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RESEARCH

Cost share-induced technological change: An analytical classical-evolutionary model

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Abstract

This paper builds on prior work by the author on cost share-induced technological change. The theoretical model views selection of candidate innovations as a capital budgeting exercise. In this paper it treats the case in which firms target an incremental rate of profit, which introduces a nonzero threshold into a "selection frontier". This presents analytical challenges, which are resolved in this paper by assuming that the probability distribution of potential increases in productivity among the set of fit innovations is normal. That permits an explicit derivation of a micro-level model of cost share-induced technological change that can be taken as a candidate functional form for an aggregate model. The model is calibrated against historical data for India, China, and the United States, three large continental economies at different levels of per capita GDP. The model is able to fit the data with reasonable fidelity, and the fitted model parameters can be given a reasonable interpretation. The paper further shows that combining cost share-induced technological change with price-setting behavior produces theoretically interesting results.

Keywords Technological change · Evolutionary · Classical · Neo-Marxian

JEL Classification E11 · E14 · O31

1 Introduction

Technological change has been recognized as a source of economic growth since at least Adam Smith. Yet, it has proven difficult to incorporate convincingly into neoclassical macroeconomic models. Famously, for Solow (1956; 1957), technological change was a residual term that nevertheless generated most of empirically observed growth, leading Abramovitz (1956, p. 11) to call the residual a "measure of our ignorance." Endogenous growth models seek to explain technological change by relying

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on non-decreasing returns to capital (broadly conceived) to drive growth (Romer, 1987; Lucas, 1988; Romer, 1990). However, they have been criticized as requiring “knife-edge” conditions to avoid either stagnation or explosive growth (Roberts and Setterfield, 2007, p. 18). A further critique is that R&D based theories of endogenous growth conflict with a pattern observed in some OECD countries of rapidly expanding R&D effort together with stagnating total factor productivity (TFP) growth. The second critique has been addressed through “semi-endogenous” (Jones, 1995) and “Schumpeterian” (Ha and Howitt, 2007) neoclassical endogenous growth models, but they do not truly resolve the first critique; semi-endogenous models lead to stagnation, while Schumpeterian models impose constant returns to scale.[1] A deeper critique is that many of the models rely on “knowledge” or “information” as a factor of production, but there is no coherent theory supporting this proposition (Mirowski, 2009).

In a separate strand of work, the theory of induced technological change, from Hicks (1932) to Kennedy (1964) and Samuelson (1965) sought to explain within neoclassical theory why labor productivity tended to grow, while capital productivity did not. However, that program had slowed considerably by the time Nordhaus (1973) effectively ended it by pointing out internal inconsistencies, in particular that the shape of the innovation possibility frontier is independent of the path taken to reach it (p. 213). Acemoglu (2002) resuscitated the program by incorporating endogenous growth theory. Yet, in addition to being a representative agent model, and therefore not truly built from a micro-theory, Acemoglu’s model yields labor-augmenting technological change only under quite restrictive conditions. He writes, “...there exists a unique equilibrium path tending to a [balanced growth path] with only labor-augmenting technical change. This result can be interpreted as either a positive or negative one: on the positive side...it is possible to construct a model where equilibrium long-run technical change is labor augmenting, even though capital-augmenting technical change is also allowed... On the negative side, it shows that this result obtains only when there is an extreme amount of state dependence in R&D” (Acemoglu, 2002, pp. 804). By contrast, in the theory presented in this paper the direction of technological change is essentially determined by price- and wage-setting norms. As shown by (Kemp-Benedict, 2022) and further argued in this paper, labor-augmenting change is one of many possibilities, but it emerges quite naturally.

Arguably, many of the problems lie within neoclassical economics itself, in particular the unwarranted assumption of an aggregate production function and the associated requirement for well-defined “factors of production” (Felipe, 2001). As an alternative, there is a long history of non-neoclassical endogenous growth theory (Kurz and Salvadori, 2003; Setterfield, 2013; Tavani and Zamparelli, 2017) that overlaps with the neoclassical tradition only through mutual reference to Kaldor (1961). Non-neoclassical contributions include the well-established link between accumulation and labor productivity growth reflected in the Kaldor-Verdoorn law (Lavoie, 2022, sec. 6.9) and the classical theory of cost share-induced technological change (Dutt, 2013). The two mechanisms are combined in the theoretical neo-Kaleckian model of Cassetti

[1] In an attempt to directly deal with the problem of having to impose constant returns to scale, Weitzman (1996, 1998) constructed a model in which constant returns to scale emerge endogenously through a combinatorial explosion of potential directions for R&D. By contrast, the model in this paper does not require constant returns to scale.

(2003) and the empirically grounded model of Hein and Tarassow (2010). Models of this type impose Harrod-neutral technological change, in which capital productivity is constant while labor productivity depends on the investment rate and distribution. In contrast, the theory applied in this paper need not impose any particular direction for technological change. Work within non-neoclassical traditions is ongoing (e.g., see Tavani and Zamparelli, 2021).

Technological change is also central to evolutionary economics, which is often critical of neoclassical theory (Nelson and Winter, 1982; Dosi and Nelson, 2018). Shiozawa (2020) provides a formal evolutionary framework for technological change, together with a critical evaluation of prevailing approaches that, like this paper, sits at the intersection of evolutionary, classical, and post-Keynesian theory. Shiozawa's contribution will be discussed further in this paper. Moreover, as Duménil and Lévy (1995) showed, classical theory can, when combined with evolutionary theory, readily explain why technological change is biased towards labor rather than capital. This point was emphasized by Foley (2003, p. 42) and was demonstrated in an agent-based model by Fanti (2021). Biased change emerges from the requirement that innovations increase the profit rate – the "viability" criterion of Okishio (1961) – which was interpreted by Kemp-Benedict (2022) as analogous to a firm's capital budgeting procedure.

An interesting implication of Duménil and Lévy's theory is that, if labor productivity depends on cost shares, then so does capital productivity. This is at odds with the assumptions in the papers cited above (Cassetti, 2003; Hein and Tarassow, 2010; Dutt, 2013) and seems to be contradicted by the common observation of at least approximately Harrod-neutral technological change. However, as demonstrated by Julius (2005, p. 109), Harrod-neutral technological change is a possible equilibrium solution of Duménil and Lévy's model, and Kemp-Benedict (2019) demonstrated that it is always an equilibrium when firms' markups are set using a target-return rule, a common if not universal pricing procedure (see Lavoie, 2022, sec. 3.6). This paper will further show it to hold in a conflict wage model. As a consequence, Harrod-neutral technological change can be derived, rather than imposed.

Prior work by the author on classical-evolutionary models (Kemp-Benedict, 2017a, 2022) substantially expanded Duménil and Lévy's original model (Duménil and Lévy, 1995, 2010). Kemp-Benedict (2022) proposed an aggregate model of technological change based on an evolutionary microeconomic model of the "NK" type. In such models, N is the number of different elements that can be changed in a product, process, or management system, while K is the number of interactions between those elements. The aggregate model, which applies to a sector composed of micro-level production units, was shown to be expressible in terms of a "generating function" that depends on cost shares. Average productivity growth rates are determined by taking partial derivatives of the generating function with respect to corresponding cost shares.

The classical-evolutionary theory in this paper differs from most other approaches to technological change in at least two important respects. First, and crucially, in evolutionary theory firms do not maximize profits. Evolutionary models are essentially process-based, and in such models agents do not maximize (Nelson, 2018, p. 15). The evolutionary process of innovation generates a cumulative advance (Dosi and Nelson, 2018), but individual firms are exploring, not optimizing, and either retaining or rejecting what they discover through diverse selection processes. Second, because of the enormous variety of possible modifications of products and processes, it is misleading to assume a set of techniques out in the world waiting to be implemented by any given firm; technologies are not "blueprints" executed in the same way by every firm (Dosi and Nelson, 2013, p. 28ff). A third distinction is shared by many non-neoclassical theories and emphasized in particular by Shiozawa (2020): if distribution is not determined by the optimal use of a given set of techniques, as is believed to be the case in marginalist theory, then price- and wage-setting becomes independent from technological change. Together, productivity growth and distribution interact along a dynamic path that depends, on one hand, on the cost share–productivity growth relationship, and price- and wage-setting behavior on the other. Cost share-induced technological change can thus be one component within a full macroeconomic model. Examples of macroeconomic closures with cost share-induced technological change are provided in the paper.

One limitation of Duménil and Lévy's classical-evolutionary theory of technological change is that it did not appear to constrain the form a cost share-dependent function for productivity growth rates might take (Julius, 2005). Another is that the model makes a highly restrictive assumption about the underlying probability distribution of productivity growth rates. A key result of Kemp-Benedict (2022) is that not only can the vector of productivity growth rates be expressed as the partial derivatives of a scalar generating function, but the form of the generating function is constrained, regardless of the underlying details. This allows candidate functions to be proposed that satisfy the constraints. Specifically, they must be first-order in the cost shares and have a positive semi-definite matrix of second derivatives. These are both local requirements that hold true in the vicinity of any value of the cost shares. A further global requirement is that certain elements of the matrix of second derivatives must go to zero when cost shares go to unity. These results were shown to apply to a direct extension of Duménil and Lévy's "selection frontier" in which profitability requires that the sum of products of cost shares and productivity growth rates exceed zero. The possibility of a non-zero threshold in the selection frontier was demonstrated but not explored.

This paper builds on the work in Kemp-Benedict (2022) in several ways. First, it constructs a selection frontier consistent with the argument by Shaikh (2016) that the incremental rate of profit is the relevant quantity, rather than the average. The resulting selection frontier has a non-zero threshold that depends on the target incremental rate of profit. While Kemp-Benedict (2022) suggested a candidate functional form for the generating function, it is challenging to extend to the case of a non-zero threshold. In this paper, an approach to developing candidate functional forms is introduced and applied. The resulting functional form is tested against empirical data and found to perform reasonably well.

As noted above, a further contribution in this paper is to show explicitly how the theory of cost share-induced technological change can be embedded in other theories. First, a single cost share-induced technological change model is combined with three different closures for a post-Keynesian model. The results demonstrate how the dynamics depend on price- and wage-setting behavior. Second, a model of target-return pricing is proposed in which R&D costs factor into the calculation of the target rate of return. Third, the model for cost share-induced technological change, which formally applies to sectors, is embedded in a multi-sector model.

Section 2 explains how capital is treated in this paper and how that differs from approaches rooted in value theory. Section 3 summarizes the theory of cost share-induced technological change as presented in the author's prior work. Section 4 presents the derivation of the selection frontier in the case of a target incremental rate of profit. Section 7 summarizes some of the insights from the paper and discusses possible extensions. Section 8 concludes. Section 5 proposes a candidate generating function assume normally-distributed potential productivity growth rates. Section 6 tests the model against empirical data.

2 Treatment of capital in this paper

Debates on the nature of capital since the 19th Century have acknowledged that it is essential to production but is also highly heterogeneous, partly but not fully malleable, priced differently in different markets, and susceptible to changing valuation. In the face of this uncomfortable reality, the rise of marginalist analysis and the extension of the theory of land rent to capital and labor created a need to find an aggregate metric for capital that was independent of price. The search for that metric spawned the famous Capital Cambridge Controversies over whether it could succeed even in theory. Those who said "no", in Cambridge, England, won the argument (Samuelson, 1966) but those who said "yes", mostly in Cambridge, Massachusetts, won in the textbooks (Harcourt et al., 2022).

One unfortunate side-effect of the long debate was that capital in macroeconomic modeling was presented as an application of value theory. In this paper, capital is always in monetary (value) terms. However, that value does not correspond to any notion of value *theory*. The theoretical focus in this paper is on behavior rather than idealized notions of what something should be worth. Behavior related to capital depends on how firms assign it a value, and how the value of their capital assets compares to the value of their output. Accordingly, in this paper the price of capital is the cost-based market price of capital goods, rather than the marginal rental rate of capital. The value of capital depends on prices and therefore cannot (and does not) serve as a "factor of production" in a neoclassical sense. Nevertheless, it does impose a cost, and firms and investors care about return on investment. The behavioral assumption behind the model is that firms and investors target a profit rate based on firms' accounts.

Following Lee (1999), the appropriate value for the capital stock is what firms calculate it to be. Firms calculate the value of their capital stock to construct their balance sheets and report depreciation to the tax authorities. Even within a single production unit, capital goods are highly heterogeneous, including plant, machinery, vehicles, and anything else for which depreciation can be reported. Capital in the classical-evolutionary theory of technological change is therefore not a single good and its value does depend on price. When the theory is embedded as a component in a larger macroeconomic model, the quantity-price relationship must be addressed. Single-good models assume capital and consumption goods take the same form and have the same price index. For any other type of model the price of capital goods must be tracked, and there may be many types of capital goods. To use the theory presented in this paper, capital productivity should be calculated as the ratio of the monetary value of sector output to the monetary value of the capital stock. In a model with multiple capital goods and multiple products, they must be weighted by their relative prices.

While this paper departs in some ways from both sides of the Cambridge Controversies, in one sense it is firmly on the Cambridge, England, side of the debate. Harcourt argues that neoclassical theories of production must define a unit of capital that is independent of distribution and prices, while classical theories must motivate a general rate of profit. He writes, "[I]n mainstream [neoclassical] theory, there is no conceptual difference between rates of profit and the rate of interest. By contrast, the conceptual difference is emphasized in the alternative [classical] approach. Profit is the return, expected and actual, on investment in capital goods. Interest is the hire price of finance" (Harcourt, 2015, p. 248). This paper takes the classical approach.

3 Classical-evolutionary cost share-induced technological change

This section presents the core theoretical elements of classical-evolutionary cost share-induced technological change. The theory builds on what this author believes to be the essence of the "classical-Marxian evolutionary" theory of cost share-induced technological change proposed by Duménil and Lévy (2010). For present purposes, their two most important findings are: 1) biased technological change emerges naturally within an evolutionary setting; 2) if the productivity growth rate of one input depends on cost shares, then so does at least one other input. The first of these two findings resolves a long-standing problem that makes their theory highly compelling as a starting point for further work. The second produces some interesting dynamics when combined with other assumptions into a complete and closed model, as shown later in this section.

Foley (2003, p. 42) provides a succinct summary of these points. He notes that in Duménil and Lévy's theory the dependence of technical change on distribution arises "in a model in which capitalist firms simply select candidate technical innovations that are thrown up by a random process..." Further, in common with Duménil and Lévy, in Foley's model both labor and capital productivity depend on cost shares. Yet, both (Foley, 2003) and (Foley et al., 2019, chap. 7) assume profit-maximizing firms face an exogenous innovation frontier. The essential evolutionary nature of Duménil and Lévy's theory was lost, together with the insights it generates.

Elaborating on Duménil and Lévy's theory, the author has shown that: (1) their model for labor and capital can be expanded to any number of inputs; (2) the functional form relating cost shares to productivity growth rates is meaningfully constrained; (3) those results hold in the aggregate across an indefinite number of "production units" (Kemp-Benedict, 2019, 2022). These points are introduced in this section.

3.1 The core theory

This section, while drawing on Duménil and Lévy (2010) for inspiration, uses the author's own notation and framing. The microeconomic conceptual model posits a "production unit" searching in the vicinity of its current practices for innovations. The production unit might be, for example, a firm, a division, or a product line. Innovations are sought that meet any of a number of fitness criteria. For example, for product innovations, firms may respond to market analysis or competitor's innovations, or they may perceive completely new opportunities. For process innovations, they may respond to data collected from practice (e.g., a lean or six sigma approach)[2] or the introduction of new machinery. Innovations can also be introduced less formally, and often are. Any innovation that passes this step is considered to be "fit." However, that is not the end of the story. For an innovation to be viable, it must not degrade the profitability of the production unit or its parent firm.[3] Thus, while *candidate* innovations pass a fitness test, to be implemented (and thus become actual innovations) they must pass a profitability test as well.

The requirement that an innovation not lower profitability is consistent with the practice of capital budgeting. In a review of capital budgeting surveys, Mukherjee and Henderson (1987) found that project proposals are initiated at lower levels in the organization and then screened and approved at higher levels, consistent with the assumption in this paper. In a longitudinal study of large firms in the UK, Pike (1996) found that an increasing fraction of firms carried out a formal financial evaluation in all the years of the study, reaching 100% by the final wave (1992). Firms used multiple techniques, with only 4% using a single method. Nearly all firms used payback period (94% in 1992), while the bulk also used internal rate of return and net present value (81% and 74% in 1992 respectively). The study found a strong increase in the use of risk appraisal techniques, such as changing the payback period or required rate of return. In a more recent and substantial study of North American firms, Graham and Harvey (2001, 2002) found internal rate of return and net present value to predominate (around 75% of firms). Payback period was still popular, but with lower uptake than in Pike's study (57% of firms).

Capital budgeting is applied to proposed investments, and therefore to the incremental impact on profits. This is consistent with the observation by Shaikh (2016) that while average profits rates differ systematically and persistently between sectors, incremental profit rates do not. In an analysis of Shaikh's dataset, Kemp-Benedict (2023) showed that while incremental rates of profits vary widely over time, their distribution is statistically similar for most sectors.[4] This paper extends the analysis in Kemp-Benedict (2019, 2022), which followed Duménil and Lévy (2010) in using the

[2] See https://asq.org/quality-resources/lean and https://asq.org/quality-resources/six-sigma at the American Society for Quality website.

[3] This is not entirely true. Sometimes production units are cross-subsidized in order to build up experience with new techniques and products. Furthermore, Mukherjee and Henderson (1987, p. 81) report that some projects can be exempt from financial justification, e.g., for public or employee safety.

[4] Time series of incremental rates of profit were treated as empirical distributions. Applying a pair-wise comparison using a two-sample Kolmogorov–Smirnov test, distributions were found to be similar for 26 of the 31 sectors in Shaikh's dataset. The five exceptions were: Real estate, Mining excluding oil, Waste, Food service, and Broadcasting.

average profit rate, to the case of incremental rates of profit. As will be shown later, the result is a selection frontier of the form

$$\sum_{i=0}^{n} \sigma_i \hat{\nu}_i > c. \tag{1}$$

In this expression, n is the number of inputs in addition to capital (e.g., $n = 1$ if the only inputs are labor and capital), the σ_i are cost shares, the ν_i are productivities, and c is a (possibly non-zero) threshold.

It is important to note that despite the superficial similarity, Duménil and Lévy's selection frontier is very different from the external innovation possibility frontiers posited by Kennedy (1964), Foley (2003), and others. As argued above, Duménil and Lévy's frontier can be interpreted as the consequence of a firm's capital budgeting procedures, which are typically carried out by the chief financial officer (CFO) or their staff. Capital budgeting is applied to proposals as they arrive; it is a filtering computation, not an optimizing one.

As firms explore in the vicinity of their own techniques and make incremental improvements, they pursue potentially divergent technological pathways. What is more, the resulting heterogeneity is crucial for evolutionary theory (Cantner, 2017). While technological pathways are constrained by the prevailing technological regime and often by commonly held dominant designs, within those constraints there is meaningful variety from one firm to the next (Dosi and Nelson, 2018, pp. 57–58).

Equation (1) can be expressed equivalently in vector form. Defining column vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ with elements σ_i and ν_i, the selection frontier can be written

$$\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} > c. \tag{2}$$

A central element of the Duménil and Lévy (2010) model is a probability distribution of productivity growth rates, which is denoted in this paper as $f(\hat{\boldsymbol{\nu}})$. Possible forms for this distribution are a central focus of the present paper. While Kemp-Benedict (2022) explicitly derived a "generating function" for productivity growth rates starting from an NK model, here it will be asserted to equal

$$\Phi(\boldsymbol{\sigma}; c) = \int d\hat{\boldsymbol{\nu}}\ f(\hat{\boldsymbol{\nu}})\,(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)\,h(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c). \tag{3}$$

In this expression, $h(x)$ is the Heaviside function, equal to one when its argument is positive and equal to zero when its argument is negative.

This is a generating function in that partial derivatives with respect to cost shares give the expected value of the corresponding productivity growth rates for fit and profitable innovations. The partial derivative with respect to σ_i is

$$\frac{\partial \Phi(\boldsymbol{\sigma}; c)}{\partial \sigma_i} = \int d\hat{\boldsymbol{\nu}}\ f(\hat{\boldsymbol{\nu}})\,\hat{\nu}_i\,h(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c) + \int d\hat{\boldsymbol{\nu}}\ f(\hat{\boldsymbol{\nu}})\,\hat{\nu}_i(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)\,\delta(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c), \tag{4}$$

where $\delta(x)$ is the Dirac delta function. The delta function is the derivative of the Heaviside function, $h'(x) = \delta(x)$, equal to zero everywhere except at $x = 0$, where it diverges. It has the property that $\int_{-\infty}^{\infty} dx\, F(x)\delta(x) = F(0)$ for any function $F(x)$. The integrand in the second integral of this equation contains a factor of the form $x\delta(x)$, so it is equal to zero. The result then becomes

$$\frac{\partial \Phi(\boldsymbol{\sigma}; c)}{\partial \sigma_i} = \int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}})\, \hat{\nu}_i\, h(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c) = E_{\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}}>c}[\hat{\nu}_i]. \tag{5}$$

This is the expected value of $\hat{\nu}_i$ over the distribution of innovations that are both *fit* – as expressed by the probability distribution $f(\hat{\boldsymbol{\nu}})$ – and *profitable* – as enforced by the Heaviside function $h(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)$. The function $\Phi(\boldsymbol{\sigma}; c)$ defined in Eq. (3) thus acts as a generating function for average productivity growth rates, as intended.

An essential result, noted above, is that even if the probability distribution $f(\hat{\boldsymbol{\nu}})$ is symmetric, and therefore unbiased towards any particular input, the expected value of productivity growth rates *is* biased, due to the selection frontier criterion $\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} > c$. This is an elegant result that follows directly from the Duménil and Lévy (2010) model: biased technological change emerges naturally in the theory, and does not need to be imposed.

The second essential result that was noted earlier – that if the productivity of one input depends on cost shares, then so must at least one other – can be demonstrated by taking the second partial derivative,

$$\frac{\partial^2 \Phi(\boldsymbol{\sigma}; c)}{\partial \sigma_i \partial \sigma_j} = \int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}})\, \hat{\nu}_i \hat{\nu}_j\, \delta(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c) = \frac{\partial}{\partial \sigma_j} E_{\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}}>c}[\hat{\nu}_i]. \tag{6}$$

Because the matrix of second partial derivatives (the Jacobian) is symmetric, if the average productivity growth rate for input i depends on cost share j then, from this equation, the productivity growth rate for input j must depend on the cost share for input i.

Pre- and post-multiplying the Jacobian by an arbitrary vector $\boldsymbol{x}$ gives

$$\sum_i \sum_j x_i x_j \frac{\partial^2 \Phi(\boldsymbol{\sigma}; c)}{\partial \sigma_i \partial \sigma_j} = \int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}})\, \left(\boldsymbol{x}'\hat{\boldsymbol{\nu}}\right)^2\, \delta(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c), \tag{7}$$

which is either positive or zero. Thus, a further finding is that the Jacobian matrix is either positive-definite or positive semi-definite. This means in particular that the diagonal of the Jacobian is positive or zero, so the change in a productivity growth rate with respect to an increase in its own cost share must be either positive or zero.

3.2 Aggregation

The presentation to this point applies to a production unit. However, as shown in Kemp-Benedict (2022), the restrictions on the generating function still hold after aggregating over production units. The argument is briefly restated here.

The presentation assumes that a sector is composed of N production units with a comparable product. Products are not identical: sectors such as textiles are highly heterogeneous, while firms in more homogeneous sectors often target different markets (for example, the motor vehicle transport sector includes both the everyday Buick and the high-end Bentley). We assume that all products in a sector are sufficiently comparable that they have a common "anchor price." Individual products may have prices far from the anchor, but all prices tend to move together. In this case, price differentials are roughly stable across production units and can be absorbed into productivities.

Following a standard growth rate decomposition (see Kemp-Benedict, 2022, for details), it is possible to show that the sector-level productivity growth rate for input i is

$$\hat{\nu}_i = \sum_{r=1}^{N} \left(a_r - b_{r,i}\right) \hat{Y}_r + \sum_{r=1}^{N} b_{r,i} \hat{\nu}_{r,i}, \tag{8}$$

where a_r is the rth unit's share of production and $b_{r,i}$ is the rth unit's share in use of input i. The first term is due to a change in composition while the second term is due to a change in productivity alone at the level of each production unit. The theory presented in this paper addresses the second term.

Following the development in (Kemp-Benedict, 2022, sec. 3.3), we propose an aggregate generating function of the form

$$\Phi(\boldsymbol{\sigma}; \{c_r\}) = \sum_{r=1}^{N} a_r \Phi_r(\boldsymbol{\sigma}_r; c_r). \tag{9}$$

Noting that the cost shares for individual production units are related to the average cost shares for the sector through

$$\sigma_{r,i} = \frac{b_{r,i}}{a_r}\sigma_i, \tag{10}$$

it follows that the partial derivative of the aggregate generating function with respect to the sector average cost share is

$$\begin{aligned} \frac{\partial}{\partial \sigma_i}\Phi(\boldsymbol{\sigma}; \{c_r\}) &= \sum_{r=1}^{N} a_r \frac{b_{r,i}}{a_r} \frac{\partial}{\partial \sigma_{r,i}} \Phi_r(\boldsymbol{\sigma}_r; c_r) \\ &= \sum_{r=1}^{N} b_{r,i} E_{\boldsymbol{\sigma}'_r \cdot \hat{\boldsymbol{\nu}} > c_r}[\hat{\nu}_i]. \end{aligned} \tag{11}$$

This is directly analogous to the expression for the sector aggregate productivity growth rate that appears in the second term of Eq. (8), but applied to the expected productivity growth rates for each production unit. Furthermore, because the Jacobian of each production unit's generating function is positive semi-definite, so is their weighted sum. This means that the aggregate generating function for the sector has the same properties as those for the individual production units.

3.3 Embedding the theory in a larger model

The Introduction emphasized that cost share-induced technological change is a mechanism that can be embedded into a variety of macroeconomic models. The purpose of this section is to demonstrate, through a series of examples, how that can be done in practice.

As earlier noted, the aggregate model applies to single sectors. Accordingly, one of the examples, in Section 3.3.3, shows how the theory can inform a multi-sector model in which each sector has intermediate inputs as well as inputs of labor and capital. However, the other examples follow common post-Keynesian practice and apply the theory to one-good models with labor and capital as the only inputs. While not strictly justified, this has two benefits: first, the presentation can be comparatively brief; second, the models can be compared to other post-Keynesian models available in the literature.

3.3.1 A one-sector model with three alternative closures

The impact of cost share-induced technological change upon macroeconomic dynamics depends on other model assumptions, particularly price- and wage-setting. In the one-good, two-input model presented in this section, prices and wages are linked through the markup. While keeping the relationship between cost shares and productivity growth rates the same, we offer three alternatives closures: constant markups, target-return pricing, and conflict wage-setting.

For this two-input model, labor has productivity λ and capital productivity κ. Using a "hat" to indicate a growth rate, the corresponding productivity growth rates are $\hat{\lambda}$ and $\hat{\kappa}$. As shown in the previous section, if the productivity growth rate for one input depends on cost shares, then so must at least one other. For two inputs, this means that we can write,

$$\hat{\lambda} = l(\pi), \quad l' < 0, \tag{12a}$$

$$\hat{\kappa} = k(\pi), \quad k' > 0. \tag{12b}$$

As will be seen below, the requirement that more than one productivity growth rate depend on cost shares yields interesting results. By contrast, many applications of cost share-induced technological change to macroeconomic theory assume that only one productivity growth rate is dependent. That is true, for example, in the survey by Dutt (2013) of endogenous technological change in classical-Marxian models and of the Kaleckian models of Cassetti (2003) and Hein and Tarassow (2010).

In addition to price- and wage-setting, each example assumes an investment function of the form

$$g = \gamma(r), \quad \gamma' > 0, \tag{13}$$

where g is the investment rate and $r = \pi\kappa$ is the profit rate. For simplicity, capital is assumed to be fully utilized, so the capacity utilization factor does not enter into the investment function.

Fixed markup

With a fixed markup m, the profit share is given by

$$\pi = \pi^* = \frac{m}{1+m}. \tag{14}$$

In this case, π is fully determined by pricing behavior, and productivity growth rates are fully determined as well: $\hat{\lambda} = l(\pi^*)$ and $\hat{\kappa} = k(\pi^*)$. Unless the markup happens to satisfy $k(\pi^*) = 0$, the profit rate $r = \pi^*\kappa$ and thus the investment rate $g = \gamma(r)$ will show a trend. If that trend is downward, then the result is a falling rate of profit. However, it depends on both the markup and the dependence of the productivity growth rates on distribution.

Target-return pricing

Under target-return pricing, markups are set such that

$$\pi\kappa = r^*, \tag{15}$$

where r^* is the target profit rate. For simplicity, it is assumed that markups are adjusted instantaneously to maintain the target profit rate (in reality, there will be a lag). If the target profit rate r^* is not changing under time, then

$$\widehat{\pi\kappa} = \hat{\pi} + \hat{\kappa} = 0, \tag{16}$$

which gives an equation of motion for the profit share,

$$\hat{\pi} = -\hat{\kappa} = -k(\pi). \tag{17}$$

This is a stabilizing dynamic, because when π increases (decreases), $k(\pi)$ also increases (decreases), so $\hat{\pi}$ decreases (increases), leading to a negative feedback. The equilibrium position satisfies $k(\pi^*) = 0$. At that point, $\hat{\kappa} = 0$, so the capital productivity is steady in equilibrium. The equilibrium level of the capital productivity is given by $\kappa^* = r^*/\pi^*$. Labor productivity grows steadily in equilibrium at the rate $l(\pi^*)$.

Over time, the profit share, capital productivity, and labor productivity can all fluctuate around their equilibrium values. However, by the assumption that the target profit rate is achieved instantaneously, the investment rate is constant, $g = \gamma(r^*)$.

Note that the equilibrium in this case produces Harrod-neutral technological change, with constant capital productivity and steady labor productivity growth. It also produces a constant profit share and constant profit rate, consistent with Kaldor's stylized facts (Kaldor, 1961).

Conflict wage determination

In the final example with this one-sector post-Keynesian model, it is more convenient to use the wage share rather than the profit share. We define functions l_w and k_w that depend on the wage share as

$$\hat{\lambda} = l_w(\omega) \equiv l(1-\omega), \quad l_w' = -l' > 0, \tag{18a}$$

$$\hat{\kappa} = k_w(\omega) \equiv k(1-\omega), \quad k_w' = -k' < 0. \tag{18b}$$

The wage share is given by the ratio between the real wage w and labor productivity,

$$\omega = \frac{w}{\lambda} \qquad \Rightarrow \qquad \hat{\omega} = \hat{w} - \hat{\lambda}. \tag{19}$$

The wage is assumed to grow at the same rate as labor productivity unless the growth in labor demand differs from the growth rate of the labor force. Specifically, denoting the growth rate of labor demand by $\hat{L}$ and the growth rate of the labor force by n,

$$\hat{w} = \hat{\lambda} + f(\hat{L} - n), \quad f(0) = 0, f' > 0. \tag{20}$$

Growth in labor demand is given by the difference between the growth of output, $\hat{Y}$, and labor productivity,

$$\hat{L} = \hat{Y} - \hat{\lambda}. \tag{21}$$

The growth rate of output is given by the sum of the investment rate and capital productivity,

$$\hat{Y} = g + \hat{\kappa}. \tag{22}$$

Finally, the investment rate is

$$g = \gamma(r) = \gamma\left((1-\omega)\kappa\right). \tag{23}$$

We can now combine the above to write

$$\hat{\omega} = f\left(\gamma\left((1-\omega)\kappa\right) + k_w(\omega) - l_w(\omega) - n\right). \tag{24}$$

When ω increases, $-l_w(\omega)$, $k_w(\omega)$ and $\gamma\left((1-\omega)\kappa\right)$ all decline (and, conversely, when ω decreases, each of those terms increases). As a consequence, this is a stabilizing dynamic. The equilibrium is reached when $f = 0$, which occurs when its argument is zero. Therefore, the equilibrium value of the wage share, ω^*, satisfies

$$\gamma\big((1-\omega^*)\kappa\big) + k_w(\omega^*) - l_w(\omega^*) = n. \tag{25}$$

The equilibrium value will change over time unless κ is constant. Taking the time derivative of this equation, using $\hat{\kappa} = k_w(\omega)$, and solving for $\dot{\omega}^*$ (where a "dot" indicates the time derivative), gives

$$\dot{\omega}^* = \frac{\kappa\gamma'}{\kappa\gamma' + l_w' - k_w'}(1-\omega^*)k_w(\omega^*). \tag{26}$$

Because γ' and l'_w are positive, while k'_w is negative, the ratio of $\kappa\gamma'$ to $\left(\kappa\gamma' + l'_w - k'_w\right)$ is also positive. The equilibrium of this equation, ω^{**}, satisfies

$$k_w(\omega^{**}) = 0. \tag{27}$$

The equilibrium is locally stable because $k'_w < 0$. However, it may not be globally stable. The model becomes unstable when k_w is sufficiently negative, specifically when $k_w < (1 - \omega^*)k'_w$. This condition may or may not be possible, depending on the form for k_w.

With this model closure, the equilibrium solution produces a constant cost share ω^{**} at which $\hat{\kappa} = 0$ and $g = \hat{\lambda} + n$. As with a target rate of return, the equilibrium is characterized by Harrod-neutral technological change and Kaldor's stylized facts of constant cost shares and a constant profit rate. What is more, it brings the investment rate in line with Harrod's natural rate.

3.3.2 Target-return pricing with R&D costs

The Introduction contrasted the dominant neoclassical theories of technological change to the non-neoclassical alternatives. The author criticized Acemoglu (2002) in particular for developing a theory that required extreme parameter assumptions to get results close to those observed in practice. However, one benefit of Acemoglu's model is that it takes explicit account of R&D expenditure. That is also true of some non-neoclassical models, notably Zamparelli (2024), who introduced both fixed-coefficient (Leontief) and neoclassical versions of a model of cost share-induced technological change in which R&D expenditure pushes the innovation possibility set outward. Similarly, Dosi et al. (2010) and Caiani et al. (2019) built agent-based models in which R&D expenditure affects the probability of discovering an innovation.

Expenditure on R&D is an example of an "enterprise cost", in contrast to direct material and labor costs, and to shop costs. Target-return pricing typically takes enterprise costs into account (Lee, 1999, p. 205). This suggests that the target-return pricing model above can be modified to include R&D expenditure. We assume R&D costs are specified as a fraction d of the value of the capital stock. In keeping with Dosi et al. (2010) and Caiani et al. (2019), the probability $\theta(d)$ of finding a fit innovation increases with R&D expenditure. As shown by Kemp-Benedict (2022, pp. 1322-3), this effect can be incorporated into a classical-evolutionary model of cost share-induced technological change by multiplying the cost share by the probability.

In the reformulated model, productivity growth rates are given by

$$\hat{\lambda} = \theta(d)l\left(\theta(d)\pi\right), \quad l' < 0, \tag{28a}$$

$$\hat{\kappa} = \theta(d)k\left(\theta(d)\pi\right), \quad k' > 0. \tag{28b}$$

The target-return formula is modified to include R&D expenditure. All variables are normalized by the value of the capital stock, so

$$\pi\kappa = r^* + d. \tag{29}$$

As before, the equilibrium for a given value of d features constant cost shares, a constant capital productivity, and steady labor productivity growth. However, the equilibrium profit share now satisfies

$$k\left(\theta(d)\pi^*\right) = 0. \tag{30}$$

A rise in R&D expenditure per unit of the capital stock translates into a rise in $\theta(d)$. Because k is increasing in its argument, to bring it back to zero, the equilibrium profit share π^* must decline. The product $\theta(d)\pi^*$ remains unchanged, so from Eq. (28a), labor productivity simply increases in line with the factor $\theta(d)$.

The equilibrium level of the capital productivity satisfies

$$\kappa^* = \frac{r^* + d}{\pi^*}. \tag{31}$$

Denoting the equilibrium profit share when $d = 0$ by π_0^*, at equilibrium the profit share is $\pi^* = \pi_0^*/\theta(d)$, so

$$\kappa^* = \frac{r^* + d}{\pi_0^*}\theta(d). \tag{32}$$

This is increasing in d, so as R&D expenditure rises, so does the equilibrium level of the capital productivity.

3.3.3 A multi-sector model

This section places the sectoral model of cost share-induced technological change within a multi-sector setting. Unlike with the previous examples, this is not a complete model. In particular, it does not track investment and therefore does not demonstrate how the value of the capital stock depends on the relative price of different capital goods. Nevertheless, even though it is incomplete, the model leads to some interesting conclusions.

Following Sraffian theory, sectoral prices are set by markup, but prices are not adjusted instantaneously. Instead, firms in sector i are assumed to set a target price p_i^*, where

$$p_i^* = \mu_i \left(c_i + \sum_{j=1}^{n} p_j a_{ji} \right). \tag{33}$$

In this expression, n is the number of sectors, $\mu_i = 1 + m_i$ is the profit margin, c_i are factor costs, such as labor and raw materials (but excluding profits), the p_j are prevailing prices, and the a_{ji} are technical coefficients. To simplify the analysis, all sectors are assumed to be “basic”, in that a chain can be constructed linking any given sector to any other given sector through a series of intermediate transactions.

Following Kemp-Benedict (2017b), firms adjust their prices towards their target price over time, expressed through an equation of motion

$$\dot{p}_i = \beta_i \left(p_i^* - p_i\right) \quad \Rightarrow \quad \hat{p}_i = \beta_i \left(\frac{p_i^*}{p_i} - 1\right). \tag{34}$$

From Eq. (33), the ratio between the target and current price can be written in terms of cost shares,

$$\begin{aligned} \frac{p_i^*}{p_i} &= \mu_i \left(\frac{c_i}{p_i} + \sum_{j=1}^{n} \frac{p_j}{p_i} a_{ji}\right) \\ &= \mu_i \left(\chi_i + \sum_{j=1}^{n} \alpha_{ij}\right). \end{aligned} \tag{35}$$

Here, $\chi_i = c_i/p_i$ is the factor cost share and $\alpha_{ij} = (p_j/p_i)a_{ji}$ is the cost share for intermediate input j. Because the sum of the cost shares is one less than the profit share π_i, the ratio p_i^*/p_i is also the ratio of the desired profit margin, μ_i, to the realized profit margin, $1/(1-\pi_i)$.

If sector i makes no use of sector j's output, then $a_{ji} = 0$ and $\alpha_{ij} = 0$. Otherwise, intermediate cost shares satisfy the following equation of motion, which follows directly from their definition,

$$\hat{\alpha}_{ij} = \hat{a}_{ji} + \hat{p}_j - \hat{p}_i. \tag{36}$$

The technical coefficients a_{ji} are inverse productivities – they are a ratio of input to output rather than the other way around. Therefore, assuming cost share-induced technological change, the change in the growth rate of the technical coefficient with respect to its own cost share must be negative;

$$\hat{a}_{ji} = f_{ji}(\chi_i, \alpha_{1i}, \alpha_{2i}, \ldots, \alpha_{ni}), \quad \text{where} \quad \frac{\partial f_{ji}}{\alpha_{ij}} < 0. \tag{37}$$

Furthermore, from the pricing dynamics assumed above,

$$\hat{p}_i = \beta_i \left[\mu_i \left(\chi_i + \sum_{k=1}^{n} \alpha_{ik}\right) - 1\right]. \tag{38}$$

This expression is increasing in cost shares, while the cost share α_{ij} does not enter the price p_j, so

$$\frac{\partial \hat{\alpha}_{ij}}{\partial \alpha_{ij}} = \frac{\partial \hat{a}_{ji}}{\partial \alpha_{ij}} - \frac{\partial \hat{p}_i}{\partial \alpha_{ij}} < 0. \tag{39}$$

That is, the direct effect of a change in intermediate cost share is stabilizing. Because there are great number of cost shares, it is possible to have at least local instability through secondary interactions, but the underlying dynamic leads to an equilibrium in which the cost shares are not changing.

An equilibrium of constant cost shares has two implications. First, from Eq. (38), inflation rates are constant unless factor costs χ_i, target profit margins μ_i, or adjustment rates β_i change. Second, setting Eq. (36) to zero gives a condition

$$\hat{a}_{ji} = \hat{p}_i - \hat{p}_j. \tag{40}$$

As noted above, this expression was derived on the assumption that a_{ji} is not equal to zero. However, because all sectors are assumed to be basic, it is possible to find a chain of intermediate links between any two sectors, for example $j \to k_1 \to k_2 \to \cdots \to k_N \to i$. After repeated intermediate exchanges, the effective coefficient linking the two sectors is

$$A_{ji} = a_{jk_1} a_{k_1 k_2} \cdots a_{k_N j}, \tag{41}$$

and its growth rate is, at equilibrium,

$$\hat{A}_{ji} = \hat{a}_{jk_1} + \hat{a}_{k_1 k_2} \cdots + \hat{a}_{k_N j} = \hat{p}_i - \hat{p}_j. \tag{42}$$

The final equation holds because each intermediate $\hat{p}_{k_n}$ enters twice, first with a positive sign and second with a negative sign, and therefore cancels out.

If inflation rates differ between sectors, then there must be at least one sector J with the lowest inflation rate. From Eq. (42), for any other sector i, $\hat{A}_{Ji} \geq 0$. This means that demand overall for sector J's output is growing faster than any other sector. That may be possible for a while, but at some point the assumption of fixed markups and price adjustment rates will break down. Costs for sector J are such that firms in that sector can lower their prices while maintaining their desired profit margins, but the situation invites a rise in profit margins. Aside from the lost opportunity to accumulate profit, the sector's market is expanding, and a rise in the profit margin will attract the investment needed for the expansion. Crucially for the cost share-induced technological change mechanism, it will also raise the sector's price and therefore narrow the inflation gap between sector J and other sectors.

The implication is that through a combination of markup-based Sraffian pricing, demand-pull inflation, and cost share-induced technological change, the economy features an equilibrium in which cost shares are constant and prices in all sectors advance at the general rate of inflation. At the same time, the model allows for deviations from equilibrium that may persist for meaningfully long times.

3.4 Comments

The examples in this section made very few assumptions about the form of the functional relationship between cost shares and productivity growth rates. Nevertheless, they illustrated some general points and generated interesting results. The main general point is that the cost-share induced technological change mechanism can, when combined with price- and wage-setting behavior, produce dynamic models in which technological change and distribution influence one another. While the dynamic systems may never reach their equilibria in practice, in classical terms the equilibria provide “centers of gravitation” or long-period positions that depends on both the technological change mechanism and price- and wage-setting behavior.

The equilibria are theoretically interesting. The simplest of the examples, a fixed markup model, can (but need not) lead to a falling rate of profit. However, that result can be seen as an argument against assuming a fixed markup. If firms see their profit rate fall, they can target a desired rate of return. The resulting equilibrium was found to feature Kaldor’s stylized facts of constant capital productivity and constant profit rate, as well as Harrod-neutral technological change. The same result was found when target-return pricing takes R&D expenditure into account. In that case the labor productivity growth rate and the level of capital productivity are increasing functions of R&D expenditure.

Political economy considerations would suggest a move from fixed markups to target-return policies if profit rates begin to fall. But a third model showed that a quite different mechanism – conflict wage-setting – can also produce an equilibrium that exhibits Kaldor’s stylized facts and Harrod-neutral technological change. The model includes a feedback on wage claims mediated through the investment rate, which leads the investment rate to Harrod’s natural rate at equilibrium.[5]

The final, multi-sector, model was incomplete as a macroeconomic model. Nevertheless, it showed how a combination of cost share-induced technological change, Sraffian pricing, and markups that change in response to demand lead to an equilibrium with stable technical coefficients and relative prices.

These results, which are of theoretical interest, emerge from dynamics and require both labor and capital productivity growth to depend on distribution. In contrast, post-Keynesian models in which only labor productivity growth is cost share-dependent tend to apply comparative statics. Examples include Cassetti (2003), who synthesized a model of conflict inflation with a theory of growth and distribution, and Hein and Tarassow (2010) who sought to understand the implications of distribution on growth, allowing for multiple channels. It is possible that allowing both labor *and* capital productivity to depend on cost shares would change the comparative statics results. For example, it would add a further dimension to the analysis of regimes by Hein and Tarassow (2010). For Cassetti (2003), if the equilibrium is characterized by constant capital productivity, then the comparative statics results may not change. However, based on the models presented above, constant capital productivity is a contingent result that depends on the way that prices and wages are set. By introducing cost share dependence for capital productivity growth, the goods-market equilibrium of Casetti’s Kaleckian model could be paired with a dynamic equilibrium resulting from the interaction between distribution and productivity.

[5] Kemp-Benedict (2020) showed in a competing claims model with the possibility of either target-return or wage conflict price-setting behavior that the incompatibility between the two mechanisms gave rise to long waves.

The multi-sector model introduced in this paper can be contrasted with that of Shiozawa (2020). Shiozawa's theory was developed over many years (see Shiozawa et al., 2019) and is quite sophisticated. The aim is to provide a firm theoretical foundation that explains how technological change induces economic growth. Many of the behavioral assumptions are compatible with those in the present paper, in particular the selection criterion in his Eq. (4-1) (p. 1011) and the co-evolution of techniques and prices (p. 1013). However, there remain important differences. In Shiozawa's model firms choose from among a given set of techniques. As explained earlier in this paper, this is a problematic assumption, particularly in an evolutionary theory. A second, more subtle difference is that while Shiozawa's minimal price theorem (pp. 997–999) assumes constant markups, the theory in this paper generates a stable equilibrium given the possibility of changing markup rates. However, taken as a whole there are many more similarities than differences between Shiozawa's theory and that in the present paper. From the point of view of this paper, the most problematic of Shiozawa's assumptions is that firms continually seek amongst an existing set of techniques and choose that with the lowest price. Instead, as Shiozawa himself notes (p. 991), "Technological evolution comprises a series of half-blind selections of 'better' production techniques." His framework would be strengthened by drawing more systematically on the evolutionary economics literature to represent that half-blind search. The theory presented in this paper provides a possible way forward, but doubtless not the only one.

4 Selection frontier for a target incremental profit rate

The starting point for determining the selection frontier in this paper is the observation by Shaikh (2016), supported by the analysis by Kemp-Benedict (2023), that incremental profit rates are similar between sectors, while average profit rates are not. This suggests a central role for incremental rates of profit. As argued above, this conclusion is further reinforced by the widespread practice of capital budgeting, which seeks to anticipate the potential profitability of new investment.

As emphasized by Shell (1967), technological change arises due both to the potential embodied in new machinery and, importantly, through disembodied learning processes. In this paper, disembodied technological change is defined as a change in output with no change in the capital stock (although the capital stock may be revalued), while embodied change is due to the addition of new capital goods (and perhaps the retirement of old capital goods).

4.1 Capital productivity

Following the treatment of capital introduced earlier in this paper, it is always represented in value form, where its value is that defined by the firm that owns it. The actual capital goods behind that single measure of value are in reality a heterogeneous mix, even within firms. Capital productivity κ is then defined as the ratio of the value of a firm's output, V, to the value of its capital stock, K. The values of V and K

are influenced by factors that are difficult to capture in an aggregate model; the strategy adopted in this paper is to find expressions that rely on details to the least extent possible.

To shed some light on the complexities of modeling value: A firm will typically produce multiple products, each aimed at a different market. For this reason, the value V of output, even at constant prices, can change through a combination of expanding output and changing composition.[6] Furthermore, assessing the value of the capital stock is a bookkeeping exercise, and different bookkeeping procedures can result in different valuations (Lee, 1999). One of the most salient distinctions is between capital valued at historical cost – what was actually paid in the past – and replacement cost – what it would cost to buy equivalent equipment in the present. As older equipment is less valuable than new, replacement cost is normally higher than resale value in nominal terms, although if manufacturing costs have declined it might be less than the inflation-adjusted original cost. What is more, a decline in resale value will typically differ from depreciation, which is defined by the tax code. Finally, neither depreciation nor the loss in resale value corresponds to the loss of productive capacity due to retirement of capital.

Given the manifold influences on V and K, the goal is to identify a general expression, specifying a minimum of details. When embedded in a larger model, details that are avoided here can be more or less fully specified. For example, in a multi-sector model with multiple investment goods (such as plant and machinery) and investment services (such as construction), the prices of each type of good and service must be tracked, and the mix of inputs specified, in order to calculate the value of investment and the capital stock. By contrast, in a single-good model, the price level of capital goods will be identical to the general price level. The goal of this paper is to identify a handful of details required to specify the selection frontier, independent of the specificities of any given model.

4.2 Change in value

Regardless of how V and K are determined, capital productivity is $\kappa \equiv V/K$. Working to first order, the change in value is

$$\Delta V \simeq K\Delta\kappa + \kappa\Delta K. \tag{43}$$

The change in the value of output can be ascribed to either disembodied or embodied change. While all innovation is ultimately constrained by incremental profitability, that associated with the purchase of new capital with a value I is the subject of capital budgeting exercises. Those are made on the basis of cash flow (Graham and Harvey, 2002), which requires an anticipated incremental value of output V_m associated with investment expenditure. The ratio $\kappa_m = V_m/I$ is the incremental capital productivity associated with investment expenditure.

[6] For example, a bakery, with no change in equipment, could increase the proportion of fine pastries in its output. Holding the level of sales steady in physical terms (as measured, say, by the mass of wheat embodied in the bakery's products), value would rise without any change in prices. Note that cost of materials would likely rise as well, so the margin may or may not change.

As a practical example of using κ_m, if anticipated operating costs as a share of revenue are denoted a, then the payback period is $I/V_m(1-a) = 1/\kappa_m(1-a)$. The ratio appears also in expressions for net present value and internal rate of return. This demonstrates that if a CFO is to carry out a capital budgeting exercise, then he or she needs to have an idea of the ratio κ_m; it is a tangible quantity of relevance to firm decision-making.

The change in the value of output associated with capital retirement and loss of productive capacity of existing equipment is less easily defined than for new investment, and is written ΔV_{exist}. Accordingly, the change in value due to changes in the existing capital stock, which we define to be embodied technological change, can be written

$$\Delta V_{\text{emb}} = \kappa_m I - \Delta V_{\text{exist}}. \tag{44}$$

Similarly, the change in the value of the capital stock can be written $\Delta K = I - \Delta K_{\text{exist}}$, where the first term is the value of new additions to the capital stock while the second term is the loss in the value of existing capital stock due to retirement, wear and tear, and loss of resale value, net of any change due to revaluation.

4.3 Productivity growth

Disembodied technological change alters the value of output without adding to the capital stock. Denoting the change in value due to disembodied technological change by ΔV_{dis}, the total change in value is $\Delta V = \Delta V_{\text{emb}} + \Delta V_{\text{dis}}$. Combining this equation with Eqs. (43) and (44) gives an expression for the growth rate of capital productivity, $\hat{\kappa} = \Delta\kappa/\kappa$,

$$\hat{\kappa} = \frac{\Delta V_{\text{dis}}}{\kappa K} + \left(\frac{\kappa_m}{\kappa} - 1\right)\frac{I}{K} + \left(\frac{\Delta K_{\text{exist}}}{K} - \frac{\Delta V_{\text{exist}}}{\kappa K}\right). \tag{45}$$

In the classical-evolutionary cost share-induced technological model, the first two terms can be treated as random variables that contribute to productivity growth rates. The first arises from disembodied technological change, while the second arises from embodied technological change. The question is how to address the final, parenthetical, term, which is a difference of two ambiguous expressions.

One approach to evaluating the final parenthetical term in Eq. (45) is to make some common and pragmatic, but not entirely realistic, assumptions. First, assume that capital stock of any vintage depreciates at a common proportional rate, δ. Further assume that the depreciation rate captures all sources of change in the value of capital and the change in productive capacity. In that case, the value of output declines at the same proportional rate δ. With these assumptions, $\Delta K_{\text{exist}} = \delta K$, while $\Delta V_{\text{exist}} = \delta V = \delta\kappa K$, and the final term in parentheses in Eq. (45) vanishes.

Another approach is to absorb the final term into disembodied technological change. The first approach showed that the final term *can* be zero. However, it might in practice be either positive or negative. One plausible assumption is that capital would tend to be retired when it is comparatively less productive, so that the final term would tend to be positive. As that would result in an increase in capital productivity without the introduction of new capital, it fits the definition of disembodied technological change, and can be absorbed into a redefined term,

$$\Delta V_{\text{dis}} + (\kappa \Delta K_{\text{exist}} - \Delta V_{\text{exist}}) \to \Delta V_{\text{dis}}. \tag{46}$$

Following either the first or second approach, it is possible to write

$$\hat{\kappa} = \hat{\kappa}_{\text{dis}} + \hat{\kappa}_{\text{emb}}, \tag{47}$$

where

$$\hat{\kappa}_{\text{dis}} = \frac{\Delta V_{\text{dis}}}{\kappa K}, \tag{48a}$$

$$\hat{\kappa}_{\text{emb}} = \left(\frac{\kappa_m}{\kappa} - 1\right) \frac{I}{K}. \tag{48b}$$

As a practical matter, this result allows the further derivation to proceed using gross quantities, $\Delta K = I$ and $\Delta V_{\text{emb}} = \kappa_m I$, because the resulting expression for $\hat{\kappa}_{\text{emb}}$ is identical to Eq. (48b).

The ratio I/K that appears in Eq. (48b) is the investment rate g. The expression in parentheses that multiplies it is the relative difference between the incremental and average capital productivity. To distinguish that quantity from a growth rate, the value will be denoted by a "check" (an inverted "hat"), $\check{\kappa}_m \equiv \kappa_m/\kappa - 1$. With this notation,

$$\hat{\kappa}_{\text{emb}} = g\check{\kappa}_m. \tag{49}$$

The total growth rate in capital productivity is then

$$\hat{\kappa} = \hat{\kappa}_{\text{dis}} + g\check{\kappa}_m \tag{50}$$

For other inputs, the calculation proceeds similarly. However, the embodied contribution depends on the incremental input-to-capital ratio $q_{i,m} = \kappa_m/\nu_{i,m}$ rather than the productivity itself. The result is that $\hat{q}_{i,\text{emb}} = g\check{q}_{i,m}$. As with κ_m, ν_i must be known for a capital budgeting exercise to take place, so the ratio q_i corresponds to actual figures used by firms. The change in productivity can be evaluated using the identity $\hat{\nu}_{i,\text{emb}} = \hat{\kappa}_{\text{emb}} - \hat{q}_{i,\text{emb}}$. Adding the contribution from disembodied change gives

$$\hat{\nu}_i = \hat{\nu}_{i,\text{dis}} + g\left(\check{\kappa}_m - \check{q}_{i,m}\right). \tag{51}$$

4.4 Targeting an incremental rate of profit

This section derives a profitability criterion when firms are targeting a general incremental rate of profit. Following Shaikh (2016), the relevant expression is the change in gross profit divided by previous-period investment. In this section, gross profit includes only direct costs, but that is not the only choice. Firms could allocate estimated shop costs or a portion of enterprise costs as well. Those costs could include prior R&D expenditure, but could also include legal or lobbying costs, CEO pay premia, and so on.

Gross profit can be calculated as the value of output, V, net of the cost of inputs but gross of depreciation. Cost of capital is captured in the target incremental rate of profit, so at this stage capital is excluded from the list of inputs. The cost of inputs is therefore given by the sum over quantities q_i multiplied by prices p_i, where the sum is over $i \in 1 \ldots n$. (Capital will be introduced into the list of inputs with index $i = 0$ at a later stage.) Gross profits Π are then

$$\Pi = V - \sum_{i=1}^{n} p_i q_i. \tag{52}$$

After an innovation, which may include a combination of embodied and disembodied technological change, profits evaluated at fixed prices and wages becomes

$$\Pi_{+1} = V_{+1} - \sum_{i=1}^{n} p_i q_{i,+1}. \tag{53}$$

As noted earlier, the value at fixed prices can change because of an overall increase in the quantity of outputs as well as a changing composition of the output. The change in profits, $\Delta\Pi$, is

$$\Delta\Pi = \Delta V - \sum_{i=1}^{n} p_i \Delta q_i. \tag{54}$$

Following the arguments in Section 4, it is possible to calculate ΔV using gross investment, so that, to first order

$$\Delta V \simeq K\Delta\kappa + \kappa\Delta K = K\Delta\kappa + \kappa I. \tag{55}$$

To calculate the change in input costs at constant prices, note that by definition the productivity of input q_i is $\nu_i = V/q_i$, while the cost share is $\sigma_i = p_i q_i/V = p_i/\nu_i$. Again working to first order,

$$p_i\Delta q_i \simeq \frac{p_i}{\nu_i}\Delta V - \frac{p_i}{\nu_i} V \frac{\Delta\nu_i}{\nu_i} = \sigma_i \left(\Delta V - V\hat{\nu}_i\right). \tag{56}$$

In the final expression, $\hat{\nu}_i = \Delta\nu_i/\nu_i$ is the productivity growth rate of input i. Substituting the expressions for ΔV and $p_i\Delta q_i$ into Eq. (54) gives

$$\Delta\Pi \simeq \left(1 - \sum_{i=1}^{n}\sigma_i\right)(K\Delta\kappa + \kappa I) + V\sum_{i=1}^{n}\sigma_i\hat{\nu}_i. \tag{57}$$

In this equation, note that one minus the sum of cost shares of inputs is the profit share π. Using the identity $V = \kappa K$, and denoting the investment rate by $g = I/K$, Eq. (57) becomes

$$\frac{\Delta\Pi}{\kappa K} \simeq \pi\left(\hat{\kappa} + g\right) + \sum_{i=1}^{n}\sigma_i\hat{\nu}_i. \tag{58}$$

At this point, the profit share and capital productivity can be identified as the zeroth elements of the vectors of cost shares and productivities: $\sigma_0 \equiv \pi$ and $\nu_0 = \kappa$. Defining the vector product so that $\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}}$ extends from 0 to n, Eq. (58) can be written

$$\frac{\Delta\Pi}{\kappa K} \simeq \pi g + \boldsymbol{\sigma}'\hat{\boldsymbol{\nu}}. \tag{59}$$

In terms of the variables introduced above, the incremental rate of profit is calculated as $r = \Delta\Pi/I$. Through a process of "turbulent arbitrage", sectoral incremental profit rates rapidly, but turbulently, adjust towards the prevailing value (Shaikh, 2016). This suggests a rule for new investment of

$$\Delta\Pi > rI \quad \Longrightarrow \quad \frac{\Delta\Pi}{\kappa K} > \frac{r}{\kappa} g. \tag{60}$$

From Eq. (59), this leads to a version of Duménil and Lévy's (1995; 2010) selection frontier,

$$\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} > \left(\frac{r}{\kappa} - \pi\right) g. \tag{61}$$

This inequality is an example of a selection frontier with a threshold. It shows that, if the average profit rate $\bar{r} = \pi\kappa$ is less than the target incremental profit rate r, then cost-weighted average productivity growth must be comparatively high in order to meet the threshold. Conversely, if the average profit rate exceeds the target incremental profit rate, then the threshold is lower.

In Shaikh's theory of turbulent arbitrage, r is "turbulently equalized" to the equity rate of return (Shaikh, 2016, p. 468). That is shown to exceed the interest rate i (Shaikh, 2016, p. 459), but is otherwise undetermined. Post-Keynesian theory provides possible explicit expressions through the concept of a firm's "finance frontier." As discussed by Lavoie (2022, pp. 142-146), various theoretical expressions for the finance frontier take the form $r = ai + bg$, with different formulations producing different values for a and b. The calibration assumes one particular form, $r = i + g/(1+\ell)$, where ℓ is the firm's leverage. This expression combines both the interest rate, thereby capturing monetary policy, and leverage, thereby capturing a measure of risk. Substituting into Eq. (61),

$$\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} > \left(\frac{i + g/(1+\ell)}{\kappa} - \pi\right) g. \tag{62}$$

In this formulation, the threshold that appears in the selection frontier is: a) increasing in the interest rate i; increasing in the investment rate g; declining in the average profit share π and capital productivity κ. A rise in the threshold means that it is harder to find profitable innovations, and therefore tends to slow the pace of innovation, other things considered. However, because embodied productivity growth is also increasing in the investment rate, a rise in the investment rate raises both sides of the inequality, with ambiguous results. What is not ambiguous is that increasing the interest rate will, other things remaining the same, tend to slow average productivity growth.

5 Candidate forms for generating functions

The rules governing the aggregate generating function are: first, that it be first order in cost shares; second, that its matrix of second derivatives (the Jacobian) be positive-definite or positive semi-definite; and third, that if the cost share of input i goes to one, then entries in the Jacobian matrix with index i go to zero. The first two conditions are local, in that they apply in the vicinity of any cost share; the last is global, in that it applies only at extreme values of cost shares.

When constructing generating functions it is important to not impose the fact that cost shares sum to one until after having taken all derivatives. Otherwise, the rules above will change (and become more complicated). The constraint on cost shares should be imposed after computing productivity growth rates by taking derivatives.

The author has proposed possible forms for generating functions in prior papers. They have all assumed a zero threshold: $c = 0$. First, a cost share-induced technological change model was introduced in Kemp-Benedict (2020) that can be derived from a generating function of the form

$$\Phi(\pi, \omega) = A\pi + B\omega + C\omega \ln \frac{\omega}{\pi}. \tag{63}$$

The corresponding growth rates are

$$\hat{\kappa} = \frac{\partial \Phi}{\partial \pi} = A - C\frac{\omega}{\pi}, \tag{64a}$$

$$\hat{\lambda} = \frac{\partial \Phi}{\partial \omega} = (B + C) + C \ln \frac{\omega}{\pi}, \tag{64b}$$

while the Jacobian is

$$J = C \begin{pmatrix} \omega/\pi^2 & -1/\pi \\ -1/\pi & 1/\omega \end{pmatrix}. \tag{65}$$

The Jacobian matrix can be shown to be positive semi-definite, with a null vector equal to the cost shares. It does not satisfy the global condition of Jacobian matrix elements going to zero when cost shares go to one, but can be used as a local approximation. This generating function is most useful in a model with a target rate of return, because in that case the equation of motion for the profit share is linear,

$$\dot{\pi} = -\pi \left(A - C\frac{\omega}{\pi} \right) = C\omega - A\pi = C - (A + C)\pi. \tag{66}$$

At the equilibrium, $\pi^* = C/(A + C)$. Given the profit share, labor productivity growth can be calculated using Eq. (64b).

It is difficult to generalize Eq. (63) to more than two inputs. For multiple inputs, Kemp-Benedict (2022) introduced a generating function that can be written

$$\Phi(\boldsymbol{\sigma}) = \boldsymbol{A}'\boldsymbol{\sigma} + \left(\sum_{i=0}^{n} B_i \sigma_i^k \right)^{\frac{1}{k}}. \tag{67}$$

This satisfies all of the conditions, both global and local, if $k \geq 1$.

Neither of the function forms above is easily extended to the case of a nonzero threshold with $c \neq 0$. Furthermore, they are clearly *ad hoc*. That is not necessarily a problem for practical applications. Many functional forms can behave similarly within a modest range of values, and the criteria of analytical tractability and modest number of free parameters help to choose between competing options. The examples above may fit those criteria, and if they perform well when fitting the model, then they may be used as long as variable values do not stray too far from the range of values used in calibration. However, the growing number of assumptions in the passage from two inputs to many, and from a zero threshold to a nonzero threshold, urges a more systematic approach.

The approach taken in this paper is to take the expression for the generating function that applies to a production unit, Eq. (3), and use it to construct candidate functional forms for aggregate generating functions. In general, the aggregate generating function, which from Eq. (9) is a weighted sum of production unit-level generating functions, will not take the same form as that of an individual production unit. Explicit aggregation may be possible, but that is a separate and challenging topic that will be taken up in future work. Instead, the firm-level expression for the generating function is used to construct candidate aggregate generating functions that are guaranteed to have the required properties.

5.1 Normally distributed potential productivity growth

Appendix A provides an alternative expression for production unit-level generating functions that is fully equivalent to Eq. (3). It is written in terms of the characteristic function of the distribution $f(\hat{\boldsymbol{\nu}})$ of productivity growth rates for fit, but not necessarily profitable, innovations. The final expression, Eq. (102), is reproduced here:

$$\Phi(\boldsymbol{\sigma}; c) = \left(F(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) + \frac{1}{2} - F(0) \right) \left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k} e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu} - c)} \varphi'_{f0}(k\boldsymbol{\sigma}). \quad (68)$$

In this expression, $\varphi_{f0}(k\boldsymbol{\sigma})$ is the characteristic function of $f(\hat{\boldsymbol{\nu}})$ with zero mean evaluated at $k\boldsymbol{\sigma}$. The function $F(x)$ is the cumulative distribution of a univariate distribution constructed from the characteristic function of $f(\hat{\boldsymbol{\nu}})$ evaluated at $k\boldsymbol{\sigma}$.

In this paper the expression above is applied to a normal distribution of potential productivity growth rates. While the generating function applies at the level of a production unit, the goal will be adopted as a candidate generating function to be used for aggregate analysis. It will be tested for suitability in an empirical analysis.

The characteristic function of the multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega}$ is a standard result, and is given by

$$\varphi_f(k\boldsymbol{\sigma}) = e^{-\frac{1}{2}\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}k^2 + ik\boldsymbol{\sigma}'\boldsymbol{\mu}}. \quad (69)$$

The expression in Appendix A requires the first derivative with the respect to k of the distribution with zero mean. That is equal to

$$\varphi'_{f0}(k\boldsymbol{\sigma}) = -\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma} k e^{-\frac{1}{2}\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma} k^2}. \tag{70}$$

Otherwise, it needs the cumulative normal distribution $N(x)$. The integral in which $\varphi'_{f0}(k\boldsymbol{\sigma})$ appears can be seen to be the inverse Fourier transform of the characteristic function, which is simply the normal distribution itself. The result is

$$\Phi(\boldsymbol{\sigma}; c) = \sqrt{\frac{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}{2\pi}} e^{-(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)^2/2\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}} + N\left(\frac{\boldsymbol{\sigma}'\boldsymbol{\mu} - c}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}}\right)\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right). \tag{71}$$

This is the generating function when the distribution of productivity growth rates is normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega}$.

As noted above, strictly speaking this applies to a production unit. However, in this paper we are using the formula for the generating function for a production unit as a candidate functional form for an aggregate model.

5.1.1 Productivity growth rates

Following the development in Kemp-Benedict (2022), and as demonstrated in Eq. (4) in the body of the paper, expressions for average productivity growth rates over profitable technologies with improved fit are found by taking partial derivatives of $\Phi(\boldsymbol{\sigma}; c)$ with respect to cost shares. The exponential in the first term of Eq. (71) and the cumulative normal distribution in the second term potentially complicate any expression for average productivity growth rates. However, their derivatives conveniently cancel, as we now show. First, write

$$x \equiv \frac{\boldsymbol{\sigma}'\boldsymbol{\mu} - c}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}}. \tag{72}$$

Next, note that a normal probability distribution function with respect to x, $n(x)$, appears in the first term in Eq. (71),

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{73}$$

With these definitions. Eq. (71) can be written

$$\Phi(\boldsymbol{\sigma}; c) = n(x)\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}} + N(x)\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right). \tag{74}$$

Taking derivatives with respect to the ith cost share σ_i gives the average productivity growth rate for profitable innovations with respect to the ith input, $\hat{\nu}_i$,

$$\hat{\nu}_i = \sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}\, n'(x)\frac{\partial x}{\partial \sigma_i} + \left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) N'(x)\frac{\partial x}{\partial \sigma_i} + n(x)\frac{(\boldsymbol{\Omega}\boldsymbol{\sigma})_i}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}} + N(x)\mu_i. \tag{75}$$

The derivative of the cumulative normal distribution is the density, so $N'(x) = n(x)$. Furthermore, the derivative of the distribution function is $n'(x) = -xn(x)$. Substitution gives

$$\hat{\nu}_i = \left(-x\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}} + \boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) n(x)\frac{\partial x}{\partial \sigma_i} + n(x)\frac{(\boldsymbol{\Omega}\boldsymbol{\sigma})_i}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}} + N(x)\mu_i. \tag{76}$$

From the definition of x in Eq. (72), the expression in parentheses vanishes, so the partial derivative $\partial x/\partial \sigma_i$ need never be evaluated. Writing the final result in vector form,

$$\hat{\boldsymbol{\nu}} = n(x)\frac{\boldsymbol{\Omega}\boldsymbol{\sigma}}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}} + N(x)\boldsymbol{\mu}, \quad \text{where} \quad x = \frac{\boldsymbol{\sigma}'\boldsymbol{\mu} - c}{\sqrt{\boldsymbol{\sigma}'\boldsymbol{\Omega}\boldsymbol{\sigma}}}. \tag{77}$$

This is the expression for average productivity growth rates under cost share-induced technological change at the level of a production unit from innovations that are both fit *and* profitable when the distribution of productivity growth rates for fit innovations is normally distributed with a mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega}$.

Because the aggregate generating function has the same properties as the generating function for a production unit, this expression can also be applied as a candidate generating function at the aggregate level. That is the procedure followed in this paper.

While Eq. (77) can be used by itself, it does not distinguish between embodied and disembodied technological change. That is, it combines changes in productivity due to the addition of new capital and the retirement of old, on one hand, and improvements with existing capital, on the other. That is treated next.

5.1.2 Combining embodied and disembodied technological change

We now return to the expression for productivity change with both embodied and disembodied technological change given in Eq. (51). If all growth rate variables (with a hat) and proportional change variables (with a check) are taken to be normally distributed, the analysis can proceed reasonably rapidly from this point. Sums of normally distributed variables are themselves normally distributed, with a mean and covariance matrix equal to the sum of the means and the covariance matrices of the component variables. For this reason, if $\check{\kappa}_m$ and $\check{q}_{i,m}$ are normally distributed, then so is their difference $\check{\nu}_{i,m} \equiv \check{\kappa}_m - \check{q}_{i,m}$. The same holds true for the sum in Eq. (51). Furthermore, if a normally distributed variable is multiplied by a scalar – in this case, g – then the resulting product is also normally distributed, with a mean equal to the scalar multiplied by the mean of the original variable and the covariance matrix equal to the square of the scalar multiplied by the original covariance matrix.

Putting all of that together, the means and covariance matrices can be written

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\text{dis}} + g\boldsymbol{\mu}_m, \tag{78a}$$

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\text{dis}} + g^2\boldsymbol{\Omega}_m. \tag{78b}$$

Assuming the selection frontier in Eq. (61), there is a nonzero threshold. The expression for the average productivity growth rate is therefore,

$$\hat{\boldsymbol{\nu}} = n(x)\frac{\left(\boldsymbol{\Omega}_{\text{dis}} + g^2\boldsymbol{\Omega}_m\right)\boldsymbol{\sigma}}{\sqrt{\boldsymbol{\sigma}'\left(\boldsymbol{\Omega}_{\text{dis}} + g^2\boldsymbol{\Omega}_m\right)\boldsymbol{\sigma}}} + N(x)\left(\boldsymbol{\mu}_{\text{dis}} + g\boldsymbol{\mu}_m\right), \tag{79}$$

where

$$x = \frac{\boldsymbol{\sigma}'\left(\boldsymbol{\mu}_{\text{dis}} + g\boldsymbol{\mu}_m\right) - (r/\kappa - \pi)\,g}{\sqrt{\boldsymbol{\sigma}'\left(\boldsymbol{\Omega}_{\text{dis}} + g^2\boldsymbol{\Omega}_m\right)\boldsymbol{\sigma}}}. \tag{80}$$

We note that linearizing these expressions in g would yield a reduced-form model with a Kaldor-Verdoorn type term. However, this paper adopts the full nonlinear expression. While a linearized model – for example, a linear model in $\boldsymbol{\sigma}$, g, and r – might provide a good fit to data, the coefficients would be challenging to interpret, whereas those of the nonlinear model are well defined. Furthermore, in the nonlinear model parameters jointly impact both labor and capital productivity because they are linked through the selection frontier. In a linearized model the underlying connection is lost.

Eqs. (79) and (80) express the model derived in this paper in its final form. The model captures the influence of the pace of investment and income distribution on both embodied and disembodied technological change. Following the evidence presented by Shaikh (2016) and Kemp-Benedict (2023), profitability is assessed through the incremental profit rate.

6 Calibrating to historical data

A candidate function form for a generating function to be applied at aggregate level must be tested for its ability to reproduce observations. The model defined by Eqs. (79) and (80) was calibrated to three countries – India, China, and the United States – using Bayesian analytical techniques. The Penn World Table (PWT) (Feenstra et al., 2015) was used as the main data source. As described below, for a two-input model (labor and capital), the PWT provides nearly all of the needed data. The target incremental profit rate r is specified as in Eq. (62).

6.1 Data

The main source of data is the Penn World Table version 10.01 (Feenstra et al., 2015), which extends at its maximum from 1951 to 2019. For many countries the time series begins much later. The relevant variables are: the labor share `labsh`; employment `empl`; the real value of the capital stock following the perpetual inventory method `rnna`; the depreciation rate `delta`; and real GDP from the national accounts `rgdpna`.

The PWT capital stock measure `rnna` was used in preference to the capital services measure `rkna` because the latter applies a different measure of value to that used in this paper. As explained in Section 4, in this paper the value of capital is determined by the prices of capital goods and services, consistent with `rnna`, whereas the value of capital services `rkna` is determined by a presumed rental rate of capital (Inklaar et al., 2019). In the model presented here, the rental rate of capital appears in the target incremental profit rate.

Real interest rates were obtained from the World Bank World Development Indicators (WDI, variable `FR.INR.RINR`). Coverage for this variable is sparse, and constitutes the main constraint on data availability. The other significant constraint is the `labsh` variable from the PWT. To generate as complete a data set as possible, the PWT records a constant labor share in years prior to and after the span of actual data. Because cost shares are crucial explanatory variables in the model developed in this paper, only years with data for the labor share are included.[7]

Preparation of country data sets followed this procedure: 1) extract data from PWT 10.01, find the earliest and latest years at which `labsh` begins to change, and truncate the series by setting them to the start and ending years; 2) set $\omega =$ `labsh` and $\pi = 1 - \omega$; 3) calculate the growth rate of capital productivity $\hat{\kappa}$ from the series `rgdpna/rnna` and the growth rate of labor productivity $\hat{\lambda}$ from the series `rgdpna/empl`; 4) estimate the investment rate g as the sum of `delta` and the year-on-year growth rate of `rnna`; 5) set the interest rate i to the real interest rate from WDI. The variables ω, π, g, and i are model inputs. The variables $\hat{\kappa}$ and $\hat{\lambda}$ are observations.

For India, data are available from 1978 through 2017. However, the deep recession in 1979 was flagged as seriously problematic in a leave-one-out cross-validation (LOO) robustness check, so the years 1978 and 1979 were omitted and the data used for calibration extends from 1980 to 2017. For China, data are available from 1992 to 2016. The 1997 recession was flagged as "slightly high" in an LOO check, but was retained in the calibration data set. For United States, data are available from 1961 through 2019.[8]

6.2 Implementation in Stan

The Bayesian calibration procedure was carried out with the R interface to Stan using `rstan` ver. 2.26.22. The Stan code is provided in Appendix B. The model structure is hierarchical, in that there are sub-groups: disembodied and embodied technological change. However, with respect to the data, this structure is latent; it is suggested by the underlying NK model but not reflected in macro-level observations. The means and covariance matrices entering Eqs. (79) and (80) characterize an unobserved probability of discovering fit candidate innovations, while observed technological change is

[7] See (Feenstra et al., 2015, AppendixB,p. 25). Additionally, if there were missing data within the series of estimated labor shares, rather than prior to the first observation or following the final observation, Feenstra et al. applied a linear interpolation. However, even though these are not observations, the linearly interpolated segments are difficult to detect, so they were included in the data set for this paper.

[8] A standard guideline is at least 10 observations per parameter. However, that guideline derives from the asymptotic behavior of linear regression models, and is not appropriate for a Bayesian analysis of a nonlinear model. Nevertheless, while keeping that caveat in mind, the values are reported here for those concerned about over-fitting: there were 6.3 observations per parameter for India, 4.2 for China, and 9.8 for the United States.

of innovations that are both fit *and* profitable, and the effects of embodied and disembodied technological change are entangled in the observed data. As discussed below, this creates some challenges to parameter identification.

Before discussing identification, we note that empirical estimates of productivity growth include changes in capacity utilization at fixed technology, whereas the theoretical model simulates changes in technology at normal utilization. The cost share-induced technological change mechanism tends to drive capital and labor productivity growth in opposite directions, although not exclusively. In contrast, changes in utilization drive estimated productivity growth in the same direction. To proxy changes in utilization, a variable γ was computed as the growth rate of the ratio of GDP to the trend in GDP as estimated with a two-sided Hodrick-Prescott filter.[9] The trend was calculated using the R package `hpfilter` ver. 1.0.2 with a parameter $\lambda = 6.25$ appropriate for annual data (Ravn and Uhlig, 2002). The variable γ is multiplied by a vector of input-specific parameters $\boldsymbol{\alpha}$.

As noted above, problems arise with identification and scale when the model is presented as in Eqs. (79) and (80). Typically, g varies by a moderate amount around an average that is on the order of 0.1; to first order, then, changes in $\boldsymbol{\mu}_{\text{dis}}$ and $\boldsymbol{\mu}_m$ can offset one another, leading to a potential degeneracy. A similar degeneracy can apply to $\boldsymbol{\Omega}_{\text{dis}}$ and $\boldsymbol{\Omega}_m$. Furthermore, while $\boldsymbol{\mu}_{\text{dis}}$ and $g\boldsymbol{\mu}_m$ are expected to be of similar orders of magnitude, on the order of 1.0, that means that $\boldsymbol{\mu}_m$, when not multiplied by g, will be on the order of 10. For $\boldsymbol{\Omega}_m$ the factor is $1/g^2$, or about 100. Such large deviations in the scale of the parameters can cause problems with calibration.

Both the scale mismatch and – to some extent – the potential degeneracy were addressed by reparameterizing the model. First, the normalized deviation of g from its median value $\overline{g}$ was calculated as $u = g/\overline{g} - 1$.[10] Then, with this definition, new vectors $\boldsymbol{\mu}_a$ and $\boldsymbol{\mu}_b$ and matrices $\boldsymbol{\Omega}_a$ and $\boldsymbol{\Omega}_b$ were defined as

$$\boldsymbol{\mu}_a = \left(\boldsymbol{\mu}_{\text{dis}} + \boldsymbol{\mu}_m \overline{g}\right), \tag{81a}$$

$$\boldsymbol{\mu}_b = \boldsymbol{\mu}_m \overline{g}, \tag{81b}$$

$$\boldsymbol{\Omega}_a = \boldsymbol{\Omega}_{\text{dis}}, \tag{81c}$$

$$\boldsymbol{\Omega}_b = \boldsymbol{\Omega}_m \overline{g}^2. \tag{81d}$$

Note that Eq. (81c) is simply a relabeling of $\boldsymbol{\Omega}_{\text{dis}}$. This definition was assumed in favor of one parallel to Eq. (81a) because $\boldsymbol{\Omega}_m(g^2 - \overline{g}^2)$ is not positive definite when $g < \overline{g}$.

9 Hamilton (2017) offered a trenchant critique of the Hodrick-Prescott filter. While his critique has been disputed (Franke et al., 2024), neither the critique nor the defense is central to the analysis in this paper. The choice was made on practical grounds. Labor and capital productivity growth rates estimated from data deviate due to fluctuations that impact both in the same direction and that are not part of the model. Because filtering the time series for productivity growth rates could distort them and give a false signal, as a compromise a proxy for the common deviations was generated by applying a filter to the time series for GDP, which is itself neither an explanatory nor a dependent variable in the model.

10 The median was used because deviations in accumulation are typically skewed, with larger negative deviations than positive ones.

The transformations in Eqs.(81) can be inverted after the model is fit, to recover $\boldsymbol{\mu}_{\text{dis}}$, $\boldsymbol{\mu}_m$, $\boldsymbol{\Omega}_{\text{dis}}$, and $\boldsymbol{\Omega}_m$. The fitted model can be expressed in terms of transformed variables as

$$\hat{\boldsymbol{\nu}} = n(x) \frac{\left[\boldsymbol{\Omega}_a + \boldsymbol{\Omega}_b (1+u)^2\right] \boldsymbol{\sigma}}{\sqrt{\boldsymbol{\sigma}' \left[\boldsymbol{\Omega}_a + \boldsymbol{\Omega}_b (1+u)^2\right] \boldsymbol{\sigma}}} + N(x) \left(\boldsymbol{\mu}_a + \boldsymbol{\mu}_b u\right) + \boldsymbol{\alpha}\gamma, \tag{82}$$

where

$$x = \frac{\boldsymbol{\sigma}' \left(\boldsymbol{\mu}_a + \boldsymbol{\mu}_b u\right) - (r/\kappa - \pi)\, g}{\sqrt{\boldsymbol{\sigma}' \left[\boldsymbol{\Omega}_a + \boldsymbol{\Omega}_b (1+u)^2\right] \boldsymbol{\sigma}}}. \tag{83}$$

Following the recommendation in the Stan User's Guide (Stan Development Team, 2022, Sec. 1.13), the covariance matrices $\boldsymbol{\Omega}_k$, where $k \in \{a, b\}$, were parameterized as scale vectors $\boldsymbol{\tau}_k$ multiplied by correlation matrices $\boldsymbol{\Sigma}_k$:

$$\boldsymbol{\Omega}_k = \tilde{\boldsymbol{\tau}}_k \boldsymbol{\Sigma}_k \tilde{\boldsymbol{\tau}}_k. \tag{84}$$

In this expression, a tilde indicates a diagonal matrix constructed from a vector.

The reparameterization renders $\boldsymbol{\mu}_a$, $\boldsymbol{\mu}_b$, $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ of similar orders of magnitude, each of them $\mathcal{O}(1)$. Moreover, the $\boldsymbol{\tau}_k$ must be positive, while the $\boldsymbol{\mu}_k$ can be either positive or negative. These expectations are reflected in the priors, shown below in Eqs. (85). The $\boldsymbol{\alpha}$ coefficients that multiply the GDP growth rate deviation γ are assigned a moderately informative prior limited to the unit interval with a peak at one-half. For other parameters, prior predictive checks[11] were used to ensure that the priors generated distributions of estimated productivity growth rates consistent with typical growth rates while allowing for possible extreme values. The priors were further adjusted to eliminate divergent transitions after Stan's warmup period.[12]

When developing the prior distributions, the following criteria were applied: (1) No bias towards embodied vs. disembodied technological change; (2) No bias towards labor vs. capital productivity growth; (3) Bias *against* cost share-based explanations of productivity growth. Criterion (1) was selected because it is unclear whether there is a systematic bias in technological change towards embodied or disembodied causal factors. Criterion (2) was selected because the Duménil and Lévy model (Dumènil and Lèvy, 1992; Duménil and Lévy, 2010) is intended to explain biased technological change even when the underlying distribution is symmetric, so a bias should not be imposed. Criterion (3) is necessary because of residual identification problems between $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$. Given the need to bias in one direction or the other, this choice was motivated by the desire to discover through the data, rather than impose through the prior, whether cost shares are significant in explaining productivity change.

Together, criteria (1) and (2) suggest that the priors for $\boldsymbol{\Sigma}_k$, $\boldsymbol{\mu}_k$, and $\boldsymbol{\tau}_k$ should be the same for $k \in \{a, b\}$ and for both labor and capital. Criterion (3) is implemented by

[11] Estimates of the dependent variable $\hat{\boldsymbol{\nu}}$ generated by drawing parameter values from the prior distribution.

[12] For more information about the "divergent transitions after warmup" warning issued by Stan's No U-Turn Sampler (NUTS), see the Stan web site (https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup).

choosing a comparatively tight prior for the $\boldsymbol{\tau}_k$ parameter with a mean value below that of the $\boldsymbol{\mu}_k$ parameter in absolute value (0.5 compared to 1.5). As a result of these criteria, any bias in the posterior towards embodied vs. disembodied change, between labor vs. capital, or towards cost share-induced technological change arises from the data rather than the prior. As a check on the differences, prior and posterior distributions for the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ parameters are provided in Appendix C.

Taking all of the above into consideration, the selected priors are:

$$\boldsymbol{\Sigma}_k \sim \text{LKJCorr}(3), \tag{85a}$$
$$\boldsymbol{\tau}_k \sim \text{Gamma}(25, 50), \tag{85b}$$
$$\boldsymbol{\mu}_k \sim \text{Normal}(-1.5, 1.0), \tag{85c}$$
$$\boldsymbol{\alpha} \sim \text{Beta}(2, 2). \tag{85d}$$

The LKJCorr distribution in Eq. (85a) is a distribution of correlation matrices provided by Lewandowski et al. (2009). The argument to the distribution must be greater than one, with smaller values leading to broader distributions; the value of 3 results in a moderately informative prior for the off-diagonal terms in the correlation matrices.

The leverage parameter ℓ is expected to be $\mathcal{O}(1)$. It was initially assigned a prior of $\ell \sim \text{Gamma}(2, 2)$. However, experience with the calibration process showed that this parameter was very weakly constrained. In calibration runs, it was fixed at $\ell = 1$.

Deviations were assumed to be normally distributed with mean zero and standard deviations σ_k. The standard deviations were assigned half-Cauchy priors, $\sigma_k \sim \text{Half-Cauchy}(0, 0.1)$.

To be clear, the normally distributed deviations are separate from the normal distribution functions that appear in the model itself in Eqs. (82) and (83). In the model, the cumulative normal distribution function and the normal density function enter as nonlinear functions of the model parameters; in the calibration, the normal distribution characterizes deviations between observations and fitted values.

Table 1 Parameter estimates for India

	Mean	s.e.	s.d.	2.50%	25%	50%	75%	97.50%	n_{eff}	$\hat{R}$
$\mu_{\text{dis},1}$	0.5	0.01	0.4	-0.1	0.2	0.4	0.7	1.4	777	1.00
$\mu_{\text{dis},2}$	-1.2	0.01	0.3	-1.8	-1.3	-1.1	-1.0	-0.7	1572	1.00
$\mu_{m,1}$	-7.0	0.06	2.0	-11.6	-8.1	-6.8	-5.7	-3.8	1033	1.00
$\mu_{m,2}$	2.4	0.06	2.3	-2.6	0.9	2.5	3.9	6.8	1731	1.00
$\Omega_{\text{dis},1,1}$	0.2	0.00	0.1	0.1	0.1	0.2	0.2	0.3	2987	1.00
$\Omega_{\text{dis},2,2}$	0.3	0.00	0.1	0.1	0.2	0.3	0.4	0.6	2180	1.00
$\Omega_{\text{dis},1,2}$	0.0	0.00	0.1	-0.2	-0.1	0.0	0.0	0.1	1098	1.00
$\Omega_{m,1,1}$	16.8	0.15	6.9	7.2	11.8	15.5	20.1	33.7	2189	1.00
$\Omega_{m,2,2}$	46.7	0.54	18.7	17.4	32.7	44.2	57.9	89.1	1217	1.00
$\Omega_{m,1,2}$	-3.5	0.23	8.5	-22.0	-8.5	-3.1	2.0	12.3	1350	1.00

Table 2 Parameter estimates for China

	Mean	s.e.	s.d.	2.50%	25%	50%	75%	97.50%	n_{eff}	$\hat{R}$
$\mu_{\text{dis},1}$	-0.4	0.01	0.5	-1.4	-0.7	-0.4	-0.1	0.7	3447	1.00
$\mu_{\text{dis},2}$	-1.0	0.01	0.3	-1.7	-1.2	-1.0	-0.7	-0.4	2723	1.00
$\mu_{m,1}$	-1.8	0.05	2.9	-7.8	-3.6	-1.7	0.1	3.8	3527	1.00
$\mu_{m,2}$	2.3	0.04	1.8	-1.4	1.2	2.4	3.5	5.9	2586	1.00
$\Omega_{\text{dis},1,1}$	0.2	0.00	0.1	0.1	0.2	0.2	0.3	0.4	3082	1.00
$\Omega_{\text{dis},2,2}$	0.3	0.00	0.1	0.1	0.2	0.3	0.4	0.6	2636	1.00
$\Omega_{\text{dis},1,2}$	0.0	0.00	0.1	-0.2	-0.1	0.0	0.1	0.2	2950	1.00
$\Omega_{m,1,1}$	6.7	0.05	2.7	2.9	4.8	6.3	8.1	13.0	2880	1.00
$\Omega_{m,2,2}$	12.1	0.11	4.8	4.8	8.7	11.4	14.9	23.6	1782	1.00
$\Omega_{m,1,2}$	0.5	0.07	3.4	-6.1	-1.8	0.6	2.8	7.1	2193	1.00

6.3 Results

The model was fitted to data for India, China, and the United States. This choice of countries offers a contrast between large continental economies classified as lower middle income, upper middle income, and high income by the World Bank.

The model was fitted with Stan using the No U-Turn Sampler (NUTS) (Hoffman and Gelman, 2014). The sampler was initialized with a specified seed to ensure reproducibility of the results. (Calibrations were also run with a random seed to check that the results did not depend significantly on the choice of seed.) Samples were drawn from four chains, each with 2000 iterations, of which the warm-up took half. The samples were not thinned, so the total number of post–warm-up draws was 4000.

The mean values for the posterior distribution are shown in Table 1 for India, Table 2 for China, and Table 3 for the United States, together with distributional information: the standard error of the mean s.e.; the standard deviation s.d.; and percentiles of the

Table 3 Parameter estimates for the United States

	Mean	s.e.	s.d.	2.50%	25%	50%	75%	97.50%	n_{eff}	$\hat{R}$
$\mu_{\text{dis},1}$	-0.3	0.01	0.3	-0.8	-0.5	-0.3	-0.1	0.3	1901	1.00
$\mu_{\text{dis},2}$	-0.4	0.01	0.4	-1.2	-0.6	-0.4	-0.1	0.3	1913	1.00
$\mu_{m,1}$	-1.3	0.11	4.6	-9.1	-4.4	-1.7	1.5	8.7	1620	1.00
$\mu_{m,2}$	-9.5	0.12	5.8	-21.6	-13.3	-9.2	-5.6	1.3	2129	1.00
$\Omega_{\text{dis},1,1}$	0.2	0.00	0.1	0.1	0.2	0.2	0.3	0.4	2994	1.00
$\Omega_{\text{dis},2,2}$	0.3	0.00	0.1	0.1	0.2	0.3	0.4	0.6	2155	1.00
$\Omega_{\text{dis},1,2}$	0.0	0.00	0.1	-0.2	-0.1	0.0	0.0	0.2	2214	1.00
$\Omega_{m,1,1}$	59.2	0.37	21.7	25.8	43.8	56.2	71.5	109.1	3358	1.00
$\Omega_{m,2,2}$	86.3	0.68	32.2	34.0	62.9	82.5	105.6	158.6	2260	1.00
$\Omega_{m,1,2}$	-2.8	0.54	23.7	-49.3	-18.7	-3.3	13.4	43.5	1894	1.00

distribution. The final two columns report Stan-specific statistics: a "crude" measure of sample size n_{eff} that adjusts for any autocorrelation induced by the sampling process; and an indicator $\hat{R}$ that approaches $\hat{R} = 1$ at convergence. For the covariance matrices, the diagonal elements are shown first and then the off-diagonal element.

As noted earlier, the expected orders of magnitudes of parameters are $\boldsymbol{\mu}_{\text{dis}} \sim \mathcal{O}(1)$, $\boldsymbol{\mu}_m \sim \mathcal{O}(10)$, $\boldsymbol{\Omega}_{\text{dis}} \sim \mathcal{O}(1)$, $\boldsymbol{\Omega}_m \sim \mathcal{O}(100)$. These expectations are partly enforced in the Stan implementation of the model, depending on the strength of the prior; they are also borne out in the table. Furthermore, entries for $\boldsymbol{\mu}_k$ can be either positive or negative, as can the off-diagonal elements of the covariance matrices $\boldsymbol{\Omega}_k$. The diagonal elements of the covariance matrices must be positive. These conditions are enforced in the Stan model implementation, and therefore also observed in the results.

Posterior predictive checks – that is, plots of simulated values for $\hat{\boldsymbol{\nu}}$ generated by Stan after the calibration – are shown in Fig. 1. The graph for each country shows capital productivity growth in the upper panel and labor productivity growth in the lower panel.

6.4 Discussion

The calibration results, together with the simulations, show that the model derived in this paper and presented in Eqs. (79) and (80) can represent macroeconomic time trends in countries at very different levels of economic development with some fidelity. In particular, in Fig. 1, despite the very different productivity growth rate trajectories in the three countries, the model was able to fit each of them reasonably well, particularly in India and the United States. The fitted model would not have anticipated the outstanding labor productivity growth in China in the first decade of the 21st Century, but that was also the decade following China's accession to the WTO, which had numerous impacts on innovation. India and the United States also experienced structural shifts and shocks, such as the oil shocks of the 1970s. However, due to data limitations, no attempt was made to model structural breaks. Data limitations also prevented out-of-sample testing.

Some patterns are notable in the graphs. For India, the labor and capital productivity growth rate trends in both the model and the data are roughly mirror images. In the model, their trajectories were driven by a generally increasing investment rate g and profit share π over the period. In contrast, in China and the United States, productivity growth rates feature more subtle dependence on the investment rate, cost shares, and interest rate.

The parameter estimates in Tables 1, 2, and 3 likewise show some regularities, although with only three countries as a basis of comparison, any observations are tentative. To interpret labor and capital productivity growth jointly, it can be useful to calculate their difference $\mu_{k,2} - \mu_{k,1}$ for $k \in \{\text{dis}, m\}$. This indicator corresponds to the mean value of the growth rate of capital per worker for fit, but not necessarily profitable, innovations in the candidate generating function.

Two points are important to keep in mind when examining this indicator. First, strictly speaking, it is the difference of means of a distribution that would apply at the level of a production unit. Here it is applied at aggregate level, so the interpretation

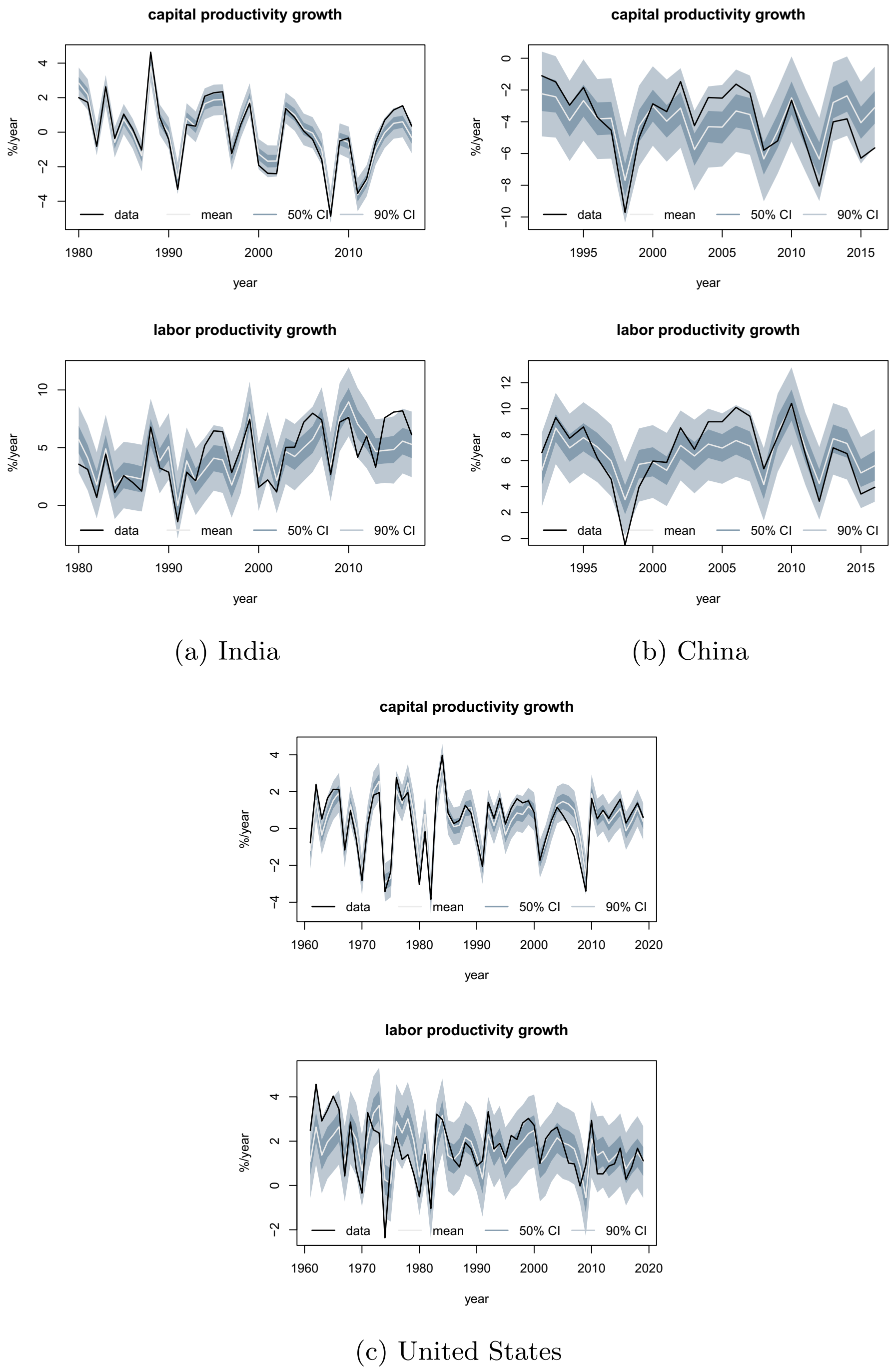

Fig. 1 Productivity growth rates: Data and simulation

is made on an "as-if" basis. That would be inappropriate if the indicator were used for further analysis, but it is used here strictly as a way to interpret results. Second, $\mu_{k,1}$ and $\mu_{k,2}$ are not the mean values of *realized* productivity growth rates; those are given by Eq. (79). Instead, they are model parameters that – if the model is correct – are the mean values for candidate innovations that have passed a fitness evaluation but have not been subjected to a capital budgeting assessment. In a mature economy in particular those values are expected to be negative: when the potential of a technology has been explored over many decades, most of the available options will be a step backward in terms of productivity.

If the difference $\mu_{k,2} - \mu_{k,1}$ is positive, then it implies capital deepening, while a negative value implies reverse capital deepening. Considering embodied technological change, the value of $\mu_{m,2} - \mu_{m,1}$ is 9.4 for India, 4.1 for China, and -8.2 for the United States. The implication is that the fit (but likely not profitable) embodied innovations accessible to India and China are capital deepening, while those accessible to the United States are reverse capital deepening. In contrast, the mean value for disembodied technological change is consistent with reverse capital deepening in India and China, and to negligible change in the capital-labor ratio in the US.

These results make sense; in a country with a high capital-labor ratio like the US, most of the alternatives for new technology will have lower capital-labor ratios, but they will not be profitable. Similarly, in a country with a comparatively low capital-labor ratio, capital-intensive technologies will be available, but introducing them may not translate into increased labor productivity. The challenge of identifying profitable technologies at the frontier can be offset by broadening search (Terjesen and Patel, 2017; Torres de Oliveira et al., 2022). This should be reflected in the variance, and it is notable that the entries for $\mathbf{\Omega}_m$ are substantially larger for the US (Table 3) than they are for India (Table 1) or China (Table 2). Regarding disembodied technological change, regardless of the country's income level, the most accessible "fit" modifications using existing capital will likely not meet the profitability criterion, a result consistent with reverse capital deepening (or no effect on capital deepening) at the mean for fit but unprofitable disembodied innovation.

In general, the prevailing technology, distribution, pace of investment, and borrowing costs combine to determine profitability. Insight can be gained by looking at the separate z-scores for disembodied and embodied technological change. These are calculated as

$$z_k \equiv \frac{\boldsymbol{\sigma}'\boldsymbol{\mu}_k}{\sqrt{\boldsymbol{\sigma}'\mathbf{\Omega}_k\boldsymbol{\sigma}}}. \tag{86}$$

Results are shown in Fig. 2. As can be seen from the figures, the estimated z-score for embodied technological change (the lower panel) is lower in the United States than in India or China. This is consistent with the US being on the technological frontier, as profitable technologies are less readily available. The z-score for disembodied technological change is higher in the US than in China and India, again consistent with the US being on the technological frontier; once a new technology is introduced, there is a learning period in which productivity gains are comparatively rapid (Grosse et al., 2015). In contrast, there are fewer opportunities for technological advance with mature technologies.

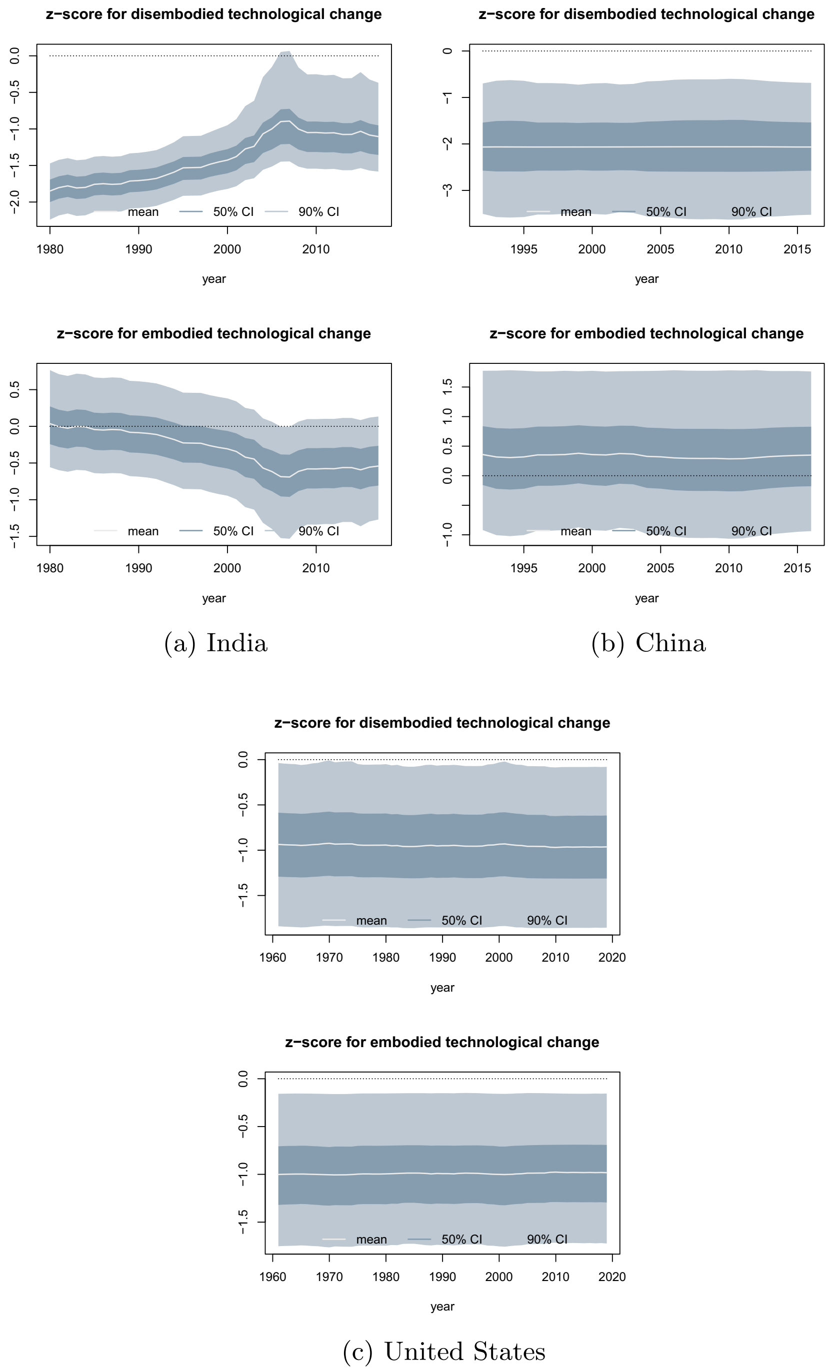

Fig. 2 Estimated z-scores

Notably, India's z-scores show opposite trends for disembodied and embodied technological change. The shifts coincide with a fall in the wage share, suggesting that, at the mean values of the distribution of fit innovations, lower wage costs tend to make embodied innovations less profitable (because they are capital deepening) and disembodied technological change more profitable (because it is reverse capital deepening). At the start of the period, India's z-scores resembled those of China, while by 2019 they had approached values typical of the US.

7 Discussion

Building on prior work of the author Kemp-Benedict (2019, 2022), which itself built on the work of Duménil and Lévy (1995, 2010), this paper offers an explicit analytical functional form for cost share-induced technological change. The functional form takes into account the observation of Shaikh (2016) that incremental rates of profit, rather than average rates of profit, are most relevant for firm finance. The proposed model was tested against empirical data in a Bayesian statistical framework and was shown to perform reasonably well, in that the model can reproduce patterns observed in the data with plausible estimates for the model parameters.

One implication of the model is that the rate of productivity growth, given by Eq. (79), depends on profitability through the variable x defined in Eq. (80). The ultimate reason for the dependence is the assumption that firms compare profitability of potential innovations to a target marginal profit rate, consistent with the observation of Shaikh (2016) and Kemp-Benedict (2023) that marginal profit rates follow similar patterns across sectors. Furthermore, in the empirical model, the target marginal profit rate is assumed to depend on the interest rate, following Lavoie's presentation of the finance frontier (Lavoie, 2022). The model therefore predicts that productivity should tend to move opposite to interest rates. This prediction was tested indirectly through the empirical analysis. To supplement that evidence, Fig. 3 directly compares the US prime rate against labor productivity growth measured as output per hour in the nonfarm business sector from 1960 to 1990. (Note that this is a different measure of labor productivity than used for model testing and shown in Fig. 1.) The selected period includes a time of extremely high interest rates, and during that time labor productivity indeed tended to decline when interest rates rose and *vice versa*.

When considering the interacting contributions of distribution, investment, and interest rate on productivity growth, any expression for cost share-induced technological change paper must be complemented by other dynamics. The model of cost share-induced technological change is intended as a component within larger models that include price and wage-setting procedures, determination of the interest rate, an investment function, and a specification of demand. Those additional dynamics will feed back upon productivity growth. Within such an expanded model, the productivity growth model presented in this paper can be used to endogenize technological change. This point was made in the paper through five comparatively simple examples. They showed that theoretically interesting results can emerge from the interaction of cost share-induced technological change with price and wage dynamics. Depending on price- and wage-setting behavior, the equilibria of example models were shown to

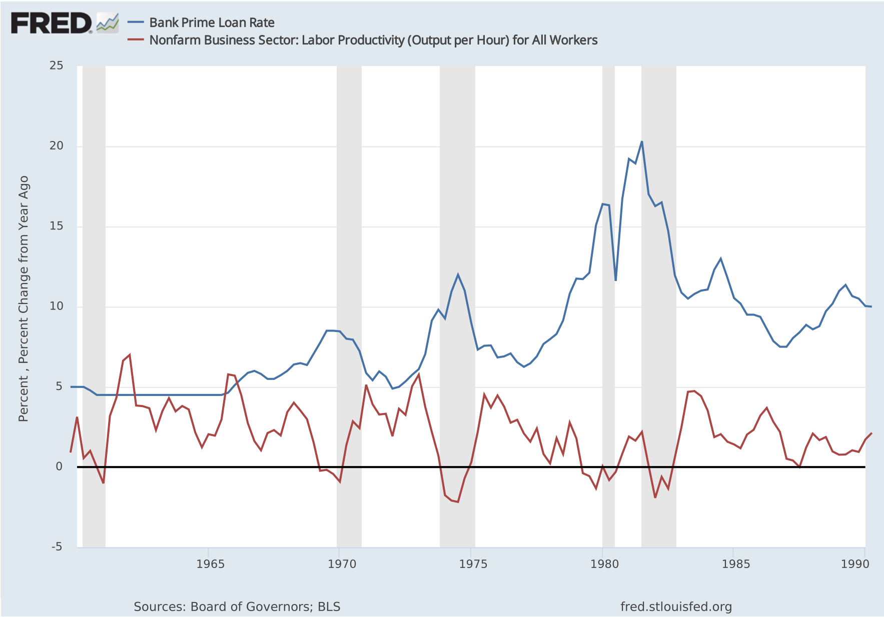

Fig. 3 Bank prime loan rate (*blue line*) and the year-on-year percent change in labor productivity (*red line*), quarterly data, from 1960 to 1990

produce Harrod-neutral technological change, convergence to Harrod's natural growth rate, and, in a multi-sector setting, stable relative prices.

Looking beyond the model in this paper, which assumed incremental technological change, productivity growth could be resuscitated through more radical change. That could be captured by "fat-tailed" distributions of potential productivity growth rather than the thin-tailed normal distribution. For the mathematical analysis in this paper to carry over, it would be necessary that those fat-tailed distributions maintain their form under linear combinations of independent randomly distributed variables, a property known as "stability" in statistics. Both stability and fat tails characterize every member of the family of Lévy-stable distributions except for the normal, which is stable but thin-tailed. Lévy-stable distributions are therefore candidates for a cost share-induced technological change model that features non-incremental improvements.

8 Conclusion

This paper expanded on the classical-evolutionary model of cost share-induced technological change introduced by Kemp-Benedict (2019, 2022), which itself built on the pioneering work of Dumènil and Lèvy (1992); Duménil and Lévy (2010). The model accounts for both embodied and disembodied technological change, and incorporates the finding from Shaikh (2016) that the relevant profitability criterion for investors is the incremental profit rate, rather than the average profit rate.

The model was calibrated to historical data for India, China, and the United States. These countries were chosen because they represent lower-middle, upper-middle, and high-income large economies. The data used for calibration are available for many countries, opening the possibility for further exploration. The fitted parameters are amenable to reasonable economic interpretation, while outputs of simulations with the fitted model were shown to have some diagnostic value.

Appendix A Generating functions from characteristic functions

This subsection demonstrates that the generating function can be written in terms of the characteristic function of the distribution $f(\hat{\boldsymbol{\nu}})$.

The derivation starts by writing the Heaviside function in Eq. (3) in terms of its Fourier transform. This is a standard result, and it is given by[13]

$$h(x) = \int_{-\infty}^{\infty} dk\, \frac{1}{2}\left(\delta(k) - \frac{i}{\pi k}\right) e^{ikx} = \frac{1}{2} - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k}\, e^{ikx}. \tag{87}$$

Substituting Eq. (87) into the expression for the generating function in Eq. (3) gives

$$\Phi(\boldsymbol{\sigma}; c) = \frac{1}{2}\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k} \int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}})\, \left(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c\right)\, e^{ik(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)}, \tag{88}$$

where $\boldsymbol{\mu}$ is the mean value of $\hat{\boldsymbol{\nu}}$ over the distribution of fit candidate innovations $f(\hat{\boldsymbol{\nu}})$. The final integral in this expression can be written

$$\begin{aligned} \int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}})\, \left(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c\right)\, e^{ik(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)} &= -i\frac{d}{dk}\int d\hat{\boldsymbol{\nu}}\; f(\hat{\boldsymbol{\nu}}) e^{ik(\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}} - c)} \\ &= -i\frac{d}{dk} e^{-ikc}\varphi_f(k\boldsymbol{\sigma}), \end{aligned} \tag{89}$$

where $\varphi_f(\cdot)$ is the (multivariate) characteristic function of the distribution $f(\hat{\boldsymbol{\nu}})$. Substituting this expression into Eq. (88), the result is

$$\Phi(\boldsymbol{\sigma}; c) = \frac{1}{2}\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k}\frac{d}{dk} e^{-ikc}\varphi_f(k\boldsymbol{\sigma}). \tag{90}$$

This is a general expression, true for any distribution $f(\hat{\boldsymbol{\nu}})$, and is fully equivalent to Eq. (3) in the body of the paper.

This formulation of the generating function can be developed further using some properties of generating functions. First, it is possible to write

$$\varphi_f(k\boldsymbol{\sigma}) = e^{ik\boldsymbol{\sigma}'\boldsymbol{\mu}}\varphi_{f0}(k\boldsymbol{\sigma}), \tag{91}$$

[13] See https://mathworld.wolfram.com/FourierTransformHeavisideStepFunction.htm at Wolfram MathWorld.

where $\varphi_{f0}(k\boldsymbol{\sigma})$ is for a distribution with zero mean. It has the properties $\varphi_{f0}(0) = 1$ and $\varphi'_{f0}(0) = 0$, where the prime indicates the first derivative with respect to k. Making this substitution and taking the derivative inside the integral sign gives

$$\Phi(\boldsymbol{\sigma}; c) = \frac{1}{2}\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) - \frac{i}{2\pi}\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right)\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi_{f0}(k\boldsymbol{\sigma}) \\ -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi'_{f0}(k\boldsymbol{\sigma}). \tag{92}$$

In the final integral, $\varphi'_{f0}(k\boldsymbol{\sigma})$ is zero at $k = 0$. As long as it goes to zero at least as fast as k – that is, it goes to zero as k^a, where $a \geq 1$ – the integral is regular and factor of $1/k$ does not play a role.

The second integral is potentially problematic because $\varphi_{f0}(k\boldsymbol{\sigma}) = 1$ at $k = 0$. To show that it, too, may be regular, write the characteristic function as the sum of a symmetric and an asymmetric part:

$$\varphi_{f0}(k\boldsymbol{\sigma}) = \varphi^{+}_{f0}(k\boldsymbol{\sigma}) + \varphi^{-}_{f0}(k\boldsymbol{\sigma}), \tag{93}$$

where

$$\varphi^{\pm}_{f0}(k\boldsymbol{\sigma}) = \frac{1}{2}\left(\varphi_{f0}(k\boldsymbol{\sigma}) \pm \varphi_{f0}(k\boldsymbol{\sigma})\right). \tag{94}$$

Then it is possible to show that

$$\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi^{+}_{f0}(k\boldsymbol{\sigma}) = 2i\int_{0}^{\infty}\frac{dk}{k}\varphi^{+}_{f0}(k\boldsymbol{\sigma})\sin\left[k(\boldsymbol{\sigma}'\boldsymbol{\mu} - c)\right], \tag{95a}$$

$$\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi^{-}_{f0}(k\boldsymbol{\sigma}) = 2\int_{0}^{\infty}\frac{dk}{k}\varphi^{-}_{f0}(k\boldsymbol{\sigma})\cos\left[k(\boldsymbol{\sigma}'\boldsymbol{\mu} - c)\right]. \tag{95b}$$

The first of these expressions is regular because $\lim_{x\to 0}\sin x/x = 1$. The second is regular as long as $\varphi^{-}_{f0}(k\boldsymbol{\sigma})$ goes to zero at $k = 0$ at least as fast as k. Note that these results hold only if $\boldsymbol{\sigma}'\boldsymbol{\mu} \neq c$. When $\boldsymbol{\sigma}'\boldsymbol{\mu} = c$ the second integral in Eq. (92) has a simple pole at $k = 0$ with a residue equal to one.

Assuming the integral is regular, define a function

$$J(b) \equiv -\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k}e^{ibk}\varphi_{f0}(k\boldsymbol{\sigma}). \tag{96}$$

In terms of this function, the generating function can be written

$$\Phi(\boldsymbol{\sigma}; c) = \frac{1}{2}\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) + \left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right)J(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) \\ -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi'_{f0}(k\boldsymbol{\sigma}). \tag{97}$$

Because the integrand in $J(b)$ is regular for $b \neq 0$, it is possible to take its derivative with respect to b,

$$J'(b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ibk}\varphi_{f0}(k\boldsymbol{\sigma}). \tag{98}$$

But this is just the inverse of a characteristic function. Specifically, it is a univariate probability distribution with mean zero that can be derived from the multivariate distribution by applying this formula. Writing that distribution function, which has mean zero, as $f_0(x)$,

$$J'(b) = f_0(b). \tag{99}$$

While the expression for $J(b)$ is regular, the demonstration relied on the fact that $b \neq 0$. For that reason, when integrating $J'(b)$ it is necessary to avoid the value $b = 0$. When $\boldsymbol{\sigma}'\boldsymbol{\mu} > c$, the integral should be taken from a small positive value and the limit taken to zero,

$$J(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\boldsymbol{\sigma}'\boldsymbol{\mu}-c} db\, J'(b) = F(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) - F(0), \tag{100}$$

where $F(x)$ is the cumulative distribution of $f(x)$. When $\boldsymbol{\sigma}'\boldsymbol{\mu} < c$, the integral should be taken from a small negative value and the limit taken to zero, again giving

$$J(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) = \lim_{\epsilon\to 0^-}\int_{\epsilon}^{\boldsymbol{\sigma}'\boldsymbol{\mu}-c} db\, J'(b) = F(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) - F(0), \tag{101}$$

Substituting into the expression for $\Phi(\boldsymbol{\sigma}; c)$ gives

$$\Phi(\boldsymbol{\sigma}; c) = \left(F(\boldsymbol{\sigma}'\boldsymbol{\mu} - c) + \frac{1}{2} - F(0)\right)\left(\boldsymbol{\sigma}'\boldsymbol{\mu} - c\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik(\boldsymbol{\sigma}'\boldsymbol{\mu}-c)}\varphi'_{f0}(k\boldsymbol{\sigma}). \tag{102}$$

This is the final, general, expression for the generating function in terms of a characteristic function.

The integral in Eq. (102) depends on the particular distribution and must be evaluated on a case-by-case basis. The other term can be seen as a smoothed version of the Heaviside function. When $\boldsymbol{\sigma}'\boldsymbol{\mu} = c$, the expression in parentheses is $1/2$. It is less than $1/2$ when $\boldsymbol{\sigma}'\boldsymbol{\mu} < c$ and greater than $1/2$ when $\boldsymbol{\sigma}'\boldsymbol{\mu} > c$, rising more or less steeply depending on the shape of the distribution. The range is from $1/2 - F(0)$ to $3/2 - F(0)$. If the distribution is symmetric, then $F(0) = 1/2$ and the range is from 0 to 1, as with the Heaviside function. But if, for example, the distribution is skewed towards negative values of $\boldsymbol{\sigma}'\hat{\boldsymbol{\nu}}$, then the minimum will be negative and the maximum will be less than one.

Appendix B Stan calibration code

```
// gaussian_model.stan
// Stan ver. 2.26

data {
  int<lower=0,upper=1> priorOnly; // Flag for using
  the prior distribution only
  int<lower=1> N; // Number of observations
  int<lower=1> D; // Number of inputs
  array[N] vector[D] y; // Observations (productivity growth rates)
  array[N] vector[D] s; // Cost shares
  vector[N] g; // Investment rate
  vector[N] kappa; // Capital productivity
  vector[N] i; // Interest rate
  vector[N] dgdp; // Deviation of GDP growth rate trend
  // Prior parameters
  vector[D] mu_prior_m;
  vector<lower=0>[D] mu_prior_s;
  vector<lower=0>[D] tau_prior_m;
  vector<lower=0>[D] tau_prior_s;
  real<lower=1> Om_prior_shape;
  vector<lower=0>[D] sigma_prior_shape;
}

transformed data {
  vector[N] profshare;
  vector[N] u;
  vector[N] r = i + g/2.0; // Leverage = 1
  real g_median;
  vector[N] g_sort;
  int mid_N_upper;
  int mid_N_lower;
  for (n in 1:N) {
    profshare[n] = s[n][1];
  }
  // Deviations from median of g (Stan v. 2.27 has median built in,
  but not 2.26) g_sort = sort_asc(g);
  mid_N_lower = (N + 1) %/% 2;
  mid_N_upper = mid_N_lower + (N + 1) % 2;
  g_median = 0.5 * (g_sort[mid_N_lower] + g_sort[mid_N_upper]);
  u = g/g_median - 1;
}

parameters {
  // Tech change mean
  vector[D] mu_a;
  vector[D] mu_b;
  // Unscaled prior for covariances
  corr_matrix[D] Om_a_unsc;
  corr_matrix[D] Om_b_unsc;
  // Scale parameters for covariances
  vector<lower=0>[D] tau_a;
  vector<lower=0>[D] tau_b;
  // Prediction error scale
  vector<lower=0>[D] sigma;
  // GDP gr deviation multiplier
  vector<lower=0,upper=1>[D] dgdp_mult;
}
```

```
transformed parameters {
  //------------------------------------------
  // Tech change distribution
  //------------------------------------------
  // Calibrated parameters
  // Scale parameters for covariances
  cov_matrix[D] Om_a = quad_form_diag(Om_a_unsc, tau_a);
  cov_matrix[D] Om_b = quad_form_diag(Om_b_unsc, tau_b);
  //------------------------------------------
  // Convenient reparameterizations
  //------------------------------------------
  matrix[N,D] Om_s; // (Om_a + Om_b * (2u + u^2)) * s
  vector[N] V; // s' * Om_s
  matrix[N,D] mu; // mu_a + mu_b * u;
  vector[N] x; // Argument for normals
  vector[N] ncdf_x; // Standard normal CDF at x
  vector[N] npdf_x; // Standard normal PDF at x
  for (n in 1:N) {
    V[n] = 0;
    for (d1 in 1:D) {
      Om_s[n,d1] = 0;
      mu[n,d1] = mu_a[d1] + u[n] * mu_b[d1];
      for (d2 in 1:D) {
        Om_s[n,d1] += (Om_a[d1,d2] + (1 + u[n])^2 * Om_b[d1,d2]) *
        s[n][d2]; V[n] += s[n][d1] * Om_s[n,d1];
      }
    }
    V[n] = sqrt(V[n]); // Calculate square root of the quadratic form
    x[n] = (s[n]' * mu[n,]')/V[n] - (r[n]/kappa[n] - profshare[n]) *
    g[n];
  }
  ncdf_x = Phi(x);
  // Divisor is sqrt(2*pi)
  npdf_x = exp(-x .* x/2)/2.506628274631000502442;
}

model {
  vector[D] alpha;
  vector[D] beta;

  mu_a ~ normal(mu_prior_m,mu_prior_s);
  mu_b ~ normal(mu_prior_m,mu_prior_s);
  alpha = (tau_prior_m ./ tau_prior_s)^2;
  beta = tau_prior_m ./ tau_prior_s^2;
  tau_a ~ gamma(alpha,beta);
  tau_b ~ gamma(alpha,beta);
  Om_a_unsc ~ lkj_corr(Om_prior_shape);
  Om_b_unsc ~ lkj_corr(Om_prior_shape);
  sigma ~ cauchy(0,sigma_prior_shape);
  dgdp_mult ~ beta(2,2);

  if (!priorOnly) {
    for (d in 1:D) {
      y[,d] ~ normal(npdf_x .* Om_s[,d] ./ V + ncdf_x .* mu[,d] +
                    dgdp_mult[d] * dgdp, sigma[d]);
    }
```

```
  } else {
    print(''Prior only'');
  }
}

generated quantities {
  // Model parameters
  cov_matrix[D] Om_m;
  cov_matrix[D] Om_d;
  vector[D] mu_m;
  vector[D] mu_d;
  real Om_m_scale;
  real Om_d_scale;
  // Simulated outputs
  array[N] vector[D] y_sim;
  // Simulated outputs
  real z_m[N];
  real z_d[N];
  // Log likelihood
  vector[N] log_lik = rep_vector(0,N);

  // Invert the fitted parameters to find the model parameters
  mu_d = mu_a - mu_b;
  mu_m = mu_b/g_median;
  Om_d = Om_a; // This is an alias, to keep the notation simple
  Om_m = Om_b/g_median^2;
  // This works for arrays of any size D >= 2
  Om_m_scale = Om_m[1,2]/sqrt(Om_m[1,1] * Om_m[2,2]);
  Om_d_scale = Om_d[1,2]/sqrt(Om_d[1,1] * Om_d[2,2]);

  for (n in 1:N) {
    z_m[n] = (s[n]' * mu_m)/sqrt(s[n]' * Om_m * s[n]);
    z_d[n] = (s[n]' * mu_d)/sqrt(s[n]' * Om_d * s[n]);
  }

  for (d in 1:D) {
    y_sim[,d] = normal_rng(npdf_x .* Om_s[,d] ./ V + ncdf_x .*
mu[,d] +
                      (1.0 - priorOnly) * dgdp_mult[d] * dgdp,
   sigma[d]);
  }

  if (!priorOnly) {
    for (d in 1:D) {
      for (n in 1:N) {
        log_lik[n] += normal_lpdf(y[n,d] | npdf_x[n] * Om_s[n,d]
/ V[n] +
                                  ncdf_x[n] * mu[n,d] +
                                  dgdp_mult[d] * dgdp[n], sigma[d]);
      }
    }
  }

}
```

Appendix C Comparison of prior and posterior distributions

This appendix contains plots of prior and posterior distributions for each component of the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ parameters. Plots for India are in Fig. 4, for China in Fig. 5, and the United States in Fig. 6. The light blue band in each plot is the 50% confidence interval, while the dashed blue line is the mean of the prior. The top two plots, labeled (a) and (b) in each figure, are draws from the prior distribution and are nearly identical between the countries. The bottom two plots, labeled (c) and (d), are posterior distributions. Any differences between the prior and posterior distributions can be attributed to the data.

The posterior distributions for the $\boldsymbol{\mu}_k$ are, for the most part, much more sharply peaked than the prior distributions, while the dispersion in the prior and posterior distributions for the $\boldsymbol{\tau}_k$ are similar. (However, notice that the scales on the x-axis

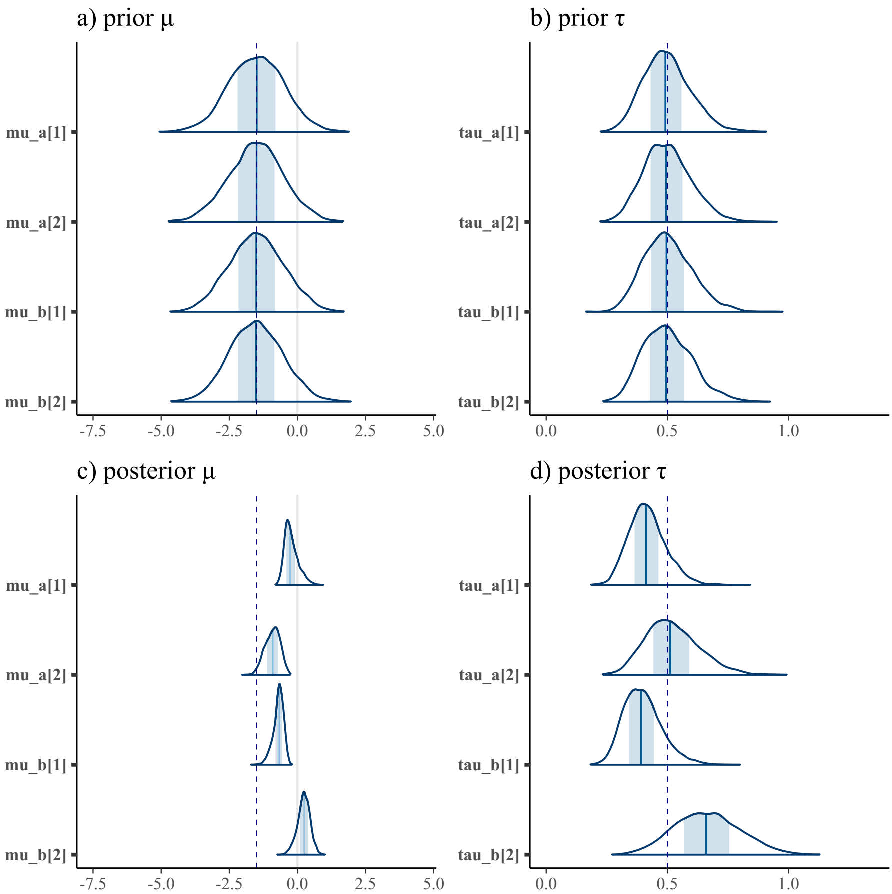

Fig. 4 Prior and posterior plots for India of the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ parameters

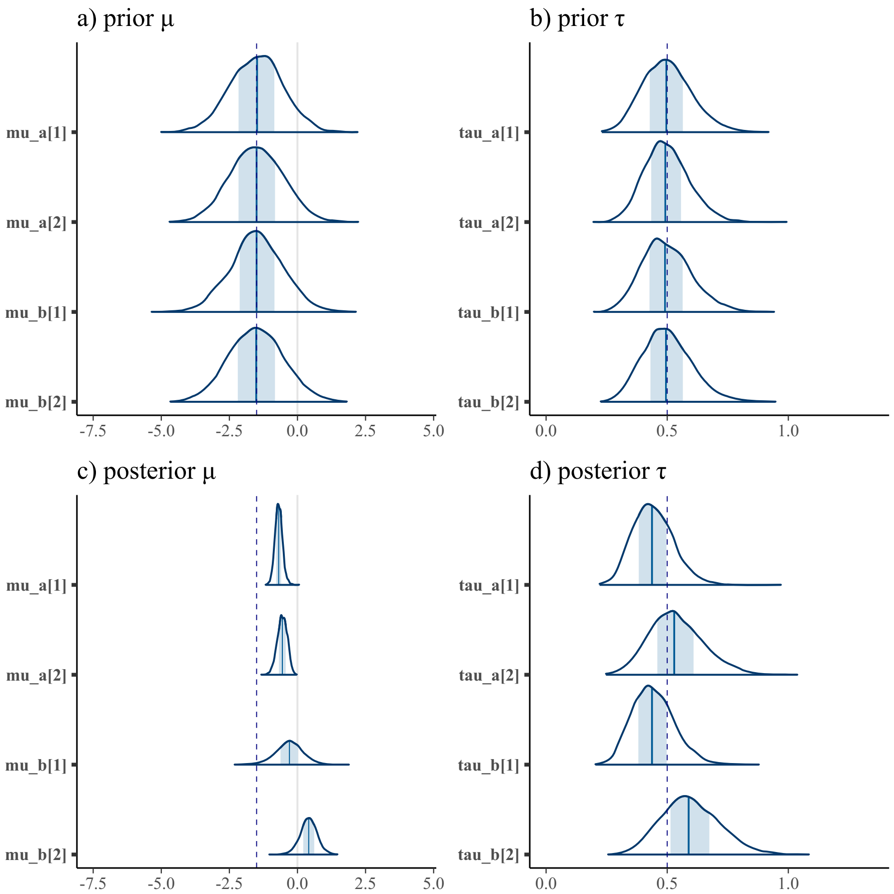

Fig. 5 Prior and posterior plots for China of the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ parameters

are quite different between the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ plots; that is because the priors for the $\boldsymbol{\tau}_k$ parameters are narrower than for the $\boldsymbol{\mu}_k$.)

One common feature to note is that the means for the $\tau_{a,2}$ and $\tau_{b,2}$ parameters in each country lie to the right of the means for $\tau_{b,1}$ and $\tau_{a,1}$ in the posterior distribution. The $\boldsymbol{\tau}_k$ parameters are associated with cost share dependence, while the second index is for labor, so this suggests that labor productivity is more sensitive to cost than is capital productivity. That is a plausible conclusion in that much innovation is aimed at reducing labor costs. Moreover, the shift is most pronounced in India, which showed the greatest variation in cost shares over the historical data period, suggesting that the shift is a real phenomenon brought out by the data. The difference is largest for $\boldsymbol{\tau}_b$, which is associated with embodied technological change, consistent with the introduction of labor-saving new machinery.

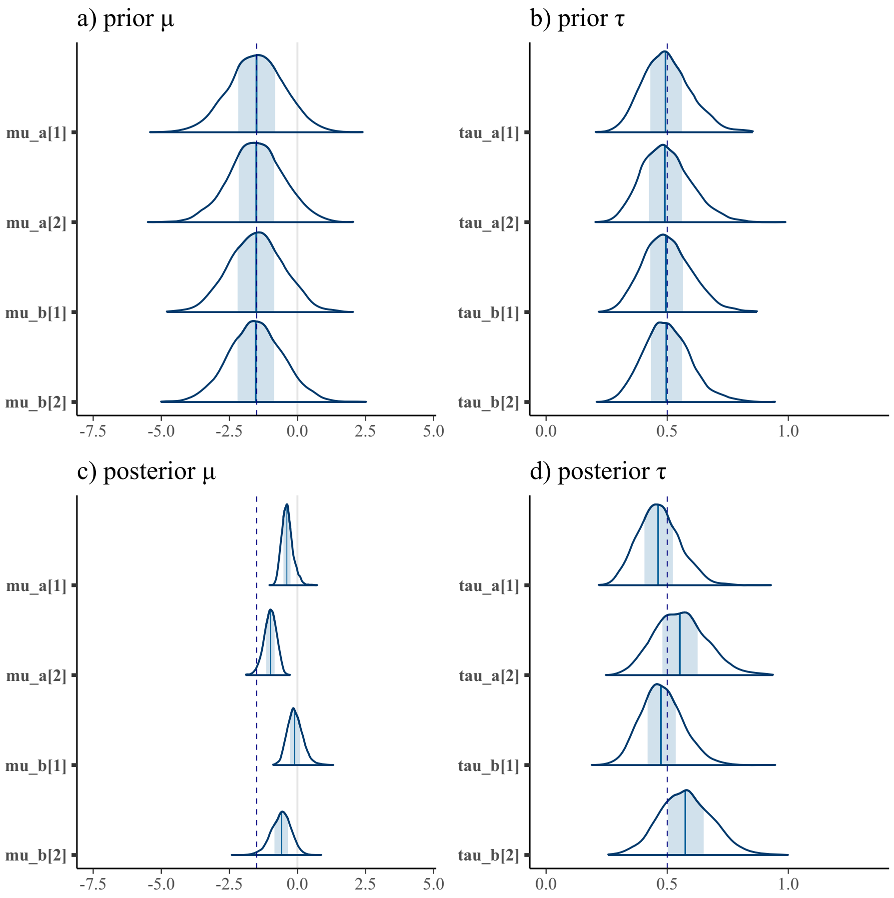

Fig. 6 Prior and posterior plots for the United States of the $\boldsymbol{\mu}_k$ and $\boldsymbol{\tau}_k$ parameters

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