

2.1.1. *Degree of Equilibria.* Fix both the player set $\mathcal{N}$ and the strategy space S. The space of games with strategy space S is then the Euclidean space $\Gamma \equiv \mathbb{R}^{N|S|}$ of all payoff functions G. Let $\mathcal{E}$ be the graph of the Nash equilibrium correspondence over Γ, i.e., $\mathcal{E} = \{\, (G, \sigma) \in \Gamma \times \Sigma \mid \sigma \text{ is a Nash equilibrium of } G \,\}$. Let $proj : \mathcal{E} \to \Gamma$ be the natural projection: $proj(G, \sigma) = G$. By the Kohlberg-Mertens Structure Theorem [17], there exists a homeomorphism $h : \mathcal{E} \to \Gamma$

such that h^{-1} is differentiable almost everywhere. Say that an equilibrium σ of a game G is *regular* if $proj \circ h^{-1}$ is differentiable and has a nonsingular Jacobian at $h(G, \sigma)$; and say that a game is *regular* if each equilibrium σ of G is regular. If an equilibrium σ is regular, then it is a *quasi-strict equilibrium*—that is, all unused strategies are inferior replies[11]—and locally, the equilibrium is a smooth, even analytic, function of the game; moreover, it is also a regular equilibrium in the space of games obtained by deleting the unused strategies or, indeed, by adding strategies that are inferior replies to the equilibrium. The set of games that are not regular is a closed subset of lower dimension—actually codimension one—in Γ and thus regular games are generic.

If an equilibrium σ of a game G is regular, then we can assign a *degree* to it that is either $+1$ or -1 depending on whether the Jacobian of $proj \circ h^{-1}$ at $h(G, \sigma)$ has a positive or a negative determinant. An inspection of the formula for the Jacobian shows that the degree of a regular equilibrium σ is the same as its degree computed in the space of games obtained by deleting the strategies that are inferior replies to σ. Therefore, if σ is a regular equilibrium of G and if $(G, \sigma) \sim (\bar{G}, \bar{\sigma})$ then $\bar{\sigma}$ is a regular equilibrium of $\bar{G}$ and it has the same degree as σ, making degree an invariant for an equivalence class.

As the Kohlberg-Mertens homeomorphism extends to the one-point compactification of $\mathcal{E}$ and Γ, and $proj \circ h^{-1}$ is homotopic to the identity on this extension, the sum of the degrees of equilibria of a regular game is $+1$.

If G is nongeneric, we can define the degree of a component of equilibria as the sum of the degrees of equilibria in a neighborhood of the component for a regular game that is in a neighborhood of G; this computation is independent of the neighborhoods chosen, as long as they are sufficiently small. The sum of the degrees of the components of equilibria of a game is $+1$.

When a component is a singleton, then the equilibrium is said to be isolated. If a game is regular, then every equilibrium is isolated. However, there are games with isolated equilibria that are not regular (see the first game of Section 4, for an example). By the degree of an isolated equilibrium, we mean its degree as a component of equilibria. If σ is an isolated equilibrium and a strategy s_n of a player n is not a best reply to σ, then it is not a best reply to any regular equilibrium that is close σ for any regular game that is close to the given game. Consequently, the degree of an isolated equilibrium can computed as its degree in the game obtained by deleting all strategies that are not best replies to the equilibrium.

2.1.2. *Index of Equilibria.* In fixed-point theory, the index of fixed points contains information about their robustness when the map is perturbed. (See McLennan [18] for an account

[11]see Ritzberger [27] or van Damme [33].

of index theory written primarily for economists.) Since Nash equilibria are obtainable as fixed points, index theory applies directly to them. For simplicity, suppose $f : U \to \Sigma$ is a differentiable map defined on a neighborhood U of Σ in $\mathbb{R}^{\sum_n |S_n|}$ and such that the fixed points of f are the Nash equilibria of a game G. Let d_f be the displacement of f, i.e., $d_f(\sigma) = \sigma - f(\sigma)$. Then the Nash equilibria of G are the zeros of d_f. Suppose now that the Jacobian of d at a Nash equilibrium σ of G is nonsingular. Then we can define the *index* of σ under f as ± 1 depending on whether the determinant of the Jacobian of d_f is positive or negative. As with the degree of equilibria, we can obtain a definition the index of a component of equilibria by considering a perturbation of its displacement that is differentiable and has finitely many zeros.

One potential problem with the definition of index is the dependence of the computation on the function f, as intuitively we would think of the index as depending only on the game G. But, under some regularity assumptions on f, we can show that the index is independent of f. Specifically, consider the class of continuous maps $F : \Gamma \times \Sigma \to \Sigma$ with the property that the fixed points of the restriction of F to $\{ G \} \times \Sigma$ are the Nash equilibria of G. Demichelis and Germano [4] show that the index of equilibria is independent of the particular map in this class that is used to compute it; Govindan and Wilson [6] show that the degree is equivalent to the index computed using one of the maps in this class, the fixed-point map defined by Gül, Pearce and Stacchetti [9]. Thus, *the index and degree of equilibria coincide*—see Demichelis and Germano [4] for an alternate, more direct, proof of this equivalence. Given these results, for a regular equilibrium, we can talk unambiguously of its index and use the term degree interchangeably with it.

2.2. **Statement of the main theorem.** The following theorem settles the Myerson-Hofbauer conjecture in the affirmative.

Theorem 2.3. *An equilibrium is sustainable iff it is isolated and its index is* $+1$.

As the sum of the indices of the equilibria of a regular game is $+1$, there is at least one with index $+1$. Thus, we have the following corollary.

Corollary 2.4. *Every regular game has at least one sustainable equilibrium.*

3. Proof of Theorem 2.3

We will present the proof in a sequence of steps, each of which will be carried out in a separate subsection.

3.1. **The index of a sustainable equilibrium.** We begin with a proof of the necessity of the conditions. Assume σ^* is a sustainable equilibrium of a game G, and let $(G, \sigma^*) \sim (\bar{G}, \bar{\sigma})$, where $\bar{\sigma}$ is the unique equilibrium of $\bar{G}$. We claim first that σ^* is isolated. Suppose to the contrary that it is not. Let $(\sigma^k)_{k\in\mathbb{N}}$ be a sequence of equilibria of G converging to σ^*, with $\sigma^k \neq \sigma^*$ for all k. Passing to a subsequence if necessary, we can assume that the support T of σ^k is fixed and that all strategies in the support are best replies to σ^*. All the profiles of strategies in T are present in $\bar{G}$. Hence, up to relabeling, σ^k is a feasible profile in $\bar{G}$ for all k. As $\bar{\sigma}$ is the unique equilibrium in $\bar{G}$, by passing to another subsequence, there is a player n and a fixed profitable pure-strategy deviation t_n from σ^k in $\bar{G}$. Obviously t_n is not available in G. By continuity, t_n is a best reply to $\bar{\sigma}$ in $\bar{G}$, which contradicts the assumption that $(\bar{G}, \bar{\sigma}) \sim (G, \sigma^*)$. Therefore, σ^* is an isolated equilibrium.

As we saw in the previous section, the index of an isolated equilibrium can be computed by deleting all strategies that are not best replies to it. Such deletions in the two games G and $\bar{G}$ yield the same game. Hence the index of σ^* in G is the same as the index of $\bar{\sigma}$ in $\bar{G}$. Since $\bar{\sigma}$ is the unique equilibrium of $\bar{G}$, its index is $+1$, and the result follows.

3.2. **Preliminaries.** The rest of the section is devoted to proving the sufficiency of the conditions. When $N = 1$, the game G reduces to a decision problem. G has a single component of equilibria, which has degree $+1$. Therefore, if G has an isolated equilibrium, it is the unique equilibrium of the game, which proves the sufficiency part of the Theorem for the case $N = 1$. For the rest of the proof, we will assume that $N \geqslant 2$. In this subsection, we introduce some key ideas that we exploit in the proof.

First, we gather a list of notational conventions to be used. Throughout Section 3 (but not in Appendix A) we use the ℓ_∞-norm on Euclidean spaces. For any subset A of a topological space X, we let $\partial_X A$ be its topological boundary and $\mathrm{int}_X(A)$ its interior. If C is a convex set in a Euclidean space, then ∂C and $\mathrm{int}(C)$ refer to the boundary and the interior of C in the affine space generated by C.

For a game G, recall that Nash [25] obtains its equilibria as fixed points of a map on the strategy space. This function, which we denote by f, is defined as follows. For each n, s_n and σ, let $\phi_{n,s_n}(\sigma) \equiv \max\{\, 0, G_n(s_n, \sigma_{-n}) - G_n(\sigma)\,\}$ and $\phi_n \equiv \prod_{s_n\in S_n} \phi_{n,s_n}$; then

$$f_{n,s_n}(\sigma) \equiv \frac{\sigma_{n,s_n} + \phi_{n,s_n}(\sigma)}{1 + \sum_{t_n\in S_n} \phi_{n,t_n}(\sigma)}.$$

For each n, $f_n(\sigma) = \sigma_n$ iff $\phi_{n,s_n} = 0$ for each $s_n \in S_n$. If $f_n(\sigma) \neq \sigma_n$ for some n and σ, then letting

$$r_n(\sigma) = \left(\sum_{t_n \in S_n} \phi_{n,t_n}(\sigma)\right)^{-1} \phi_n(\sigma)$$

and

$$\lambda_n(\sigma) = \frac{1}{1 + \sum_{t_n \in S_n} \phi_{n,t_n}(\sigma)},$$

we have

$$f_n(\sigma) = \lambda_n(\sigma)\sigma_n + (1 - \lambda_n(\sigma))r_n(\sigma).$$

Thus $f_n(\sigma)$ is an average of σ and a mixed strategy $r_n(\sigma)$; $r_n(\sigma)$ has the following properties: (i) it assigns a positive probability to a pure strategy iff it does strictly better than σ_n—in particular, it assigns zero probability to some strategy in the support of σ_n, as $f_n(\sigma) \neq \sigma_n$; (ii) it assigns the highest probabilities to the best replies to σ.

Our proof technique is to show that for each isolated $+1$ equilibrium σ^* of G, we can embed G in a game $\bar{G}$ where σ^* is the unique equilibrium and the newly added strategies are inferior replies to σ^*.

3.3. **A Simple property of isolated equilibria.** From now on fix a game G and let σ^* be an isolated equilibrium with index $+1$. For each n, let S_n^* be the support of σ_n^*. Our objective in this subsection is to record the following simple, and yet consequential, property of σ^*.

Proposition 3.1. *There exists $\bar{\varepsilon} > 0$ such that: if $\sigma \neq \sigma^*$ is an equilibrium of G, then there exist two different players n_1, n_2, such that for $i = 1, 2$, there exists $s_{n_i} \in S_{n_i}^*$ with $\sigma_{n_i,s_{n_i}} < \sigma^*_{n_i,s_{n_i}} - \bar{\varepsilon}$.*

Proof. If the statement is not true, there exist a sequence $k \to \infty$, a corresponding sequence σ^k of equilibria converging to some σ, and a player n such that: (i) $\sigma^k \neq \sigma^*$ for all k; and (ii) for all $m \neq n$ and $s_m \in S_m^*$, $\sigma^k_{m,s_m} \geqslant \sigma^*_{m,s_m} - k^{-1}$. Therefore, $\sigma_m = \sigma_m^*$ for all $m \neq n$. But $\sigma_n \neq \sigma_n^*$, as σ^* is isolated. This implies that $\lambda\sigma + (1 - \lambda)\sigma^*$ is an equilibrium for all $\lambda \in [0, 1]$, again contradicting the fact that σ^* is isolated. Thus, there exists $\bar{\varepsilon}$ with the stated property. $\square$

The remainder of the proof will make use of the following notation: for all $s_n \in S_n^*$ and $\delta > 0$, let B_n^δ be the set of $\sigma_n \in \Sigma_n$ such that $\sigma_{n,s_n} \geqslant \sigma^*_{n,s_n} - \delta$. Let B^δ be the set of σ such that σ_n is not in B_n^δ for at most one n. (N.B. $\prod_{n \in \mathcal{N}} B_n^\delta$ is a proper subset of B^δ). Since $B^\delta = \bigcup_n (\Sigma_n \times \prod_{m \neq n} B_m^\delta)$ is the union of finitely many closed sets, it is closed. Fix $\bar{\varepsilon} > 0$

according to Proposition 3.1 and $\varepsilon \in (0, \bar{\varepsilon})$. Denote $V \equiv B^{\bar{\varepsilon}}$ and $U \equiv B^{\varepsilon}$. Note that U is contained in the interior of V.

3.4. Killing all fixed points of f other than σ^*.

From the viewpoint of fixed-point theory, our problem amounts to embedding Σ as a proper face of a polytope $\bar{\Sigma}$, extending f (the Nash map) to a function $\bar{f}$ on it, and then modifying $\bar{f}$ such that its only fixed point is σ^*. From a game-theoretic viewpoint, there is an additional problem introduced by the caveat that $\bar{f}$ should, in a sense, be realizable as a fixed-point map of a game $\bar{G}$ that embeds G—i.e., a map whose fixed points are the equilibria of game $\bar{G}$. In this subsection, we solve the first problem partially, by constructing a map f^0 that coincides with f on V and that has no fixed points outside it. We will later use this map to construct the embedding $\bar{G}$.

If necessary by adding a strictly dominated strategy for each player, we can assume that σ_n^* belongs to $\partial\Sigma_n$ for each n. (Recall that sustainability and index are properties of equivalence classes of isolated equilibria, so that the addition of such strategies is harmless.)

Proposition 3.2. *There is a continuous function $f^0 : \Sigma \to \Sigma$ such that $f^0 \restriction_V = f \restriction_V$, and σ^* is the unique fixed point of f^0.*

Proof. Let $X \equiv \Sigma \setminus \mathrm{int}_\Sigma(V)$. The boundary of X is relative to the affine space generated by Σ, i.e., $\partial X \equiv (\partial\Sigma \setminus V) \cup \partial_\Sigma V$. We claim now that $(X, \partial X)$ is homeomorphic to a ball with boundary. Indeed, the desired homeomorphism can be constructed as follows. Pick a completely-mixed strategy-profile σ^0 such that $\sigma^0_{n,s_n} < \sigma^*_{n,s_n} - \bar{\varepsilon}$, for all n and $s_n \in S_n^*$. (Such a choice is possible since σ_n^* belongs to the boundary of Σ_n and if necessary, we can decrease the $\bar{\varepsilon}$ defined above.) The set X is star-convex at σ^0: for each $\sigma \in X$, $\lambda\sigma + (1-\lambda)\sigma^0 \in X$ for all $\lambda \in [0,1]$. Therefore, there is now a closed ball around σ^0 in $\Sigma \setminus \partial\Sigma$ that is contained in $X \setminus \partial X$ that is homeomorphic to X using radial projections from σ^0.

Define $\tilde{f} : \Sigma \to \Sigma$ as follows. First, using Urysohn's lemma, construct a continuous function $\alpha : \Sigma \to [0,1]$ that is zero on V and positive everywhere else. Then, letting τ_n^0 be the barycenter of Σ_n for each n, define $\tilde{f}(\sigma) = (1-\alpha(\sigma))f(\sigma) + \alpha(\sigma)\tau^0$. The function $\tilde{f}$ equals f on V and therefore σ^* is the unique fixed point of $\tilde{f}$ in V. The set $\Sigma \setminus V$ is mapped by $\tilde{f}$ to $\Sigma \setminus \partial\Sigma$. Hence all the other fixed points of $\tilde{f}$ belong to $X \setminus \partial X$.

For each n, let A_n be the hyperplane in $\mathbb{R}^{S_n}$ through the origin and with normal $(1, \ldots, 1)$, and let $A = \prod_n A_n$. The map $d_{\tilde{f}}$ maps Σ into A. As the index of σ^* is $+1$, the sum of the indices of the other components of fixed points of $\tilde{f}$, which are contained in $X \setminus \partial X$, is zero. Therefore, $d_{\tilde{f}} : (X, \partial X) \to (A, A - 0)$ has degree zero. By the Hopf Extension Theorem (cf.

Corollary 8.1.18, [32]) there exists a map $\hat{d}^0$ from X to $A - 0$ such that its restriction to ∂X coincides with $\tilde{d}$. Extend $\hat{d}^0$ to the whole of Σ by letting it be $d_{\tilde{f}}$ outside X, i.e., on $V \setminus \partial_\Sigma V$.

For each $\sigma \in \Sigma$, there exists $\lambda \in (0, 1]$ such that $\sigma - \lambda \hat{d}^0(\sigma) \in \Sigma$: indeed, this is obvious for $\sigma \in X \setminus \partial X$, since σ belongs to the interior of Σ; for $\sigma \in \partial X \cup V$, we can take $\lambda = 1$ as $\hat{d}^0 = d_{\tilde{f}}$. Now for each σ, let $\lambda(\sigma)$ be the largest $\lambda \in [0, 1]$ such that $\sigma - \lambda(\sigma)\hat{d}^0(\sigma) \in \Sigma$. Note that the map $\sigma \mapsto \lambda(\sigma)$ is continuous. Define $f^0 : \Sigma \to \Sigma$ by $f^0(\sigma) = \sigma - \lambda(\sigma)\hat{d}^0(\sigma)$. The map f^0 is continuous, coincides with f on V, and has σ^* as its unique fixed point. $\square$

3.5. **Example.** We now introduce a running example, where we can carry out our construction numerically, and which we hope will aid in the understanding of the proof. The example differs from the text in one somewhat irrelevant respect: we focus on symmetric strategies, as it reduces the dimension of the problem and allows us to perform a two-dimensional graphical analysis as well.

The game we study is a two-player coordination game given below.

	L	R
L	$(1, 1)$	$(0, 0)$
R	$(0, 0)$	$(1, 1)$

Given our restriction to symmetric strategies, we will dispense with the subscript for players in the notation (here and throughout the paper when we work with this example). Thus, a symmetric mixed-strategy profile is represented by one number, $x \in [0, 1]$, where x is the probability of playing L. There are two pure strategy equilibria: $x = 1$ and $x = 0$, both of which have index $+1$; and there is a mixed equilibrium, $x = 1/2$, which has index -1. The restriction of the Nash map f to symmetric strategies allows us to represent it as a function from $[0, 1]$ to itself. By computation, we obtain:

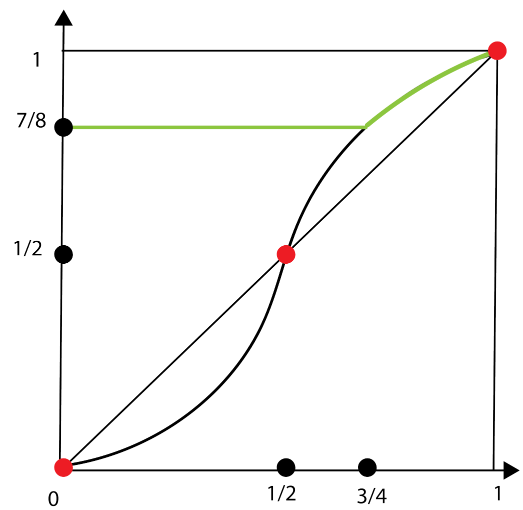
$$f(x) \equiv \begin{cases} \frac{x}{1-2x^2+x} & \text{if } x \in [0, 1/2] \\ \frac{x-2x^2+3x-1}{1-2x^2+3x-1} & \text{if } x \in (1/2, 1] \end{cases}$$

The graph of f, with its three fixed points corresponding to the three equilibria of the game, is illustrated below in Figure 1. Let $V \equiv [2/3, 1]$. We can directly construct a map f^0 as in the previous section, whose only fixed point is $x = 1$ (see the graph of f^0 in green in Figure 1).[12]

$$f^0(x) \equiv \begin{cases} \frac{7}{8} & \text{if } x \in [0, 2/3] \\ \frac{x-2x^2+3x-1}{1-2x^2+3x-1} & \text{if } x \in (2/3, 1] \end{cases}$$

[12] From Figure 1, one can observe that it is impossible to construct a map which coincides with f in the neighbourhood of $x = 1/2$ (the equilibrium with index -1) and which has 1/2 as the unique fixed point.

FIGURE 1. Graphs of f (black) and f^0 (green)



3.6. **A parametrized family of perturbed games.** Ideally, we would like a game G^0 such that f^0 obtained in Proposition 3.2 is the Nash map of G^0. This seems to be too strong a property to hold. However, a continuously perturbed game of our original game G can be constructed in such a way that f^0 does constitute an approximate best-reply. The proof of Proposition 3.4 makes this point precise.

Definition 3.3. Given a vector $h \in \prod_{n \in \mathcal{N}} \mathbb{R}^{S_n}$, let $G \oplus h$ be the game where the payoff to player n from a profile $s \in S$ is $G_n(s) + h_{n,s_n}$.

Proposition 3.4. *There is a continuous function $g : \Sigma \to \prod_{n \in \mathcal{N}} \mathbb{R}^{S_n}$ such that: (1) $g(\sigma) = 0$ for all $\sigma \in U$; (2) σ is an equilibrium of $G \oplus g(\sigma)$ iff $\sigma = \sigma^*$.*

Proof. For each n, let Z_n be $(d_{f_n^0})^{-1}(0) \cap (\Sigma \setminus \mathrm{int}_\Sigma(U))$. Let Z_n^1 be the complement of Z_n in $\Sigma \setminus \mathrm{int}_\Sigma(U)$. Define $r_n^0 : Z_n^1 \to \partial\Sigma_n$ as follows. For each $\sigma \in Z_n^1$, let $r_n^0(\sigma)$ be the unique point in $\partial\Sigma_n$ on the ray from σ_n through $f_n^0(\sigma)$, i.e., it is the unique point of the form $(1-\alpha)\sigma_n + \alpha f_n^0(\sigma)$ for $\alpha \geqslant 1$ that belongs to $\partial\Sigma_n$. If $\sigma \in Z_n^1$, then there exists some t_n that is in the support of σ_n but not of $r_n^0(\sigma)$. For each n, s_n, let Z_{n,s_n}^+ be the closure of the set of $\sigma \in Z_n^1$ for which $r_{n,s_n}^0(\sigma) \geqslant r_{n,t_n}^0(\sigma)$ for all $t_n \in S_n$. If $f^0(\sigma) = f(\sigma)$ (the Nash map) and $\sigma \in Z_n^1$, then $r_n^0(\sigma)$ equals $r_n(\sigma)$ as defined in subsection 3.2; therefore, $\sigma \in Z_{n,s_n}^+$ iff s_n is a best reply to σ in G.

We are now ready to define the function $g(\sigma)$. In doing so, we repeatedly invoke Urysohn's lemma to construct functions that are zero on a closed set and positive outside it. First, let $v_n(\sigma) = \max_{s_n} G_n(s_n, \sigma_{-n})$. Second, let $\beta^1_n : \Sigma \to [0,1]$ be a continuous function that is zero on Z_n and positive everywhere else. Third, for each n, s_n, let $\beta^2_{n,s_n} : \Sigma \to [0,1]$ be a continuous function that is one on Z^+_{n,s_n} and strictly smaller than one elsewhere. Finally, let $\beta^3 : \Sigma \to [0,1]$ be a continuous function that is one on $\Sigma \setminus \mathrm{int}_\Sigma(V)$, zero on U and strictly positive everywhere else. For each n, s_n and σ, define:

$$g_{n,s_n}(\sigma) = \beta^3(\sigma)\beta^2_{n,s_n}(\sigma)[v_n(\sigma) - G_n(s_n, \sigma_{-n}) + \beta^1_n(\sigma)].$$

If $\sigma \in U$, then $g(\sigma) = 0$ as $\beta^3(\sigma) = 0$; and σ is an equilibrium of $G \oplus g(\sigma)$ iff $\sigma = \sigma^*$.

Suppose $\sigma \notin U$. Since σ^* is the only fixed point of f^0, there exists some n such that $f^0_n(\sigma) \neq \sigma_n$. For this n, there exists s_n such that $\sigma \in Z^+_{n,s_n}$ (take s_n s.t. $r^0_{n,s_n}(\sigma) \geqslant r^0_{n,t_n}(\sigma)$ for all $t_n \in S_n$); and there is t_n in the support of σ_n but not in the support of $r^0_n(\sigma)$. This implies $\beta^2_{n,s_n}(\sigma) = 1$, while $\beta^2_{n,t_n}(\sigma) < 1$.

If $\sigma \notin V$, then $\beta^3(\sigma) = 1$ and therefore

$$G_n(s_n, \sigma_{-n}) + g_{n,s_n}(\sigma) = v_n(\sigma) + \beta^1_n(\sigma) > v_n(\sigma) + \beta^2_{n,t_n}(\sigma)\beta^1_n(\sigma) \geqslant G_n(t_n, \sigma_{-n}) + g_{n,t_n}(\sigma),$$

showing that σ is not an equilibrium of $G \oplus g(\sigma)$.

If $\sigma \in V \setminus U$, then as f^0 coincides with f, s_n is a best reply against σ while t_n is not. Thus $G_n(s_n, \sigma_{-n}) = v_n(\sigma)$ and $G_n(t_n, \sigma_{-n}) < v_n(\sigma)$. Since $\beta^3(\sigma) > 0$, we obtain that

$$\begin{aligned} G_n(s_n, \sigma_{-n}) + g_{n,s_n}(\sigma) &= v_n(\sigma) + \beta^3(\sigma)\beta^1_n(\sigma) &> v_n(\sigma) + \beta^3(\sigma)\beta^2_{n,t_n}(\sigma)\beta^1_n(\sigma) \\ & &> G(t_n, \sigma_{-n}) + g_{n,t_n}(\sigma), \end{aligned}$$

and again σ is not an equilibrium of $G \oplus g(\sigma)$. Thus the function g has the desired properties. □

3.7. **Example.** We continue with the example of subsection 3.5. We will construct the function g of the previous section. (Recall our convention of dropping the player subscript for terms like Z_n, Z^1_n.) What we are after is a function $g : [0,1] \to \mathbb{R}^{\{L,R\}}$ such that $x = 1$ is the only (symmetric) equilibrium of the game $G \oplus g(x)$. Let $U \equiv [3/4, 1]$. First recall that f^0 has no fixed point outside U, so Z is empty and $Z^1 = [0, 3/4]$. For each $x \in [0, 3/4]$, $f^0(x) > x$, which implies that $r^0_n(x) = 1$ so that $Z^+_L = [0, 3/4]$. Now, by computation one gets

$$v_n(x) \equiv \begin{cases} 1 - x & \text{if } x \in [0, 1/2] \\ x & \text{if } x \in (1/2, 1]. \end{cases}$$

Because Z is empty, we can set $\beta^1(\cdot)$ to be a constant function equal to $\delta > 0$. The map $\beta^3(\cdot)$ will be defined as follows:

$$\beta^3(x) \equiv \begin{cases} 1 & \text{if } x \in [0, 2/3] \\ -12x + 9 & \text{if } x \in (2/3, 3/4) \\ 0 & \text{if } x \in (3/4, 1]. \end{cases}$$

There is no need to introduce the function $\beta^2(\cdot)$ in this example. Putting these ingredients together, we can define g as follows: $g_L(x) = \beta^3(x)[v_1(x) - x + \delta]$ and $g_R \equiv 0$. We show that if the payoffs are now perturbed according to the bonus function g for each player, the only remaining equilibrium is $x = 1$. Let x be an equilibrium of the perturbed game. If $x \in [0, 2/3]$, $\beta^3(x) = 1$, so if player 1 plays x, player 2 gets δ more than the best payoff $v_2(x)$ in the unperturbed game from playing L whereas by playing R he will not get more than $v_2(x)$. Therefore, $x = 1$, which is a contradiction. On the other hand, if $x \in [2/3, 1]$, then L is already the strict best-reply in the original game and since g is nonnegative, it follows that $x = 1$ is the unique equilibrium of the perturbed game.

3.8. **The embedding $\bar{G}^\delta$.** In this subsection we construct the embedding $\bar{G}^\delta$ of G, for which σ^* is the unique equilibrium. The construction adds a finite number of mixed strategies as pure strategies for each player and defines their payoffs to eliminate all other equilibria.[13]

Choose $0 < \varepsilon^* < \varepsilon$ such that $\sigma^*_{n,s_n} > \varepsilon^*$ for each n and $s_n \in S^*_n$. Let $U^*_n \equiv B^{\varepsilon^*}_n$ for each $n \in \mathcal{N}$, and let $U^* \equiv B^{\varepsilon^*}$. The set U^* is a proper subset of U (and a closed subset contained in the interior of V). The set $\Sigma \setminus \mathrm{int}_\Sigma(U^*)$ is compact, $g(\cdot)$ is continuous, and no $\sigma \in \Sigma \setminus \mathrm{int}_\Sigma(U^*)$ is an equilibrium of $G \oplus g(\sigma)$, as shown in Proposition 3.4. Hence, there exists $\eta > 0$ such that no $\sigma \in \Sigma \setminus \mathrm{int}_\Sigma(U^*)$ is an equilibrium of $G \oplus g$ for any g with $\|g - g(\sigma)\| \leqslant \eta$. Choose $1/3 > \zeta > 0$ such that:

(i) $\|g(\sigma) - g(\sigma')\| \leqslant \eta$, if $\|\sigma - \sigma'\| \leqslant \zeta$.

(ii) For each n, the distance between U^*_n and $\partial_{\Sigma_n} U_n$ is greater than ζ.

For each n, choose an arbitrary object 0^*_n (not in S^*_n). Let Θ_n be the set of distributions over $\prod_{m \neq n}(S^*_m \cup \{\, 0^*_m \,\})$. For each n and $m \neq n$, let $\Theta_{n,m} = \Delta(S^*_m \cup \{0^*_m\})$, and for $\theta_n \in \Theta_n$ let $\theta_{n,m}$ be the marginal of θ_n on $S^*_m \cup \{0^*_m\}$. Let $\theta^0_n = (0^*_m)_{m \neq n}$. For each n, take a triangulation $\mathcal{T}_n$ of $\Sigma_n \times \Theta_n$ with the following properties: (See Appendix A for the details.)

(iii) The only vertices in $\Sigma_n \times \{\, \theta^0_n \,\}$ of $\mathcal{T}_n$ are pure strategies (s_n, θ^0_n), $s_n \in S_n$;

[13] von Schemde and von Stengel have an explicit bound on the number of strategies they add, which is three times the number of pure strategies in G. In our construction, we have no way of obtaining such a bound: the construction depends on the fixed-point map f^0 obtained in subsection 3.4, which in turn relies on an existence result, the Hopf Extension Theorem.

(iv) Letting Θ_n^1 be the face of Θ_n where θ_n^0 has zero probability, if $T_n \in \mathcal{T}_n$ is a simplex either with a face in $\Sigma_n \times \Theta_n^1$, or shares a face with such a simplex, then the diameter of T_n is less than ζ;

(v) There exists a convex function $\varrho_n : \Sigma_n \times \Theta_n \to \mathbb{R}_+$ such that: (v.i) $\varrho_n(\lambda x + (1-\lambda)y) = \lambda\varrho_n(x) + (1-\lambda)\varrho_n(y)$ iff x and y belong to a simplex T_n of $\mathcal{T}_n$; (v.ii) $\varrho_n^{-1}(0) = \Sigma_n \times \{\, \theta_n^0 \,\}$.

Let $\bar{S}_n^0$ be the set of vertices of the triangulation $\mathcal{T}_n$. Let $\bar{S}_n^1 \equiv \bar{S}_{n+1}^0$ (we are treating players' names as integers modulo N, thus $N+1=1$). A typical element of $\bar{S}_n^0$ is a pair (σ_n, θ_n) in $\Sigma_n \times \Theta_n$ that is a vertex of $\mathcal{T}_n$. We fix a pure strategy $t_n^0 \in S_n$ for each n. These pure strategies will be used below to define the perturbation of payoffs π_n for each player n. We denote a typical element of $\bar{S}_n^1$ by $(\sigma_{n,n+1}, \theta_{n,n+1})$, which is a vertex in $\mathcal{T}_{n+1}$. For $i = 0,1$, let $\bar{\Sigma}_n^i$ be the set of mixtures over $\bar{S}_n^i$. The pure strategy set of player n in the game $\bar{G}^\delta$ in normal form is $\bar{S}_n \equiv \bar{S}_n^0 \times \bar{S}_n^1$. The set of mixed strategies is denoted $\bar{\Sigma}_n$. For each mixed strategy $\bar{\sigma}_n$, and $i=0,1$, we let $\bar{\sigma}_n^i$ be the marginals over $\bar{S}_n^i$. Define $\bar{S} \equiv \prod_n \bar{S}_n$ and $\bar{\Sigma} \equiv \prod_n \bar{\Sigma}_n$. Also, let $\bar{S}^i \equiv \prod_n \bar{S}_n^i$ and $\bar{\Sigma}^i \equiv \prod_n \bar{\Sigma}_n^i$ for $i = 0,1$.

Fix $\delta > 0$. We will now define the payoff function $\bar{G}^\delta$. For each n, let $\mathcal{T}_n^1$ be the collection of simplices of $\mathcal{T}_n$ that have nonempty intersection with $\Sigma_n \times \Theta_n^1$. Given a pure strategy profile $\bar{s} \in \bar{S}$ with $\bar{s}_n = (\sigma_n, \theta_n, \sigma_{n,n+1}, \theta_{n,n+1})$ for each n, the payoff $\bar{G}_n^\delta(\bar{s})$ has five distinct components:

$$\bar{G}_n^\delta(\bar{s}) = G_n(\sigma) + \sum_{s_n \in S_n} g_{n,s_n}^1(\bar{s}_{-n})\sigma_{n,s_n} + \gamma_n(\theta_n, \sigma_{-n}) + \pi_n(\bar{s}_n^1, \bar{s}_{n+1}^0) - \delta\varrho_n(\bar{s}_n^0).$$

We define each of the components, as follows: let $\gamma_{n,m} : \Theta_{n,m} \times \Sigma_m \to \mathbb{R}$ be the bilinear function such that

$$\gamma_{n,m}(s_m, \sigma_m) = \begin{cases} 0, & s_m = 0_m^*, \\ 1 - (\sigma_{m,s_m}^* - \epsilon^*)^{-1}\sigma_{m,s_m}, & s_m \neq 0_m^*. \end{cases}$$

Let $\gamma_n : \Theta_n \times \Sigma_{-n} \to \mathbb{R}$ be the function

$$\gamma_n(\theta_n, \sigma_{-n}) = \sum_{m \neq n} \gamma_{n,m}(\theta_{n,m}, \sigma_m).$$

The function ϱ_n in the last term is the convex function obtained from (v) above. We will specify the other two terms.

$$g_{n,s_n}^1(\bar{s}_{-n}) = \xi_n(\bar{s}_{-n}^0) g_{n,s_n}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n-1,n}, \sigma_{n+1}, \ldots, \sigma_N),$$

where $\xi_n(\bar{s}^0_{-n})$ is one if for each $m \neq n$, $\bar{s}^0_m$ is a vertex of some simplex in $\mathcal{T}^1_m$; otherwise it is zero. For each n let $\pi_n \colon \bar{S}^1_n \times \bar{S}^0_{n+1} \to \{-1, 0\}$ be the function

$$\pi_n(\bar{s}^1_n, \bar{s}^0_{n+1}) = \begin{cases} 0, & \bar{s}^0_{n+1} \in \mathcal{T}^1_{n+1} \text{ and } \bar{s}^1_n = \bar{s}^0_{n+1}, \\ 0, & \bar{s}^0_{n+1} \notin \mathcal{T}^1_{n+1} \text{ and } \bar{s}^1_n = (t^0_{n+1}, \theta^0_{n+1}), \\ -1, & \text{otherwise.} \end{cases}$$

The definition of $\bar{G}^\delta$ clearly implies that it embeds G.

Remark 3.5. We want to make a couple of remarks about the payoffs. First, the function π_n incentivizes player n to mimic player $n+1$ whenever the latter is choosing a strategy close to $\Sigma_{n+1} \times \Theta^1_{n+1}$: if $n+1$ randomizes over the vertices of a simplex $T_{n+1} \in \mathcal{T}^1_{n+1}$, then player n's best replies must be among the vertices of the simplex. This will be a crucial property, since the choices in $\Sigma_{n-1,n}$ play a role in the evaluation of the bonus function g^1_n. The idea is that whenever the bonus function g^1_n is active, meaning that all players $m \neq n$ randomize over the vertices of a simplex T_m in $\mathcal{T}^1_m$, then each pure best-reply for player n must choose a vertex of T_{n+1}. On the other hand, if player $n+1$ is randomizing over $S_{n+1} \times \{ \theta^0_{n+1} \}$ then it follows that the unique best-reply for player n is to choose the previously fixed strategies $\bar{s}^1_n = (t^0_{n+1}, \theta^0_{n+1})$.

Our second remark concerns the nature of the payoffs for mixed strategies. For n and each $i = 0, 1$, there is a linear map $p^i_n : \bar{\Sigma}_n \to \Sigma_{n+i} \times \Theta_{n+i}$ that sends each pure $\bar{s}_n$ to the corresponding mixed strategy in $\Sigma_{n+i} \times \Theta_{n+i}$. For each n, the first and the third terms of the payoffs depend on $\bar{\sigma}$ only through their images under $p^0 = \prod_{n \in \mathcal{N}} p^0_n$; the fourth term depends on all the information in $\bar{\sigma}^1_n$ and $\bar{\sigma}^0_{n+1}$. The second term depends on $\bar{\sigma}_n$ only through p^0_n, but requires the entire information in $\bar{\sigma}_{-n}$, while the last term requires the information in $\bar{\sigma}^0_n$.

3.9. **Wrapping up the proof.** Let $\bar{\sigma}^*$ be the profile where for each player n, the marginal on $\bar{\Sigma}^0_n$ is (σ^*, θ^0_n) and the marginal on $\bar{\Sigma}^1_n$ is $(t^0_{n+1}, \theta^0_{n+1})$. For $\delta \geqslant 0$, $\bar{\sigma}^*$ is an equilibrium of $\bar{G}^\delta$, and $(G, \sigma^*) \sim (\bar{G}^\delta, \bar{\sigma}^*)$. We will now show that for δ sufficiently small, this is the only equilibrium of $\bar{G}^\delta$, which completes the proof.

Say that a strategy $\bar{\sigma}_n$ is *admissible* for player n if the support of its marginal $\bar{\sigma}^0_n \in \bar{\Sigma}^0_n$ is the set of vertices of a simplex T_n in $\mathcal{T}_n$. Observe that for any $\delta > 0$, every best reply for player n is admissible. Indeed, the first three components of n's payoff function depend on n's strategy only through its projection to $\Sigma_n \times \Theta_n$ and the fourth is independent of these choices. Therefore, any two strategies for n that project under p^0_n to the same point in $\Sigma_n \times \Theta_n$ yield the same payoffs for these four terms, leaving the fifth to decide which one is better. But the map ϱ_n is convex, and it is linear precisely on the simplices of $\mathcal{T}_n$, which then forces each best reply to be a mixture over the vertices of a simplex of $\mathcal{T}_n$.

Proposition 3.6. *For $\delta = 0$, assume that the only admissible equilibrium of $\bar{G}^\delta$ is $\bar{\sigma}^*$. Then for sufficiently small $\delta > 0$, $\bar{\sigma}^*$ is the only equilibrium of $\bar{G}^\delta$.*

Proof. To prove this claim, suppose that we have a sequence $\bar{\sigma}^\delta$ of equilibria of $\bar{G}^\delta$ converging to some equilibrium $\bar{\sigma}^0$ of $\bar{G}^0$, then as we saw above $\bar{\sigma}^\delta$ must be admissible, and hence also its limit $\bar{\sigma}^0$. As we have assumed that $\bar{\sigma}^*$ is the unique admissible equilibrium of $\bar{G}^0$, $\bar{\sigma}^0 = \bar{\sigma}^*$. Observe now that for each n, every pure best reply in $\bar{G}^0$ to $\bar{\sigma}^*$ is of the form $(s_n, \theta_n^0, t_{n+1}^0, \theta_{n+1}^0)$ where $s_n \in S_n$ is a best reply to σ^*; and this property holds for best replies to $\bar{\sigma}^\delta$, for small δ. Thus for each such δ, and for each n, $\bar{\sigma}_n^\delta$ is of the form $(\sigma_n^\delta, \theta_n^0, t_{n+1}^0, \theta_{n+1}^0)$, where σ_n^δ is a best reply to σ^* in G. In other words, σ^δ is an equilibrium of G. As σ^δ converges to σ^* and as σ^* is an isolated equilibrium of G, $\sigma^\delta = \sigma^*$ for all small δ. Thus the claim follows. $\square$

The proof of Theorem 2.3 is now concluded with the following proposition:

Proposition 3.7. *The profile $\bar{\sigma}^*$ is the only admissible equilibrium in $\bar{G}^0$.*

Proof. Fix an admissible equilibrium $\bar{\sigma}$ with marginals $(\bar{\sigma}^0, \bar{\sigma}^1) \in \bar{\Sigma}^0 \times \bar{\Sigma}^1$ of the game $\bar{G}^0$. For each n, let (σ_n, θ_n) and $(\sigma_{n,n+1}, \theta_{n,n+1})$ be the image of $\bar{\sigma}_n$ under p_n^0 and p_n^1, resp.. Also, let T_n be the simplex of $\mathcal{T}_n$ generated by the support of $\bar{\sigma}_n^0$ for each n. We now analyze the three possible cases:

Case 1: Suppose first for each n, T_n belongs to $\mathcal{T}_n^1$. For each n, θ_n assigns probability less than ζ, which is smaller than one, to θ_n^0. Also, for at least two n, $\sigma_n \notin \mathrm{int}_{\Sigma_n}(U_n^*)$: indeed, otherwise there is one player n all of whose opponents m are choosing in $\mathrm{int}_{\Sigma_n}(U_m^*)$, making θ_n^0 the unique optimal choice, which is impossible. Thus, $\sigma_n \notin \mathrm{int}_{\Sigma_n}(U_n^*)$ for at least two n, i.e., $\sigma \notin \mathrm{int}_{\Sigma}(U^*)$ and, hence, σ is not an equilibrium of $G \oplus g(\sigma)$. For each n, and each $\bar{s}_{-n}$ in the support of $\bar{\sigma}_{-n}$, $\xi_n(\bar{s}_{-n}^0) = 1$ as $\bar{s}_m^0$ is a vertex of the simplex T_m, which is in $\mathcal{T}_m^1$, for each m; because of the function π_n, the optimality of $\bar{\sigma}_{n-1}^1$ implies that each $\bar{s}_{n-1}^1$ in the support of $\bar{\sigma}_{n-1}^1$ is a vertex of T_n. Therefore, for each $\bar{s}_{-n}$ in the support of $\bar{\sigma}_{-n}$, $\|g^1(\bar{s}_{-n}) - g(\sigma)\| \leqslant \eta$ and then $\|g^1(\bar{\sigma}_{-n}) - g(\sigma)\| \leqslant \eta$. As σ is not an equilibrium of $G \oplus g(\sigma)$, by the choice of η in subsection 3.8, it is not an equilibrium of $G \oplus g^1(\bar{\sigma})$, which contradicts the fact that $\bar{\sigma}$ is an equilibrium of $\bar{G}^0$.

Case 2: Now suppose that for exactly one n, say $n = 1$, T_n does not belong to $\mathcal{T}_n^1$. Then, θ_1^0 has positive probability under θ_1. Therefore, because of the definition of γ_n, $\sigma_n \in U_n^*$ for $n > 1$, i.e., $\sigma \in U^*$. For $n > 1$, the fact that $\sigma_n \in U_n^*$ and T_n belongs to $\mathcal{T}_n^1$ imply that for each $\bar{s}_n^0 = (\sigma_n, \theta_n)$ in the support of $\bar{\sigma}_n^0$, σ_n belongs to U_n (as the diameter of T_n is less than ζ, which is smaller than the distance between U^* and $\partial_\Sigma U$). Thus, $g_1^1(\sigma) = 0$. We will now show that $g_n^1(\sigma) = 0$ for $n > 1$. The payoff function π_n for each $n \neq N$ forces each strategy $\bar{s}_n^1 = (\sigma_{n,n+1}, \theta_{n,n+1})$ in the support of $\bar{\sigma}_n^1$ to be a vertex of T_{n+1} and hence $\sigma_{n,n+1}$ is in U_{n+1}. Therefore, for $n > 1$, $\sigma_{n-1,n} \in U_n$. Recall that $g_n(\cdot)$ was constructed to be 0 on U. Consequently, for each $n > 1$, $g_n^1(\bar{s}_{-n}) = 0$ for each $\bar{s}_{-n}$ in the support of $\bar{\sigma}_{-n}$, i.e., $g_n^1(\bar{\sigma}_{-n}) = 0$. The fact that $g^1(\sigma) = 0$, implies that $\sigma = \sigma^*$. Optimality of θ_n for $n > 1$ now requires that it assign probability one to θ_n^0. This is a contradiction: since $T_n \in \mathcal{T}_n^1$, its diameter is smaller than ζ (and, hence, smaller than one), putting it at positive distance from $\Sigma_n \times \{ \theta_n^0 \}$.

Case 3: Finally, suppose that for at least two players n, T_n does not belong to $\mathcal{T}_n^1$. Then, again because of γ_n, for each n, $\sigma_n \in U_n^*$. We claim that for each n, $g_n^1(\bar{s}_{-n}) = 0$ for each $\bar{s}_{-n}$ in the support of $\bar{\sigma}_{-n}$. Indeed, if for some $m \neq n$, T_m has no vertex in $\mathcal{T}_m^1$, then ξ_n is zero by construction at each $\bar{s}_{-n}$ in the support and we are done. Otherwise, if for each $m \neq n$, T_m has a vertex in $\mathcal{T}_m^1$ then, letting $\bar{s}_m^0 = (\sigma_m, \theta_m)$ be an arbitrary vertex of T_m, it follows from the fact that the diameter of each T_m is less than ζ that $\sigma_m \in U_m$, which implies $\sigma \in U$. Therefore, $g_n^1(\cdot)$ is again zero on the support of $\bar{\sigma}_{-n}$. It follows that σ is an equilibrium of G, i.e., $\sigma = \sigma^*$, making $\theta_n = \theta_n^0$. Finally, optimality of $\bar{\sigma}_n^1$ implies that it is $(t_{n,n+1}^0, \theta_{n,n+1}^0)$, as it yields zero with others yielding -1 (cf. Definition of π_n). Thus, $\bar{\sigma} = \bar{\sigma}^*$, which concludes the proof. $\square$

3.10. **Example.** In the context of the example of subsection 3.5: $\Theta = \Delta(\{0^*\} \cup \{L\})$. We identify Θ with $[0,1]$ and an element $\theta_1 \in [0,1]$ denotes the probability of L. Let $\gamma(0^*, x) = 0$ and $\gamma(L, x) = 1 - \frac{8x}{7}$. Then define $\gamma(\theta, x) = \theta_1 \gamma(L, x) + (1-\theta_1)\gamma_1(0^*, x)$. The graph of $\gamma(L, x)$ is depicted below in red in Figure 2. Let $\varepsilon^* = 1/8$.

We can now triangulate the set $\Sigma \times \Theta \equiv [0,1] \times [0,1]$ of each player as in Figure 3. The horizontal axis represents $\Theta = [0,1]$ and the vertical axis $\Sigma = [0,1]$.

Payoffs are defined in the exact same way as in subsection 3.8 with the following modifications: the function $\pi \equiv 0$, since there is no need to duplicate the strategy set of each player (by our construction, g_n depends only on σ_{-n}); the function ξ is equal to 1 at all vertices of the triangulation that lie on the face $\theta = 1$.

Paralleling the proof of subsection 3.8, we show that the only admissible equilibrium of the perturbed game $\bar{G}^\delta$ with $\delta = 0$ is the (symmetric) equilibrium (θ, x), where $x = 1$ and $\theta = 0$. To see this, let (x, θ) be an equilibrium. Suppose first $x < 7/8$. Then $\theta = 1$ is a strict best-reply. The support of the equilibrium (θ, x) is then a subset of one of the 1-dimensional

FIGURE 2. Graph of $\gamma(L,\cdot)$ (red)

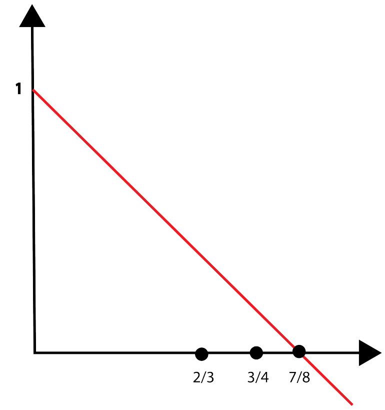
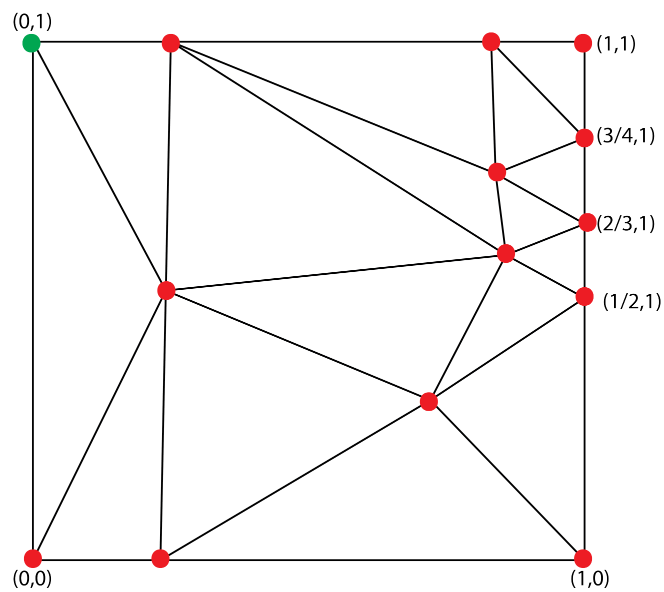


FIGURE 3. Triangulation of $[0,1]^2$



simplices that subdivide $\{1\}\times[0,1]$. By our construction of g and the fact that it is linear in each of these simplices, it follows that $x = 1$, which is a contradiction. Therefore, $x \geqslant 7/8$. Recall that in the original game $x = 1$ is a strict best-reply if $x \geqslant 7/8$. Since $g_L \geqslant 0$ (whereas $g_R \equiv 0$), it follows that $x = 1$ is a strict-best reply in $\bar{G}^\delta$. Finally, given $x = 1$, $\theta = 0$ is the optimal choice in Θ.

4. Deleting Unused Best Replies

In the definition of equivalence between game-equilibrium pairs (G,σ) and $(\bar{G},\bar{\sigma})$, we insisted that G and $\bar{G}$ be the same game once we delete all strategies that are inferior replies to σ and $\bar{\sigma}$, resp. We could have weakened the requirement by allowing the deletion of unused best replies as well, i.e., that the games be the same once we restrict them to the support of their respective equilibria. For generic games, of course, these two notions coincide. But, for nongeneric games, as we show now using examples, allowing for the deletion of unused best replies leads to an unsatisfactory concept of equivalence.

Consider first the following bimatrix game:

$G_3 =$

	l	r
t	$(1,1)$	$(0,0)$
b	$(0,0)$	$(0,0)$

The equilibrium (b,r) is in dominated strategies and, clearly, an unreasonable equilibrium. Deleting the strategies t and l, which are unused best replies against this equilibrium, produces a trivial game where (b,r) is now the only solution.

One might conjecture that the misbehavior in the above game stems from the fact that the equilibrium (b,r) has index zero. But that is not the case, as the following 3-player game $G_{4.1}$ shows. Player 1's strategy set is $\{(T,t),(T,b),B\}$; 2's strategy set is $\{(L,l),(L,r),R\}$; 3's strategy set is $\{W,E^w,E^\varepsilon\}$. The payoffs are:

$W:$

		L: l	L: r	R
T	t	$(6,6,1)$	$(0,0,1)$	$(3,3,0)$
	b	$(0,0,1)$	$(6,6,1)$	
B		$(3,0,1)$		$(0,3,1)$

$E^w:$

		L: l	L: r	R
T	t	$(-3,0,4)$	$(1,4,0)$	$(1,0,1)$
	b	$(1,4,0)$	$(1,4,0)$	
B		$(3,0,0)$		$(0,3,0)$

$E^e:$

		L: l	L: r	R
T	t	$(1,4,0)$	$(1,4,0)$	$(3,0,1)$
	b	$(1,4,0)$	$(-3,0,4)$	
B		$(3,0,0)$		$(0,3,0)$

This game has a unique equilibrium, $\bar{\sigma}$, in which player 1 mixes uniformly between (T,t) and (T,b); player 2 mixes uniformly between (L,l) and (L,r); and player 3 plays W. If we delete B, R, E^w, E^e, all of which are unused best replies, we get the game $G_{4.2}$:

$G_{4.2} =$

	l	r
t	$(6,6,1)$	$(0,0,1)$
b	$(0,0,1)$	$(6,6,1)$

where the equilibrium $\bar{\sigma}$ is now regular but reverses its index to -1.

Our final example shows that there are problems even when an equilibrium is unique and in pure strategies: deleting some unused best replies results in a game where this equilibrium is not even isolated. In the game $G_{5.1}$ below

$$W: \begin{array}{c|c|c|} & L & R \\ \hline T & (1,1,1) & (1,1,0) \\ \hline B & (1,0,1) & (0,1,1) \\ \hline \end{array} \qquad E: \begin{array}{c|c|c|} & L & R \\ \hline T & (0,1,1) & (1,0,1) \\ \hline B & (1,0,0) & (0,1,0) \\ \hline \end{array}$$

(T, L, W) is the unique equilibrium.[14] Deleting the unused replies B and E, but not R, gives us the following game $G_{5.2}$:

$$\begin{array}{c|c|c|} & L & R \\ \hline T & (1,1,1) & (0,1,0) \\ \hline \end{array}$$

where now there is an interval of equilibria.

It is noteworthy that, when we view game $G_{5.2}$ as a two-player game, the equilibrium (T, L) (part of a connected component) is made unique by adding players and strategies. Similarly, in the two-player game G_1 above, the -1 index regular equilibrium in which player 1 mixes uniformly between (T, t) and (T, b) and player 2 mixes uniformly between (L, l) and (L, r) has been made unique by adding players and strategies, leaving open the question of how general such a property is.[15] Observe that without adding players, it is impossible to make a non $+1$ equilibrium unique in a two player game by adding only strategies, because a unique equilibrium of a two player game is necessarily quasi-strict (Norde [26]).

5. Composition of Games

Suppose $f_i : X_i \to X_i$ is a self-map with an isolated fixed point x_i, for $i = 1, 2$. Then (x_1, x_2) is an isolated fixed point of $f_1 \times f_2$ and, by the *multiplication property* of the index of fixed points (see [18]), its index is the product of the indices of the x_i's. This fact has important implications for sustainability, as we will now see.

Suppose $(G^1, \mathcal{N}^1, S^1)$ and $(G^2, \mathcal{N}^2, S^2)$ are two games with disjoint player sets. The composition $G^1 \otimes G^2$ of the two games is the game where:

(1) The player set is $\mathcal{N}^1 \cup \mathcal{N}^2$.
(2) For $i = 1, 2$, and $n^i \in \mathcal{N}^i$,
 (a) n^i's strategy set is as in G^i;
 (b) n^i's payoffs depend only on the strategies of players in $\mathcal{N}^i$ and are exactly as in G^i.

Obviously the Nash equilibria of $G^1 \otimes G^2$ are the products of Nash equilibria of the component games, i.e., a strategy profile (σ^1, σ^2) in $G^1 \otimes G^2$ is a Nash equilibrium iff for each i, σ^i is a

[14]This is a mild strengthening of the example in Brandt and Fischer [2], where the unique equilibrium—which also fails to be quasi-strict—was in mixed strategies.

[15]We thank Joseph Hofbauer for raising this issue.

Nash equilibrium of G^i. The Decomposition Axiom for games, formulated by Mertens [20], imposes a similar requirement for solution concepts that refine Nash equilibria: the solutions of $G^1 \otimes G^2$ are the products of solutions of the G^i's. This axiom, which is a version of the small worlds axiom, is relevant from a practical viewpoint. Say that we are interested in analyzing the game G^1 and intend to use a particular refinement. If the refinement satisfies this axiom, we can be assured that it is sufficient to just look at G^1, even if it might be the case that G^1 is one component of a larger game. A violation, on the other hand, would imply that we have to construct the "metagame" of which this game is a part and then analyze it.

It is straightforward to verify that most single-valued solution concepts like perfection, properness, and sequential equilibria satisfy the Decomposition Axiom. Unfortunately, a corresponding result does not hold for sustainable equilibria. To be more specific, we have the following proposition describing the relationship between the sustainable equilibria of a game G^1 and those of its composition with another game G^2. Its proof follows from the fact that if σ^i is an isolated equilibrium of G^i for $i = 1, 2$, then (σ^1, σ^2) is an isolated equilibrium of $G^1 \otimes G^2$ and its index is the product of the indices of the σ^i's, thanks to the multiplication property for index.

Proposition 5.1. *Let σ^1 be an isolated equilibrium of a game G^1. The profile σ^1 has index $+1$ (resp. -1) iff for any game-equilibrium pair (G^2, σ^2) where σ^2 is an isolated equilibrium of G^2 with index $+1$ (resp. -1) the product (σ^1, σ^2) is a sustainable equilibrium of $G^1 \otimes G^2$.*

The Decomposition Axiom has two parts. First, compositions of solutions of the G^i's are solutions of $G^1 \otimes G^2$. Second, projections of solutions from $G^1 \otimes G^2$ to the G^i's are solutions. If the solution concept we use is sustainability (on the domain of regular games), then the first part of the axiom is satisfied, as shown by the part of the proposition relating to $+1$ equilibria. The other part of the proposition shows that sustainability does not satisfy the second part of the axiom.

The violation of the Decomposition Axiom is a significant drawback of sustainability. Suppose G^1 is a Battle-of-Sexes game. The completely mixed equilibrium of G^1 is not sustainable. But compose G^1 with another Battle-of-Sexes game G^2, and take the product of the two mixed equilibria, we then obtain a sustainable equilibrium! Thus, every regular equilibrium is the projection of a sustainable equilibrium of a game obtained by composition. One way to address the problem is to say that sustainability is too restrictive and that we should relax the equivalence relation to be more permissive, thus enlarging the equivalences classes of game-equilibrium pairs. Another approach, which we take in the next section, is to make the equivalence relation more demanding, and thus refining sustainability, so that the Decomposition Axiom holds.

6. Strongly Sustainable Equilibria

Two key ideas are involved in the definition of sustainability. The first is a notion of equivalence between game-equilibrium pairs; and the second is the selection of equivalence classes that contain a game-equilibrium pair where the equilibrium is unique. In our refinement of sustainability, we strengthen the criterion for equivalence to accommodate not just the decomposition property, but a related property called the Small Worlds Axiom by Mertens [21], while still retaining the key property of existence.

Definition 6.1. Let $G = (\mathcal{N}, (S_n)_{n\in\mathcal{N}}, (G_n)_{n\in\mathcal{N}})$ be a finite game. We say that a nonempty subset $\mathcal{N}^0$ of the player set $\mathcal{N}$ is *strategically independent in* G if for each $n^0 \in \mathcal{N}^0$, the payoff function G_{n^0} is independent of the coordinates of players $n \in \mathcal{N} \setminus \mathcal{N}^0$.

Let G be a game and let $\mathcal{N}^0$ be a strategically independent subset of players in G. There is now a well-defined game $G(\mathcal{N}^0)$ among the players in $\mathcal{N}^0$ that is induced by G. If $\mathcal{N} \setminus \mathcal{N}^0$ is also strategically independent, then G is the composition of the two games $G(\mathcal{N}^0)$ and $G(\mathcal{N} \setminus \mathcal{N}^0)$. But even if $\mathcal{N} \setminus \mathcal{N}^0$ is not a strategically independent set of players, we could still analyze the game $G(\mathcal{N}^0)$ by itself if we accept the Small Worlds Axiom, which asks that the solutions of $G(\mathcal{N}^0)$ be the projections of the solutions of G..

Before introducing the definition of strong sustainability, recall that in the definition of the equivalence relation introduced in Section 2, we allowed for a relabeling of players and strategies in an equivalent game-equilibrium pair.

Definition 6.2. Two game-equilibrium pairs (G, σ) and $(\bar{G}, \bar{\sigma})$ are *strongly equivalent*, denoted $(G, \sigma) \approx (\bar{G}, \bar{\sigma})$, if:

(1) $(G, \sigma) \sim (\bar{G}, \bar{\sigma})$;
(2) A subset $\mathcal{N}^0 \subset \mathcal{N}$ is strategically independent in G iff the relabeled set of players $\bar{\mathcal{N}}^0 \subset \bar{\mathcal{N}}$ is strategically independent in $\bar{G}$.

Strong equivalence strenghtens the initial equivalence relation by considering equivalent game-equilibrium pairs that respect strategic independence of subset of players. With this equivalence notion in hand, we have the following solution concept.

Definition 6.3. An equilibrium σ is *strongly sustainable* if it is strongly equivalent to a pair $(\bar{G}, \bar{\sigma})$ where $\bar{\sigma}$ is the unique equilibrium of $\bar{G}$.

We now introduce a characterization of strongly sustainable equilibria which allows us to prove its existence in a meaningful class of games.

Theorem 6.4. *An equilibrium σ of a game G is strongly sustainable iff for each strategically independent subset $\mathcal{N}^0$ of players, $\sigma_{\mathcal{N}^0}$ is isolated and has index $+1$ in the game $G(\mathcal{N}^0)$.*

Proof. We first prove the sufficiency of the condition. Let σ be a strongly sustainable equilibrium of G. Let $(\bar{G}, \bar{\sigma})$ be strongly equivalent to (G, σ) with $\bar{\sigma}$ the equilibrium of $\bar{G}$. Let $\mathcal{N}^0$ be a strategically independent set of players in G. Let $\bar{\mathcal{N}}^0$ be the corresponding set in $\bar{G}$. As $\bar{\sigma}$ is the unique equilibrium of $\bar{G}$, $\bar{\sigma}_{\bar{\mathcal{N}}^0}$ is the unique equilibrium of $\bar{G}(\bar{\mathcal{N}}^0)$. Obviously, $(G(\mathcal{N}^0), \sigma_{\mathcal{N}^0}) \sim (\bar{G}(\bar{\mathcal{N}}^0), \bar{\sigma}_{\bar{\mathcal{N}}^0})$ and thus the index of $\sigma_{\mathcal{N}^0}$ is $+1$, which completes the proof.

We now prove the necessity of the condition. Let $\mathcal{P} \subset 2^{\mathcal{N}}$ be the collection of all strategically independent subsets of players. Let $\mathcal{P}^* \subseteq \mathcal{P}$ be the collection of all elements of $\mathcal{P}$ that are minimal under set inclusion, i.e., $M \in \mathcal{P}$ belongs to $\mathcal{P}^*$ iff no proper subset of M is strategically independent. Clearly, the elements of $\mathcal{P}^*$ are disjoint.

For each $\mathcal{N}^0 \in \mathcal{P}^*$, since $\sigma_{\mathcal{N}^0}$ has index $+1$ and is isolated, Theorem 2.3 allows us to obtain an embedding $H^{\mathcal{N}^0}$ of the game among players in $\mathcal{N}^0$ in which $\sigma_{\mathcal{N}^0}$ is the unique equilibrium. Let $\mathcal{N}^1 = \mathcal{N} \setminus \cup_{\mathcal{N}^0 \in \mathcal{P}^*} \mathcal{N}^0$. If $\mathcal{N}^1$ is nonempty, fixing $\sigma_{\mathcal{N}^0}$ for each $\mathcal{N}^0 \in \mathcal{P}^*$, apply Theorem 2.3 to the game among players in $\mathcal{N}^1$, to obtain an embedding $H^{\mathcal{N}^1}$ of this game where $\sigma_{\mathcal{N} \setminus \cup_{\mathcal{N}^0 \in \mathcal{P}^*} \mathcal{N}^0}$ is the unique equilibrium. We can now define an embedding $\bar{G}$ for game G as follows: for $\bar{s} \in S \subset \bar{S}$, the payoff to all players in $\bar{G}$ is identical to that in G; if $\bar{s} \in \bar{S} \setminus S$, the payoffs to players in $\mathcal{N}^0 \in P^*$ are identical to $H^{\mathcal{N}^0}$ (i.e., they are independent of strategies chosen by players outside $\mathcal{N}^0$). For $n \in \mathcal{N}^1$, the payoff is $H_n^{\mathcal{N}^1}(\bar{s}^1)$, where $\bar{s}^1 = (\bar{s}_n)_{n \in \mathcal{N}^1}$. It is easily checked that (G, σ) is strongly equivalent to $(\bar{G}, \bar{\sigma})$ and that $\bar{\sigma}$ is the unique equilibrium of $\bar{G}$. □

The characterization of Theorem 6.4 allows us to establish that any regular game, or more generally, any game which has finitely many equilibria with indices $+1, -1$ or 0, has a strongly sustainable equilibrium. The proof of the corollary shows a technique to identify strongly sustainable equilibria in such games.

Corollary 6.5. *Let G be a finite game whose equilibria are all isolated with index $+1$, -1 or 0. Then G has a strongly sustainable equilibrium.*

Proof. As in the necessity proof of Theorem 6.4, let $\mathcal{P}^*$ be the collection of minimally-strategically independent subsets of players of G. For each $\mathcal{N}^0 \in \mathcal{P}^*$, fix a $+1$ equilibrium $\sigma_{\mathcal{N}^0}$ of the game $G(\mathcal{N}^0)$. If, again in the notation of the previous proof, $\mathcal{N}^1$ is nonempty, pick a $+1$ equilibrium $\sigma_{\mathcal{N}^1}$ of the game among $\mathcal{N}^1$ when the strategies of the others are fixed at $(\sigma_{\mathcal{N}^0})_{\mathcal{N}^0 \in \mathcal{P}^*}$. The strategy profile consisting of these various components has index +1 and by Theorem 6.4 is strongly sustainable. □

7. Conclusion

Most existing refinement concepts that are guaranteed to exist, such as perfection [30], properness [24], and stable equilibria [14, 17, 20] only refine equilibria of non-generic games, i.e., every equilibrium of a regular game survives these refinement criteria.[16] On the other hand, the procedure proposed by Harsanyi and Selten [11] *selects* a unique equilibrium in every game. Theorem 2.3 shows that sustainability lies somewhere in between refinement and selection: it slices by half the set of predictions in regular games, as it discards equilibria with index −1. Strongly sustainable equilibria further refines this prediction by requiring a selection of a subset of the +1 equilibria.

One should note that extending the concept of sustainability (or strong sustainability) to the universe of all games is not a trivial matter. It is fairly easy to construct non-generic games where no equilibrium is sustainable. For example, consider the (in effect) one-player game $G_{5.2}$, where the set of equilibria is a connected component. There is no way to make any of these equilibria unique by adding strategies that are inferior replies. Therefore, any extension of sustainability to non-generic games needs to allow for solutions to be sets of equilibria, if we are to have existence. It is then natural to require that solutions be connected: as argued by Mertens [20], this requirement keep our selections "minimal" and, in addition, if the game is a generic extensive-form game, each solution would have a unique prediction in terms of the equilibrium outcome.

A natural definition of sustainability for connected sets of equilibria calls for such a set to be sustainable if it is the entire set of equilibria of a game obtained by adding or deleting strategies that are inferior replies to *every* equilibrium in the set. In such a definition, the component in which the candidate solution lies would survive these alterations and, hence, the connected sets would have to be entire components.[17] Moreover, a sustainable component should also have index +1 (for the same reason that sustainability for individual equilibria implies that their index is +1). As there are games where no component has index +1 (see [12] Fig. 8, or [28] p. 325) there seems to be no reasonable way to extend sustainability to non-generic games without running afoul of the existence criterion.

[16] An exception is persistent equilibrium [16] which eliminates the mixed equilibrium in the battle of the sexes; but, as noted by Myerson, it violates invariance [1].

[17] A strict subset U of an equilibrium component C can never be made the unique equilibrium set in a game $\hat{G}$ obtained from G by adding/deleting inferior replies. The proof is the same as the one showing that an equilibrium that is not isolated is not sustainable.

One way around this problem is to extend sustainability by an axiomatic procedure, rather than by a definition using the game-theoretic property that it attempts to capture. In a previous version of this paper [8], we offered an axiomatization of the concept of sustainable equilibrium in terms of elementary properties, which we then used to obtain a definition of sustainability to all games (and an axiomatic characterization in terms of index). More specifically, we noticed that sustainable equilibria in regular games could be characterized as the minimal solution concept satisfying Independence of Irrelevant Alternatives and Existence (i.e., whenever the game has a unique equilibrium, it must be the solution). We then proposed an extension of the solution concept of sustainable equilibria to all games which assigned all the positive index components of equilibria as solutions to a game. Furhtermore, we showed that any solution which extends sustainable equilibrium from regular games and satisfies Connectedness and Uniform Robustness[18] must select among the positive index components of equilibria.

We then noticed that sustainable equilibrium fails the Decomposition Axiom—a failure that is an immediate consequence of the multiplication property of the index, once we have Theorem 2.3. We have proposed a refinement (strong sustainability) that corrects this problem while adhering to the spirit of the definition of sustainability. Our previous characterization of sustainability in terms of positive index components suggests that an extension of this refinement to all games selects among positive index components. Extending the concept of strong sustainability to all games while satisfying other desirable axioms (e.g., invariance, decomposition and backward induction) is a challenging problem that requires futher investigation, which we aim to do in future work.

References

[1] Balkenborg, D. (1992): "The Properties of Persistent Retracts and Related Concepts," Ph.D. Thesis, University of Bonn.

[2] Brandt, F., and F. Fischer (2008): "On the Hardness and Existence of Quasi-Strict Equilibria," SAGT, LNCS 4997, 291–302.

[3] De Loera, J.A., J. Rambau, and F. Santos (2010): *Triangulations: Structures for Algorithms and Applications*. Berlin: Springer Verlag.

[4] Demichelis, S., and F. Germano (2000): "On the Indices of Zeros of Vector Fields," *Journal of Economic Theory*, 94, 192–218.

[5] Demichelis, S., and K. Ritzberger (2003): "From Evolutionary to Strategic Stability," *Journal of Economic Theory*, 113, 51–75.

[18]A solution concept for games is *Robust* if nearby games (in terms of payoff perturbations) have nearby solutions. Uniform Robustness is a combination of the requirements of Invariance and Robustness. See section 5 in [8] for the results and extensive discussion.)

[6] Govindan, S., and R. Wilson (1997): "Equivalence and Invariance of the Degree and Index of Nash Equilibria," *Games and Economic Behavior*, 21, 56–61.
[7] Govindan, S., and R. Wilson (2005): "Essential Equilibria," *Proceedings of the National Academy of Sciences*, 102, 15706–11.
[8] Govindan, S., R. Laraki and L. Pahl (2021): "On Sustainable Equilibria," *arXiv:2005.14094*, v2.
[9] Gül, F., D. Pearce, and E. Stacchetti (1993): "A Bound on the Proportion of Pure Strategy Equilibria in Generic Games," *Mathematics of Operations Research*, 18, 548–52.
[10] Harsanyi, J.C. (1975): "The tracing procedure: A Bayesian approach to defining a solution for n-person noncooperative games," *International Journal of Game Theory*, 4, 61–94.
[11] Harsanyi, J.C., and R. Selten (1988): *A General Theory of Equilibrium Selection in Games*. Cambridge, MA: MIT Press.
[12] Hauk, E., and S. Hurkens (2002): "On Forward Induction and Evolutionary and Strategic Stability," *Journal of Economic Theory*, 106 (1), 66-90.
[13] Herings, P.J-J. and R. Peeters (2010): "Homotopy methods to compute equilibria in game theory," *Economic Theory*, 42, 119–156.
[14] Hillas, J. (1990): "On the Definition of the Strategic Stability of Equilibria," *Econometrica*, 58, 1365–1390.
[15] Hofbauer, J. (2003): "Some Thoughts on Sustainable/Learnable Equilibria," in *Conference Book of the XV Italian Meeting of Game Theory and Applications*. Urbino, IT: Universitá Degli Studi di Urbino.
[16] Kalai, E., and D. Samet (1984): "Persistent Equilibria," *International Journal of Game Theory*, 13, 129–141.
[17] Kohlberg, E., and J.-F. Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003–37.
[18] McLennan, A. (2018): *Advanced Fixed Point Theory for Economics*. Singapore: Springer.
[19] McLennan, A. (2016): The Index +1 principle. Mimeo.
[20] Mertens J.-F. (1989): "Stable Equilibria—A Reformulation, Part I. Definition and Basic Properties," *Mathematics of Operations Research*, 14, 575–625.
[21] Mertens J.-F. (1989): "Stable Equilibria—A Reformulation, Part II. Discussion of the Definition and Further Results," *Mathematics of Operations Research*, 16, 694–753.
[22] Myerson, R.B., and J. Weibull (2015): "Tenable Strategy Blocks and Settled Equilibria," *Econometrica*, 83, 943–976.
[23] Myerson, R.B. (1996): "Sustainable Equilibria in Culturally Familiar Games," in *Understanding Strategic Interaction: Essays in Honor of Reinhard Selten*, edited by W. Albers et al, Berlin: Springer Verlag, 111–21.
[24] Myerson, R. (1978): "Refinements of the Nash Equilibrium Concept," *International Journal of Game Theory*, 7, 73–80.
[25] Nash, J.F. (1951): "Noncooperative Games," *Annals of Mathematics*, 54, 286–95.
[26] Norde, H. (1999): "Bimatrix Games Have a Quasi-Strict Equilibria," *Mathematical Programming*, 85, 35–49.
[27] Ritzberger, K. (1994): "The Theory of Normal Form Games from the Differentiable Viewpoint," *International Journal of Game Theory*, 23, 207–236.
[28] Ritzberger, K. (2002): *Foundations of Non-Cooperative Game Theory*. Oxford, U.K.: Oxford University Press.
[29] Schelling, T. (1960): *Strategy of Conflict*. Cambridge: Harvard University Press.

[30] Selten, R. (1975): "Re-examination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25–55.

[31] Shapley, L.S. (1974): "A note on the Lemke-Howson algorithm," *Mathematical Programming Study 1: Pivoting and Extensions*, 175-189.

[32] Spanier, E.H. (1966): *Algebraic Topology.* New York: McGraw Hill. Reprinted in New York: Springer Verlag, 1989.

[33] van Damme, E. (1987): *Stability and Perfection of Nash Equilibria.* Berlin: Springer Verlag.

[34] von Schemde, A., and B. von Stengel (2008): "Strategic Characterization of the Index of an Equilibrium," SAGT 2008, LNCS 4997, 242–54.

[35] von Schemde, A. (2005): *Index and Stability in Bimatrix Games.* Berlin: Springer Verlag.

Appendix A. Delaunay Triangulations

Here we construct a triangulation $\mathcal{T}_n$ of $\Sigma_n \times \Theta_n$ for each n with the properties stated in subsection 3.8.

A.1. **Preliminaries.** A *simplex* T in $\mathbb{R}^d$ is the convex hull of affinely independent points $x_0, x_1, \ldots, x_k$ ($k \leqslant d$); a *face* of T is the convex hull of a subset of the points x_i. A *triangulation* $\mathcal{T}$ of a polytope $C \subset \mathbb{R}^d$ is a finite collection of simplices T in $\mathbb{R}^d$ such that: (i) if $T \in \mathcal{T}$, so is every face of T; (ii) the intersection of two simplices in $\mathcal{T}$ is a face of both (possibly empty); (iii) the union of the simplices in $\mathcal{T}$ equals C.

Throughout this Appendix, we use the ℓ_2-norm, unless we specify differently. Take a finite collection of points $\{ x_0, x_1, \ldots, x_k \}$ in $\mathbb{R}^d$ (where k is now an arbitrary positive integer) such that its convex hull C is d-dimensional. Suppose that the x_i's are in *general position for spheres* in $\mathbb{R}^d$—i.e., no subcollection of $d+2$ of these points lie on any $(d-1)$-sphere (centered at any point and of any radius) in $\mathbb{R}^d$. We can construct a triangulation of the convex hull C, called the *Delaunay triangulation*, as follows (cf. De Loera et al. [3] for details). Let D be the convex hull of the set of points $(x_i, \|x_i\|^2) \in \mathbb{R}^{d+1}$, $i = 0, 1, \ldots k$. Let D_0 be the lower envelope of D. The natural projection $(x, y) \mapsto x$ from D_0 to $\mathbb{R}^d$ is C and D_0 is the graph of a piece-wise linear and convex function $\varrho : C \to \mathbb{R}$ with the property that the subsets on which ϱ is linear are simplices, whose projections then yield the simplices of a triangulation of C.

There is a dual representation of the Delaunay triangulation, known as the *Voronoi Diagram*, which works as follows. For each $i = 0, 1, \ldots k$, let P_i be the polyhedron in $\mathbb{R}^d$ consisting of points y in $\mathbb{R}^d$ such that $\|y - x_i\| \leqslant \|y - x_j\|$ for all $j \neq i$. We then have a polyhedral complex (which is like a simplicial complex but with polyhedra rather than simplices) where the maximal polyhedra are the P_i. There is an edge between two vertices

x_i and x_j in the Delaunay triangulation iff the polyhedra P_i and P_j have a nonempty intersection. Also, the intersection of $d+1$ of these polyhedra when nonempty is a single point (because of genericity), which is then the center of a ball that contains $d+1$ points of the collection on its boundary and no other point in the ball itself—these $d+1$ points span a d-dimensional simplex in the Delaunay triangulation.

For our purposes, we need a triangulation with the diameter of certain simplices to be smaller than ζ, as specified in Subsection 3.8. To obtain that, we need an auxiliary construction. Let C be a full-dimensional polytope in $\mathbb{R}^d$. Let B_0 be a proper face of C and let H be a hyperplane that strictly separates B_0 from the vertices of C that are not in B_0. Let B be the intersection of C with the halfspace generated by H that contains B_0 in its interior, i.e., B is of the form $C \cap H_-$ where $H_- = \{\, x \in \mathbb{R}^d \mid a \cdot x \leqslant b \,\}$ and $a \cdot x < b$ for all $x \in B_0$. Let X be the set of vertices of C and let Y be the set of vertices of B that are not in X. Suppose the vertices in $X \cup Y$ satisfy the following property:

(P) For each $0 < k \leqslant d$, and each collection $v_0, \ldots, v_{k+1}$ of vertices in $X \cup Y$ that are not affinely independent, the intersection of the Voronoi polyhedra of these $k+2$ vertices is empty.

This assumption implies, a.o., by taking $k = d$, that the vertices in $X \cup Y$ are in general position for a Delaunay triangulation of C. Beyond that, the stronger assumption allows us to obtain refined Delaunay triangulations of C. Specifically, we have:

Claim A.1. *Assume $X \cup Y$ satisfy property (P). If $Z \subseteq C$ is such that each point $z \in Z$ is in generic position in the face of C or $H \cap C$ that contains it in its interior—i.e., outside a nowhere dense subset of this face—then the collection $X \cup Y \cup Z$ is in general position for spheres and we can construct a Delaunay triangulation with this set of vertices.*

Proof. Suppose the points in Z are chosen generically and suppose $v_0, \ldots, v_{d+1}$ is a collection of vertices in $X \cup Y \cup Z$. There exists $0 < k \leqslant d$ such that, after permuting the labels of the collection if necessary, $v_0 = \sum_{i=1}^{k+1} \lambda_i v_i$, $\lambda_i \neq 0$ for all $i > 0$ and $\sum_i \lambda_i = 1$. By Property (P), the Voronoi polyhedra of the vertices do not intersect if all the vertices belong to $X \cup Y$. If there is some vertex in the collection that belongs to Z, then we can assume that v_0 belongs to Z. If $v_1, \ldots, v_{k+1}$ do not span a face of C or $H \cap C$, then clearly we can perturb v_0 so that it does not lie in the affine space of the other v_i's, which contradicts the assumption that v_0 was chosen in a generic position. On the other hand, if $v_1, \ldots, v_{k+1}$ do span a face of C or $H \cap C$, then for generic choice of v_0 and also of the v_i's in the list $v_1, \ldots, v_{k+1}$ that are not in $X \cup Y$, their Voronoi polyhedra do not intersect. $\square$

Thus, Property (P) allows to us refine the initial Delaunay triangulation of C (involving vertices $X \cup Y$).

Let $\delta > 0$ be such that $\|x - y\| \geqslant \delta/2$, for all $x \in B$ and vertices $y \in C \setminus B$. Let X_δ be a finite collection of points in C such that:

(i) $X \cup Y \subset X_\delta$ and $X_\delta \cap (C \setminus B) \subset X$;
(ii) For $x \in B$, there is a point $x_\delta \in B \cap X_\delta$ such that $\|x - x_\delta\| < \delta/2$ and x_δ belongs to the face of B that contains x in its interior;
(iii) Every point in $\mathrm{int}_C(B) \cap X_\delta$ is at least $\delta/2$ from $\partial_C B$;
(iv) The points in X_δ are in general position for spheres.

Call $\mathcal{T}_\delta$ the associated Delaunay triangulation of C. Our objective is now to prove the following:

Claim A.2. *The triangulation $\mathcal{T}_\delta$ above achieves two properties:*

(v) *Every simplex with vertices in B has diameter at most δ;*
(vi) *Every simplex of $\mathcal{T}_\delta$ that has a vertex outside B does not intersect $\mathrm{int}_C(B)$.*

Proof. We first establish an auxiliary result for the proof of the claim. Define $r : \mathbb{R}^d \to B$ by letting $r(x)$ be the point in B that is closest to x. If $r(x) \neq x$, $r(x)$ belongs to a proper face of B, and then we can write $r(x)$ as $x - p$ where p is a normal for a supporting hyperplane at $r(x)$ with $p \cdot r(x) \geqslant p \cdot y$ for all $y \in B$. If in addition $r(x) \in \mathrm{int}_C(B)$, then $r(x)$ is at the boundary of C and so $p \cdot r(x) \geqslant p \cdot y$ for all $y \in C$ as well. Suppose $r(x) \neq x$ and let $r(x) = x - p$. Let y be a point such that $p \cdot y \leqslant p \cdot r(x)$. Let z be the nearest-point projection of y onto the line from x through $r(x)$. Then

$$\|x - y\|^2 = (\|x - r(x)\| + \|r(x) - z\|)^2 + \|z - y\|^2 \geqslant \|r(x) - x\|^2 + \|r(x) - y\|^2,$$

with the inequality being an equality iff $z = r(x)$, i.e., $p \cdot y = p \cdot r(x)$.

We are now ready to prove that the traingulation $\mathcal{T}_\delta$ satisfies the properties. Let x_δ be a point in $X_\delta \cap B$ and let x be a point in $\mathbb{R}^d$ that belongs to the Voronoi polyhedron $P(x_\delta)$ of x_δ. We claim that $\|r(x) - x_\delta\| < \delta/2$. If $r(x) = x$, this follows directly from Property (ii) of X_δ. Suppose that $r(x) \neq x$. Then $r(x)$ belongs to the interior of a proper face B' of B and as we saw in the last paragraph, $r(x)$ can be written as $x - p$. By definition of $r(x)$, $p \cdot x_\delta \leqslant p \cdot r(x)$ and thus: $\|x - x_\delta\|^2 \geqslant \|r(x) - x\|^2 + \|r(x) - x_\delta\|^2$. By Property (ii), there exists y_δ in $B' \cap X_\delta$ such that $\|r(x) - y_\delta\| < \delta/2$. Obviously $p \cdot y_\delta = p \cdot r(x)$ and since $x \in P(x_\delta)$, it follows that $\|x - x_\delta\|^2 \leqslant \|x - y_\delta\|^2 < \|r(x) - x\|^2 + \delta^2/4$; therefore, $\|r(x) - x_\delta\| < \delta/2$, as claimed. Observe that we proved that $\|x - x_\delta\|^2 < \|r(x) - x\|^2 + \delta^2/4$, a fact we will use below.

From the above paragraph, for each $x_\delta \in X_\delta \cap B$ and each $x \in P(x_\delta)$, the distance between $r(x)$ and x_δ is less than $\delta/2$; We claim finally that the diameter of each simplex with vertices in B is less than δ. Indeed, letting x_δ and y_δ be two vertices of a simplex in B, then their Voronoi cells intersect, so we can take x in the intersection. Since $r(x)$ is of distance $\delta/2$ from x_δ and $\delta/2$ from y_δ, x_δ and y_δ are distant less than δ. This concludes the proof that $\mathcal{T}_\delta$ satisfies (v).

We now prove that $\mathcal{T}_\delta$ satisfies (vi): for this, it is sufficient to show that the intersection of $P(x_\delta)$ and $P(y_\delta)$ is empty for all $x_\delta \in \mathrm{int}_C(B) \cap X_\delta$ and y_δ in $X_\delta \setminus B$. Take such a pair x_δ, y_δ. Fix $x \in P(x_\delta)$. If $r(x) = x$, then $\|x - x_\delta\| < \delta/2$, while by the definition of δ, $\|x - y_\delta\| \geqslant \delta/2$ and thus $x \notin P(y_\delta)$. Suppose $r(x) \neq x$. Since $x_\delta \in \mathrm{int}_C(B)$, by Property (iii) of the set X_δ, $r(x)$ cannot belong to $\partial_C B$, since in this case the distance between x_δ and $r(x)$ is greater than $\delta/2$. Therefore, $r(x)$ belongs to a face of C. Writing $r(x)$ as $x - p$, we then have that p is a normal to a hyperplane containing one of the faces of C and thus $p \cdot y_\delta \leqslant p \cdot r(x)$. Hence, $\|x - y_\delta\|^2 \geqslant \|r(x) - x\|^2 + \|r(x) - y_\delta\|^2 \geqslant \|r(x) - x\|^2 + \delta^2/4$ by the definition of δ, while as we saw in the previous paragraph, $\|x - x_\delta\|^2 < \|r(x) - x\|^2 + \delta^2/4$; thus again $x \notin P(y_\delta)$ and we are done. □

A.2. **Construction of the Triangulation.** For our problem of triangulating $\Sigma_n \times \Theta_n$, we recall the properties that the triangulation should satisfy:

(i) The only vertices of the triangulation in $\Sigma_n \times \{\, \theta_n^0 \,\}$ of $\mathcal{T}_n$ are pure strategies (s_n, θ_n^0), $s_n \in S_n$;

(ii) Letting Θ_n^1 be the face of Θ_n where θ_n^0 has zero probability, if $T_n \in \mathcal{T}_n$ is a simplex either with a face in $\Sigma_n \times \Theta_n^1$, or shares a face with such a simplex, then the diameter of T_n is less than ζ;

(iii) There exists a convex function $\varrho_n : \Sigma_n \times \Theta_n \to \mathbb{R}_+$ such that: (iii.i) $\varrho_n(\lambda x + (1-\lambda) y) = \lambda \varrho_n(x) + (1-\lambda)\varrho_n(y)$ iff x and y belong to a simplex T_n of $\mathcal{T}_n$; (iii.ii) $\varrho_n^{-1}(0) = \Sigma_n \times \{\, \theta_n^0 \,\}$.

Let $\hat{S}_n^1 \equiv S_n \times \{\, \theta_n^0 \,\}$. Let T_n be the set of vertices of Θ_n and $\hat{S}_n^2 \equiv \{\, s_n^0 \,\} \times (T_n \setminus \{\, \theta_n^0 \})$, where s_n^0 is an arbitrary fixed pure strategy in S_n. Let $\hat{S}_n \equiv \hat{S}_n^1 \cup \hat{S}_n^2 \setminus \{(s_n^0, \theta_n^1)\}$ where θ_n^1 is a vertex of Θ_n that is different from θ_n^0. Note that $d \equiv \dim(\Sigma_n \times \Theta_n) = |S_n| + |T_n| - 2 = |\hat{S}_n|$. For each $\hat{s}_n^1 \in \hat{S}_n^1$, let $x(\hat{s}_n^1)$ be the unit vector in $\mathbb{R}^{\hat{S}_n}$ for the coordinate $\hat{s}_n^1$; for each $\hat{s}_n^2 \in \hat{S}_n^2$, let $x(\hat{s}_n^2)$ be a point in $\mathbb{R}^{\hat{S}_n}$ to be determined later. Let $X \equiv \{x(\hat{s}_n^2)\}_{\hat{s}_n^2 \in \hat{S}_n^2}$.

Define an affine function $F_n^X : \Sigma_n \times \Theta_n \to \mathbb{R}^{\hat{S}_n}$ as follows: for each $\hat{s}_n \in \hat{S}_n$, $F_n^X(\hat{s}_n) = x(\hat{s}_n)$; for a vertex (s_n, θ_n) of $\Sigma_n \times \Theta_n$ that is not in $\hat{S}_n^1 \cup \hat{S}_n^2$, define $F_n^X(s_n, \theta_n) = F_n^X(s_n, \theta_n^0) +$

$F_n^X(s_n^0, \theta_n) - F_n^X(s_n^0, \theta_n^0)$. The map F_n^X extends to the whole of $\Sigma_n \times \Theta_n$ by linear interpolation. If the collection $X \cup \{x(\hat{s}_n^1)\}_{\hat{s}_n^1 \in \hat{S}_n^1}$ is affinely independent (which holds for an open and dense set of choices for X), then F_n^X is an affine homeomorphism with its image $C(X) \equiv F_n^X(\Sigma_n \times \Theta_n)$ and the dimension of $C(X)$ is $|\hat{S}_n|$.

Let H be a hyperplane in $\mathbb{R}^{S_n \times T_n}$ that strictly separates $\Sigma_n \times \Theta_n^1$ from $\Sigma_n \times \{\theta_n^0\}$ and such that the distance between $H \cap (\Sigma_n \times \Theta_n)$ and $\Sigma_n \times \Theta_n^1$ is less than ζ. Every vertex of $H \cap (\Sigma_n \times \Theta_n)$ is of the form $(1 - \varepsilon_{s_n,\theta_n})(s_n, \theta_n) + \varepsilon_{s_n,\theta_n}(s_n, \theta_n^0)$ for some $s_n \in S_n$, $\theta_n^0 \neq \theta_n \in \Theta_n$ and $0 < \varepsilon_{s_n,\theta_n} < \zeta$. The hyperplane could be defined equivalently by choosing $|S_n| + |T_n|$ of the coordinates $\varepsilon_{s_n,\theta_n}$. Let $B_0(X) = F_n^X(\Sigma_n \times \Theta_n^1)$ and let $B(X) = F_n^X(H_- \cap (\Sigma_n \times \Theta_n))$, where H_- is the halfspace that contains $\Sigma_n \times \Theta_n^1$.

Let $\bar{X}$ be the set of vertices of $C(X)$ and let $\bar{Y}$ be the set of vertices of $C(X) \cap F_n^X(H \cap (\Sigma_n \times \Theta_n))$. We claim now that if the choice of the vectors in X is generic and if the hyperplane H is in generic position, i.e., if the collection of $\varepsilon_{s_n,\theta_n}$ is chosen outside a nowhere dense set, then $\bar{X} \cup \bar{Y}$ satisfies property (P); consequently there exists a Delaunay triangulation of $C(X)$ that can be refined according to (P). To prove this claim, let $v_0, \ldots, v_{k+1}$ be a collection of vertices each belonging to either $\Sigma_n \times \Theta_n$ or its intersection with H and such that $v_0 = \sum_{i=1}^{k+1} v_i$, $\lambda_i \neq 0$ for all $i > 0$ and $\sum_i \lambda_i = 1$. We have to show that the Voronoi polyhedra of the v_i's do not intersect. We divide the proof in three cases: assume to begin with that all the vertices belong to $\Sigma_n \times \Theta_n$. Then, $k + 2 = 2J$ for some integer $J > 1$ and, after a relabeling of the vertices, $\sum_{i=1}^{J} v_i = \sum_{i=J+1}^{2J} v_i$; of these there are as many vertices in $\hat{S}_n^1$ among the first J as there are in the second; and there is at least one vertex on each side of the equality that does not belong to $\hat{S}_n^1$. If the Voronoi polyedra of the $F_n^X(v_i)$'s contain a point y in common, then letting c be the distance between y and each of the $F_n^X(v_i)$'s, elementary algebra shows that $2F_n^X(v_i) \cdot y = \|F_n^X(v_i)\| + \|y\| - c^2$ for each i. Since F_n^X is affine, it follows that $\sum_{i=1}^{J} \|F_n^X(v_i)\|^2 = \sum_{i=J+1}^{2J} \|F_n^X(v_i)\|^2$, an equality that holds only for a nongeneric choice of X. Therefore, the Voronoi polyhedra of the $F_n^X(v_i)$'s do not intersect.

Now suppose that all the v_i's belong to $H \cap C$. Then the nearest-point projection of the v_i's to $\Sigma_n \times \Theta_n^1$ span a face. The Voronoi polyhedra of the projections do not have a point in common, as we saw in the previous paragraph. Therefore, if the $\varepsilon_{s_n,\theta_n}$'s are small, the Voronoi polyhedra of the v_i's do not intersect either.

Finally, there remains to consider the case where one of them, which we can assume to be v_0, belongs to H. There must be at least two v_i's in $\Sigma_n \times \Theta_n$ as the hyperplane H does not contain any vertex of $\Sigma_n \times \Theta_n$. In particular, there are at most $k \leqslant d$ vertices in the collection of v_i's that belong to H. Therefore, we can perturb the $\varepsilon_{s_n,\theta_n}$ corresponding to

v_0 but not the others, to ensure that the Voronoi polyhedra do not intersect. Thus we have verified our claim that for generic X and H, the vertices in $\bar{X} \cup \bar{Y}$ allows to construct a Delaunay triangulation of $C(X)$ satisfying (P).

Take now a generic set X with the property that the norm of each $x(\hat{s}_n^2)$ is strictly greater than one for $\hat{s}_n^2 \in \hat{S}_n^2$. Since F_n^X is an affine homeomorphism, $\|x - y\|_\infty \leqslant M\|F_n^X(x) - F_n^X(y)\|$ for some $M > 0$ and for all $x, y \in \Sigma_n \times \Theta_n$. Let $\delta > 0$ be smaller than $M\zeta$. Using the construction of a subdivision described above, we now have a triangulation $\mathcal{T}_\delta$ of $C(X)$ where each point in $\mathrm{int}_C(B(X))$ belongs to a simplex with diameter less than δ/M, giving us properties (i) and (ii). As for property (iii), if ϱ_n is the convex function associated to the Delaunay triangulation $\mathcal{T}_\delta$, the composition $\varrho_n \circ F_n^X$ is convex and linear precisely on each cell of the triangulation of $\Sigma_n \times \Theta_n$ induced by the inverse mapping $(F_n^X)^{-1} : C(X) \to \Sigma_n \times \Theta_n$. Our convex function takes value one on $\Sigma_n \times \{\, \theta_n^0 \,\}$ and is strictly above one elsewhere. Subtracting now 1 from $\varrho_n \circ F_n^X$ we have a convex function satisfying property (iii).

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