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Ghazanfar, S., Ahmed, N., Iqbal, M.S. et al. (5 more authors) (2024) Analysis of multi-wave solitary solutions of (2+1)-dimensional coupled system of Boiti–Leon–Pempinelli. Scientific Reports, 14 (1). 20234. ISSN 2045-2322

https://doi.org/10.1038/s41598-024-67698-z scientific reports

OPEN

Analysis of multi-wave solitary solutions of (2+1)-dimensional coupled system of Boiti–Leon–Pempinelli

Sidra Ghazanfar[1], Nauman Ahmed[1,6], Muhammad Sajid Iqbal[2,3], Syed Mansoor Ali[4], Ali Akgül[5,10,11], Shah Muhammad[7], Mubasher Ali[8] & Murad Khan Hassani[9]✉

This work examines the (2+1)-dimensional Boiti–Leon–Pempinelli model, which finds its use in hydrodynamics. This model explains how water waves vary over time in hydrodynamics. We provide new explicit solutions to the generalized (2+1)-dimensional Boiti–Leon–Pempinelli equation by applying the Sardar sub-equation technique. This method is shown to be a reliable and practical tool for solving nonlinear wave equations. Furthermore, different types of solitary wave solutions are constructed: w-shaped, breather waved, chirped, dark, bright, kink, unique, periodic, and more. The results obtained with the variable coefficient Boiti–Leon–Pempinelli equation are stable and different from previous methods. As compared to their constant-coefficient counterparts, the variable-coefficient models are more general here. In the current work, the problem is solved using the Sardar Sub-problem Technique to produce distinct soliton solutions with parameters. Plotting these graphs of the solutions will help you better comprehend the model. The outcomes demonstrate how well the method works to solve nonlinear partial differential equations, which are common in mathematical physics.With the help of this method, we may examine a variety of solutions from significant physical perspectives.

Keywords Solitons, Sardar sub-equation technique, Exact wave structures

In the past few decades, the research of traveling wave solutions explored by researchers has gained considerable attention. It includes the solutions of non-linear partial differential equations (NPDEs) which play a key role in the study of non-linear physical phenomena arising in many fields of engineering and sciences i.e., mathematical physics[1], technical arena[2], plasma physics[3], ocean engineering[4], tsunami waves[5], etc. NPDEs have great potential for applications in various fields, therefore, these equations have the advantage of getting the special attention of researchers to find their analytical and numerical solutions. In recent times, many researchers in mathematics and physics have established various methods of constructing and analyzing exact traveling wave solutions of different non-linear problems such as Hirota's bilinear transformation method[6–8], the extended Exp-expansion method[9], the new extended direct algebraic method[10], the variational iteration method[11], the semi-inverse variational principle[12], the generalized Kudryashov technique[13], the sine-Gordon method[14], the Cole-Hopf transformation method[15], the Adomian decomposition method[16], the traveling wave scheme[17], A special kind of distributive product[18], the Bäcklund transformation method[19].

Solitons are the fascinating aspect of nonlinear physical events. The solitonic concept is accessible due to ethical balance and nonlinearity of concentration. Many scholars have conducted studies on solitary wave solutions as mentioned above.

[1]Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan. [2]Department of Academic Affairs, School of Leadership and Business, Oryx Universal College With Liverpool John Moores University (UK), 12253 Doha, Qatar. [3]Department of Humanities and Basic Science, MCS, NUST, Islamabad, Pakistan. [4]Department of Physics and Astronomy, College of Science,, King Saud University, P.O. BOX 2455, 11451 Riyadh, Saudi Arabia. [5]Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey. [6]Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon. [7]Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, 11451 Riyadh, Saudi Arabia. [8]Department of Electronic and Electrical Engineering, University of Sheffield, Sheffield, SYK, England. [9]Department of Mathematics, Ghazni University, Ghazni, Afghanistan. [10]Department of Computer Engineering, Biruni University, 34010 Topkapı Istanbul, Turkey. [11]Department of Mathematics, Near East Unive99138,rsity, Mathematics Research Center, 99138 Near East Boulevard, Nicosia /Mersin 10, Turkey. ✉email: mhassani@gu.edu.af

Recently, an effective method has been introduced, called the Sardar sub-equation technique (SSET)[20]. Our primary emphasis is developing various wave soliton solutions, such as bright, singular, dark-bright, kink, dark, w-shaped, chirped, breather wave, and periodic wave solitons. The method under consideration is more universal than the others listed above. Similarly, these findings help us recognize the dynamic performance of various physical configurations. Furthermore, these results are positive, unique, and precise, and they may help illuminate particular non-linear natural phenomena in non-linear mathematical models.

This work has a few obvious limitations, such as the that it is usually only appropriate for a specific class of nonlinear PDEs,it might not be able to solve more complex equations, it might struggle with highly nonlinear terms,the solution it does provide may take particular forms, it frequently yields solutions under specific conditions, it frequently calls for the use of symbolic computational tools, it can be difficult to incorporate initial and boundary conditions into this method, the solutions it produces sometimes be non-trivial, and there may be easier solutions available.

In this work, the following coupled system of (2+1)-Dimensional Boiti–Leon–Pempinelli (BLP) equations[21] has been taken into consideration.

$$\begin{aligned} U_{yt} &=(U^2 - U_x)_{xy} + 2\, V_{xxx}, \\ V_t &=V_{xx} + 2\, U\, V_x, \end{aligned} \tag{1}$$

which was initially introduced by Boiti et al.[22]. Many mathematicians studied this system and developed precise explicit solutions using various methods. The Boiti–Leon–Pempinelli equation has drawn a lot of attention from researchers in the past ten years since it is used to explain the wave propagation of incompressible fluids in plasma physics, fluid dynamics, ocean engineering, astrophysics, and aerodynamics.More relevant material can also be studied in[23–30].

Wazwaz and Mehanna's precise traveling wave system suggestions of the system (1). System (1)'s lump-type solutions, Lie point symmetries, and some precise answers to some other algebraic equations with new optical, lump wave, breather, periodic, and other multi-wave solutions can also be found with some additional precise solutions to the Eq. (1). There are some novel traveling wave system solutions of (1) that are provided in this article. Some other novel exact traveling wave solutions were presented in[31–34].

It is a crucial system for describing how the horizontal velocity component of waves in an infinitely narrow channel with constant depth changes over time. The horizontal velocity and height of the water wave are related to the velocity components $U(x, y, t)$ and $V(x, y, t)$, respectively. Eq. (1) is a member of a group of equations that explain how water waves move through channels with a constant depth. The Sinh-Gordon equation can be generalized as the (2+1)-Dimensional equation in Eq. (1), and it can be transformed into the (anti)-Burgers equations, for example, in Mu et al.[35], the model was taken into account using Hamiltonians, the Bäcklund transform, Lax Pair, and the Painlevé integrability.

More related research work can also be studied in this regard e.g.,[36–41].

The terms "horizontal velocity" and "height" relate to two essential characteristics that characterize the motion of water waves. The pace at which individual water particles move horizontally during the propagation of a wave is known as the horizontal wave velocity. The motion of the water particles in a wave is either elliptical or circular. This motion in the direction of wave propagation is composed of the horizontal velocity. It shows the rate at which the wave is propagating laterally. Both the vertical and horizontal components of the particle motion affect the wave's real speed. While, the vertical distance between a water wave's highest point, the crest, and its lowest point, the trough, determines the height of the wave. An indicator of a water wave's energy is its height. Greater heights and energy are carried by larger waves. The amplitude of the wave, or the maximum displacement of the water particles from their undisturbed position, is correlated with the wave's height. The wave's energy is determined by its amplitude. To comprehend and forecast wave behavior, a water wave's height and horizontal velocity work together. Wind speed, water depth, and the distance at which the wind has blown are some variables that affect them. An understanding of these factors is essential in disciplines like marine science, coastal engineering, and oceanography. This work is novel in itself that has not been done before. We can take this work so far till we are able to present these specific solitons/solutions in the form of physical interpretations.

The soliton-type solutions provided in this paper are beneficial for those who are physically related to it. These solutions can be applied in a variety of situations. Regarding its all-purpose applications, nevertheless, a few of these solutions can be useful to all physicists working in the field of soliton solutions for PDEs. Even though we have offered many solutions. Furthermore, since we have found the exact solutions in the absence of auxiliary data, there are infinitely many solutions. Since we haven't connected the problem to any initial or boundary conditions, the physicist must determine which solution best fits the available information.

Problem statement

Using SSET, the following traveling wave transformation is used to create strong and authentic solitons of the BLP system

$$U(x, y, t) = U(\eta), \quad V(x, y, t) = V(\eta), \quad \text{where } \eta = x + y - ct. \tag{2}$$

Here c is the real constant to be determined. Applying Eq. (2) in Eq. (1) to get the following form of ODE (ordinary differential equation) of the given system

$$\begin{aligned} -cU'' &= (U^2 - U')'' + 2V''', \\ -cV' &= V'' + 2UV', \end{aligned} \tag{3}$$

By integrating and assembling the above system, we can write the following equation as

$$U'' - 2U^3 - 3cU^2 - c^2U = 0. \tag{4}$$

This will lead us to the exact solutions of the given BLP system. Moreover, some general insights or constraints of the equations which are worth mentioning here are its boundary conditions, nature of the equation and, its stability analysis.

Mathematical details of Sardar sub-equation technique

This thorough and straightforward method is used by many experts to discover solitons and other wave solutions to the given issue. This technique can provide precise responses for a class of NPDEs. The given system of equations can be included by following the procedures below.

Step I: Considering the NPDE as follows

$$O(P, P_t, P_x, P_{xx}, P_{tt}, P_{xt}, P_{xxx}, \ldots) = 0, \tag{5}$$

where $P = P(x, t)$ is the unknown function, O is a polynomial of $P(x, t)$ and its derivatives with respect to x and t. Now applying the traveling wave transformation

$$P(x, t) = P(\eta), \quad \eta = \alpha x + \beta t, \tag{6}$$

where α, and β are the unknown constants to be determined later.

Using the above transformation, Eq. (5) is converted to the following ODE (ordinary differential equation),

$$Q(\Psi, \Psi', \Psi'', \Psi''', \ldots) = 0, \tag{7}$$

where Q is the function of $\Psi(\eta)$ and its derivatives and its superscripts designate ordinary derivatives w.r.t η

Step II: Solution of Eq. (7) is then formulated as

$$\Psi(\eta) = \sum_{n=0}^{N} c_n M^n(\eta), \quad c_n \neq 0, \tag{8}$$

where $c_n (0 \leq n \leq N)$ are real constants and $M(\eta)$ satisfies the ODE of the following form

$$M'(\eta) = \sqrt{\mu + \nu M(\eta)^2 + M(\eta)^4}. \tag{9}$$

Here μ and ν are real constants and Eq. (9) presents the following solutions:

- If $\nu > 0$ and $\mu = 0$, then

$$\begin{aligned} M_1^{\pm} &= \pm\sqrt{-pq\nu}\, \mathrm{sech}_{pq}(\sqrt{\nu}\,\eta), \\ M_2^{\pm} &= \pm\sqrt{pq\nu}\, \mathrm{csch}_{pq}(\sqrt{\nu}\,\eta), \end{aligned} \tag{10}$$

  where

$$\mathrm{sech}_{pq}(\eta) = \frac{2}{pe^{\eta} + qe^{-\eta}}, \quad \mathrm{csch}_{pq}(\eta) = \frac{2}{pe^{\eta} - qe^{-\eta}}.$$

- If $\nu < 0$ and $\mu = 0$, then

$$\begin{aligned} M_3^{\pm} &= \pm\sqrt{-pq\nu}\; \sec_{pq}(\sqrt{-\nu}\,\eta), \\ M_4^{\pm} &= \pm\sqrt{-pq\nu}\; \csc_{pq}(\sqrt{-\nu}\,\eta), \end{aligned} \tag{11}$$

  where

$$\sec_{pq}(\eta) = \frac{2}{pe^{\iota\eta} + qe^{-\iota\eta}}, \quad \csc_{pq}(\eta) = \frac{2}{pe^{\iota\eta} - qe^{-\iota\eta}}.$$

- If $\nu < 0$ and $\mu = \frac{\nu^2}{4}$, then

$$
\begin{aligned}
M_5^{\pm} &= \pm\sqrt{\frac{-\nu}{2}}\tanh_{pq}\Big(\sqrt{\frac{-\nu}{2}}\,\eta\Big),\\
M_6^{\pm} &= \pm\sqrt{\frac{-\nu}{2}}\coth_{pq}\Big(\sqrt{\frac{-\nu}{2}}\,\eta\Big),\\
M_7^{\pm} &= \pm\sqrt{\frac{-\nu}{2}}\left(\tanh_{pq}(\sqrt{-2\nu}\,\eta)\pm\iota\sqrt{pq}\,\mathrm{sech}_{pq}(\sqrt{-2\nu}\,\eta)\right),\\
M_8^{\pm} &= \pm\sqrt{\frac{-\nu}{2}}\left(\coth_{pq}(\sqrt{-2\nu}\,\eta)\pm\sqrt{pq}\,\mathrm{csch}_{pq}(\sqrt{-2\nu}\,\eta)\right),\\
M_9^{\pm} &= \pm\sqrt{\frac{-\nu}{8}}\left(\tanh_{pq}\Big(\sqrt{\frac{-\nu}{8}}\,\eta\Big)+\coth_{pq}\Big(\sqrt{\frac{-\nu}{8}}\,\eta\Big)\right),
\end{aligned}
\tag{12}
$$

where

$$
\tanh_{pq}(\eta)=\frac{pe^{\eta}-qe^{-\eta}}{pe^{\eta}+qe^{-\eta}},\quad \coth_{pq}(\eta)=\frac{pe^{\eta}+qe^{-\eta}}{pe^{\eta}-qe^{-\eta}}.
$$

- If $\nu > 0$ and $\mu = \frac{\nu^2}{4}$, then

$$
\begin{aligned}
M_{10}^{\pm} &= \pm\sqrt{\frac{\nu}{2}}\tan_{pq}\Big(\sqrt{\frac{\nu}{2}}\,\eta\Big),\\
M_{11}^{\pm} &= \pm\sqrt{\frac{\nu}{2}}\cot_{pq}\Big(\sqrt{\frac{\nu}{2}}\,\eta\Big),\\
M_{12}^{\pm} &= \pm\sqrt{\frac{\nu}{2}}\left(\tan_{pq}(\sqrt{2\nu}\,\eta)\pm\sqrt{pq}\,\sec_{pq}(\sqrt{2\nu}\,\eta)\right),\\
M_{13}^{\pm} &= \pm\sqrt{\frac{\nu}{2}}\left(\cot_{pq}(\sqrt{2\nu}\,\eta)\pm\sqrt{pq}\,\csc_{pq}(\sqrt{2\nu}\,\eta)\right),\\
M_{14}^{\pm} &= \pm\sqrt{\frac{\nu}{8}}\left(\tan_{pq}\Big(\sqrt{\frac{\nu}{8}}\,\eta\Big)+\cot_{pq}\Big(\sqrt{\frac{\nu}{8}}\,\eta\Big)\right),
\end{aligned}
\tag{13}
$$

where

$$
\tan_{pq}(\eta)=-\iota\frac{pe^{\iota\eta}-qe^{-\iota\eta}}{pe^{\iota\eta}+qe^{-\iota\eta}},\quad \cot_{pq}(\eta)=\iota\frac{pe^{\iota\eta}+qe^{-\iota\eta}}{pe^{\iota\eta}-qe^{-\iota\eta}}.
$$

The above listed functions are the generalized forms of trigonometric and hyperbolic functions with parameters p and q. If we take the values of p and q to be 1, then the above functions become known functions.

Step III: By balancing the capital, we can determine the number N. Using this value of N, we get an algebraic equation in the shape of $M^n(\eta)$ by substituting Eq. (8) into Eq. (7), which we balance by setting the powers of $M^n(\eta),\ n=(0,1,2,\ldots)$ to zero, resulting in a set of algebraic equations.

Step IV: This system of equations provides the necessary inputs and the precise answer to the provided equation.

Execution of the technique

The traveling wave solution to the Boiti–Leon–Pempinelli System is created in this part using SSET. Using homogeneous balance principle, we balance the equations U'' and U^3 to get the value of N and found to be 1.

Equation (8) is reduced by the equilibrium formula into

$$
U = a_0 + a_1 T(\eta), \tag{14}
$$

where a_0 and a_1 are the constants to determine. Substituting Eq. (14), Eq. (4) into Eq. (9) to get a polynomial in the form of $T^n(\eta)$. Equating the powers of $T^n(\eta),\ (n=0,1,2,3)$ to zero to get the algebraic equations in the form of a_0, a_1, ν and μ.

Where

$$
T'(\eta)=\sqrt{\xi+uM(\eta)^2+M(\eta)^4}. \tag{15}
$$

The system of equations is as

$$
\begin{aligned}
&T^3: 2\,a_1 - 2\,{a_1}^3 = 0,\\
&T^2: -6\,a_0{a_1}^2 - 3\,c{a_1}^2 = 0,\\
&T^1: -2\,c^2a_1 - 6\,{a_0}^2a_1 - 6\,ca_0a_1 + a_1u = 0,\\
&T^0: -2\,{a_0}^3 - 2\,c^2a_0 - 3\,c{a_0}^2 = 0.
\end{aligned}
\tag{16}
$$

We discovered the following results by analyzing the above system of equations

$$a_0 = \frac{-1}{2}c, \quad a_1 = -1. \tag{17}$$

Using these values in Eqs. (9),(14) and (17) along with Eq. (2), we summarized the results for functions U along with their corresponding V as follows:

1. If $u > 0$ and $\xi = 0$, then

$$\begin{aligned} U_1 &= \frac{-c}{2} \pm \sqrt{-pqu}\,\mathrm{sech}_{pq}\big(\sqrt{u}\eta\big),\\ U_2 &= \frac{-c}{2} \pm \sqrt{pqu}\,\mathrm{csch}_{pq}\big(\sqrt{u}\eta\big), \end{aligned} \tag{18}$$

$$\begin{aligned} V_1 &= \frac{c \pm \sqrt{-pqu}\Big(-12\left(\cosh_{pq}\left(\sqrt{u}\eta\right)\right)^2 - 3c^2 \pm \sqrt{-pqu}\cosh_{pq}\left(\sqrt{u}\eta\right) + c - pqu\Big)}{48\big(\cosh_{pq}\left(\sqrt{u}\eta\right)\big)^3},\\ V_2 &= \frac{-c}{48} - \sqrt{pqu}\,\mathrm{csch}_{pq}\big(\sqrt{u}\eta\big)\Big(-12 + 3c^2 - \sqrt{pqu}\,\mathrm{csch}_{pq}\big(\sqrt{u}\eta\big) + c \pm pqu\left(\mathrm{csch}_{pq}\big(\sqrt{u}\eta\big)\right)^2\Big). \end{aligned} \tag{19}$$

In summary, the constraints of the above equations include:

- $u > 0$ and $\xi = 0$ (parameter u must be a positive real number).
- $pq < 0$ to ensure the square root term is real for U_1, V_1 and, V_2 and, $pq > 0$ to ensure the square root term is real for U_2.

2. If $u < 0$, and $\xi = 0$, then

$$\begin{aligned} U_3 &= \frac{-c}{2} \pm \sqrt{-pqu}\,\sec_{pq}\big(\sqrt{-u}\eta\big),\\ U_4 &= \frac{-c}{2} \pm \sqrt{-pqu}\,\csc_{pq}\big(\sqrt{-u}\eta\big), \end{aligned} \tag{20}$$

$$\begin{aligned} V_3 &= \frac{c \pm \sqrt{-pqu}\Big(-12\left(\cos_{pq}\left(\sqrt{-u}\eta\right)\right)^2 - 3c^2 \pm \sqrt{-pqu}\,\cos_{pq}\left(\sqrt{-u}\eta\right) + c - pqu\Big)}{48\big(\cos_{pq}\left(\sqrt{-u}\eta\right)\big)^3},\\ V_4 &= \frac{c}{48} \pm \sqrt{-pqu}\,\csc_{pq}\left(\sqrt{-u}\eta\right)\Big(-12 - 3c^2 \pm \sqrt{-pqu}\,\csc_{pq}\left(\sqrt{-u}\eta\right) + c - pqu\left(\csc_{pq}\left(\sqrt{-u}\eta\right)\right)^2\Big). \end{aligned} \tag{21}$$

In summary, the constraints of the above equations are:

- $u < 0$ and $\xi = 0$ (parameter u must be a negative real number).
- $pq > 0$ to ensure the square root term is real for all U_3, U_4, V_3 and V_4.
- $\cos_{pq}(\sqrt{-u}\,\eta) \neq 0$ (to avoid division by zero in the denominator of V_3).

3. If $u < 0$ and $\xi = \frac{u^2}{4}$, then

$$\begin{aligned} U_5 &= \frac{-1}{2} \pm \sqrt{\frac{-u}{2}}\tanh_{pq}\left(\sqrt{\frac{-u}{2}}\,\eta\right),\\ U_6 &= \frac{-1}{2} \pm \sqrt{\frac{-u}{2}}\coth_{pq}\left(\sqrt{\frac{-u}{2}}\,\eta\right),\\ U_7 &= \frac{-c}{2} \pm \sqrt{\frac{-u}{2}} \pm \left(\tanh_{pq}\left(\sqrt{-2\,u\eta}\right) \pm \iota\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-2\,u\eta}\right)\right), \end{aligned} \tag{22}$$

$$
\begin{aligned}
U_8 =& \frac{-c}{2} \pm \sqrt{\frac{-u}{2}} \pm \left(\coth_{pq}\left(\sqrt{-2\,u}\eta\right) \pm \sqrt{pq}\,\text{csch}_{pq}\left(\sqrt{-2\,u}\eta\right)\right),\\
U_9 =& \frac{-c}{2} \pm \frac{\sqrt{-2u}}{4} \pm \left(\tanh_{pq}\left(\frac{\sqrt{-2u}}{4}\eta\right) + \sqrt{pq}\coth_{pq}\left(\frac{\sqrt{-2u}}{4}\eta\right)\right),\\
V_5 =& \frac{c}{24} \pm \frac{1}{2}\tanh_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right)\left(-6\sqrt{-2u} + 3\,c^2 \pm \frac{u}{2}\tanh_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right) + c \pm \frac{\sqrt{-2u^3}}{4}\left(\tanh_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right)\right)^2\right),\\
V_6 =& \frac{c}{24} \pm \frac{1}{2}\coth_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right)\left(-6\sqrt{-2u} + 3\,c^2 \pm \frac{u}{2}\coth_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right) + c \pm \frac{\sqrt{-2u^3}}{4}\left(\coth_{pq}\left(\sqrt{\frac{-u}{2}}\eta\right)\right)^2\right),\\
V_7 =& -\frac{11c}{48} \pm \sqrt{\frac{-u}{2}}\tanh pq\left(\sqrt{-2\,u}\eta\right) \pm \iota\sqrt{pq}\,\text{sech}_{pq}\left(\sqrt{-2\,u}\eta\right) + \frac{c^2}{8} \pm \frac{u}{4}\tanh_{pq}\left(\sqrt{-2\,u}\eta\right)\\
&\pm \iota^2 pq\left(\text{sech}_{pq}\left(\sqrt{-2\,u}\eta\right)\right)^2 \pm \frac{(-2\,u)^{3/2}}{8}\tanh\left(\sqrt{-2\,u}\eta\right) \pm \iota^3 pq^{3/2}\left(\text{sech}_{pq}\left(\sqrt{-2\,u}\eta\right)\right)^3,\\
V_8 =& -\frac{11c}{48} \pm \sqrt{\frac{-u}{2}}\coth pq\left(\sqrt{-2\,u}\eta\right) \pm \sqrt{pq}\,\text{csch}_{pq}\left(\sqrt{-2\,u}\eta\right) + \frac{c^2}{8} \pm \frac{u}{4}\coth_{pq}\left(\sqrt{-2\,u}\eta\right)\\
&\pm pq\left(\text{csch}_{pq}\left(\sqrt{-2\,u}\eta\right)\right)^2 \pm \frac{(-2\,u)^{3/2}}{8}\coth\left(\sqrt{-2\,u}\eta\right) \pm pq^{3/2}\left(\text{csch}_{pq}\left(\sqrt{-2\,u}\eta\right)\right)^3,\\
V_9 =& -\frac{11c}{48} \pm \frac{\sqrt{-2\,u}}{4}\left(\tanh_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right) + \coth_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right)\right) + \frac{c^2}{8} \pm \frac{u}{16}(\tanh_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right)\\
&+ \coth_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right))^2 \pm \frac{1}{64}(-2\,u)^{3/2}\left(\tanh_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right) + \coth_{pq}\left(\frac{\sqrt{-2\,u}}{4}\eta\right)\right)^3.
\end{aligned}
\tag{23}
$$

In summary, the constraints of the above equations are given below:

- $u < 0$ and $\xi = \frac{u^2}{4}$ (parameter u must be a negative real number).
- pq should be defined appropriately for the hyperbolic tangent, cotangent, secant and cosecant functions.
- The expression should be well-defined for the given values of $\sqrt{\frac{-\mu}{2}}\,\eta$ and, $\xi = \frac{u^2}{4}$.

4. If $u > 0$ and $\xi = \frac{u^2}{4}$, then

$$
\begin{aligned}
U_{10} =& \frac{-c}{2} \pm \sqrt{\frac{u}{2}}\tan_{pq}\left(\sqrt{\frac{u}{2}}\eta\right),\\
U_{11} =& \frac{-c}{2} \pm \sqrt{\frac{u}{2}}\cot_{pq}\left(\sqrt{\frac{u}{2}}\eta\right),\\
U_{12} =& \frac{-c}{2} \pm \sqrt{\frac{u}{2}}\left(\tan_{pq}\left(\sqrt{2u}\,\eta\right) \pm \sqrt{pq}\sec_{pq}\left(\sqrt{2u}\,\eta\right)\right),\\
U_{13} =& \frac{-c}{2} \pm \sqrt{\frac{-u}{2}}\left(\cot_{pq}\left(\sqrt{2u}\,\eta\right) \pm \sqrt{pq}\csc_{pq}\left(\sqrt{2u}\,\eta\right)\right),\\
U_{14} =& \frac{-c}{2} \pm \frac{\sqrt{2u}}{4}\left(\tan_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right) + \cot_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right)\right),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
V_{10} =& \frac{c}{48} \pm \frac{1}{\sqrt{2}} \tan_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right)\left(-12\sqrt{u} - 3c^2 \pm \frac{u}{\sqrt{2}} \tan_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right) + c \pm u^{3/2}\left(\tan_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right)\right)^2\right),\\
V_{11} =& \frac{c}{48} \pm \frac{1}{\sqrt{2}} \cot_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right)\left(-12\sqrt{u} - 3c^2 \pm \frac{u}{\sqrt{2}} \cot_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right) + c \pm u^{3/2}\left(\cot_{pq}\left(\sqrt{\frac{u}{2}}\,\eta\right)\right)^2\right),\\
V_{12} =& -\frac{5c}{24} \pm \sqrt{\frac{u}{2}} \tan_{pq}\left(\sqrt{2u}\,\eta\right) \pm \sqrt{pq}\sec_{pq}\left(\sqrt{2u}\eta\right) - \frac{c^2}{8} \pm \frac{u}{4}\tan_{pq}\left(\sqrt{2u}\eta\right) \pm pq\left(\sec_{pq}\left(\sqrt{2u}\eta\right)\right)^2\\
&\pm \frac{\sqrt{2u^3}}{8}\tan_{pq}\left(\sqrt{2u}\eta\right) \pm pq^{3/2}\left(\sec_{pq}\left(\sqrt{2u}\eta\right)\right)^3,\\
V_{13} =& -\frac{11c}{48} \pm \sqrt{\frac{-u}{2}} \cot_{pq}\left(\sqrt{2u}\,\eta\right) \pm \sqrt{pq}\csc_{pq}\left(\sqrt{2u}\eta\right) + \frac{c^2}{8} \pm \frac{u}{4}\cot_{pq}\left(\sqrt{2u}\eta\right) \pm pq\left(\csc_{pq}\left(\sqrt{2u}\eta\right)\right)^2\\
&\pm \frac{\sqrt{-2u^3}}{8}\cot_{pq}\left(\sqrt{2u}\eta\right) \pm pq^{3/2}\left(\csc_{pq}\left(\sqrt{2u}\eta\right)\right)^3,\\
V_{14} =& -\frac{5c}{24} \pm \frac{\sqrt{2u}}{4}\left(\tan_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right) + \cot_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right)\right) - \frac{c^2}{8} \pm \frac{u}{16}\left(\tan_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right) + \cot_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right)\right)^2\\
&\pm \frac{\sqrt{2u^3}}{64}\left(\tan_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right) + \cot_{pq}\left(\frac{\sqrt{2u}}{4}\eta\right)\right)^3.
\end{aligned}
\tag{25}
$$

In summary, the constraints of the above equations are:

- $u > 0$ and $\xi = \frac{u^2}{4}$ (parameter u must be a positive real number).
- pq should be defined appropriately for the tangent, cotangent, secant and cosecant functions.
- The expression should be well-defined for the given values of $\sqrt{\frac{\mu}{2}}\,\eta$ and, $\xi = \frac{u^2}{4}$.

Graphical behavior

Various types of solitons are shown below, each displaying the graphical behavior of the solution to the issue mentioned above.

For Case I

For Case II

For Case III

For Case IV

The parameters used to generate these figures are listed below.

Results and discussion

The surface and contour plots of each solution using each condition stated in the technique are shown in the graphs above where Fig. 1 shows breather wave singular solitonic behavior of U of the given coupled system using the conditions in case I represented in Eq. (19), Fig. 2 shows the surface and contour plots of V like rogue wave (singular) solitons represented in Eq. (19) of the same case I, Fig. 3 represents the surface and contour plots showing w-shaped dark-bright solitons in the form of U using the condition given in case II represented in Eq. (21), Fig. 4 signifies the surface and contour plots representing periodic function solitons in the form of V represented in Eq. (21) of the same case II, Fig. 5 shows the plots as kink soliton type behavior of U of the given system of equations represented in Eq. (23) under conditions of case III, Fig. 6 displays the surface and contour plots signifying kink solitons with non-topological (bright) background in the form of V of Eq. (23) of the same case III, Fig. 7 represents the surface and contour plots of chirped periodic solitons of U represented in Eq. (25) under condition of case IV, Fig. 8 signifies the surface and contour plots representing dark-bright solitons in the form of V of the given system represented in Eq. (25) of the same case IV.

Physical interpretation

Discovering NPDE solutions is critical for comprehending the underlying physical processes. Solitons are important in mathematics and physics because they keep their shape and velocity constant while propagating. The modulation instability of the carrier wave train requires distinguishing between topological (dark) and non-topological (bright) solitons. Topological solitons occur when the carrier wave is unsustainable due to long-wave modulations, whereas non-topological solitons occur when the carrier wave is modulationally consistent.

Rogue waves, often known as freakish or killer waves, have progressed from maritime folklore to a recognized phenomenon. These waves, which are twice the magnitude of the surrounding waves, are unpredictable and frequently appear from other directions than the current wind and waves. The study of rogue waves advances our understanding of extraordinary phenomena in fluid dynamics.

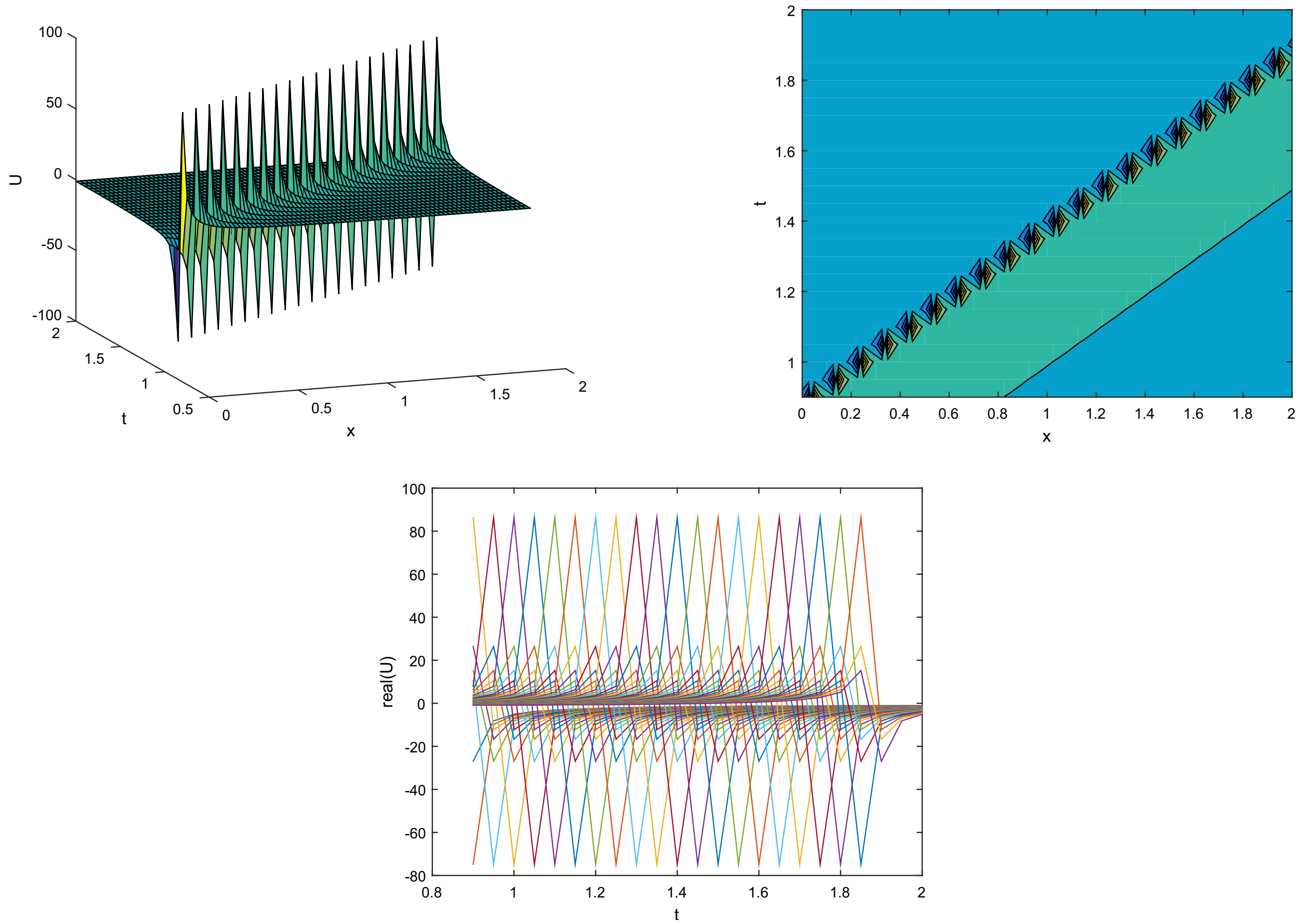

Figure 1. Above plottings are associated to Case I function U.

Conclusions

This article presents new and interesting optical soliton solutions to the (2+1)-Dimensional Coupled System of the BLP equations using the analytical method of SSET. Our main goal in writing this article is to assess the BLP system using this methodology for the first time. This is a relatively new method that yields several new soliton solutions for the system being studied. The method is incredibly effective and easy to use. The results are given as hyperbolic, rational, and trigonometric functions. As we can see, this method provides a powerful, efficient,

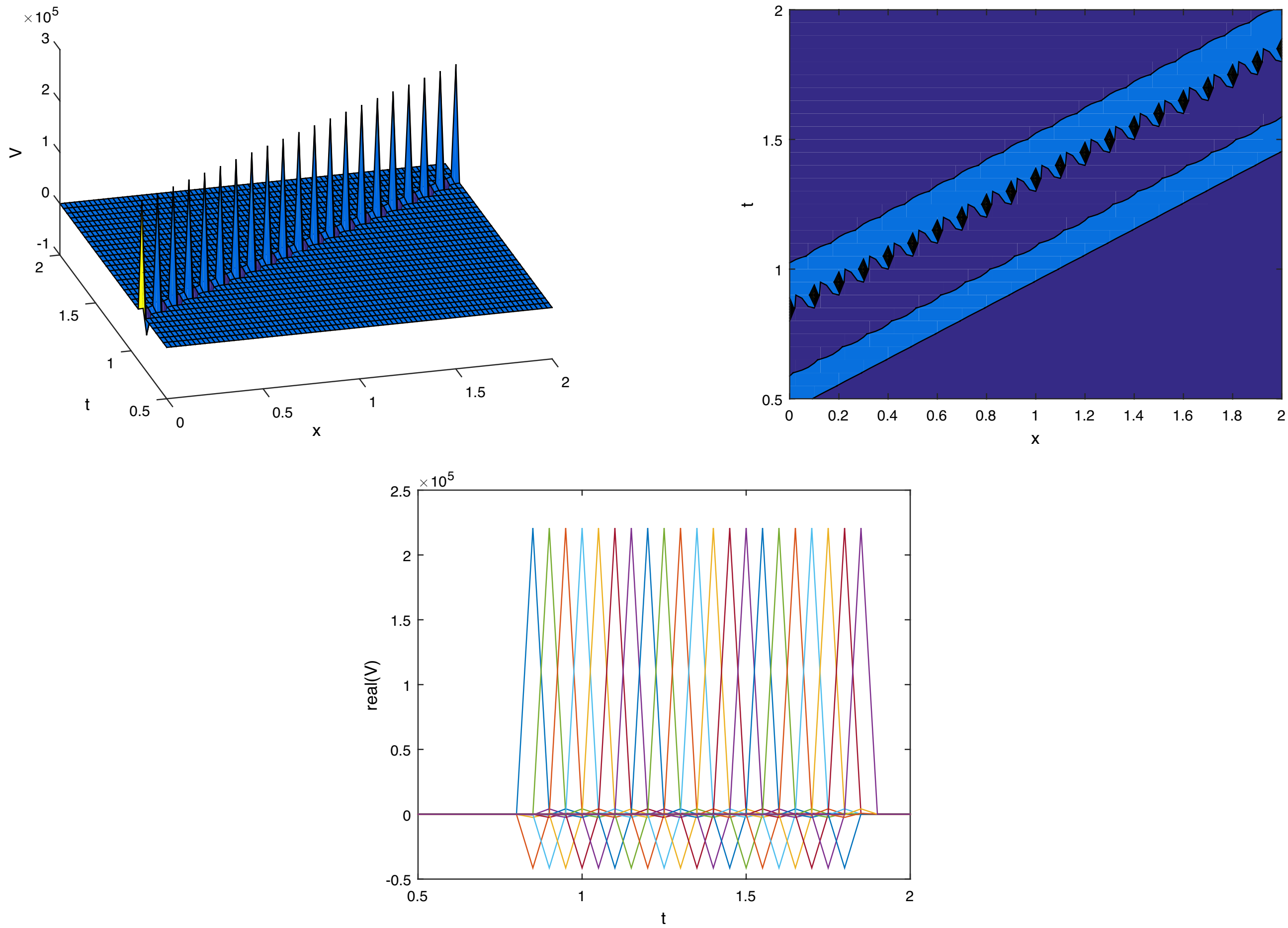

Figure 2. Graphical representation of function V Case I.

and simple tool for solving a range of nonlinear PDEs that are included in many models in the fields of natural science and engineering. The results may have practical applications and explain water waves in domains such as optics, linked circuits, elastic rods, shallow water with long wavelengths, and marine engineering. Lastly, 3D and contour plots of these solutions are produced using Maple. Bright, dark, periodic, chirped, breather-waved, singular, w-shaped, and breather-waved solitons are the results of this approach. We have investigated the forms and directions of the different solitons using the generated graphs.

As we all know, in the field of integrable systems, there is no general method to solve the analytical solution of NPDEs. The symbol calculation method based on neural networks proposed by Zhang et al. (see,[42–45]) open up a general symbolic computing path for the analytic solution of NPDEs, and lays the foundation for the universal method of symbolic calculation of analytical expression. The problems studied in this paper can be solved by using this method in future work.

This work is novel in itself. Within the framework of analytical solutions, researchers can choose our solution for numerical analysis. This methodology distinguishes itself from other ways by providing a systematic approach, comprehensive applicability, efficiency and brevity, the generation of many solutions, the reduction of equations to simpler ones, and integration with other techniques.

Graphing parameters

The graphs in this article were created using the settings listed below.

Figure 1: $c = 2; u = 2.5; p = 1; q = -0.09; y = 1.$

Figure 2: $c = 2; u = -2.5; p = 1; q = -0.09; y = 1.$

Figure 3: $c = 2; u = -1.5; p = 2; q = 1; y = 2.$

Figure 4: $c = -5; u = 2.5; p = 1; q = -0.09; y = 5.$

Figure 5: $c = 2; u = 2.5; p = 0.8; q = -0.09; y = 1.$

Figure 6: $c = 2; u = -2.5; p = 1; q = -0.09; y = 1.$

Figure 7: $c = 2; u = -5.5; p = 2; q = 1; y = 2.$

Figure 8: $c = -5; u = 2.5; p = 1; q = 0.09; y = 5.$

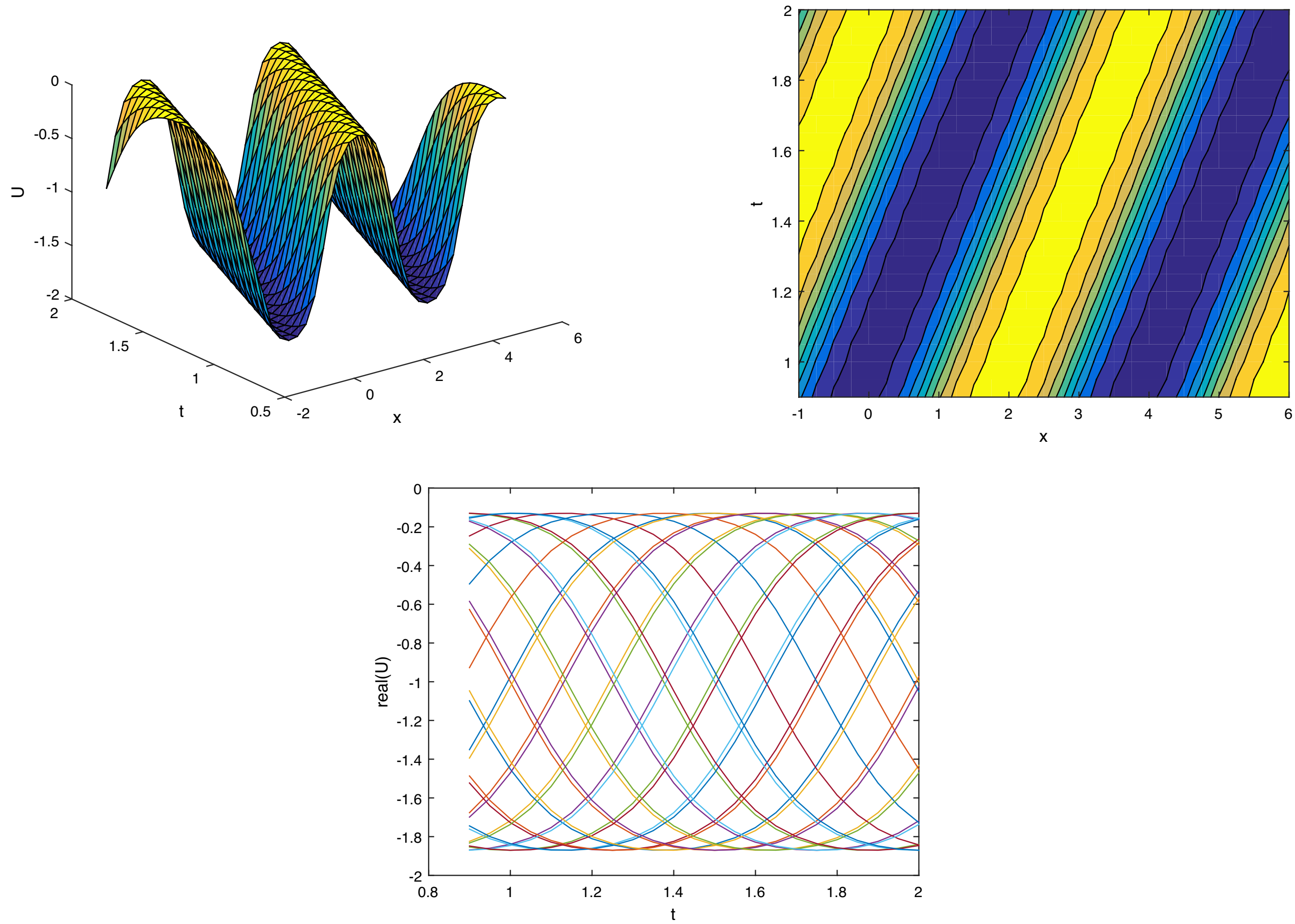

Figure 3. Plots associated to Case II function U.

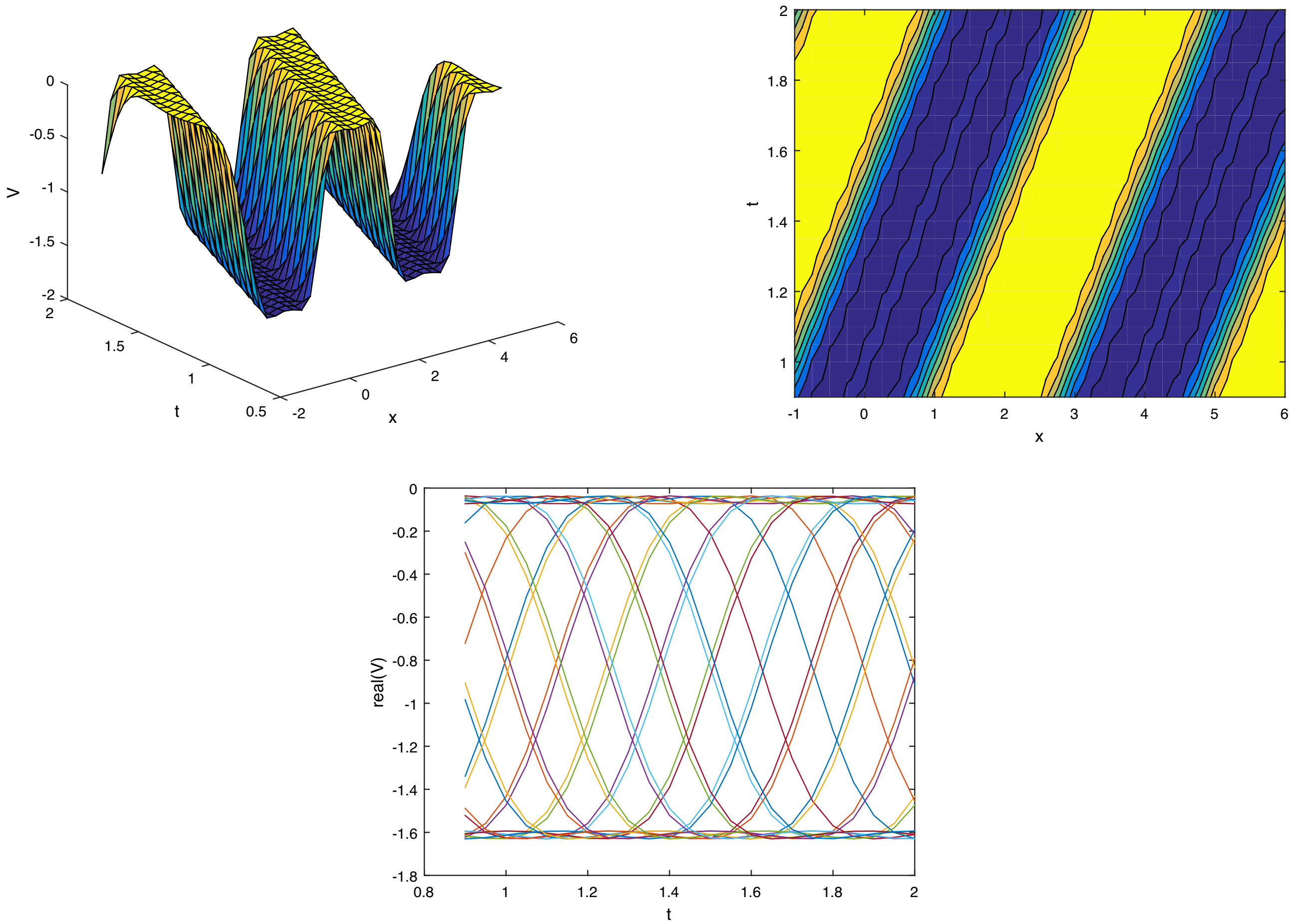

Figure 4. Graphing of exact solutions linked with Case II function V.

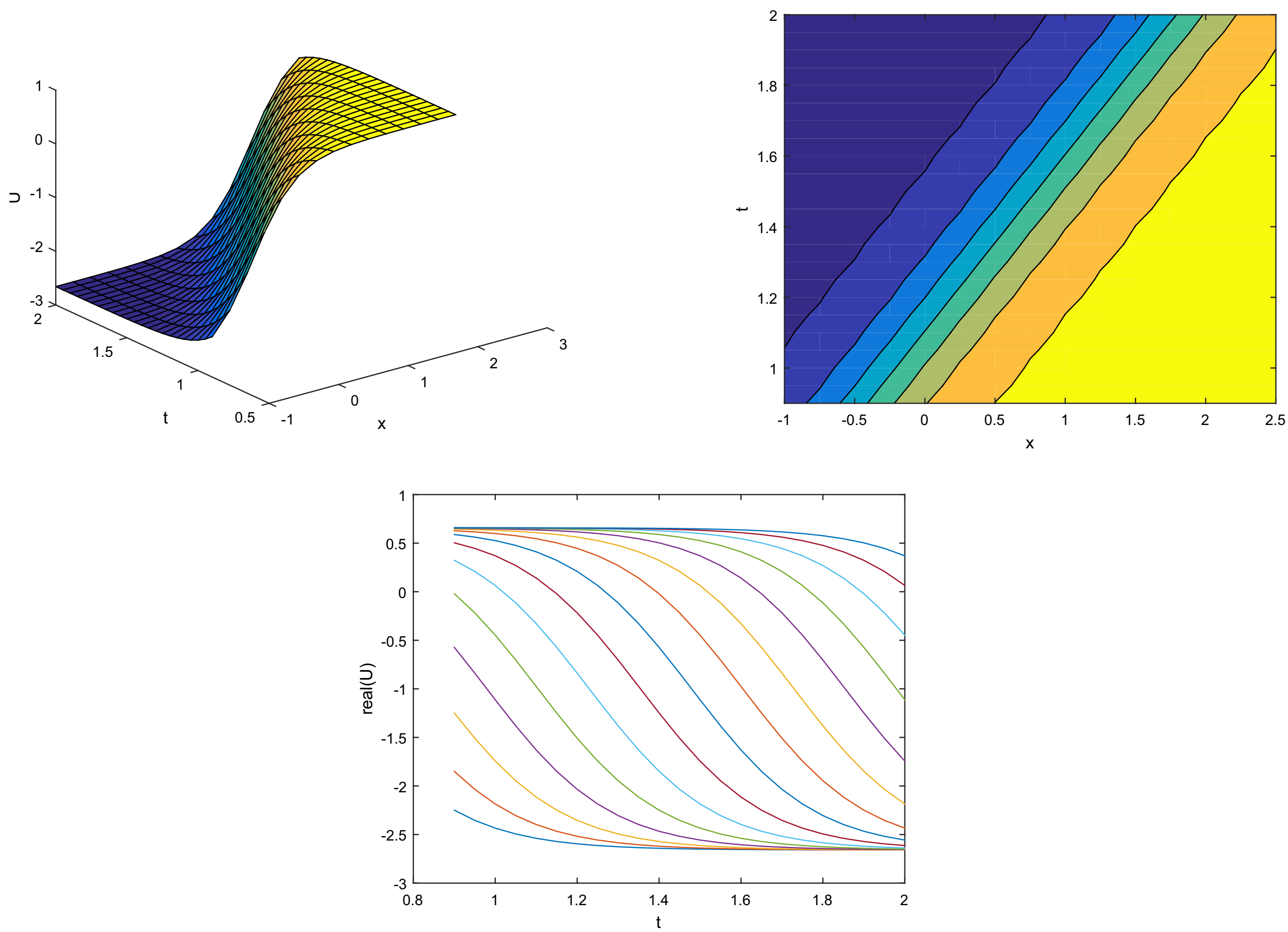

Figure 5. Plots representing Case III function U.

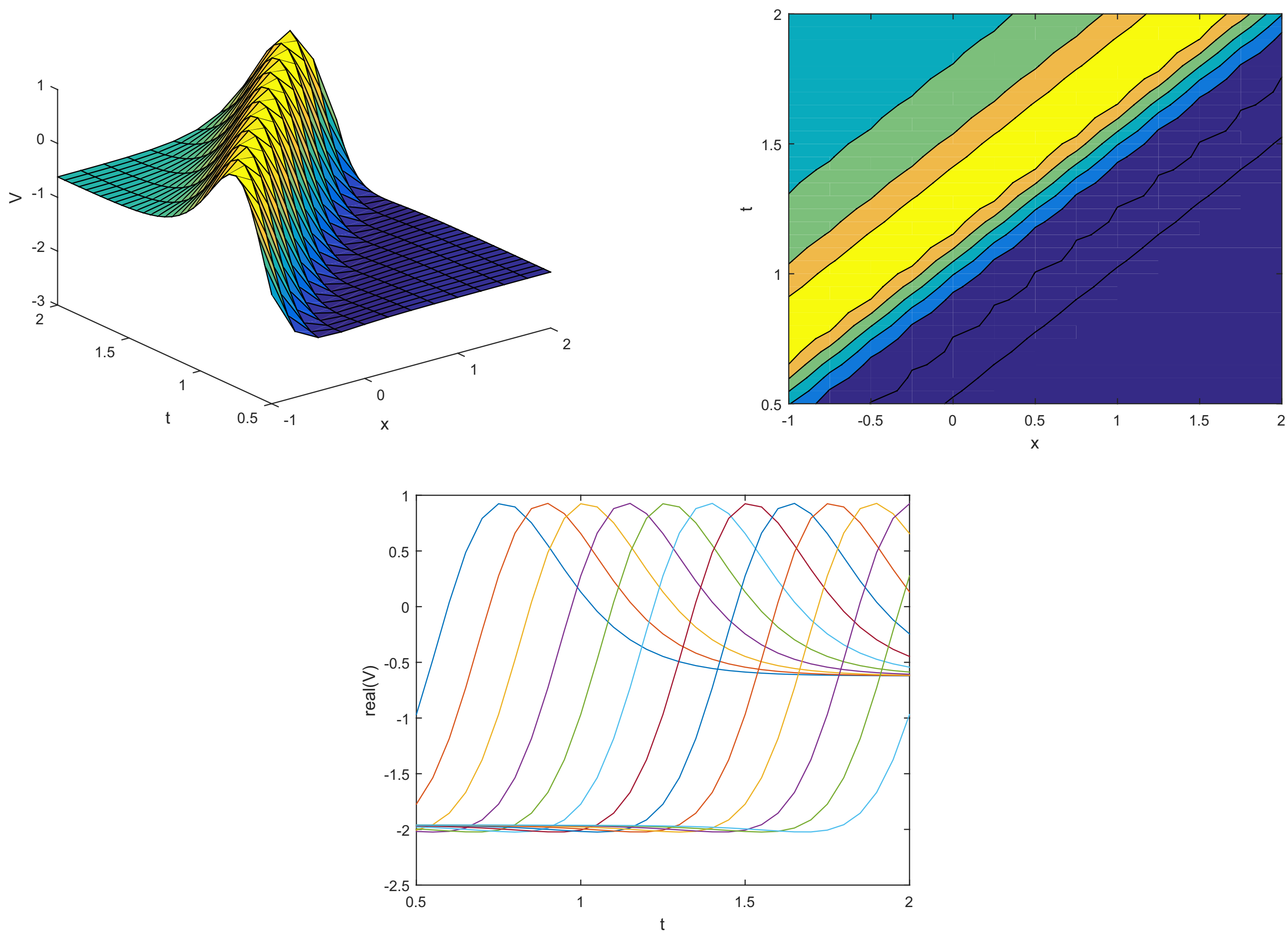

Figure 6. Surface and contour plots representing Case III function V.

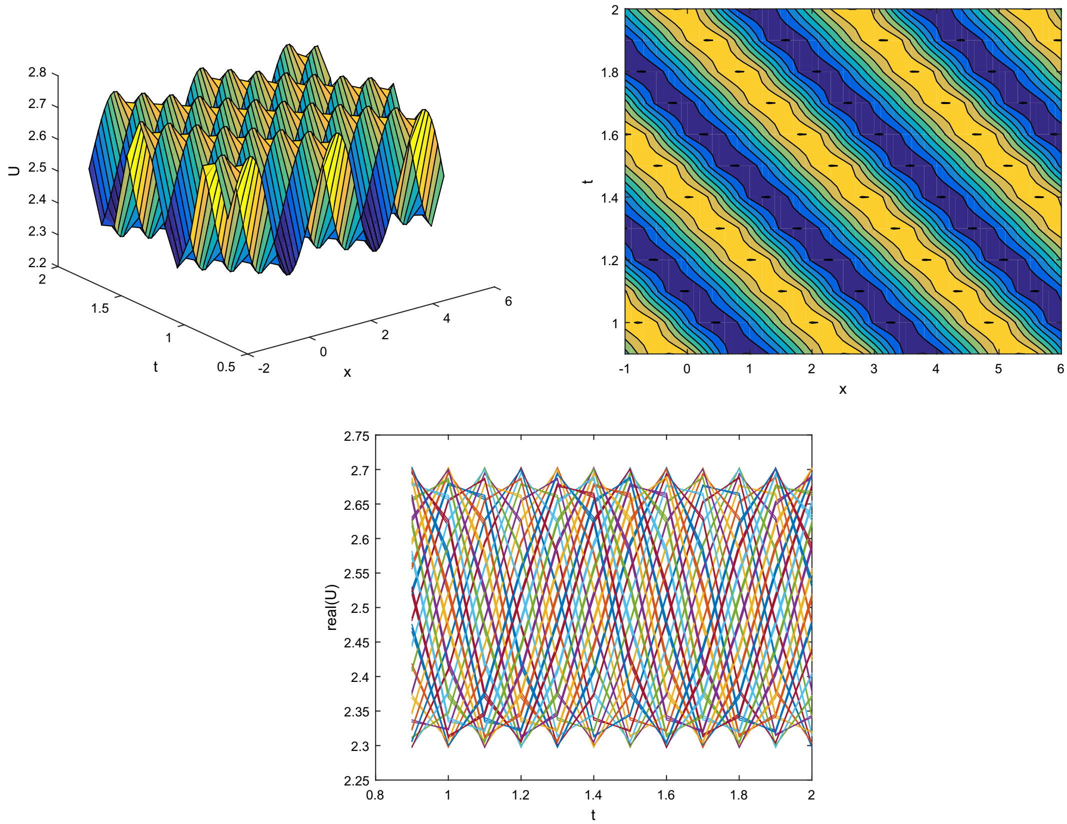

Figure 7. Representation of exact solutions of Case IV function U.

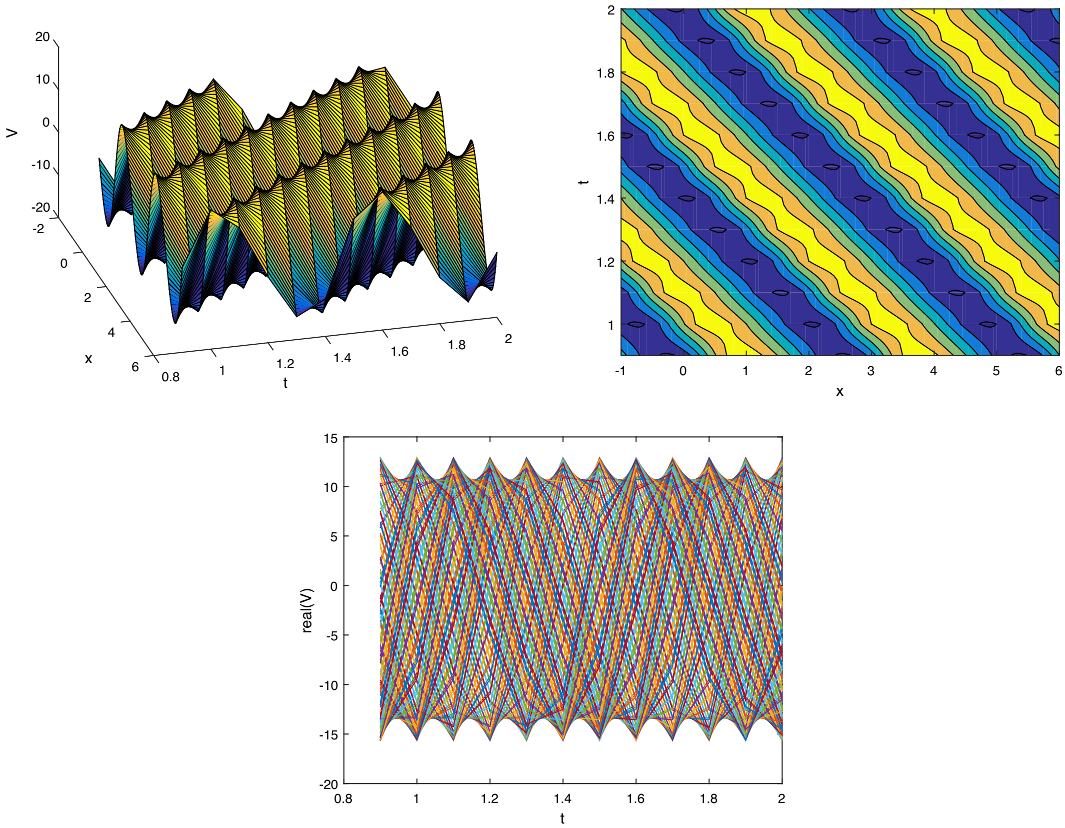

Figure 8. Plots signifying Case IV function V.

Data availibility

Data will be provided by corresponding author on reasonable request.

Received: 28 May 2024; Accepted: 15 July 2024
Published online: 30 August 2024

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Acknowledgements
This research is funded by "Researchers Supporting Project number (RSPD2024R733), King Saud University, Riyadh, Saudi Arabia.

Author contributions
S.G. and N.A. and M.S.I. and S.M.A. wrote the main manuscript text and A.A. and S.M. and M.A. and M.K.H. prepared Figs. 1-8. All authors reviewed the manuscript.

Competing interests
The authors declare no competing interests.

Additional information
Correspondence and requests for materials should be addressed to M.K.H.