Equi-normalized Robust Positively Invariant Sets for Linear Difference Inclusions

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Abstract

This note establishes the characterization, existence and uniqueness of equi-normalized polytopic robust positively invariant sets for linear difference inclusions. The computation of this set results in a nonconvex optimization problem. Although this may be reformulated exactly as a mixed integer linear programme, we propose a more practical and tractable alternative in the form of a fixed-point iteration based on linear programming. Convergence of the algorithm is established.

Key words: Robust Positive Invariance, Linear Difference Inclusions, Optimization.

1 Introduction

Set invariance is an important technique in the analysis and synthesis of systems under constraints and uncertainty [1], the popularity of which has been particularly amplified by its multifaceted use in model predictive control [2]. The theory and computation of invariant sets in various settings have led to a number of fundamental contributions addressing, *inter alia*, maximality [3–7] and minimality [6, 8] issues as well as approximation [9,10] and representation [7,11]. A more detailed overview of numerous important contributions in set invariance and its use in model predictive control can be found in the comprehensive monographs [1, 2].

A conceptually flexible and computationally promising notion of equi-normalized polytopic robust positively invariant sets has emerged recently [12–14]. The theory of these sets for linear systems is reasonably complete: characterization and existence, in the setting of linear systems controlled by continuous and positively homogeneous state feedback, were addressed in [12], while the uniqueness aspect was addressed in [13] for the setting of autonomous linear dynamics. Computational issues are also well understood: for a given design collection of points, the equi-normalized polytopic robust positively invariant set can be computed by a standard fixed point iteration [12] or by a single linear programme when the dynamics are linear [13]. A systematic design of suitable collections of points that generate equi-normalized polytopic robust positively invariant sets for linear dynamics has also been recently developed [14]. The importance of these sets stems from their uniqueness and minimality, with respect to set inclusion, over all polytopic robust positively invariant sets generated by a considered design collection of points.

It is well known—see e.g. monographs [1, 2] and a recent article [15] for a detailed discussion and numerous references—that linear difference inclusions represent an important and frequently deployed mathematical model for parametrically uncertain linear dynamics. It is also well known that the computation of robust positively invariant sets for linear difference inclusions is a significant challenge [1]. This is one of the drivers for this note. Another driver is that the versatility of the equi-normalized polytopic robust positively invariant sets would be considerably enhanced if it were possible to generalize the related theoretical and computational aspects to this setting. The main contribution of this note is therefore the study of equi-normalized polytopic robust positively invariant sets for linear difference inclusions. In terms of theory, the characterization, existence and uniqueness of such sets is established. In terms of computation, we show that the natural formulation of the problem of finding the equi-normalized robust postively invariant set is nonconvex. Although this may be recast, exactly, as a mixed-integer linear programming problem, we show that a fixed-point iteration, based on linear programming, offers a tractable and more practical alternative

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with guaranteed convergence.

Note Structure: Section 2 outlines the setting, notions, and objectives. Section 3 establishes the characterization, existence and uniqueness of the equi-normalized robust positively invariant sets for linear difference inclusions, while Section 4 addresses the corresponding computations. Section 5 provides a concluding discussion.

Nomenclature, Definitions and Conventions: The sets of non-negative integers and reals are denoted by $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$. A compact and convex subset $\mathcal{S}$ of $\mathbb{R}^n$ that contains origin in its interior is a proper C-set in $\mathbb{R}^n$. The intersection of finitely many closed half-spaces is a polyhedral set. A polytopic set is a bounded polyhedral set. The Minkowski sum of nonempty sets $\mathcal{S}_1, \mathcal{S}_2$ in $\mathbb{R}^n$ is

$$\mathcal{S}_1 + \mathcal{S}_2 := \{s_1 + s_2 \ : \ s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2\}.$$

The image of a set $\mathcal{S}$ under a matrix M of compatible dimensions, or a scalar M, is

$$M\mathcal{S} := \{Ms \ : \ s \in \mathcal{S}\}.$$

The convex hull of a set $\mathcal{S}$ is denoted by $\mathrm{convh}(\mathcal{S})$; for a finite set of points $\mathcal{S}_\mathcal{D} = \{s_i \in \mathbb{R}^n \ : \ i \in \mathcal{I}\}$,

$$\mathrm{convh}(\mathcal{S}_\mathcal{D}) := \{\sum_{i\in\mathcal{I}} \lambda_i s_i \ : \ \forall i \in \mathcal{I},\ \lambda_i \geq 0,\ \sum_{i\in\mathcal{I}} \lambda_i = 1\}.$$

For a finite set of matrices $\mathcal{M}_\mathcal{D} = \{M_i \in \mathbb{R}^{n\times n} \ : \ i \in \mathcal{I}\}$,

$$\mathrm{convh}(\mathcal{M}_\mathcal{D}) := \{\sum_{i\in\mathcal{I}} \lambda_i M_i \ : \ \forall i \in \mathcal{I},\ \lambda_i \geq 0,\ \sum_{i\in\mathcal{I}} \lambda_i = 1\}.$$

The spectral radius $\rho(M)$ of a matrix $M \in \mathbb{R}^{n\times n}$ is the largest absolute value of its eigenvalues. The joint spectral radius $\rho(\mathcal{M}_\mathcal{D})$ of a finite set of matrices $\mathcal{M}_\mathcal{D} = \{M_i \in \mathbb{R}^{n\times n} \ : \ i \in \mathcal{I}\}$ is

$$\rho(\mathcal{M}_\mathcal{D}) := \lim_{k\to\infty} \sup\{\|M_{i_1} M_{i_2} \cdots M_{i_k}\|^{1/k} \ : \ M_{i_j} \in \mathcal{M}_\mathcal{D}\}.$$

The support function $\mathrm{h}(\mathcal{S}, \cdot)$ of a nonempty closed convex subset $\mathcal{S}$ of $\mathbb{R}^n$ is given, for all $x \in \mathbb{R}^n$, by

$$\mathrm{h}(\mathcal{S}, x) := \sup_y \{x^\top y \ : \ y \in \mathcal{S}\}.$$

A nonempty closed convex subset $\mathcal{S}$ of $\mathbb{R}^n$ can be represented in terms of its support function $\mathrm{h}(\mathcal{S}, \cdot)$ as

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ \forall z \in \mathbb{R}^n,\ z^\top x \leq \mathrm{h}(\mathcal{S}, z)\}.$$

We do not distinguish between a variable and its vectorized form. Once introduced, Assumptions are utilized in the subsequent discussion without explicit referencing.

2 Preliminaries

We consider an uncertain discrete time system modelled as a linear difference inclusion

$$x^+ \in \mathcal{F}(x) \text{ with}$$
$$\mathcal{F}(x) := \{Ax + w \ : \ A \in \mathrm{convh}(\mathcal{A}_\mathcal{D}),\ w \in \mathcal{W}\}, \quad (2.1)$$

where $x \in \mathbb{R}^n$ and $x^+ \in \mathbb{R}^n$ are the current and successor states, respectively, $w \in \mathbb{R}^n$ is the current disturbance, $A \in \mathbb{R}^{n\times n}$ is an admissible state transition matrix, and $\mathcal{A}_\mathcal{D}$ is a finite set of matrices in $\mathbb{R}^{n\times n}$. We consider a setting characterized by the following two assumptions.

Assumption 1 *For a finite index set $\mathcal{I} := \{1, 2, \ldots, q\}$,*

$$\mathcal{A}_\mathcal{D} := \{A_i \in \mathbb{R}^{n\times n} \ : \ i \in \mathcal{I}\}, \quad (2.2)$$

and the finite set of matrices $\mathcal{A}_\mathcal{D}$ is strictly stable, i.e., its joint spectral radius $\rho(\mathcal{A}_\mathcal{D})$ is strictly less than 1.

Assumption 2 *The set $\mathcal{W}$ is a proper C-set in $\mathbb{R}^n$.*

2.1 Robust Positively Invariant Sets

A proper C-set $\mathcal{S}$ in $\mathbb{R}^n$ is a robust positively invariant set for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in (2.1) if and only if for all $x \in \mathcal{S}$, all $A_i \in \mathcal{A}_\mathcal{D}$ and all $w \in \mathcal{W}$ it holds that $A_i x + w \in \mathcal{S}$. Thus, the robust positive invariance property of a proper C-set $\mathcal{S}$ in $\mathbb{R}^n$ can be expressed by a system of set inclusions

$$\forall i \in \mathcal{I},\ A_i\mathcal{S} + \mathcal{W} \subseteq \mathcal{S}, \quad (2.3)$$

or by an equivalent system of the functional inequalities

$$\forall z \in \mathbb{R}^n,\ \forall i \in \mathcal{I},\ \mathrm{h}(\mathcal{S}, A_i^\top z) + \mathrm{h}(\mathcal{W}, z) \leq \mathrm{h}(\mathcal{S}, z), \quad (2.4)$$

which can be equivalently rewritten in a compact form

$$\forall z \in \mathbb{R}^n,\ \max_{i\in\mathcal{I}} \mathrm{h}(\mathcal{S}, A_i^\top z) + \mathrm{h}(\mathcal{W}, z) \leq \mathrm{h}(\mathcal{S}, z). \quad (2.5)$$

A proper C-set $\mathcal{S}$ in $\mathbb{R}^n$ is the minimal, with respect to set inclusion, closed convex robust positively invariant set for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in (2.1) if and only if it verifies the fixed point set equation

$$\mathrm{convh}(\bigcup_{i\in\mathcal{I}} A_i\mathcal{S}) + \mathcal{W} = \mathcal{S}, \quad (2.6)$$

or, equivalently, if and only if its support function $\mathrm{h}(\mathcal{S}, \cdot)$ verifies the following fixed point functional equation

$$\forall z \in \mathbb{R}^n,\ \max_{i\in\mathcal{I}} \mathrm{h}(\mathcal{S}, A_i^\top z) + \mathrm{h}(\mathcal{W}, z) = \mathrm{h}(\mathcal{S}, z). \quad (2.7)$$

2.2 Polytopic Robust Positively Invariant Sets

A polytopic proper C-set $\mathcal{P}$ in $\mathbb{R}^n$ can be represented as

$$\mathcal{P} = \{x \in \mathbb{R}^n \ : \ \forall j \in \mathcal{J},\ p_j^\top x \leq \mathrm{h}(\mathcal{P}, p_j)\}, \qquad (2.8)$$

in which $\mathcal{J}$ is a finite index set, $\{p_j \in \mathbb{R}^n \setminus \{0\} \ : \ j \in \mathcal{J}\}$ is a finite collection of points and, for all $j \in \mathcal{J}$, $0 < \mathrm{h}(\mathcal{P}, p_j) < \infty$. A polytopic proper C-set $\mathcal{P}$ in $\mathbb{R}^n$ is a robust positively invariant set for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in (2.1) if and only if

$$\forall j \in \mathcal{J},\ \max_{i \in \mathcal{I}} \mathrm{h}(\mathcal{P}, A_i^\top p_j) + \mathrm{h}(\mathcal{W}, p_j) \leq \mathrm{h}(\mathcal{P}, p_j). \quad (2.9)$$

A polytopic proper C-set $\mathcal{P}$ in $\mathbb{R}^n$ is an equi-normalized robust positively invariant set for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in (2.1) if and only if

$$\forall j \in \mathcal{J},\ \max_{i \in \mathcal{I}} \mathrm{h}(\mathcal{P}, A_i^\top p_j) + \mathrm{h}(\mathcal{W}, p_j) = \mathrm{h}(\mathcal{P}, p_j). \quad (2.10)$$

2.3 Main Objectives

The objective of this note is to address the characterization, existence, uniqueness and computation of equi-normalized polytopic robust positively invariant sets for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in (2.1).

Specifically, for any $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_r) \in \mathbb{R}^r_{\geq 0}$, we consider a polytope

$$\mathcal{P}(\varphi) := \{x \in \mathbb{R}^n \ : \ \forall j \in \mathcal{J},\ p_j^\top x \leq \varphi_j\} \text{ with } \mathcal{J} := \{1, 2, \ldots, r\}, \qquad (2.11)$$

in which the r normal vectors to the defining halfspaces

$$\{p_j \in \mathbb{R}^n \setminus \{0\} \ : \ j \in \mathcal{J}\}. \qquad (2.12)$$

are already chosen and meet the following assumption.

Assumption 3 *For the finite collection of points $\{p_1, \ldots, p_r\}$ there exists a $\varphi = \widehat{\varphi} \in \mathbb{R}^r_{\geq 0}$ such that $\mathcal{P} = \mathcal{P}(\widehat{\varphi})$ satisfies* (2.9).

This assumption requires the normals $\{p_1, \ldots, p_r\}$ to be chosen in order that the set $\mathcal{P}(\varphi)$ can be made robust positively invariant by suitable choice of φ; existence, but not necessarily knowledge, of $\widehat{\varphi}$ is assumed. The design of a suitable set of points is discussed in [14], albeit in the context of linear dynamics; design in the linear difference inclusion case is a topic for future research.

The objective is then to determine the $\varphi \in \mathbb{R}^r_{\geq 0}$ in order that the polytopic proper C-set $\mathcal{P}(\varphi)$ satisfies the *equi-normalized* form of the robust positive invariance condition (2.10) (i.e., condition (2.9) with equality):

$$\forall j \in \mathcal{J}, \mathrm{h}(\mathcal{P}(\varphi), p_j) = \varphi_j, \text{ and} \qquad (2.13)$$
$$\max_{i \in \mathcal{I}} \mathrm{h}(\mathcal{P}(\varphi), A_i^\top p_j) + \mathrm{h}(\mathcal{W}, p_j) = \mathrm{h}(\mathcal{P}(\varphi), p_j). \quad (2.14)$$

Such a φ, if it exists, defines a polytope $\mathcal{P}(\varphi)$ that is *minimal*, with respect to set inclusion, among all robust positively invariant polytopic sets generated by the same points $\{p_1, \ldots, p_r\}$.

3 Characterization, Existence and Uniqueness

To compactly discuss the characterization, existence and uniqueness of the solution to the fixed point equation (2.13) and (2.14), let, for all $j \in \mathcal{J}$,

$$\psi_j := \mathrm{h}(\mathcal{W}, p_j) \text{ and } \psi := (\psi_1, \psi_2, \ldots, \psi_r), \qquad (3.1)$$

as well as, for all $j \in \mathcal{J}$ and all $\varphi \in \mathbb{R}^r_{\geq 0}$,

$$f_j(\varphi) := \max_{i \in \mathcal{I}} \mathrm{h}(\mathcal{P}(\varphi), A_i^\top p_j) \text{ and } g_j(\varphi) := f_j(\varphi) + \psi_j, \text{ and}$$
$$f(\varphi) := (f_1(\varphi), f_2(\varphi), \ldots, f_r(\varphi)) \text{ and}$$
$$g(\varphi) := (g_1(\varphi), g_2(\varphi), \ldots, g_r(\varphi)). \qquad (3.2)$$

We now outline the most relevant facts that enable an adequate utilization of the results established in [12,13].

Proposition 1 *Suppose Assumptions 1, 2 and 3 hold. Consider the vector $\psi \in \mathbb{R}^r$ and the functions $f(\cdot)$ and $g(\cdot)$ defined in* (3.1) *and* (3.2)*, respectively.*

(i) $0 < \psi \leq \widehat{\varphi}$ *and, for all* $\varphi \in \mathbb{R}^r_{\geq 0}$, $0 \leq f(\varphi) < \infty$ *and* $0 < \psi \leq g(\varphi) < \infty$.
(ii) $f(\cdot) : \mathbb{R}^r_{\geq 0} \to \mathbb{R}^r_{\geq 0}$ *and* $g(\cdot) : \mathbb{R}^r_{\geq 0} \to \mathbb{R}^r_{\geq 0}$ *are continuous and monotonically non-decreasing.*
(iii) $f(\cdot) : \mathbb{R}^r_{\geq 0} \to \mathbb{R}^r_{\geq 0}$ *is positively homogeneous of the first degree, i.e., for all* $\varphi \in \mathbb{R}^r_{\geq 0}$ *and all* $\alpha \in \mathbb{R}_{\geq 0}$, $f(\alpha\varphi) = \alpha f(\varphi)$.
(iv) *For all* $\varphi \in \mathbb{R}^r_{\geq 0}$ *and all* $j \in \mathcal{J}$, $\mathrm{h}(\mathcal{P}(f(\varphi)), p_j) = f_j(\varphi)$ *and* $\mathrm{h}(\mathcal{P}(g(\varphi)), p_j) = g_j(\varphi)$.

In light of Proposition 1(*iv*), the fixed point equation (2.13) with condition (2.14) is equivalently given by

$$f(\varphi) + \psi = \varphi \quad \text{i.e.,} \quad g(\varphi) = \varphi, \qquad (3.3)$$

which yields directly the following characterization.

Theorem 1 *Suppose Assumptions 1, 2 and 3 hold. A polytopic proper C-set $\mathcal{P}(\phi)$ in $\mathbb{R}^n$ parameterized as in* (2.11) *and specified as in* (2.8) *is an equi-normalized robust positively invariant set for the linear difference inclusion $x^+ \in \mathcal{F}(x)$ specified in* (2.1) *if and only if $\phi \in \mathbb{R}^r_{\geq 0}$ verifies the fixed point equation* (3.3).

The existence of the fixed point verifying (3.3) follows from the Brouwer fixed point theorem [16, Corollary in Page 176] by an argument in the spirit of [12, Theorem 1].

Theorem 2 *Suppose Assumptions 1, 2 and 3 hold. Consider the function* $g(\cdot)$ *defined in* (3.2) *and let*

$$\mathcal{G} := \{\varphi \in \mathbb{R}^r \ : \ 0 \le \varphi \le \widehat{\varphi}\}.$$

There exists a $\phi \in \mathcal{G}$ *such that* $g(\phi) = \phi$.

The identified fixed point is, in fact, unique, which can be verified by making an adequate, and relatively direct, use of the arguments of the proof of [13, Theorem 3].

Theorem 3 *Suppose Assumptions 1, 2 and 3 hold. Consider the function* $g(\cdot) \ : \ \mathbb{R}^r_{\ge 0} \to \mathbb{R}^r_{\ge 0}$ *defined by* (3.2). *There exists a unique* $\phi \in \mathbb{R}^r_{\ge 0}$ *such that* $g(\phi) = \phi$, *which is also such that* $0 < g(0) = \psi \le g(\phi) = \phi \le g(\widehat{\varphi}) \le \widehat{\varphi}$.

The relevance of the proper C-polytope $\mathcal{P}(\phi)$, where $\phi = g(\phi)$, is summarized by an analogue of [12, Corollary 1].

Corollary 1 *Suppose Assumptions 1, 2 and 3 hold and let* $\phi \in \mathbb{R}^q_{\ge 0}$ *be the unique fixed point of* (3.3). *The proper* C*-polytope* $\mathcal{P}(\phi)$ *is the unique equi-normalized and the minimal, with respect to set inclusion, robust positively invariant set for the linear difference inclusion specified by* (2.1), *which is generated via* (2.11) *by the collection of points* $\{p_1, \dots, p_r\}$.

4 Computation

In the setting of linear dynamics, the computation of the fixed point ϕ via a single linear programme was developed in [13] and slightly expanded in [14]. In our setting, an optimization problem $\mathfrak{P}$ for the computation of the unique solution $\phi \in \mathbb{R}^q_{\ge 0}$ of the fixed point equation (3.3) takes the form

$$\begin{aligned} &\text{maximize } \varsigma_1 + \varsigma_2 + \ldots + \varsigma_r \\ &\text{with respect to } \{\varsigma_j \in \mathbb{R}\}_{j\in\mathcal{J}} \text{ and } \{x_j \in \mathbb{R}^n\}_{j\in\mathcal{J}} \\ &\text{subject to } \forall j \in \mathcal{J},\ \varsigma_j \le \max_{i\in\mathcal{I}} p_j^\top A_i x_j, \\ &\text{and } \forall j \in \mathcal{J},\ \forall k \in \mathcal{J},\ p_k^\top x_j \le \varsigma_k + \psi_k. \end{aligned}$$

If $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_r)$ is formed from the $\{\varsigma_j^0 \in \mathbb{R}\}_{j\in\mathcal{J}}$ part of the maximizer of the optimization problem $\mathfrak{P}$ by setting, for all $j \in \mathcal{J}$, $\sigma_j := \varsigma_j^0$, then the unique fixed point $\phi \in \mathbb{R}^q_{\ge 0}$ solving (3.3) is recovered by setting $\phi = \sigma + \psi$. Namely, by construction, for all $j \in \mathcal{J}$, $\max_{i\in\mathcal{I}} \mathrm{h}(\mathcal{P}(\sigma+\psi), A_i^\top p_j) = \sigma_j$, $\mathrm{h}(\mathcal{W}, p_j) = \psi_j$ and $\mathrm{h}(\mathcal{P}(\sigma+\psi), p_j) = \sigma_j + \psi_j = \phi_j$. Despite this, the optimization problem $\mathfrak{P}$ is not a linear programme because the constraints, for all $j \in \mathcal{J}$, $\varsigma_j \le \max_{i\in\mathcal{I}} p_j^\top A_i x_j$, define the hypograph of a set of convex functions.

Problem $\mathfrak{P}$ may be recast as a mixed-integer linear programming problem by associating a binary variable with each $i \in \mathcal{I}$ in each of the r constraints $\varsigma_j \le \max_{i\in\mathcal{I}} p_j^\top A_i x_j$. Nonetheless, we propose a tractable and more practical alternative to nonlinear or mixed-integer optimization in the form of the fixed-point iteration

$$\forall k \in \mathbb{N},\ \varphi_{k+1} = g(\varphi_k) \text{ i.e., } \varphi_{k+1} = f(\varphi_k) + \psi, \quad (4.1)$$

with φ_0 suitably defined. In these dynamics, determination of ψ as specified in (3.1), given the set $\mathcal{W}$, is via convex optimization prior to initialization of the iterations. Likewise, given any φ, $g(\varphi)$ and $f(\varphi)$ defined in (3.2) may be evaluated by first solving rq linear programming problems specified, for each $j \in \mathcal{J}$ and each $i \in \mathcal{I}$, by

$$\mathrm{h}(\mathcal{P}(\varphi), A_i^\top p_j) = \max_x \{p_j^\top A_i x \ : \ \forall \ell \in \mathcal{J}, p_\ell^\top x \le \varphi_\ell\}$$

and then taking, for each $j \in \mathcal{J}$, the maximum over $i \in \mathcal{I}$ of $\mathrm{h}(\mathcal{P}(\varphi), A_i^\top p_j)$, i.e., for each $j \in \mathcal{J}$,

$$f_j(\varphi) = \max\{\mathrm{h}(\mathcal{P}(\varphi), A_1^\top p_j), \dots, \mathrm{h}(\mathcal{P}(\varphi), A_q^\top p_j)\}. \quad (4.2)$$

The following, main result of this section establishes convergence of this iteration from different initial points φ_0.

Theorem 4 *Suppose Assumptions 1, 2 and 3 hold.*

(1) If $\varphi_0 = \psi$ *then, for all* $k \in \mathbb{N}$,

$$\psi \le \varphi_k \le \varphi_{k+1} = g(\varphi_k) \le \widehat{\varphi}.$$

(2) For any $\alpha \ge 1$, *if* $\varphi_0 = \alpha\widehat{\varphi}$ *then, for all* $k \in \mathbb{N}$,

$$\psi \le \varphi_{k+1} = g(\varphi_k) \le \varphi_k \le \alpha\widehat{\varphi}.$$

Moreover, in either case, the sequence $\{\varphi_k\}_{k\ge 0}$ *converges monotonically to the unique solution* ϕ *of the fixed point equation* (3.3), *which is such that* $\psi \le g(\phi) = \phi \le \widehat{\varphi}$. *Furthermore, if for some* $k \in \mathbb{N}$, $\varphi_{k+1} = \varphi_k$ *then* $\phi = \varphi_k$. *Moreover, for all* $\varepsilon > 0$, *there exists a finite* $k^* \in \mathbb{N}$ *such that, for all* $k \in \mathbb{N},\ k \ge k^*$, *it holds that* $|\varphi_k - \phi|_\infty \le \varepsilon$.

The *sandwich theorem* helps then to establish asymptotic convergence from an arbitrary initial point.

Theorem 5 *Suppose Assumptions 1, 2 and 3 hold. For any* $\alpha \ge 1$, *if* φ_0 *is such that* $\psi \le \varphi_0 \le \alpha\widehat{\varphi}$, *then* $\{\varphi_k\}_{k\ge 0}$ *generated by* (4.1) *converges asymptotically to the unique solution* ϕ *of the fixed point equation* (3.3).

5 Discussion

The fixed-point iteration results in a convergence to the unique solution ϕ of the fixed point equation (3.3) via

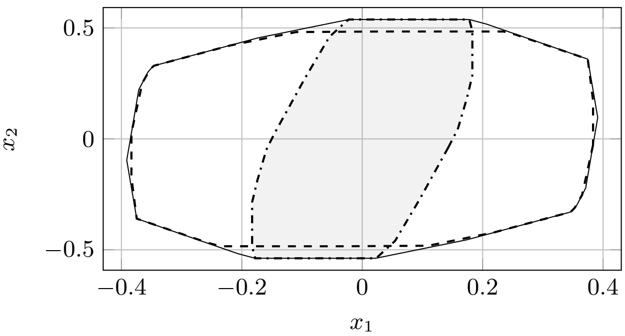

Fig. 1. Illustration of $\mathcal{P}(\phi)$ (solid), $A_1\mathcal{P}(\phi) + \mathcal{W}$ (dashed) and $A_2\mathcal{P}(\phi) + \mathcal{W}$ (shaded) for the $(q, r) = (2, 50)$ case.

solving a sequence of linear programming problems. Table 1 demonstrates this convergence for the LDI system (2.1) with $\mathcal{A}_D = \text{convh}(\{A_1, A_2\})$, with

$$A_1 = \begin{bmatrix} 0.7216 & 0.0119 \\ 0.0117 & 0.7096 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.0254 & 0.1530 \\ 0.2633 & 0.7272 \end{bmatrix},$$

and $\mathcal{W} = \{w : \|w\|_\infty \leq 0.1\}$. The points $\{p_1, \ldots, p_r\}$ are chosen so that $\mathcal{P}(1)$ is the regular r-sided polygon in $\mathbb{R}^2$; it is verified that $\widehat{\varphi} = 1$ meets Assumption 3 in this case. Figure 1 illustrates the equi-normalized set $\mathcal{P}(\phi)$.

The set $\mathcal{A}_\mathcal{D}$ was augmented with further randomly generated A_i matrices in a manner that $\mathcal{A}_\mathcal{D}$ remained strictly stable. Likewise, the set $\mathcal{P}(1)$ was refined with additional, uniformly spaced points to make an $r = 50$-sided regular polygon. The results are shown in Table 1. The number of iterations required for $q = 4$, $r = 50$ is comparable to the $(q, r) = (2, 8)$ case. With $q = 8$, the set $\mathcal{P}(1)$ does not meet the robust positive invariance condition and more iterations were required.

In comparison to these results, we found that the mixed integer linear programming formulation scales poorly and performs unpredictably: although the $(q, r) = (2, 8)$ and $(8, 8)$ problems solved in, respectively, 0.1 s and 20 s using CPLEX 22.1.1, the $(2, 50)$ problem required 18.5 h to solve on the same platform.

This note has addressed the characterization, existence, uniqueness and computation of the equi-normalized polytopic robust positively invariant sets for the linear difference inclusions, which opens up space for their utilization for the design of improved tube model predictive controllers. Additional relevant lines for future research include a systematic design of suitable collection of points $\{p_1, \ldots, p_r\}$, and exploitation of the dual form of the support functions in order to provide an accelerated fixed-point iteration.

Table 1
Results for different q (number of vertices in the system description) and r (number of halfspaces in the description of $\mathcal{P}$). The termination criterion was $|\varphi_{k+1} - \varphi_k|_\infty < 1e^{-6}$.

(q, r)	$(2, 8)$	$(2, 50)$	$(4, 8)$	$(4, 50)$	$(8, 8)$	$(8, 50)$
Iterations	41	47	43	43	95	99
LPs per iter.	16	100	32	200	64	400
φ_0 meets A3	Yes	Yes	Yes	Yes	No	No

Appendix A: Proof of Proposition 1. (i) These facts follow from (3.1) and (3.2). (ii) By the virtue of [12, Proposition 1], for each $j \in \mathcal{J}$ and each $i \in \mathcal{I}$, the functions $\varphi \mapsto \text{h}(\mathcal{P}(\varphi), A_i^\top p_j)$ are continuous and monotonically non-decreasing over $\mathbb{R}^r_{\geq 0}$. Thus, in light of [17, Proposition 1.26(c)], for each $j \in \mathcal{J}$, the functions $\varphi \mapsto f_j(\varphi) = \max_{i \in \mathcal{I}} \text{h}(\mathcal{P}(\varphi), A_i^\top p_j)$ and $\varphi \mapsto g_j(\varphi) = f_j(\varphi) + \psi_j$ are continuous over $\mathbb{R}^r_{\geq 0}$, and, due to their definitions, the functions $\varphi \mapsto f_j(\varphi)$ and $\varphi \mapsto g_j(\varphi)$ are also monotonically non-decreasing over $\mathbb{R}^r_{\geq 0}$. (iii) Clearly, for all $\varphi \in \mathbb{R}^r_{\geq 0}$ and all $\alpha \in \mathbb{R}_{\geq 0}$, $\mathcal{P}(\alpha\varphi) = \alpha\mathcal{P}(\varphi)$, and, in turn, for each $j \in \mathcal{J}$, $\text{h}(\mathcal{P}(\alpha\varphi), p_j) = \text{h}(\alpha\mathcal{P}(\varphi), p_j) = \alpha\,\text{h}(\mathcal{P}(\varphi), p_j)$. By the same token, for all $\varphi \in \mathbb{R}^r_{\geq 0}$ and all $\alpha \in \mathbb{R}_{\geq 0}$, and for each $j \in \mathcal{J}$ and each $i \in \mathcal{I}$, $\text{h}(\mathcal{P}(\alpha\varphi), A_i^\top p_j) = \alpha\,\text{h}(\mathcal{P}(\varphi), A_i^\top p_j)$. In turn, for all $\varphi \in \mathbb{R}^r_{\geq 0}$ and all $\alpha \in \mathbb{R}_{\geq 0}$, and for each $j \in \mathcal{J}$, $\max_{i \in \mathcal{I}} \text{h}(\mathcal{P}(\alpha\varphi), A_i^\top p_j) = \alpha \max_{i \in \mathcal{I}} \text{h}(\mathcal{P}(\varphi), A_i^\top p_j)$. Thus, for all $\varphi \in \mathbb{R}^r_{\geq 0}$ and all $\alpha \in \mathbb{R}_{\geq 0}$, $f(\alpha\varphi) = \alpha f(\varphi)$, as claimed. (iv) These facts follow from (2.11) and (3.2).

Appendix B: Proof of Theorem 2. By monotonicity of the function $g(\cdot)$, for all $\varphi \in \mathcal{G}$, $0 < g(0) = \psi \leq g(\varphi) \leq g(\widehat{\varphi}) \leq \widehat{\varphi}$, where $g(\widehat{\varphi}) \leq \widehat{\varphi}$ by Assumption 3. Thus, $g(\cdot)$ maps $\mathcal{G}$ to itself. Since $g(\cdot)$ is continuous, $\mathcal{G}$ is a convex and compact subset of $\mathbb{R}^r_{\geq 0}$, and $g(\cdot)$ maps $\mathcal{G}$ to itself, the Brouwer fixed point theorem guarantees the existence of the claimed point $\phi \in \mathcal{G}$ such that $g(\phi) = \phi$.

Appendix C: Proof of Theorem 4. By Theorem 2, $g(\cdot)$ maps $\mathcal{G}$ to itself. Thus, $\varphi_0 \in \mathcal{G}$ implies $\varphi_k \in \mathcal{G}$ for all $k \in \mathbb{N}$. If, for any $k \in \mathbb{N}$, $\varphi_k \leq \varphi_{k+1}$ then, by monotonicity of $g(\cdot)$, $\varphi_{k+1} = g(\varphi_k) \leq g(\varphi_{k+1}) = \varphi_{k+2}$. Thus, $\varphi_k \leq \varphi_{k+1}$ implies $\varphi_{k+1} \leq \varphi_{k+2}$. Clearly, for $0 < \varphi_0 = \psi$, $g(0) = \psi = \varphi_0 \leq \varphi_1 = g(\psi) \leq \widehat{\varphi}$, so $\varphi_0 \leq \varphi_1$. Claim (1) then follows by induction. Thus, the sequence $\{\varphi_k\}_{k \geq 0}$ is monotonically nondecreasing and every $\varphi_k \in \mathcal{G}$. Since $\mathcal{G}$ is compact and $g(\cdot)$ is continuous, it follows that $\{\varphi_k\}_{k \geq 0}$ converges to a limit $\overline{\varphi}$ such that $\overline{\varphi} = g(\overline{\varphi})$. By Theorem 3 such a point is the unique fixed point ϕ and, hence $\{\varphi_k\}_{k \geq 0}$ converges monotonically from below to ϕ. Claim (2) follows by applying the same arguments starting from the fact that if $\varphi_{k+1} \leq \varphi_k$ then $\varphi_{k+2} \leq \varphi_{k+1}$, and then noting that, for any $\alpha \geq 1$, $\varphi_0 = \alpha\widehat{\varphi}$ implies $\varphi_1 \leq \varphi_0$. The finite convergence assertion is hopefully clear, while the numerical convergence assertion follows directly from the fact that the sequence $\{\varphi_k\}_{k \geq 0}$, from either initial point, is convergent.

Appendix D: Proof of Theorem 5. Let $\underline{\varphi}_k$, φ_k, $\overline{\varphi}_k$

denote the iterates from the respective initial points $\psi =: \underline{\varphi}_0, \varphi_0$ (such that $\psi \leq \varphi_0 \leq \alpha\widehat{\varphi}$) and $\alpha\widehat{\varphi} =: \overline{\varphi}_0$ with $\alpha \geq 1$. Since $\underline{\varphi}_0 \leq \varphi_0 \leq \overline{\varphi}_0$ monotonicity of $g(\cdot)$ yields $\underline{\varphi}_k \leq \varphi_k \leq \overline{\varphi}_k$ for all $k \in \mathbb{N}$. Hence, since $g(\cdot)$ is continuous, $\mathcal{G}$ is compact, and $\{\underline{\varphi}_k\}_{k\geq 0}$ and $\{\overline{\varphi}_k\}_{k\geq 0}$ are convergent to the same limit $\phi = g(\phi) \in \mathcal{G}$, the sequence $\{\varphi_k\}_{k\geq 0}$ is also convergent to the same limit $\phi = g(\phi) \in \mathcal{G}$.

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