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Stochastic dynamics on product manifolds: twenty five years after

In memory of Yuri Kondratiev

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Abstract

We consider an infinite system of stochastic differential equations in a compact manifold $\mathcal{M}$. The equations are labeled by vertices of a geometric graph with unbounded vertex degrees and coupled via nearest neighbour interaction. We prove the global existence and uniqueness of strong solutions and construct in this way stochastic dynamics associated with Gibbs measures describing equilibrium states of a (quenched) system of particles with positions forming a typical realization of a Poisson or Gibbs point process in $\mathbb{R}^d$.

Contents

1 Introduction

In 1997, Ukrainian Mathematical Journal published the paper "Infinite systems of stochastic differential equations and some lattice models on compact Riemannian manifolds" by Sergio Albeverio, Yuri Kondratiev and AD [2]. That paper initiated a series of works on stochastic analysis on infinite product manifolds, and its applications in statistical mechanics of interacting particle systems on integer lattices, see [3, 4, 5] and references therein.

In those works, the main object of our study was an infinite system of stochastic differential equations (SDE) labelled by vertices of the integer lattice $\mathbb{Z}^d$, on a compact Riemannian manifold $\mathcal{M}$. Such a system can be, at least heuristically, understood as a single SDE on the Cartesian power $\mathcal{M}^{\mathbb{Z}^d}$. The latter is a compact topological space but it does not possess any structure of a Hilbert or Banach manifold, which makes it impossible to directly apply the known theory of SDEs on such manifolds, like in [10], [13]. In the case where the equations are coupled via nearest neighbours pair interaction, or, more generally, uniformly bounded interaction, this equation can be considered and solved in the space of weighted sequences in a Euclidean space $\mathbb{R}^n$ containing $\mathcal{M}$ as a subset, with a specially chosen weight sequence. This weight sequence gives also a possibility to introduce manifold-like structures on $\mathcal{M}^{\mathbb{Z}^d}$.

On the other hand, the last two decades have witnessed an increasing interest in the so-called configuration space analysis (beware- this is an ambiguous term), that is, analysis on the space $\Gamma(\mathbb{R}^d)$ of locally finite subsets γ of a Euclidean space $\mathbb{R}^d$ (or, for that matter, a Riemannian manifold or metric space), endowed by a Poisson or Gibbs measure, see [7, 8] and e.g. [19, 20] for more recent developments and literature review.

For a typical configuration $\gamma \in \Gamma$ and fixed $r > 0$, the number n_x of elements in the set

$$\gamma_x = \gamma_x(r) := \{y \in \gamma : |x - y| < r\} \tag{1}$$

is finite but unbounded, in contrast to the case of $\gamma = \mathbb{Z}^d$, where $\sup_{x\in\mathbb{Z}^d} n_x < \infty$. The numbers n_x, $x \in \gamma$, can be interpreted as vertex degrees of the geometric graph γ_r with the vertex set γ and $x, y \in \gamma$ connected by an edge iff $|x - y| < r$.

A natural question is whether we can construct a reasonable analysis on the product manifold $\mathcal{M}^\gamma$ instead of $\mathcal{M}^{\mathbb{Z}^d}$. It boils down to considering a system of SDEs with unbounded number of coupled equations, which cannot be in general solved in any weighted space with a fixed weight sequence. In [18, 17, 14, 15], an approach to such systems in Euclidean spaces, both deterministic and stochastic, has been developed. It implements an extension of the classical Ovsyannikov method for ODEs, see e.g. [22], and allows to prove the existence and uniqueness of global strong solutions in a scale of expanding Hilbert (or Banach) spaces $(X_\alpha)_{\alpha\in\mathcal{A}}$, where $\mathcal{A}$ is an interval and $X_\alpha \subset X_\beta$ for $\alpha < \beta$. Here X_α is the space of weighted sequences with elements $\sigma_x \in \mathbb{R}^n$, indexed by $x \in \gamma$, and exponential weights $e^{-\alpha|x|}$. The price to pay is that the solution with an initial value in X_α will live in X_β with $\beta > \alpha$. However, the situation somewhat improves for product manifolds. Indeed, for compact $\mathcal{M}$ we have $\mathcal{M}^\gamma \subset X_\alpha$ for any α, which gives a possibility to prove global well-posedness of the corresponding system of SDEs.

Mathematically, the pair $(\gamma, (\sigma_x)_{x\in\gamma})$ is an element of the marked configuration space $\Gamma(\mathbb{R}^d, \mathcal{M})$, see e.g. [16, 19, 20, 21] for rigorous definition and properties of such spaces and their applications in statistical mechanics.

From the physical point of view, $(\gamma, (\sigma_x)_{x\in\gamma}) \in \Gamma(\mathbb{R}^d, \mathcal{M})$ represents a collection of particles with positions $x \in \mathbb{R}^d$ and internal parameters (spins) $\sigma_x \in \mathcal{M}$, see for example [29], [25, Sec. 11], [12] and [20, 21], in relation to modelling of non-crystalline (amorphous) substances, e.g. ferrofluids and amorphous magnets.

2 Preliminaries

2.1 SDE on a compact manifolds

Let $\mathcal{M}$ be a compact connected N-dimensional Riemannian manifold. Consider the following (heuristic) stochastic differential equation on $\mathcal{M}$:

$$d\xi(t) = a(\xi(t))dt + B(\xi(t)) \circ dW(t),\ \xi(t) \in \mathcal{M},\ t \geq 0, \tag{2}$$

where $\circ\, dW(t)$ stands for the Stratonovich differential of a Wiener process $W(t)$ in $\mathbb{R}^N$, $a : \mathcal{M} \to T\mathcal{M}$ and $B : \mathcal{M} \times \mathbb{R}^N \to T\mathcal{M}$ are, respectively, vector and operator fields on $\mathcal{M}$, so that $B(\sigma) \in \mathcal{L}(\mathbb{R}^N, T_\sigma\mathcal{M})$, $\sigma \in \mathcal{M}$. Here and in what follows $\mathcal{L}$ denotes the space of bounded linear operators. We assume that a and B belong to C^1 and C^2, respectively.

For simplicity, we assume that the diffusion operator B satisfies the equality

$$G^{-1}(\sigma) = B(\sigma)B^*(\sigma),$$

where the operator field $G(\sigma) : T_\sigma\mathcal{M} \to T^*_\sigma\mathcal{M}$, $\sigma \in \mathcal{M}$, defines the Riemannian structure in $\mathcal{M}$. Observe that such B always exists. Then the (formal) generator H of the process ξ is given by the differential expression

$$H\phi(\sigma) = \frac{1}{2}\Delta\phi(\sigma) + (\nabla\phi(\sigma), a(\sigma) + b(\sigma))_{T_\sigma\mathcal{M}}\,, \tag{3}$$

where $\phi \in C^2(\mathcal{M})$, $(\cdot,\cdot)_{T_\sigma\mathcal{M}}$ is the scalar product in $T_\sigma\mathcal{M}$, Δ is the Laplace-Beltrami operator, ∇ and denotes the gradient and b is the Stratonovich correction term, that is, a vector field defined by

$$b(\sigma) = \frac{1}{2} tr(B'(\sigma)B(\sigma)).$$

There are at least two ways of rigorous understanding of what a solution of equation (2) is - one via an SDE on the corresponding orthogonal frame bundle and another one via an embedding of $\mathcal{M}$ into a Euclidean space. It is known that, for any $n \geq 2N$, there exists a smooth embedding $\varphi : \mathcal{M} \to \mathbb{R}^n$. It induces an embedding of the tangent bundle $T\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^n$, so that $T_\sigma\mathcal{M} \subset \mathbb{R}^n$, $\sigma \in \mathcal{M}$. Consider the corresponding normal bundle $\nu\mathcal{M}$ with fibres $\nu_\sigma\mathcal{M}$ defined as orthogonal complements of $T_\sigma\mathcal{M}$ in $\mathbb{R}^n$. It is well-known ([23]) that there exists $\rho > 0$ and a neighborhood

$$U_\rho := \{(\sigma, v) \in \nu\mathcal{M} : |v| < \rho\}$$

of the zero section $(\mathcal{M}, 0)$ of $\nu\mathcal{M}$, which is diffeomorphic to the neighborhood

$$N_\rho := \bigcup_{\sigma\in\mathcal{M}} \{y \in \mathbb{R}^n : |y - \sigma| < \rho\}$$

of $\mathcal{M}$ in $\mathbb{R}^n$.

The set N_ρ is called the tubular neighborhood of radius ρ of $\mathcal{M}$ in $\mathbb{R}^n$. We will use it in order to extend coefficients a and B to $\mathbb{R}^n$. Fix $\tilde{\rho} < \rho$ and a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ with support in N_r such that $F(\sigma) = 1,\ \sigma \in N_{\tilde{\rho}}$. Define

$$\tilde{a} : \mathbb{R}^n \to \mathbb{R}^n \text{ and } \tilde{B} : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^n)$$

by formulae

$$\tilde{a}(z) := a(\sigma_z)F(z),\ \ \tilde{B}(y) = B(\sigma_z)F(z),$$

where (σ_z, v_z) is the image of $z \in N_\rho$ in U_ρ.

We can now consider the SDE

$$d\xi(t) = \tilde{a}(\xi(t))dt + \tilde{B}(\xi(t)) \circ dW(t),\ \ t \geq 0,$$

in the Stratonovich form in $\mathbb{R}^n$. Its rigorous Ito form is

$$d\xi(t) = \left(\tilde{a}(\xi(t)) + \frac{1}{2} tr(\tilde{B}'(\sigma)\tilde{B}(\sigma))\right) dt + \tilde{B}(\xi(t))dW(t),\ \ t \geq 0. \qquad (4)$$

The coefficients of equation (4) are globally Lipschitz. Therefore, it has a global strong solution $\xi_\sigma(t)$, $t \geq 0$, for any initial data $\sigma \in \mathbb{R}^n$. The following result is well-known.

Theorem 1 *[23] For any $\sigma \in \mathcal{M}$, the process $\xi_\sigma(t)$, $t \geq 0$, does not leave $\mathcal{M}$. It is independent of the choice of the Euclidean space $\mathbb{R}^N$, tubular neighborhood N_ρ and function F.*

The process $\{\xi_\sigma(t), t \geq 0\}$ on $\mathcal{M}$ will be called the strong solution of SDE (2) with initial value $\sigma \in \mathcal{M}$. It defines a Markov semigroup T_t on $C(\mathcal{M})$ via the standard formula

$$T_t u(\sigma) := \mathbb{E}\left(u(\xi_\sigma(t))\right), \ t > 0,$$

with the generator given by formula (3).

2.2 Infinite systems of SDEs on compact manifolds via embedding in a Hilbert space

In what follows we will use *bar* to denote sequences of elements of $\mathcal{M}$ indexed by elements of γ, e.g. $\bar{\xi} = (\xi_x)_{x\in\gamma}$, $\bar{\sigma} = (\sigma_x)_{x\in\gamma}$.

We consider now an infinite system of coupled SDEs in $\mathcal{M}$ of the form

$$d\xi_x(t) = f_x(\bar{\xi}(t))dt + \Phi_x(\bar{\xi}(t)) \circ dW_x(t), \ x \in \gamma, \tag{5}$$

where $\gamma \subset \mathbb{R}^d$ is a locally finite (countable) set (configuration) and $W = (W_x)_{x\in\gamma}$ is a collection of independent Wiener processes in $\mathbb{R}^N$. We assume that the drift coefficient has the form

$$f_x(\bar{\sigma}) = \sum_{\substack{y\in\gamma \\ y\neq x}} \varphi_{xy}(\sigma_x, \sigma_y) + \psi_x(\sigma_x), \tag{6}$$

where ψ_x is a C^1-vector field on $\mathcal{M}$, the C^1 mappings $\varphi_{xy} : \mathcal{M} \times \mathcal{M} \to T\mathcal{M}$ are such that $\varphi(\sigma_1, \sigma_2) \in T_{\sigma_1}\mathcal{M}$ and have finite range, that is, $\varphi_{xy} \equiv 0$ whenever $|x - y| \geq r$ for a fixed $r > 0$, The latter condition implies that, for any $x \in \gamma$, the sum in (6) has finite number n_x of non-zero elements, where n_x is defined in (1), that is,

$$n_x = \#\left\{y \in \gamma : |x - y| < r\right\}. \tag{7}$$

The diffusion coefficient Φ_x is supposed to have a simpler "diagonal" form, that is,

$$\Phi_x(\bar{\sigma}) = B(\sigma_x),$$

where B is as in (2).

Moreover, we assume that the first derivatives of φ_{xy} and ψ_x are bounded uniformly in $x, y \in \gamma$. For instance,

$$\varphi_{xy} = \varphi \mathbf{1}_r(|x - y|), \tag{8}$$

where $\mathbf{1}_r$ is the indicator of the open ball of radius r in $\mathbb{R}^N$ and $\varphi \in C^1(\mathcal{M} \times \mathcal{M}, T\mathcal{M})$, and $\psi_x = \psi$, a fixed C^1 vector field on $\mathcal{M}$, for all $x, y \in \gamma$.

We will now construct an extension of the mappings φ_{xy} to the Euclidean space $\mathbb{R}^n$, similar to Section 2.1. Define

$$\widetilde{\varphi}_{xy}(z_1, z_2) = \varphi_{xy}(z_1, z_2)F(z_1)F(z_2),\ \widetilde{\psi}_x(z_1) = \psi_x(z_1)F(z_1),\ z_1, z_2 \in \mathbb{R}^n,$$

where (σ_k, v_k) is the image of $z_k \in N_r$ in U_r, $k = 1, 2$. It is clear that the mappings $\widetilde{\varphi}_{xy} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are globally Lipschitz. Similar to (6), set

$$\tilde{f}_x(\bar{\sigma}) = \sum_{\substack{y \in \gamma \\ y \neq x}} \widetilde{\varphi}_{xy}(\sigma_x, \sigma_y) + \widetilde{\psi}_x(\sigma_x), \quad \tilde{\Phi}_x(\bar{\sigma}) = \widetilde{B}(\sigma_x).$$

We can now rewrite system (5) in the form of the following single (heuristic) SDE in the space of sequences $(\mathbb{R}^n)^\gamma := \{\bar{\sigma} = (\sigma_x)_{x\in\gamma},\ \sigma_x \in \mathbb{R}^n\}$:

$$d\bar{\xi}(t) = \widetilde{f}(\bar{\xi}(t))dt + \widetilde{\Phi}(\bar{\xi}(t)) \circ dW(t), \tag{9}$$

where $\widetilde{f}(\bar{\sigma}) = \left(\tilde{f}_x(\bar{\sigma})\right)_{x\in\gamma}$, $\widetilde{\Phi}(\bar{\sigma})$ is an infinite block-diagonal matrix with diagonal elements $\widetilde{\Phi}_x(\bar{\sigma}), x \in \gamma$, and $W(t) = (W_x(t))_{x\in\gamma}$. To give this equation a rigorous meaning and solve it, we need to restrict it to a Hilbert or Banach subspace of $(\mathbb{R}^n)^\gamma$. A natural idea, explored in [2, 5, 3], is to use the Hilbert space of weighted l_2 sequences

$$\mathcal{X}_{\bar{p}} := \left\{ \bar{\sigma} \in (\mathbb{R}^n)^\gamma : \sum_{x\in\gamma} |\sigma_x|^2 p_x < \infty \right\},$$

where $\bar{p} = (p_x)_{x\in\gamma} \in l_1$ is a fixed weight sequence, that is, $p_x > 0$ and $\sum_{x\in\gamma} p_x < \infty$. Consider also the standard l_2 space

$$\mathcal{H} := \left\{ \bar{\sigma} \in (\mathbb{R}^n)^\gamma : \sum_{x\in\gamma} |\sigma_x|^2 < \infty \right\} \tag{10}$$

and introduce the space $HS(\mathcal{H}, \mathcal{X}_{\bar{p}})$ of Hilbert-Schmidt operators $\mathcal{H} \to \mathcal{X}_{\bar{p}}$. It is clear that the embedding $\mathcal{H} \subset \mathcal{X}_{\bar{p}}$ is Hilbert-Schmidt and therefore $\widetilde{\Phi}(\bar{\sigma}) \in HS(\mathcal{H}, \mathcal{X}_{\bar{p}})$. Here we identify $\widetilde{\Phi}(\bar{\sigma})$ with a liner operator $\mathcal{H} \to \mathcal{X}_{\bar{p}}$ acting as

$$\left(\widetilde{\Phi}(\bar{\sigma})\bar{u}\right)_x := \widetilde{\Phi}_x(\bar{\sigma})\bar{u}_x,\ x \in \gamma.$$

Observe now that for any weight sequence $\bar{p} \in l_1$ we have the embedding

$$\mathcal{M}^\gamma \subset \mathcal{X}_{\bar{p}}.$$

The space $\mathcal{M}^\gamma$ endowed by the metric $\rho_{\bar{p}}$ induced from $\mathcal{X}_{\bar{p}}$ will be denoted by $\mathcal{M}^\gamma_{\bar{p}}$. The following facts were proved in [3]:

- metric space $\mathcal{M}^\gamma_{\bar p}$ is complete; its topology coincides with the product topology of $\mathcal{M}^\gamma$;
- all metrics $\rho_{\bar p}$, $\bar p \in l_1$, are equivalent;
- the convergence in $\mathcal{M}^\gamma_{\bar p}$ coincides with the component-wise convergence;
- $\mathcal{M}^\gamma_{\bar p}$ is not a Hilbert manifold in the proper sense; in particular, it does not admit a tubular neighborhood in $\mathcal{X}_{\bar p}$.

We can now revisit system (5) and equation (9). The following result was proved in [2], see also [5, 3].

Theorem 2 *Assume that $\gamma = \mathbb{Z}^d$. Then:*

(1) there exists a weight sequence $\bar p \in l_1$ such that the map

$$\widetilde{f} : \mathcal{X}_{\bar p} \to \mathcal{X}_{\bar p}$$

is of the C^1_b class; moreover, the map

$$\widetilde{\Phi} : \mathcal{X}_{\bar p} \to HS(\mathcal{H}, \mathcal{X}_{\bar p})$$

is of the C^2_b class for any weight sequence $\bar p \in l_1$;

(2) equation (9) has a unique strong solution $\Xi(t)$, $t > 0$, in $\mathcal{X}_{\bar p}$;

(3) for any initial data $\Xi(0) \in \mathcal{M}^\gamma$ the solution $\Xi(t)$ does leave $\mathcal{M}^\gamma$ and generates a Markov process in $\mathcal{M}^\gamma$.

Remark 3 *In the aforementioned papers, somewhat more general drift coefficients were considered. Namely, it was allowed to have the form*

$$f_k(\bar\sigma) = \sum_{\Lambda \subset \mathcal{F}_k(\mathbb{Z}^d)} \varphi_\Lambda(\sigma_\Lambda), \;\; k \in \mathbb{Z}^d,$$

where $\mathcal{F}_k(\mathbb{Z}^d)$ is the collection of all finite subsets of $\mathbb{Z}^d$ containing k and $\varphi_\Lambda \in C^1(\mathcal{M}^\Lambda)$, $\sigma_\Lambda := \bar\sigma \upharpoonright_{\mathcal{M}^\Lambda}$. The main requirement in this case is that

$$\sup_{k \in \mathbb{Z}^d} \|f_k\| < \infty,$$

which obviously holds for f of the form (6).

Remark 4 *The structure of the set γ is not important for the result above. It can be easily extended to the case of general γsuch that*

$$\sup_{x\in\gamma} n_x < \infty. \tag{11}$$

The latter condition is crucial. If it fails, the coefficients of equation (9) will not be Lipschitz continuous in any fixed weighted space $\mathcal{X}_{\bar{p}}$. However, under certain conditions, they will satisfy more general Lipschitz condition in an appropriate scale of Hilbert spaces. That will allow us to solve equation (9) using the generalization of the Ovsyannikov method developed in [18, 17, 14, 15].

2.3 Ovsyannikov method and existence of solutions in a scale of Hilbert spaces

In this section we introduce the general framework we will be using in order to solve system (5) in the case where condition (11) fails, that is, $\sup_{x\in\gamma} n_x = \infty$, following [18, 17, 14, 15]. Let us consider a family $\mathfrak{B}$ of Banach spaces B_α indexed by $\alpha \in \mathcal{A} := [\alpha_*, \alpha^*]$ with fixed $0 \le \alpha_*, \alpha^* < \infty$, and denote by $\|\cdot\|_{B_\alpha}$ the corresponding norms.

Definition 5 *The family $\mathfrak{B}$ is called a scale if*

$$B_\alpha \subset B_\beta \text{ and } \|u\|_{B_\beta} \le \|u\|_{B_\alpha} \text{ for any } \alpha < \beta,\ u \in X_\alpha,$$

where the embedding means that B_α is a dense vector subspace of B_β.

We will use the following notations:

$$\overline{B} := \bigcup\nolimits_{\alpha\in[\alpha_*,\alpha^*)} B_\alpha, \quad \underline{B} := \bigcap\nolimits_{\alpha\in(\alpha_*,\alpha^*]} B_\alpha.$$

Definition 6 *For two scales $\mathfrak{B}_1$, $\mathfrak{B}_2$ (with the same index set) and a constant $q > 0$ we introduce the class $\mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ of (generalized Lipschitz) maps $g\colon \overline{B}_1 \to \overline{B}_2$ such that*

(1) $g(B_{1,\alpha}) \subset B_{2,\beta}$ for any $\alpha < \beta$;

(2) there exists constant $L > 0$ such that

$$\|g(u) - g(v)\|_{B_{2,\beta}} \le \frac{L}{|\beta-\alpha|^{1/q}} \|u - v\|_{B_{1,\alpha}} \tag{12}$$

for any $\alpha < \beta$ and $u, v \in B_{1,\alpha}$.

We will write $\mathcal{GL}_q(\mathfrak{B}) := \mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ if $\mathfrak{B}_1 = \mathfrak{B}_2 =: \mathfrak{B}$.

Remark 7 *$g \in \mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$ generates a map $\underline{B}_1 \to \underline{B}_2$.*

Observe that (12) implies the linear growth condition

$$\|g(u)\|_{B_{2,\beta}} \leq \frac{K}{|\beta - \alpha|^{1/q}} \left(1 + \|u\|_{B_{1,\alpha}}\right), \ u \in B_{1,\alpha},$$

for some constant K and any $\alpha < \beta$. Without loss of generality we assume that $K = L$.

In what follows, we will use the following three main scales:

(1) the scale $\mathfrak{X}$ of separable Hilbert spaces X_α;

(2) the scale $\mathfrak{H}$ of spaces

$$H_\alpha \equiv HS(\mathcal{H}, X_\alpha) := \{\text{Hilbert-Schmidt operators } \mathcal{H} \to X_\alpha\}, \quad (13)$$

for a fixed separable Hilbert space $\mathcal{H}$;

(3) the scale $\mathfrak{Z}^p_T$ of Banach spaces $Z^p_{\alpha,T}$ of progressively measurable random processes $u : [0,T] \to X_\alpha$ with finite norm

$$\|u\|_{Z^p_{\alpha,T}} := \sup_{t\in[0,T]} \left(\mathbb{E}\,\|u(t)\|^p_{X_\alpha}\right)^{1/p},$$

defined on a suitable filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Consider the SDE

$$du(t) = f(u(t))dt + \Phi(u(t))dW(t), \ t \in [0,T] \quad (14)$$

with initial condition u_0, where $W(t), \ t \leq T$, is a fixed cylinder Wiener process in $\mathcal{H}$ (cf. (13)) defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with coefficients acting in the scale $\mathfrak{X}$ for a fixed $p \geq 2$.

The following theorem was proved in [14].

Theorem 8 (Existence and uniqueness) *Assume that $f \in \mathcal{GL}_q(\mathfrak{X})$ and $\Phi \in \mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$, $q > 2$ and $u(0) = u_0 \in X_\alpha$, $\alpha \in \mathcal{A}$. Then, for any $T > 0$, the following holds:*

(1) equation (14) has a unique strong solution $u \in Z^2_{\alpha^*,T}$;

(2) $u \in Z^p_{\beta,T}$ for any $p \in [2,q)$ and $\beta > \alpha$;

(3) u has a continuous modification that satisfies (14);

(4) for any $\beta > \alpha$ we have

$$||u||_{Z^p_\beta} \le E^{(p)}\left(c(T), \beta-\alpha, q^{-1}\right)(1+||u_0||_{Z^p_\alpha})^p,$$

where

$$E^{(p)}(t,\varepsilon,q) := 1+\sum_{n=1}^{\infty}\frac{t^n}{\varepsilon^{qn}}\frac{n^{qn}}{(n!)^{1/p}} < \infty$$

and $c(T) = a_p \max(T, T^{1/2})$, $a_p > 0$;

(5) formula

$$T_t\phi(\sigma) = \mathbb{E}\left(\phi(u(t))\,|u(0)=\sigma\right), \ \sigma \in X_\alpha,\, t \ge 0, \tag{15}$$

defines a strongly continuous semigroup in $C_b(X_{\alpha+})$, where $X_{\alpha+} := \bigcap_{\beta>\alpha} X_\beta$ is endowed by the projective limit topology (which makes it a Polish space).

We shall now replace condition (12) by a stronger condition.

Definition 9 *For two scales $\mathfrak{B}_1$, $\mathfrak{B}_2$ (with the same index set) and a constant $q > 0$ we introduce the class $\mathcal{GC}^1_q(\mathfrak{B}_1, \mathfrak{B}_2)$ of maps $g\colon \overline{B}_1 \to \overline{B}_2$ such that*

(1) $g(B_{1,\alpha}) \subset B_{2,\beta}$ for any $\alpha < \beta$; and

(2) $g : B_{1,\alpha} \to B_{2,\beta}$ is strongly differentiable and there exists constant $L > 0$ such that

$$\|g'(u)v\|_{B_{2,\beta}} \le \frac{L}{|\beta-\alpha|^{1/q}}\,\|v\|_{B_{1,\alpha}} \tag{16}$$

for any $\alpha < \beta$ and $u, v \in B_{1,\alpha}$. Here $g'(u) \in \mathcal{L}(B_{1,\alpha}, B_{2,\beta})$ is the derivative of g.

It is clear that condition (16) implies (12). It also implies that $g'(u) \in \mathcal{GL}_q(\mathfrak{B}_1, \mathfrak{B}_2)$, as a linear operator $B_{1,\alpha} \to B_{2,\beta}$.

Assume now that $f \in \mathcal{GC}^1_q(\mathfrak{X})$, $\Phi \in \mathcal{GC}^1_q(\mathfrak{X}, \mathfrak{H})$ and let $u \in Z^2_{\alpha^*,T}$ be the unique solution of equation (14). Let us fix a vector field $h : X_\alpha \to X_\alpha$ and consider the equation

$$d\eta(t) = a(t)\eta(t)dt + A(t)\eta(t)dW(t), \ \eta(0) = h(u(0)), \tag{17}$$

where $a(t) := f'(u(t))$ and $A(t) := \Phi'(u(t))$. Then $a(t) \in \mathcal{GL}_q(\mathfrak{X})$ and $B(t) \in \mathcal{GL}_q(\mathfrak{X}, \mathfrak{H})$, uniformly in t. Thus there exists the unique strong solution $\eta(t) \in X_\beta$, $t > 0$, of (17), for any $\beta > \alpha$. Standard arguments similar to those in [18] show that

$$\eta(t) = u'_h(t),$$

where $u'_h(t)$ is the square mean derivative of the solution $u(t)$ w.r.t. the initial condition along vector field h.

Theorem 10 *Formula (15) defines the operator*

$$T_t : C_b^1(X_\beta) \to C_b^1(X_\alpha)$$

for any $\alpha < \beta$, and we have the estimate

$$\|T_t \phi\|_{C_b^1(X_\alpha)} \le E^{(p)}\left(c(T), \beta - \alpha, q^{-1}\right) \|\phi\|_{C_b^1(X_\beta)}. \tag{18}$$

Proof. Follows from Theorem 8 applied to the system (14), (17) considered as a single equation in the scale $(X_\alpha \times X_\alpha)_{\alpha \in \mathcal{A}}$. ■

3 Well-posedness of a system of SDEs on a graph with unbounded vertex degrees

In order to be able to apply the theory of the previous section to system (5), we need the following condition on the configuration γ.

Condition 11 *There exist constants $q > 2$ and $a \equiv a(\gamma, r, q) > 0$ such that*

$$n_x \le a\,(1 + |x|)^{1/q} \tag{19}$$

for all $x \in \gamma$, where n_x is defined in (7), cf. (1).

Remark 12 *Condition (19) holds if γ is a typical realization of a Poisson or Gibbs (Ruelle) point process in X. For such configurations, the following stronger (logarithmic) bound holds:*

$$n_x(\gamma) \le c(\gamma)\,[1 + \log(1 + |x|)]^{1/2}\, r^d,$$

see e.g. [30] and [26, p. 1047]. Thus (19) holds for any $q > 0$.

The solution of equation (9) will live in the scale of Hilbert spaces

$$X_\alpha := \left\{ \bar{\sigma} \in (\mathbb{R}^n)^\gamma : \|\bar{\sigma}\|_\alpha := \sqrt{\sum_{x\in\gamma} |\sigma_x|^2 e^{-\alpha|x|}} < \infty \right\},$$

$\alpha \in \mathcal{A} = [\alpha_*, \alpha^*]$, where the parameters α_*,$\alpha^* > 0$ are chosen in an arbitrary way and fixed. In the notations of Section 2.2, these spaces correspond to weight sequences $\bar{p}_\alpha = (e^{-\alpha|x|})_{x\in\gamma}$, that is, $X_\alpha = \mathcal{X}_{\bar{p}_\alpha}$. We consider the corresponding spaces $\mathcal{GC}^1_p(\mathfrak{X})$ and $\mathcal{GC}^1_p(\mathfrak{X}, \mathfrak{H})$ with $\mathcal{H}$ defined in (10), cf. Definition 6. Observe that $W(t) := (W_x(t))_{x\in\gamma}$ is a cylinder Wiener process in $\mathcal{H}$.

Lemma 13 *We have* $\widetilde{f} \in \mathcal{GC}^1_p(\mathfrak{X})$ *and* $\widetilde{\Phi} \in \mathcal{GC}^1_p(\mathfrak{X}, \mathfrak{H})$.

Proof. The proof is similar to the proof of Lemma 5.4 in [14]. ■

Now we can return to the discussion of system (5). We can write it in the form (9) as in Section 2.2 and apply the results of Section 2.3. Recall that we have the embedding $\mathcal{M}^\gamma \subset X_\alpha$ for any $\alpha > 0$.

Theorem 14 *Consider equation (9) with initial condition* σ *and assume that* $\bar{\sigma} \in \mathcal{M}^\gamma$. *Than it has a unique strong solution* $\bar{\xi}(t) \in \mathcal{M}^\gamma$, $t > 0$. *This solution has a continuous modification that satisfies (5), in the sense of Section 2.1. Formula*

$$T_t\phi(\bar{\sigma}) = \mathbb{E}\left(\phi(\bar{\xi}(t)) \,\middle|\, \bar{\xi}(0) = \bar{\sigma}\right) \tag{20}$$

defines a strongly continuous Markov semigroup in $C(\mathcal{M}^\gamma)$. *Its generator* H *has the form*

$$H\phi(\bar{\sigma}) = \frac{1}{2}\sum_{x\in\gamma} \Delta_x\phi(\bar{\sigma}) + \sum_{x\in\gamma} \left(f(\bar{\sigma}), \nabla_x\phi(\bar{\sigma})\right)_{T_{\sigma_x}\mathcal{M}}.$$

Proof. We have $\bar{\sigma} \in \mathcal{M}^\gamma \subset X_{\alpha_*}$. It follows directly from Theorem 8 that equation (9) has a unique strong solution $\bar{\xi} \in Z^p_\beta$ for any $\beta > \alpha_*$. The components $\xi_x(t)$ of this solution do not leave manifold $\mathcal{M}$ and thus satisfy system (5). ■

This result implies of course that, for each $x \in \gamma$, equation (5) has a path-continuous strong solution, which is unique in the class of progressively measurable square-integrable processes.

Remark 15 *For a configuration* γ *as in Remark 12, the statement of the theorem above holds for any* $p \geq 2$.

4 Gibbs measures, Dirichlet forms and stochastic dynamics

In this section, we use the results above to construct stochastic dynamics associated with a Gibbs measure ν on $\mathcal{M}^\gamma$. This is a stochastic process Ξ that leaves ν invariant, and the corresponding semigroup coincides with the semigroup generated by the Dirichlet form of ν, see [9]. We will construct Ξ as a solution of system (5) with the drift given by the logarithmic derivative of ν. To make the paper self-contained, we start with a definition of Gibbs measures on $\mathcal{M}^\gamma$.

4.1 Definition of Gibbs measures

We are interested in Gibbs measures describing equilibrium states of a (quenched) system of particles with positions $\gamma \subset X = \mathbb{R}^d$ and spin space $\mathcal{M}$, defined by pair and single-particle potentials $W_{xy} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ such that $W_{xy} \equiv 0$ if $|x-y| < r$, and $V_x : \mathcal{M} \to \mathbb{R}$, respectively, $x, y \in \gamma$. We assume for simplicity that, for all $x, y \in X$, $W_{xy} = W\mathbf{1}_r(|x-y|)$, cf. (8), and $V_x = V$ for some $W \in C^2(\mathcal{M} \times \mathcal{M})$ and $V \in C^2(\mathcal{M})$.

There are at least two ways of explaining what the corresponding Gibbs measure is. The first one is the standard Dobrushin-Lanford-Ruelle (DLR) approach used in statistical mechanics [24, 27], where Gibbs measures (states) are constructed by means of their local conditional distributions (constituting the so-called Gibbsian specification). The second approach, more functional analytical, is based on the integration by parts formula, see [28, 1, 6, 4].

We start with a brief outline of the DLR approach. Let $\mathcal{F}(\gamma)$ be the collection of all finite subsets of $\gamma \in \Gamma(X)$. For any $\eta \in \mathcal{F}(\gamma)$, $\bar{\sigma}_\eta = (\sigma_x)_{x\in\eta} \in \mathcal{M}^\eta$ and $\bar{z}_\gamma = (z_x)_{x\in\gamma} \in \mathcal{M}^\gamma$ define the relative local interaction energy

$$E_\eta(\bar{\sigma}_\eta \,|\bar{z}_\gamma) = \sum_{\{x,y\}\subset\eta} W_{xy}(\sigma_x, \sigma_y) + \sum_{\substack{x\in\eta \\ y\in\gamma\setminus\eta}} W_{xy}(\sigma_x, z_y).$$

The corresponding specification kernel $\Pi_\eta(d\bar{\sigma}_\gamma \,|\bar{z}_\gamma)$ is a probability measure on $\mathcal{M}^\gamma$ of the form

$$\Pi_\eta(d\bar{\sigma}_\gamma|\bar{z}_\gamma) = \ \mu_\eta(d\bar{\sigma}_\eta|\bar{z}_\gamma) \otimes \delta_{\bar{z}_{\gamma\setminus\eta}}(d\bar{\sigma}_{\gamma\setminus\eta}),$$

where

$$\mu_\eta(d\bar{\sigma}_\eta|\bar{z}_\gamma) := Z(\bar{z}_\gamma)^{-1}\exp\ \left[-E_\eta(\bar{\sigma}_\eta \,|\bar{z}_\gamma)\right] \bigotimes_{x\in\eta} e^{-V(\sigma_x)} d\sigma_x$$

is a probability measure on $\mathcal{M}^\eta$. Here $Z(\bar{z}_\gamma)$ is the normalizing factor and $\delta_{\bar{z}_{\gamma\setminus\eta}}(d\bar{\sigma}_{\gamma\setminus\eta})$ is the Dirac measure on $\mathcal{M}^{\gamma\setminus\eta}$ concentrated on $\bar{z}_{\gamma\setminus\eta}$. The family $\{\Pi_\eta(d\bar{\sigma}_\eta|\bar{z}_\gamma),\ \eta \in \mathcal{F}(\gamma), \bar{z}_\gamma \in \mathcal{M}^\gamma\}$ is called the Gibbsian specification (see e.g. [24, 27]).

A probability measure ν on $\mathcal{M}^\gamma$ is said to be a Gibbs measure associated with the potentials W and V if it satisfies the DLR equation

$$\nu(B) = \int_{\mathcal{M}^\gamma} \Pi_\eta(B|\bar{z})\nu(d\bar{z}), \quad B \in \mathcal{B}(\mathcal{M}^\gamma), \tag{21}$$

for all $\eta \in \mathcal{F}(\gamma)$. For a given $\gamma \in \Gamma(X)$, by $\mathcal{G}(\mathcal{M}^\gamma)$ we denote the set of all such measures. It is known that $\mathcal{G}(\mathcal{M}^\gamma) \neq \emptyset$, see e.g. [24].

To explain the second approach, we start with the following observation. Let $\mathcal{FC}^1(\mathcal{M}^\gamma) := \cup_{\eta\in\mathcal{F}(\gamma)} C^1(\mathcal{M}^\eta)$ and $\mathcal{F}Vect^1(\mathcal{M}^\gamma) := \bigcup_{\eta\in\mathcal{F}(\gamma)} Vect^1(\mathcal{M}^\eta)$ be the classes of cylinder continuously differentiable real-valued functions and vector fields on $\mathcal{M}^\gamma$, respectively. For any cylinder vector field $Z \in \mathcal{F}Vect^1(\mathcal{M}^\gamma)$, any measure $\nu \in \mathcal{G}(\mathcal{M}^\gamma)$ satisfies the following integration by parts (IBP) formula:

$$\int_{\mathcal{M}^\gamma} \sum_{x\in\gamma} (\nabla_x \phi(\sigma)\, , Z_x(\sigma))_x \nu(d\sigma) = -\int_{\mathcal{M}^\gamma} \beta_Z^\nu(\sigma)\phi(\sigma)\nu(d\sigma). \tag{22}$$

Here $\beta_Z^\nu \in \mathcal{F}C^1(\mathcal{M}^\gamma)$ is defined by the formulae

$$\beta_Z^\nu(\sigma) = \sum_{x\in\gamma} [(\beta_x^\nu(\sigma)\, , Z_x(\sigma))_x + \mathrm{div}\, \mathrm{Z_x}(\sigma)]$$

and

$$\beta_x^\nu(\sigma) = \nabla_{\sigma_x} U_x(\sigma),\ U_x := \sum_{\substack{y\in\gamma \\ y\neq x}} W_{xy} + V_x. \tag{23}$$

The vector field $\beta^\nu := (\beta_x^\nu)_{x\in\gamma} \in \mathcal{F}Vect^1(\mathcal{M}^\gamma)$ is called the (vector) logarithmic derivative of ν. The IBP formula (22) for Gibbs measures from $\mathcal{G}(\mathcal{M}^\gamma)$ is well-known in the case of $\gamma = \mathbb{Z}^d$, see [4]. For a general $\gamma \in \Gamma(\mathbb{R}^d)$, it can be proved in a similar way, using the DLR equation (21) and integration by parts formula for measures $\Pi_\eta(d\bar{\sigma}|\bar{z})$.

Moreover, it is known that, for $\gamma = \mathbb{Z}^d$, the class $\mathcal{G}(\mathcal{M}^\gamma)$ can be characterized by (22), that is, any measure on $\mathcal{M}^\gamma$ satisfying the integration by parts formula with logarithmic derivative given by (23) is a Gibbs measure of the class $\mathcal{G}(\mathcal{M}^\gamma)$, see [4] for the proof, which is based on the ideas of earlier works [28, 1, 6] and is likely to work for a general $\gamma \in \Gamma(\mathbb{R}^d)$, too.

4.2 Dirichlet forms and stochastic dynamics

Consider a measure $\nu \in \mathcal{G}(\mathcal{M}^\gamma)$ and define the pre-Dirichlet form

$$\mathcal{E}(\phi,\psi) = \frac{1}{2}\int_{\mathcal{M}^\gamma} \sum_{x\in\gamma} (\nabla_x\phi(\bar\sigma)\,, \nabla_x\psi(\bar\sigma))_{T_{\bar\sigma_x}\mathcal{M}}\nu(d\bar\sigma),\ \ \phi,\psi \in \mathcal{FC}^2(\mathcal{M}^\gamma).$$

The integration by parts formula (22) implies that $\mathcal{E}$ is closable and generates the classical Dirichlet form associated with ν, see [9]. Its generator in $L^2(\mathcal{M}^\gamma,\nu)$ is the Friedrichs extension of an operator defined on $\mathcal{FC}^2(\mathcal{M}^\gamma)$ by the expression

$$H_\nu\phi(\bar\sigma) = -\frac{1}{2}\sum_{x\in\gamma}\Delta_x\phi(\bar\sigma) - \frac{1}{2}\sum_{x\in\gamma}(\beta^\nu_x(\bar\sigma), \nabla_x\phi(\bar\sigma))_{T_{\bar\sigma_x}\mathcal{M}}\,. \tag{24}$$

It generates the semigroup

$$T^\nu_t := e^{-tH_\nu},\ t\geq 0, \text{ in } L^2(\mathcal{M}^\gamma,\nu).$$

Let us now consider system (5) with $f_x = \frac{1}{2}\beta^\nu_x$. It satisfies conditions of Theorem 14, with $\varphi_{xy}(z_1,z_2) = \nabla_{z_1}W_{xy}(z_1,z_2)$, $x\neq y$, and $\psi_x(z) = \nabla V(z)$. Thus, according to Theorem 14, it has the unique strong solution $\xi(t),\ t>0$, on $\mathcal{M}^\gamma$, and Markov generator of the process ξ coincides with H_ν on $\mathcal{FC}^2(\mathcal{M}^\gamma)$. However, the latter fact does not, in general, imply that the L^2 semigroup T^ν coincides on $C(\mathcal{M}^\gamma)$ with Markov semigroup (20). The lacking ingredient here is the uniqueness of the L^2 semigroup generated by the pre-Dirichlet operator $\big(H_\nu, \mathcal{FC}^2(\mathcal{M}^\gamma)\big)$, which we prove in the next section.

4.3 Uniqueness of the stochastic dynamics

Theorem 16 *For any Gibbs measure $\mu\in\mathcal{G}(\mathcal{M}^\gamma)$, the pre-Dirichlet operator $(H_\mu, \mathcal{FC}^2(\mathcal{M}^\gamma))$ is essentially self-adjoint in $L^2(\mathcal{M}^\gamma,\mu)$.*

Proof. The proof uses the parabolic criterion of self-adjointness [11] and is an adaptation of the scheme of [2] to our framework.

We introduce the notation $W_{xx} := V_x$ and approximate potentials W_{xy} and $W_{xx} = V_x$ by smooth functions $W^n_{xy}\in C^\infty(\mathcal{M}^2)$ such that $W^n_{xy}\equiv 0$ if $|x-y|<r$, and $W^n_{xx}\in C^\infty(\mathcal{M})$, respectively, so that

$$\left\|W^n_{xy} - W_{xy}\right\|_{C^2} \leq e^{-d_{xy}}e^{-n},\ n\in\mathbb{N},$$

for all $x,y\in\gamma$, where $d_{xy} := \max(|x|\,,|y|)$ Similar to (23), set

$$U^n_x := \sum_{\substack{y\in\gamma\\ d_{xy}\leq n}} W^n_{xy} \text{ and } \beta^n_x = \nabla_x U^n_x.$$

Observe that $\beta^n_x = 0$ if $|x| > n$.

For any $n \in \mathbb{N}$, we define the differential operator

$$H^n u(\bar\sigma) = -\frac{1}{2}\sum_{x\in\gamma} \Delta_x u(\bar\sigma) - \frac{1}{2}\sum_{x\in\gamma} \left(\beta^n_x(\bar\sigma)\,,\nabla_x u(\bar\sigma)\right)_x$$

on the domain $\mathcal{FC}^2(\mathcal{M}^\gamma) \subset L^2(\mathcal{M}^\gamma, \mu)$.

According to the parabolic criterion of self-adjointness [11], the following two conditions are sufficient for the essential self-adjointness of operator $(H_\mu, \mathcal{FC}^2(\mathcal{M}^\gamma))$ in $L^2(\mathcal{M}^\gamma, \mu)$:

(i) for any $n \in \mathbb{N}$ and $v \in \mathcal{FC}^2(\mathcal{M}^\gamma)$, the Cauchy problem

$$\frac{d}{dt}u_n(t) + H^n u_n(t) = 0,\ t \in [0,1]\,,\ u_n(0) = v, \tag{25}$$

has a strong solution $u_n(t) \in \mathcal{FC}^2(\mathcal{M}^\gamma)$;

(ii) we have the convergence

$$\int_0^1 \|(H_\mu - H^n)\, u_n(t)\|_{L^2(\mathcal{M}^\gamma,\mu)}\, dt \to 0,\ n \to \infty. \tag{26}$$

It is clear that, for any fixed $n \in \mathbb{N}$ and $v \in \mathcal{FC}^2(\mathcal{M}^\gamma)$, Cauchy problem (25) is finite-dimensional. Therefore there exists its classical solution $u_n(t) \in \mathcal{FC}^2(\mathcal{M}^\gamma)$, which is a C^1 function of t. Therefore Condition (i) holds.

In order to prove the convergence (26), we fix $n \in \mathbb{N}$ and consider the SDE system

$$d\xi^n_x(t) = a^n_x(\xi^n(t))dt + B_x(\xi^n_x(t)) \circ dW_x(t),\ \xi^n_x(0) = \sigma_x \in \mathcal{M},\ t \geq 0, \tag{27}$$

where $a^n_x(\bar\sigma) = \beta^n_x(\bar\sigma) - b(\sigma_x),\ x \in \gamma$.

System (27) is essentially finite dimensional and thus the existence of its strong solution follows from the general theory of SDEs on compact manifolds. However, we would like to obtain a uniform in n estimate of the solution. For this, we observe that the corresponding extended coefficients $\widetilde{a^n}$ belong to the class $\mathcal{GC}_q(\mathfrak{X})$ with some constant L independent of $n \in \mathbb{N}$. Denote by $\tilde\xi_n(t)$ the solution the corresponding SDE and let and T^n_t be the corresponding semigroup acting in the scale $\mathfrak{X}$. Then the uniform estimate

$$\|T^n_t \tilde v\|_{C^1_b(X_\alpha)} \leq E^{(p)}\left(c(T), \beta - \alpha, q^{-1}\right) \|\tilde v\|_{C^1_b(X_\beta)}\,,\ \alpha < \beta,$$

holds for any $n \in \mathbb{N}$, cf. (18). Thus the derivative of the function $\hat{u}_n(t) = T_t^n \tilde{v}$ in the direction of a vector field $\mathfrak{h} : X_\alpha \to X_\alpha$ satisfies the estimate

$$|\hat{u}_n'(t,\bar{\sigma})\mathfrak{h}(\bar{\sigma})| \le C \|\tilde{v}\|_{C_b^1(X_\beta)} \|\mathfrak{h}(\bar{\sigma})\|_{X_\alpha}.$$

Observe that the restriction of $\hat{u}_n(t)$ to $\mathcal{M}^\gamma$ coincides with the solution $u_n(t)$. Then, for $\bar{\sigma} \in \mathcal{M}^\gamma$, we obtain the estimate

$$|u_n'(t,\bar{\sigma})\mathfrak{h}(\bar{\sigma})| \le C \|\tilde{v}\|_{C_b^1(X_\beta)} \|\mathfrak{h}(\bar{\sigma})\|_{X_\alpha}. \tag{28}$$

In particular, consider a collection $h = (h_x)_{x\in\gamma}$ of mappings $h_x : \mathcal{M}^{\gamma_x} \to T\mathcal{M}$ such that $h_x(\bar{\sigma}) \in T_{\sigma_x}\mathcal{M}$ and consider $\mathfrak{h}$ with its x-components $\mathfrak{h}_x = \tilde{h}_x$, where $\tilde{h}_x$ is the extension of h to $(\mathbb{R}^n)^{\gamma_x}$. Then, applying (28), we see that

$$\left|\sum_{x\in\gamma} (\nabla_x u_n(t,\bar{\sigma}), h(\sigma_x))_{T_{\sigma_x}\mathcal{M}}\right|^2 \le C^2 \|\tilde{v}\|^2_{C_b^1(X_\beta)} \sum_{x\in\gamma} e^{-\alpha|x|} |h_x(\bar{\sigma})|^2_{T_{\sigma_x}\mathcal{M}}.$$

We can now come back to the proof of (26). We have

$$\begin{aligned} |(H_\mu - H^n)\, u_n(t,\bar{\sigma})|^2 &= \left|\sum_{x\in\gamma} (\beta_x(\bar{\sigma}) - \beta_x^n(\bar{\sigma}), \nabla_x u_n(t,\bar{\sigma}))_{T_{\sigma_x}\mathcal{M}}\right|^2 \\ &\le C^2 \|\tilde{v}\|^2_{C_b^1(X_\beta)} \sum_{x\in\gamma} e^{-\alpha|x|} |\beta_x(\bar{\sigma}) - \beta_x^n(\bar{\sigma})|^2_{T_{\sigma_x}\mathcal{M}}. \end{aligned} \tag{29}$$

Taking into account formulae (23), we obtain the inequality

$$\begin{aligned} |\beta_x(\bar{\sigma}) - \beta_x^n(\bar{\sigma})|_{T_{\sigma_x}\mathcal{M}} &= \left|\sum_{y\in\gamma} \nabla_x W_{xy} - \sum_{y\in\gamma,\ d_{xy}\le n} \nabla_x W_{xy}^n\right|_{T_{\sigma_x}\mathcal{M}} \\ \le \sum_{y\in\gamma,\ d_{xy}\le n} &\left|\sum_{y\in\gamma} \nabla_x W_{xy} - \nabla_x W_{xy}^n\right|_{T_{\sigma_x}\mathcal{M}} + \sum_{y\in\gamma,\ d_{xy}>n} |\nabla_x W_{xy}|_{T_{\sigma_x}\mathcal{M}} \end{aligned}$$

Then

$$\begin{aligned} \sup_{\bar{\sigma}\in\mathcal{M}^\gamma} |\beta_x(\bar{\sigma}) - \beta_x^n(\bar{\sigma})|_{T_{\sigma_x}\mathcal{M}} &\le e^{-n} \sum_{y\in\gamma, \rho(x,y)<r} e^{-d_{xy}} + \\ &+ \sup_{\bar{\sigma}\in\mathcal{M}^\gamma} \sum_{y\in\gamma,\ d_{xy}>n} |\nabla_x W_{xy}|_{T_{\sigma_x}\mathcal{M}} \le n_x e^{-n} + c n_x \delta(|x| > n), \end{aligned}$$

where $c := \|W\|_{C^2(\mathcal{M}^2)} + \|V\|_{C^2(\mathcal{M})}$. So

$$\sup_{\bar\sigma\in\mathcal{M}^\gamma} |\beta_x(\bar\sigma) - \beta_x^n(\bar\sigma)|^2_{T_{\sigma_x}\mathcal{M}} \le 2n_x^2 e^{-2n} + 2c^2 n_x^2 \delta(|x|>n)$$

and

$$\sup_{\bar\sigma\in\mathcal{M}^\gamma} \sum_{x\in\gamma} e^{-\alpha|x|} |\beta_x(\bar\sigma) - \beta_x^n(\bar\sigma)|^2_{T_{\sigma_x}\mathcal{M}} \le e^{-2n}\sum_{x\in\gamma} n_x^2 e^{-\alpha|x|} + 2c^2 \sum_{x\in\gamma,|x|>n} n_x^2 e^{-\alpha|x|}. \quad (30)$$

Let us now prove that $\sum_{x\in\gamma} n_x^2 e^{-\alpha|x|} < \infty$. We have

$$\sum_{x\in\gamma} n_x^2 e^{-\alpha|x|} \le \sum_{m\in\mathbb{N}} e^{-\alpha m} \sum_{x\in\gamma,|x|\le m} n_x^2 \le \sum_{m\in\mathbb{N}} e^{-\alpha m} \max_{x\in\gamma,|x|\le m} n_x \times n_0(m),$$

where $n_0(m)$ is the number of elements of γ in the ball $B(m)$ of radius m centered at 0. For any m, this ball can be covered by km^d balls of radius r, for some constant k independent of m, which implies that $n_0(m) \le cm^d \max_{x\in\gamma,|x|\le m} n_x$. Taking into account estimate (19), we obtain

$$\sum_{x\in\gamma} n_x^2 e^{-\alpha|x|} \le \sum_{m\in\mathbb{N}} e^{-\alpha m} km^d \left(\max_{x\in\gamma,|x|\le m} n_x\right)^2 \le ka^2 \sum_{m\in\mathbb{N}} e^{-\alpha m} m^d a^2 (1+m)^{2/q} < \infty,$$

which, together with (30), implies that

$$\sup_{\bar\sigma\in\mathcal{M}^\gamma} \sum_{x\in\gamma} e^{-\alpha|x|} |\beta_x(\bar\sigma) - \beta_x^n(\bar\sigma)|^2_{T_{\sigma_x}\mathcal{M}} \to 0, n\to\infty.$$

So we have from (29)

$$\sup_{\bar\sigma\in\mathcal{M}^\gamma} |(H_\mu - H^n)\, u_n(t,\bar\sigma)| \to 0,\ n\to\infty,$$

which implies (26) and completes the proof. ■

Corollary 17 *There exists a unique strongly continuous semigroup of self-adjoint operators in $L^2(\mathcal{M}^\gamma,\nu)$ such that the restriction of its generator on $\mathcal{FC}^2(\mathcal{M}^\gamma)$ is given by formula (24). This semigroup coincides on $C(\mathcal{M}^\gamma)$ with Markov semigroup (20).*

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