RECTANGULAR SHRINKING TARGETS FOR $\mathbb{Z}^m$ ACTIONS ON TORI: WELL AND BADLY APPROXIMABLE SYSTEMS

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Abstract. In this paper we investigate the shrinking target property for irrational rotations. This was first studied by Kurzweil (1951) and has received considerable interest of late. Using a new approach, we generalize results of Kim (2007) and Shapira (2013) by proving a weighted effective analogue of the shrinking target property. Furthermore, our results are established in the much wider S-arithmetic setting.

1. Introduction

The shrinking target problem for a dynamical system typically seeks to study visits of orbits of the dynamical system to some neighbourhoods of a point (regarded as 'targets') whose sizes are 'shrinking' over 'time' as the orbits evolve. It is a manifestation of the chaotic behaviour of the dynamical system and is closely connected to questions in Diophantine approximation. This paper is devoted to perhaps the oldest and simplest example of the shrinking target problem, namely the case of irrational rotations on the circle, or more generally, $\mathbb{Z}^m$ actions on tori. The 'targets' in this case are the neighbourhoods of a point on the torus and the problem is to hit shrinking balls around this point by orbits of an irrational rotation. This problem is closely connected to *inhomogeneous Diophantine approximation.* The subject can be said to have started with a result of Kurzweil [9] who provided a necessary and sufficient condition on the rotation angle to ensure that a Borel-Cantelli style '0-1' law holds in this setting. More recently, in [5], Kim proved the following beautiful result.

Theorem 1.1. *Let θ be any irrational number. For almost every $s \in \mathbb{R}$ we have*

$$\liminf_{n\to\infty} n \min_{b\in\mathbb{Z}} |n\theta - b - s| = 0.$$

Kim used techniques from the theory of continued fractions, specifically the Ostrowski expansion, and also obtained further refinements in [6]. See also [12, 8] and [10] for recent developments in this direction. A higher dimensional version of Kim's result was proved by Shapira [11] using the tools of homogeneous dynamics, namely the classification of divergent orbits for diagonal flows on the space of lattices and considerations from the geometry of numbers. The irrationality condition, imposed by Kim in the one dimensional case, is replaced with a *non-singularity* condition in higher dimensions, see Definition 1.2 below. However, note that non-singularity coincides with irrationality in the one dimensional setting. We will

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provide a bi-fold generalization of Kim's result as well as Shapira's result by firstly establishing the corresponding shrinking target property for rectangular targets (defined by weights) and secondly considering the much wider S-arithmetic setting. Our methods differ from those of Kim and Shapira; we rely on a softer approach using the technology of 'constant invariant limsup sets' developed in our paper [1].

In recent years, the shrinking target problem for $\mathbb{Z}^m$ actions on tori has received considerable attention from both the dynamical systems as well as the number theory community. In addition to the works cited above, [2, 13, 4, 7, 8] deal with Hausdorff dimension questions related to circle rotations and inhomogeneous Diophantine approximation. There have also been several developments on the 'badly approximable' problem beginning with [3].

1.1. **Real tori: a special case.** We begin by stating a special case of our main theorem, namely a weighted generalization of Shapira's theorem. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_{>0}$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m_{>0}$ be n- and m-tuples of non negative real numbers that will be referred to as *weights*. We will assume throughout that

$$\sum_{i=1}^{n} \tau_i = \sum_{j=1}^{m} \eta_j \, .$$

Given vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we define the following quasi-norms associated with the weights:

$$|\mathbf{x}|_{\boldsymbol{\tau}} := \max_{1 \le i \le n} |x_i|^{1/\tau_i} \qquad \text{and} \qquad |\mathbf{y}|_{\boldsymbol{\eta}} := \max_{1 \le j \le m} |y_j|^{1/\eta_j} \, .$$

In weighted Diophantine approximation, one replaces the usual supremum norms with weighted quasi-norms as above. With this notation, we can now state a weighted analogue of Dirichlet's theorem. The proof follows from the usual argument using Minkowski's convex body theorem.

Weighted Dirichlet's theorem: *Let $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ be as above. Then for any real $m \times n$ matrix $\boldsymbol{\alpha} = (\alpha_{i,j})$ and any $H > 1$ there exists $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n}$ with $\mathbf{a} \neq \mathbf{0}$ such that*

$$|\mathbf{a}\boldsymbol{\alpha} - \mathbf{b}|_{\boldsymbol{\tau}} < H^{-1} \qquad \textit{and} \qquad |\mathbf{a}|_{\boldsymbol{\eta}} \le H \, . \tag{1}$$

Similarly, one can define a weighted analogue of singular vectors and non-singular vectors.

Definition 1.2. An $m \times n$ real matrix $\boldsymbol{\alpha}$ will be called $(\boldsymbol{\tau}, \boldsymbol{\eta})$-*singular* if for any $\varepsilon > 0$ there exists H_0 such that for all $H > H_0$ there exists $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n}$ with $\mathbf{a} \neq \mathbf{0}$ satisfying

$$|\mathbf{a}\boldsymbol{\alpha} - \mathbf{b}|_{\boldsymbol{\tau}} < \varepsilon H^{-1} \qquad \text{and} \qquad |\mathbf{a}|_{\boldsymbol{\eta}} \le H \, . \tag{2}$$

Otherwise, $\boldsymbol{\alpha}$ will be called $(\boldsymbol{\tau}, \boldsymbol{\eta})$-*non-singular.*

We are now ready to state the following weighted generalisation of the Kim-Shapira theorem.

Theorem 1.3. *Let $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ be as above and $\boldsymbol{\alpha}$ be any real $m \times n$ matrix which is $(\boldsymbol{\tau}, \boldsymbol{\eta})$-non-singular. Then for almost every $\boldsymbol{\gamma} \in \mathbb{R}^n$ we have that for any $\varepsilon > 0$ there are infinitely many $H \in \mathbb{N}$ such that*

$$|\mathbf{a}\boldsymbol{\alpha} - \mathbf{b} - \boldsymbol{\gamma}|_{\boldsymbol{\tau}} < \varepsilon H^{-1} \qquad \textit{and} \qquad |\mathbf{a}|_{\boldsymbol{\eta}} \le H \, , \tag{3}$$

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or equivalently

$$\liminf_{|\mathbf{a}|_{\boldsymbol{\eta}}\to\infty} |\mathbf{a}|_{\boldsymbol{\eta}} \min_{\mathbf{b}\in\mathbb{Z}^n} |\mathbf{a}\boldsymbol{\alpha} - \mathbf{b} - \boldsymbol{\gamma}|_{\boldsymbol{\tau}} = 0\,. \tag{4}$$

Note that while it may be possible to obtain Theorem 1.3 from the techniques employed in [11] (by choosing the functions $a_{n,m}(t)$ and $P_{n,m}(\boldsymbol{x},\boldsymbol{y})$ appearing in [11, Section 2.2] appropriately), our methods are entirely different. Even in the unweighted case, our method gives a new proof of [11, Theorem 3.4]. In Section 3 we sketch a proof of Theorem 1.3. As mentioned earlier, our main theorem is considerably more general and includes the S-arithmetic setting. We now introduce this setup and state our main result.

2. S-arithmetic setup and the main theorem

Let S be a collection of valuations of $\mathbb{Q}$ with cardinality $l \geq 1$. Let $S^\star = S\backslash\{\infty\}$. Let $\mathbb{Q}_S$ denote the set of S-arithmetic points, that is

$$\mathbb{Q}_S := \prod_{\nu\in S}\mathbb{Q}_\nu,$$

where $\mathbb{Q}_\nu$ is the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_\nu$. In particular, if $\nu \in S^\star$ is a prime number p, then $\mathbb{Q}_\nu = \mathbb{Q}_p$ is the field of p-adic numbers, and if $\nu = \infty$, then $\mathbb{Q}_\nu = \mathbb{R}$. Let

$$\mathbb{Z}_S := \prod_{\nu\in S}\mathbb{Z}_\nu\,,$$

where for $\nu \in S^\star$ we have that $\mathbb{Z}_\nu = \{x \in \mathbb{Q}_\nu : |x|_\nu \leq 1\}$ is the set of ν-adic integers, and $\mathbb{Z}_\infty = [0,1)$. Let μ_S be the S-arithmetic Haar measure, normalised by $\mu_S(\mathbb{Z}_S) = 1$. Note that μ_S is simply the product of the measures μ_ν over each $\mathbb{Q}_\nu$ normalised so that $\mu_\nu(\mathbb{Z}_\nu) = 1$ for $\nu \in S^\star$ with μ_∞ being Lebesgue measure over $\mathbb{R} = \mathbb{Q}_\infty$, *i.e.*

$$\mu_S = \prod_{\nu\in S}\mu_\nu\,.$$

Similarly, in higher dimensional settings, associated with systems of linear forms, let $\mu_{S,m\times n}$ denote the Haar measure on $\mathbb{Q}_S^{mn}$, normalised by $\mu_{S,m\times n}(\mathbb{Z}_S^{mn}) = 1$. We will denote any $\mathbf{x} \in \mathbb{Q}_S^{mn}$ as $\mathbf{x} = (\mathbf{x}_\nu) = (\mathbf{x}_\nu)_{\nu\in S}$, where $\mathbf{x}_\nu = (x_{i,j,\nu}) \in \mathbb{Q}_\nu^{mn}$, that is $(x_{i,j,\nu})$ is an $m \times n$ matrix over $\mathbb{Q}_\nu$ for each ν. Let

$$\omega = \begin{cases} m+n & \text{if } \infty \notin S, \\ m & \text{if } \infty \in S. \end{cases} \tag{5}$$

Let $\boldsymbol{\tau} = (\tau_{i,\nu})_{1\leq i\leq n,\,\nu\in S}$ and $\boldsymbol{\eta} = (\eta_\ell)_{1\leq\ell\leq\omega}$ be two collections of positive real numbers, which will be referred to as *weights*, such that

$$\sum_{i=1}^{n}\sum_{\nu\in S}\tau_{i,\nu} = \omega = \sum_{\ell=1}^{\omega}\eta_\ell\,. \tag{6}$$

Given weights $\boldsymbol{\tau} = (\tau_{i,\nu})_{1\leq i\leq n,\nu\in S}$, we define the $\boldsymbol{\tau}$-semi-norm of $\mathbf{x} \in \mathbb{Q}_S^n$ as

$$|\mathbf{x}|_{\boldsymbol{\tau}} = \max_{\nu\in S}\max_{1\leq i\leq n}|x_{i,\nu}|_\nu^{1/\tau_{i,\nu}}.$$

Also, when $\infty \in S$, we will separate out the weights at the infinite and finite places of S by introducing $\boldsymbol{\tau}_\infty := (\tau_{i,\infty})_{1\leq i\leq n}$ and $\boldsymbol{\tau}^\star = (\tau_{i,\nu})_{1\leq i\leq n,\nu\in S^\star}$. The following is an S-arithmetic version of Dirichlet theorem with weights, see for example [1, Corollary 6.3].

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Theorem 2.1. *Let $\boldsymbol{\tau} = (\tau_{i,\nu})_{1\le i\le n,\,\nu\in S}$ and $\boldsymbol{\eta} = (\eta_\ell)_{1\le\ell\le\omega}$ be two collections of weights satisfying* (6)*. Then for any $\mathbf{x} = (x_{i,j,\nu}) \in \mathbb{Z}_S^{mn}$ and any $H > 1$ there exists $(\mathbf{a},\mathbf{b}) = (a_1,\dots,a_m,b_1,\dots,b_n) \in \mathbb{Z}^{m+n}\setminus\{\mathbf{0}\}$ satisfying*

$$|a_1x_{i,1,\nu} + \cdots + a_mx_{i,m,\nu} - b_i|_\nu \le \nu H^{-\tau_{i,\nu}} \qquad (1\le i\le n,\ \nu\in S^\star) \tag{7}$$

$$|a_1x_{i,1,\nu} + \cdots + a_mx_{i,m,\nu} - b_i|_\nu < H^{-\tau_{i,\nu}} \qquad (1\le i\le n,\ \nu=\infty) \quad \text{if } \infty\in S \tag{8}$$

$$|a_j| \le H^{\eta_j} \qquad (1\le j\le m)\,, \tag{9}$$

$$|b_i| \le H^{\eta_{m+i}} \qquad (1\le i\le n) \qquad \text{if } \infty\notin S\,. \tag{10}$$

Remark 2.2. In the above theorem if $\infty\in S$ or

$$\eta_{m+i} < \sum_{\nu\in S}\tau_{i,\nu} \qquad \text{for all } 1\le i\le n \tag{11}$$

then for large enough H for any $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}\setminus\{\mathbf{0}\}$, inequalities (7)—(10) imply that $\mathbf{a}\neq\mathbf{0}$.

Next, we define weighted singular and non-singular matrices in the S-arithmetic setting.

Definition 2.3. Let $\boldsymbol{\alpha} = (\alpha_{i,j,\nu}) \in \mathbb{Z}_S^{mn}$ be given. Then $\boldsymbol{\alpha}$ will be called $(\boldsymbol{\tau},\boldsymbol{\eta})$-*singular* if for any $\varepsilon>0$ there exists $H_0>0$ such that for all $H>H_0$ there exists $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}\setminus\{\mathbf{0}\}$ satisfying

$$|\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}|_{\boldsymbol{\tau}} < \varepsilon H^{-1} \qquad \text{and} \qquad H \ge \begin{cases} |(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}} & \text{if } \infty\notin S \\ |\mathbf{a}|_{\boldsymbol{\eta}} & \text{if } \infty\in S. \end{cases} \tag{12}$$

Otherwise, $\boldsymbol{\alpha}$ will be called $(\boldsymbol{\tau},\boldsymbol{\eta})$-*non-singular.*

Now we are fully ready to state our main Theorem:

Theorem 2.4. *Let $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ be as above and $\boldsymbol{\alpha}$ be any $m\times n$ matrix with entries in $\mathbb{Q}_S$. Suppose $\boldsymbol{\alpha}$ is $(\boldsymbol{\tau},\boldsymbol{\eta})$-non-singular. Then for almost every $\boldsymbol{\gamma}\in\mathbb{Q}_S^n$ we have that for any $\varepsilon>0$ there are infinitely many $H\in\mathbb{N}$ such that for some $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}$ we have that*

$$|\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}-\boldsymbol{\gamma}|_{\boldsymbol{\tau}} < \varepsilon H^{-1} \qquad \text{and} \qquad H \ge \begin{cases} |(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}} & \text{if } \infty\notin S, \\ |\mathbf{a}|_{\boldsymbol{\eta}} & \text{if } \infty\in S, \end{cases}. \tag{13}$$

Equivalently, for almost every $\boldsymbol{\gamma}\in\mathbb{Q}_S^n$ we have that

$$\liminf_{\mathbf{a}\in\mathbb{Z}^m} |\mathbf{a}|_{\boldsymbol{\eta}} \min_{\mathbf{b}\in\mathbb{Z}^n} |\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}-\boldsymbol{\gamma}|_{\boldsymbol{\tau}} = 0 \qquad \text{if } \infty\in S, \tag{14}$$

and

$$\liminf_{(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}} |(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\, |\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}-\boldsymbol{\gamma}|_{\boldsymbol{\tau}} = 0 \qquad \text{if } \infty\notin S. \tag{15}$$

Remark 2.5. If we have finitely many solutions $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}$ or $\mathbf{a}\in\mathbb{Z}^m$ for $\infty\notin S$ and $\infty\in S$ respectively, to (13) for infinitely many H, then $\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}=\boldsymbol{\gamma}$ for some $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}$, and there are only countably many such $\boldsymbol{\gamma}$. Thus, (14)/(15) follows immediately from (13). The converse in this equivalence is obtained even easier by setting H to be $|(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}$ or $|\mathbf{a}|_{\boldsymbol{\eta}}$, depending on the case we are in, rounded to the next integer.

Prior to embarking onto the proof of the main theorem, in the next section, we sketch a proof of Theorem 1.3. It reveals the main ideas while avoiding some of the necessarily technical parts of the proof of Theorem 2.4. In section 4 we introduce a key measure theoretic tool from our paper [1]. Section 5 is devoted to the proof of the main theorem.

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3. A sketch of a proof of Theorem 1.3

In what follows $\ll$ will denote the inequality $\le$ with an unspecified constant factor, and $\asymp$ will mean both $\ll$ and $\gg$. Let $\boldsymbol{\alpha}$ be $(\boldsymbol{\tau},\boldsymbol{\eta})$-non-singular. Then for arbitrarily large $t>0$ the first Minkowski minima of the lattice

$$g_t u_{\boldsymbol{\alpha}} \mathbb{Z}^{m+n}$$

is bounded below by a fixed constant, where

$$g_t = \operatorname{diag}\{e^{\tau_1 t},\dots,e^{\tau_n t},e^{-\eta_1 t},\dots,e^{-\eta_m t}\}$$

and

$$u_{\boldsymbol{\alpha}} = \begin{pmatrix} \boldsymbol{\alpha} & I_m \\ I_n & 0 \end{pmatrix}.$$

Fix one of these t. Then, by Minkowski's second theorem, the last minimum of $g_t u_{\boldsymbol{\alpha}}\mathbb{Z}^{m+n}$ is bounded above by a constant, say $C_0>0$ independent of t, and we have a basis of this lattice

$$\mathbf{v}_\ell = g_t u_{\boldsymbol{\alpha}}(\mathbf{a}_\ell,\mathbf{b}_\ell)^T \qquad (1\le \ell\le m+n)$$

of vectors in the ball of radius C_0. Then, there are real numbers $x_1,\dots,x_{m+n}$ such that

$$\sum_{\ell=1}^{m+n} x_\ell \mathbf{v}_\ell = g_t(\boldsymbol{\gamma},\mathbf{0})^T.$$

Round each x_ℓ to the nearest non-zero integer y_ℓ and define

$$(\mathbf{a},\mathbf{b})^T = \sum_{\ell=1}^{m+n} y_\ell(\mathbf{a}_\ell,\mathbf{b}_\ell)^T \in \mathbb{Z}^{m+n}.$$

Then

$$\left|g_t u_{\boldsymbol{\alpha}}(\mathbf{a},\mathbf{b})^T - g_t(\boldsymbol{\gamma},\mathbf{0})^T\right|_\infty = \left|\sum_{\ell=1}^{m+n} y_\ell g_t u_{\boldsymbol{\alpha}}(\mathbf{a}_\ell,\mathbf{b}_\ell)^T - g_t(\boldsymbol{\gamma},\mathbf{0})^T\right|_\infty = \left|\sum_{\ell=1}^{m+n} y_\ell \mathbf{v}_\ell - g_t(\boldsymbol{\gamma},\mathbf{0})^T\right|_\infty \ll 1$$

which is the same as the system

$$\left|\sum_{j=1}^{m} a_j\alpha_{i,j} - b_i - \gamma_i\right| \ll e^{-\tau_i t} \qquad (1\le i\le n)$$
$$|a_j| \ll e^{\eta_j t} \qquad (1\le j\le m).$$

Setting $H\asymp e^t$ we get the following version of Dirichlet's theorem.

Twisted weighted asymptotic Dirichlet for non-singular matrices: *Let $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ be as above, and $\boldsymbol{\alpha}$ be any $(\boldsymbol{\tau},\boldsymbol{\eta})$-non-singular real $m\times n$ matrix. Then there is a constant $C>0$ depending on $\boldsymbol{\alpha}$ only and an increasing unbounded sequence of $H_i>0$ such that for every $H=H_i$ and for every $\boldsymbol{\gamma}\in\mathbb{R}^n$ there exists $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{m+n}$ such that*

$$|\mathbf{a}\boldsymbol{\alpha}-\mathbf{b}-\boldsymbol{\gamma}|_{\boldsymbol{\tau}} < CH^{-1} \qquad \text{and} \qquad |\mathbf{a}|_{\boldsymbol{\eta}}\le H. \tag{16}$$

Furthermore, for almost all $\boldsymbol{\gamma}$ there are infinitely many different $(\mathbf{a},\mathbf{b})$ arising from (16).

Theorem 1.3 now follows as a rather straightforward application of Lemma 4.1 below (which in turn is Lemma 5.7 from [1]). The rather simple details can also be inferred from the proof of the same ilk in the S-arithmetic setting given in §5.3.

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4. An auxiliary result

In this section we introduce an auxiliary result from [1]. Fix an integer $n \geq 1$, and for each $1 \leq i \leq n$ let (X_i, d_i, μ_i) be a metric space equipped with a σ-finite Borel regular measure μ_i. Let (X, d, μ) be the product space with $X = \prod_{i=1}^n X_i$, $\mu = \mu_1 \times \cdots \times \mu_n$ being the product measure, and

$$d(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \max_{1\leq i\leq n} d_i(x_i^{(1)}, x_i^{(2)}), \qquad \text{where } \mathbf{x}^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)}) \in X \text{ for } j = 1, 2.$$

Lemma 4.1 ([1] Lemma 5.7)**.** *Let $n \in \mathbb{N}$. For each $1 \leq j \leq n$ let (X_j, d_j, μ_j) be a metric measure space equipped with a σ-finite doubling Borel regular measure μ_j. Let $X = \prod_{j=1}^n X_j$ be the corresponding product space, $d = \max_{1\leq j\leq n} d_j$ be the corresponding metric, and $\mu = \prod_{j=1}^n \mu_j$ be the corresponding product measure. Let $(S_i)_{i\in\mathbb{N}}$ be a sequence of subsets of* $\operatorname{supp}\mu$ *and $(\boldsymbol{\delta}_i)_{i\in\mathbb{N}}$ be a sequence of positive n-tuples $\boldsymbol{\delta}_i = (\delta_i^{(1)}, \ldots, \delta_i^{(n)})$ such that $\delta_i^{(j)} \to 0$ as $i \to \infty$ for each $1 \leq j \leq n$. Let*

$$\Delta_n(S_i, \boldsymbol{\delta}_i) = \{\mathbf{x} \in X : \exists\, \mathbf{a} \in S_i \;\; d_j(a_j, x_j) < \delta_i^{(j)} \; \forall\; 1 \leq j \leq n\},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a} = (a_1, \ldots, a_n)$. Then, for any $\boldsymbol{C} = (C_1, \ldots, C_n)$ and $\boldsymbol{c} = (c_1, \ldots, c_n)$ with $0 < c_j \leq C_j$ for each $1 \leq j \leq n$

$$\mu\left(\limsup_{i\to\infty} \Delta_n(S_i, \boldsymbol{C}\boldsymbol{\delta}_i) \;\setminus\; \limsup_{i\to\infty} \Delta_n(S_i, \boldsymbol{c}\boldsymbol{\delta}_i)\right) = 0, \tag{17}$$

where $\boldsymbol{c}\boldsymbol{\delta}_j = (c_1\delta_i^{(1)}, \ldots, c_n\delta_i^{(n)})$ and similarly $\boldsymbol{C}\boldsymbol{\delta}_j = (C_1\delta_i^{(1)}, \ldots, C_n\delta_i^{(n)})$.

One can also look at [1, Theorem 4.1] for a more general statement.

5. Proof of Theorem 2.4

First, we prove the following theorem which is a twisted weighted asymptotic Dirichlet for non-singular matrices.

Theorem 5.1. *Let S, $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ be as above, and $\boldsymbol{\alpha}$ be any $(\boldsymbol{\tau}, \boldsymbol{\eta})$-non-singular $m \times n$ matrix over $\mathbb{Q}_S$. Then there is a constant $C > 0$ depending on $\boldsymbol{\alpha}$ only and an increasing unbounded sequence $(H_i)_{i\in\mathbb{N}}$ of positive integers such that for every $i \in \mathbb{N}$ and any $\boldsymbol{\gamma} \in \mathbb{Q}_S^n$ there exists $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n} \setminus \{\mathbf{0}\}$ such that*

$$|\mathbf{a}\boldsymbol{\alpha} - \mathbf{b} - \boldsymbol{\gamma}|_{\boldsymbol{\tau}} < CH_i^{-1} \qquad \text{and} \qquad H_i \geq \begin{cases} |(\mathbf{a}, \mathbf{b})|_{\boldsymbol{\eta}} & \text{if } \infty \notin S \\ |\mathbf{a}|_{\boldsymbol{\eta}} & \text{if } \infty \in S. \end{cases} \tag{18}$$

In what follows, given a lattice $\Gamma \subset \mathbb{R}^{m+n}$ and a convex body $B \subset \mathbb{R}^{m+n}$ symmetric about the origin,

$$\lambda_1(\Gamma, B) \leq \cdots \leq \lambda_{m+n}(\Gamma, B) \tag{19}$$

will denote the Minkowski minima of Γ with respect to B, that is

$$\lambda_i(\Gamma, B) := \inf\left\{\lambda > 0 : \operatorname{rank}(\Gamma \cap \lambda B) \geq i\right\}.$$

5.1. **Proof of Theorem 5.1 in the case $\infty \notin S$.** Without loss of generality, we will assume that $\boldsymbol{\alpha} \in \mathbb{Z}_S^{mn}$ and $\boldsymbol{\gamma} \in \mathbb{Z}_S^n$. For any given $\varepsilon > 0$ and $H > 1$ define

$$\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H) := \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n} \ : \ |\mathbf{a}\boldsymbol{\alpha} + \mathbf{b}|_{\boldsymbol{\tau}} < \varepsilon H^{-1}, \ |(\mathbf{a}, \mathbf{b})|_\nu \leq \tfrac{1}{\nu} \quad \text{for all } \nu \in S \right\}. \tag{20}$$

It is readily seen, as a consequence of the strong triangle inequality for every $\nu \in S$, that $\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H)$ is a sub-lattice of $\mathbb{Z}^{m+n}$. Furthermore, observe that

$$\mathrm{Vol}\big(\mathbb{R}^{m+n}/\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H)\big) \ \leq \ \varepsilon^{-m-n} H^\omega \prod_{\nu \in S} \nu^{m+2n} \,. \tag{21}$$

Next, let us consider the set

$$K_H := \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{m+n} \ : \ |(\mathbf{a}, \mathbf{b})|_{\boldsymbol{\eta}} \leq H \right\},$$

which is obviously convex and symmetric about the origin. Observe that

$$\mathrm{Vol}(K_H) = 2^{m+n} H^\omega \,. \tag{22}$$

Since $\boldsymbol{\alpha}$ is $(\boldsymbol{\tau}, \boldsymbol{\eta})$-non-singular, there exists $\varepsilon > 0$ and an unbounded subset $\mathcal{H}$ of positive real numbers such that for any $H \in \mathcal{H}$ system (12) does not have any non-zero solution $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n}$. This implies that

$$\lambda_1(\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H), K_H) \geq 1 \qquad \text{for all } H \in \mathcal{H} \,. \tag{23}$$

By Minkowski's second theorem on successive minima, we also have that

$$\mathrm{Vol}(K_H) \prod_{\ell=1}^{m+n} \lambda_\ell(\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H), K_H) \leq 2^{m+n} \mathrm{Vol}(\mathbb{R}^{m+n}/\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H)).$$

Hence, by Equations (19), (21), (22) and (23), for any $H \in \mathcal{H}$ we have that

$$\lambda_{m+n}(\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H), K_H) \ \leq \ C_0 := \varepsilon^{-m-n} \prod_{\nu \in S} \nu^{m+2n} \,. \tag{24}$$

Fix any $H \in \mathcal{H}$. Then, by (24), there are $m+n$ linearly independent vectors $\{(\mathbf{a}^{(\ell)}, \mathbf{b}^{(\ell)})\}_{\ell=1}^{m+n}$ in $\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H) \cap C_0 K_H$. Hence, by (24) and the definitions of K_H and $\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H), K_H)$, we have that

$$\begin{aligned} &|\mathbf{a}^{(\ell)}\boldsymbol{\alpha} + \mathbf{b}^{(\ell)}|_{\boldsymbol{\tau}} < \varepsilon H^{-1} \\ &|a_j^{(\ell)}| \leq C_0 H^{\eta_j} \qquad\qquad (1 \leq j \leq m) \\ &|b_i^{(\ell)}| \leq C_0 H^{\eta_{m+i}} \qquad\quad (1 \leq i \leq n) \\ &|(\mathbf{a}^{(\ell)}, \mathbf{b}^{(\ell)})|_\nu \leq \tfrac{1}{\nu} \qquad\qquad (\nu \in S). \end{aligned} \tag{25}$$

Recall that for $H \in \mathcal{H}$ the system (12) does not have a non-zero integer solution. Therefore, for every $\ell = 1, \ldots, m+n$, we have that $|(\mathbf{a}^{(\ell)}, \mathbf{b}^{(\ell)})|_{\boldsymbol{\eta}} > H$. This means that

$$|a_{j^\star}^{(\ell)}| > H^{\eta_{j^\star}} \ \text{for some } 1 \leq j^\star \leq m \quad \text{ or } \quad |b_{i^\star}^{(\ell)}| > H^{\eta_{m+i^\star}} \ \text{for some } 1 \leq i^\star \leq n.$$

Let us denote this $j^\star$ or $i^\star$, which depends on ℓ, as $\ell^\star$. So we can rewrite the above equation as follows,

$$|b_{\ell^\star}^{(\ell)}| > H^{\eta_{m+\ell^\star}}, \ \text{or } |a_{\ell^\star}^{(\ell)}| > H^{\eta_{l^\star}}. \tag{26}$$

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Now, fix the unique solution $(x^{(\ell)})_{\ell=1}^{m+n} \in \mathbb{Q}_S^{m+n}$ to the linear equation

$$(\mathbf{0},\boldsymbol{\gamma}) = \sum_{\ell=1}^{m+n} x^{(\ell)}(\mathbf{a}^{(\ell)},\mathbf{b}^{(\ell)})\,, \tag{27}$$

which exists since the vectors $\{(\mathbf{a}^{(\ell)},\mathbf{b}^{(\ell)})\}_{\ell=1}^{m+n}$ are linearly independent.

Suppose that $|b_{1\star}^{(1)}| > H^{\eta_{m+1\star}}$, which is one of the two possibilities in (26) for $\ell = 1$. See Remark 5.2 as to why we can assume this without loss of generality. Then for $\ell = 1,\dots,m{+}n$, by the strong approximation theorem we choose $r^{(\ell)} \in \mathbb{Q}$ such that

$$\begin{aligned} &\begin{cases} |r^{(\ell)} - 2\prod_{\nu\in S}\nu| \le \prod_{\nu\in S}\nu \text{ if } b_{1\star}^{(\ell)} > 0, \\ |r^{(\ell)} + 2\prod_{\nu\in S}\nu| < \prod_{\nu\in S}\nu, \text{ if } b_{1\star}^{(\ell)} \le 0, \end{cases} \\ &|r^{(\ell)} - x^{(\ell)}|_S \le 1, \\ &|r^{(\ell)}|_\sigma \le 1 \quad \text{for all primes } \sigma \notin S. \end{aligned} \tag{28}$$

Now, define

$$\mathbf{a} := \sum_{\ell=1}^{m+n} r^{(\ell)}\mathbf{a}^{(\ell)} \qquad \text{and} \qquad \mathbf{b} := \sum_{\ell=1}^{m+n} r^{(\ell)}\mathbf{b}^{(\ell)}\,, \tag{29}$$

which are thus rational vectors. Then

$$\mathbf{a}\boldsymbol{\alpha} + \mathbf{b} = \sum_{\ell=1}^{m+n} r^{(\ell)}(\mathbf{a}^{(\ell)}\boldsymbol{\alpha} + \mathbf{b}^{(\ell)}).$$

First, we claim that $\mathbf{a} \in \mathbb{Z}^m$. From the last inequalities of (28), (29) and the fact that $\mathbf{a}^{(\ell)}$ and $\mathbf{b}^{(\ell)}$ are integer vectors, we get that

$$|\mathbf{a}|_\sigma \le 1 \quad \text{and} \quad |\mathbf{b}|_\sigma \le 1 \qquad \text{for all primes } \sigma \notin S. \tag{30}$$

Note that, by (27),

$$\sum_{\ell=1}^{m+n} x^{(\ell)}\mathbf{a}^{(\ell)} = \mathbf{0}\,.$$

Therefore, by the left hand side of (29), we get that

$$\mathbf{a} = \sum_{\ell=1}^{m+n} (r^{(\ell)} - x^{(\ell)})\mathbf{a}^{(\ell)}. \tag{31}$$

This, together with the second inequality in (28), implies that $|\mathbf{a}|_S \le 1$. Combining this with (30) implies that $\mathbf{a} \in \mathbb{Z}^m$.

Next, by (27) and (29), we have that

$$\begin{aligned} \mathbf{a}\boldsymbol{\alpha} + \mathbf{b} - \boldsymbol{\gamma} &= \sum_{\ell=1}^{m+n} r^{(\ell)}(\mathbf{a}^{(\ell)}\boldsymbol{\alpha} + \mathbf{b}^{(\ell)}) - \sum_{\ell=1}^{m+n} x^{(\ell)}\mathbf{b}^{(\ell)} - \sum_{\ell=1}^{m+n} x^{(\ell)}\mathbf{a}^{(\ell)}\boldsymbol{\alpha} \\ &= \sum_{\ell=1}^{m+n} (r^{(\ell)} - x^{(\ell)})(\mathbf{a}^{(\ell)}\boldsymbol{\alpha} + \mathbf{b}^{(\ell)}). \end{aligned}$$

Hence, using the first inequalities in (25) and the second inequalities in (28), we get from the above equation that

$$|\mathbf{a}\boldsymbol{\alpha} + \mathbf{b} - \boldsymbol{\gamma}|_{\boldsymbol{\tau}} \leq \varepsilon H^{-1}. \tag{32}$$

Now we write $\mathbf{b} = \mathbf{a}\boldsymbol{\alpha} + \mathbf{b} - \boldsymbol{\gamma} + \boldsymbol{\gamma} - \mathbf{a}\boldsymbol{\alpha}$. Using (30) and (32) together with the fact that $\mathbf{a} \in \mathbb{Z}^m$, $\boldsymbol{\gamma} \in \mathbb{Z}_S^n, \boldsymbol{\alpha} \in \mathbb{Z}_S^{mn}$, we conclude that $\mathbf{b} \in \mathbb{Z}^n$.

Next, we claim that $(\mathbf{a}, \mathbf{b}) \neq \mathbf{0}$. Note that

$$b_{1^\star} = \sum_{\ell=1}^{m+n} r^{(\ell)} b_{1^\star}^{(\ell)}.$$

By the first inequalities in Equation (28), $r^{(\ell)}$ is positive if $b_{1^\star}^{(\ell)}$ is positive and $r^{(\ell)}$ is non positive if $b_{1^\star}^{(\ell)}$ is so. Now, since $|b_{1^\star}^{(1)}| > H^{\eta_{m+1^\star}}$, we are guaranteed that $\mathbf{b} \neq 0$, which confirms our claim. Moreover, we get that $|b_{1^\star}| > H^{\eta_{m+1^\star}}$, which implies there exist infinitely many $(\mathbf{a}, \mathbf{b})$ as solutions.

The last thing to show is that

$$|(\mathbf{a}, \mathbf{b})|_{\boldsymbol{\eta}} \ll H. \tag{33}$$

By the second and third inequalities in Equation (25) and the first inequality in Equation (28) we get that

$$|a_j| \leq 3 \prod_{\nu \in S} \nu(n+m)\varepsilon^{-m-n} \prod_{\nu \in S} \nu^{m+n} H^{\eta_k}, 1 \leq j \leq m.$$

Similarly, we get

$$|b_i| \leq 3 \prod_{\nu \in S} \nu(n+m)\varepsilon^{-m-n} \prod_{\nu \in S} \nu^{m+n} H^{\eta_{m+i}}, 1 \leq i \leq n.$$

Thus in the view of Equation (32), Equation (33), and the discussion from above, the proof is complete.

Remark 5.2. We can assume $|b_{1^\star}^{(1)}| > H^{\eta_{m+1^\star}}$ without loss of generality. If instead we had $|a_{1^\star}^{(1)}| > H^{\eta_{1^\star}}$, we will choose $r^{(\ell)}$ in Equation (28) accordingly. Then a similar process as in the proof above will imply $\mathbf{a} \neq \mathbf{0}$.

5.2. **Proof of Theorem 5.1 in the case $\infty \in S$.** . Let us denote $S^\star = S \setminus \infty$. Without loss of generality, we assume $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\star, \boldsymbol{\alpha}_\infty) \in \mathbb{Z}_{S^\star}^{mn} \times [0,1)^{mn}$ and $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_\star, \boldsymbol{\gamma}_\infty) \in \mathbb{Z}_{S^\star}^{n} \times [0,1)^{n}$. Let us define the lattice

$$\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H) := \{(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{m+n} \; : \; |\mathbf{a}\boldsymbol{\alpha}_\star + \mathbf{b}|_{\boldsymbol{\tau}^\star} < \varepsilon H^{-1}, |\mathbf{a}|_\nu \leq \frac{1}{\nu} \; \forall \nu \in S^\star\}.$$

Next, let us consider the convex set

$$K_H := \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{m+n} \; : \; |\mathbf{a}\boldsymbol{\alpha}_\infty + \mathbf{b}|_{\boldsymbol{\tau}_\infty} < \varepsilon H^{-1}, |\mathbf{a}|_{\boldsymbol{\eta}} \leq H\}.$$

If $\boldsymbol{\alpha}$ is $(\boldsymbol{\tau}, \boldsymbol{\eta})$-*non-singular* then there exists $\varepsilon > 0$ and an unbounded subset $\mathcal{H}$ of positive real numbers such that for any $H \in \mathcal{H}$ system system (12) does not have any non-zero solution. This implies that,

$$\lambda_1(\Gamma_{\boldsymbol{\alpha}}(\varepsilon, H), K_H) > 1 \text{ for all } H \in \mathcal{H}. \tag{34}$$

Note that

$$\mathrm{Vol}(\mathbb{R}^{m+n}/\Gamma_{\boldsymbol{\alpha}}(\varepsilon,H)) \leq \varepsilon^{-(l-1)n} H^{m-\sum_{k=1}^{n}\tau_{k,\infty}} \prod_{\nu\in S^\star} \nu^m,$$

and

$$\mathrm{Vol}(K_H) = 2^n\varepsilon^n H^{-\sum_{k=1}^{n}\tau_{k,\infty}} 2^m H^m.$$

Hence by Equation (34), we have for any $H \in \mathcal{H}$,

$$\lambda_{m+n}(\Gamma_{\boldsymbol{\alpha}}(\varepsilon,H),K_H) \leq \varepsilon^{-ln} \prod_{\nu\in S^\star} \nu^m. \tag{35}$$

Fix any $H\in\mathcal{H}$. We get $m+n$ many linearly independent $\{(\mathbf{a}^{(\ell)},\mathbf{b}^{(\ell)})\}_{l=1}^{m+n} \in \Gamma_{\boldsymbol{\alpha}}(\varepsilon,H)$ such that

$$\begin{aligned} &|\mathbf{a}^{(\ell)}\boldsymbol{\alpha}_\star + \mathbf{b}^{(\ell)}|_{\boldsymbol{\tau}^\star} < \varepsilon H^{-1},\\ &|\mathbf{a}^{(\ell)}\boldsymbol{\alpha}_\infty + \mathbf{b}^{(\ell)}|_{\boldsymbol{\tau}_\infty} < \varepsilon^{-ln}\prod_{\nu\in S^\star}\nu^m \varepsilon H^{-1},\\ &|a_k^{(\ell)}| \leq \varepsilon^{-ln}\prod_{\nu\in S^\star}\nu^m H^{\eta_k},\ \ 1\leq k\leq m\\ &|\mathbf{a}^{(\ell)}|_\nu \leq \frac{1}{\nu}\ \ \forall\,\nu\in S^\star. \end{aligned} \tag{36}$$

Since $\{(\mathbf{a}^{(\ell)},\mathbf{b}^{(\ell)})\}$ forms a basis in $\mathbb{Q}_S^{m+n}$, there exists a unique solution $\mathbf{x}=(x^\ell)_{l=1}^{m+n}\in\mathbb{Q}_S^{m+n}$ to the following system.

$$(D,\underbrace{\mathbf{0}\ldots\mathbf{0}}_{m-1},\boldsymbol{\gamma}) = (x^{(1)},\cdots,x^{(m+n)})\begin{bmatrix} \mathbf{a}^{(1)} & \cdots & \mathbf{a}^{(m+n)}\\ \mathbf{a}^{(1)}\alpha+\mathbf{b}^{(1)} & \cdots & \mathbf{a}^{(m+n)}\alpha+\mathbf{b}^{(m+n)}\end{bmatrix}^T, \tag{37}$$

where $D\in\mathbb{Q}_S, D_\nu = 0, \forall \nu\in S^\star$, and

$$D_\infty = H^{\eta_1} + \prod_{\nu\in S^\star}\nu\sum_{l=1}^{m+n}|a_1^{(\ell)}|. \tag{38}$$

Hence, we have

$$\boldsymbol{\gamma} = \sum_{\ell=1}^{m+n} x^{(\ell)}(\mathbf{a}^{(\ell)}\alpha+\mathbf{b}^{\ell}),\ \mathbf{0} = \sum_{\ell=1}^{m+m} x_\nu^{(\ell)}\mathbf{a}^{(\ell)}, \forall\,\nu\in S^\star,\ \ \text{and}\ \ D_\infty = \sum_{\ell=1}^{m+n} x_\infty^{(\ell)} a_1^{(\ell)}.$$

Using the strong approximation theorem, for $\ell = 1,\cdots,m+n$ we get $r^{(\ell)}\in\mathbb{Q}$ such that

$$\begin{aligned} &|r^{(\ell)} - x_\infty^{(\ell)}| \leq \prod_{\nu\in S^\star}\nu,\\ &|r^{(\ell)} - x^{(\ell)}|_{S^\star} \leq 1,\\ &|r^{(\ell)}|_\nu \leq 1 \quad \text{for all}\ \ \text{primes}\ \nu\notin S. \end{aligned} \tag{39}$$

Now, let us take $\mathbf{a} := \sum_{\ell=1}^{m+n} r^{(\ell)}\mathbf{a}^{(\ell)},\ \mathbf{b} := \sum_{\ell=1}^{m+n} r^{(\ell)}\mathbf{b}^{(\ell)}$.
Then

$$\mathbf{a}\boldsymbol{\alpha}+\mathbf{b} = \sum_{\ell=1}^{m+n} r^{(\ell)}(\mathbf{a}^{(\ell)}\boldsymbol{\alpha}+\mathbf{b}^{(\ell)}).$$

First, we claim that $\mathbf{a}\in\mathbb{Z}^m$. From the last inequalities in Equation (39), we get

$$|\mathbf{a}|_\nu \leq 1, |\mathbf{b}|_\nu \leq 1\ \forall\nu\notin S. \tag{40}$$

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Also by Equation (37), $\mathbf{a} = \sum_{\ell=1}^{m+n}(r^{(\ell)} - x_\nu^{(\ell)})\mathbf{a}^{(\ell)}$, $\nu \in S^\star$. This, together with the second inequality in Equation (39), implies

$$|\mathbf{a}|_{S^\star} \le 1. \tag{41}$$

Hence by Equation (40), and Equation (41), we conclude $\mathbf{a} \in \mathbb{Z}^m$.

Since $\mathbf{a}\boldsymbol{\alpha} + \mathbf{b} - \boldsymbol{\gamma} = \sum_{\ell=1}^{m+n}(r^{(\ell)} - x^{(\ell)})(\mathbf{a}^{(\ell)}\boldsymbol{\alpha} + \mathbf{b}^{(\ell)})$, using the first inequality in Equation (36) and the second inequality in Equation (39), we get

$$|\mathbf{a}\boldsymbol{\alpha}_\star + \mathbf{b} - \boldsymbol{\gamma}_\star|_{\boldsymbol{\tau}^\star} \le \varepsilon H^{-1}. \tag{42}$$

Also, the second inequality in Equation (36) and the first inequality in Equation (39), we get

$$|\mathbf{a}\boldsymbol{\alpha}_\infty + \mathbf{b} - \boldsymbol{\gamma}_\infty|_{\boldsymbol{\tau}_\infty} \le \varepsilon^{-ln} \prod_{\nu\in S^\star} \nu^m \varepsilon H^{-1}\left((n+m)\prod_{\nu\in S^\star}\nu\right)^{\frac{1}{\tau'_\infty}}, \tag{43}$$

where $\tau'_\infty = \min_{i=1}^n \tau_{i,\infty}$. Since $\mathbf{b} = \mathbf{a}\boldsymbol{\alpha} + \mathbf{b} - \boldsymbol{\gamma} + \boldsymbol{\gamma} - \mathbf{a}\boldsymbol{\alpha}$, using (42) and (42) together with the fact that $\mathbf{a} \in \mathbb{Z}^m$, $\boldsymbol{\gamma}_\star \in \mathbb{Z}^n_{S^\star}, \boldsymbol{\alpha}_\star \in \mathbb{Z}^{mn}_{S^\star}$, we conclude that $\mathbf{b} \in \mathbb{Z}^n$.

Next, we claim that

$$|\mathbf{a}|_{\boldsymbol{\eta}} \ll H. \tag{44}$$

Since $a_k = \sum_{\ell=1}^{m+n}(r^{(\ell)} - x_\infty^{(\ell)})a_k^{(\ell)}$ for $k = 2, \cdots, n$, by the third inequality in Equation (36) and the first inequality in Equation (39), we get that

$$|a_k| \le (n+m)\varepsilon^{-ln}\prod_{\nu\in S^\star}\nu^{m+1}H^{\eta_k}, 2 \le k \le m.$$

Also $a_1 = \sum_{\ell=1}^{m+n} r^{(\ell)}a_1^{(\ell)}$ which implies that $a_1 - D_\infty = \sum_{\ell=1}^{m+n}(r^{(\ell)} - x_\infty^{(\ell)})a_1^{(\ell)}$. Thus, similarly to above,

$$|a_1| \le (n+m)\varepsilon^{-ln}\prod_{\nu\in S^\star}\nu^{m+1}H^{\eta_1} + D_\infty \ll H^{\eta_1}.$$

Note that $D_\infty = \sum_{\ell=1}^{m+n}(x_\infty^{(\ell)} - r^{(\ell)})a_1^{(\ell)} + \sum_{\ell=1}^{m+n} r^{(\ell)}a_1^{(\ell)}$ and so

$$H^{\eta_1} + \prod_{\nu\in S^\star}\nu\sum_{\ell=1}^{m+n}|a_1^{(\ell)}| \overset{(38)}{=} D_\infty \le \prod_{\nu\in S^\star}\nu\sum_{\ell=1}^{m+n}|a_1^{(\ell)}| + |\sum_{\ell=1}^{m+n} r^{(\ell)}a_1^{(\ell)}|$$

implying $H^{\eta_1} \le |\sum_{\ell=1}^{m+n} r^{(\ell)}a_1^{(\ell)}| = |a_1|$. Thus, there are infinitely many $\mathbf{a} \neq \mathbf{0}$ as solutions. Therefore, in the view of Equation (42), Equation (43), and Equation (44) the proof is complete.

5.3. **Proof of Theorem 2.4 using Theorem 5.1.** Let

$$\phi(\mathbf{a},\mathbf{b}) := \begin{cases} \mathbf{a} & \text{if } \infty \notin S, \\ (\mathbf{a},\mathbf{b}) & \text{if } \infty \in S. \end{cases} \tag{45}$$

For any $\delta > 0$, let us define

$$B_{\phi(\mathbf{a},\mathbf{b})}(\delta,\boldsymbol{\tau}) := \left\{\boldsymbol{\gamma} \in \mathbb{Z}^n_S \;\middle|\; \begin{cases} |\mathbf{a}\boldsymbol{\alpha} + \mathbf{b} + \boldsymbol{\gamma}|_{\boldsymbol{\tau}} & \text{if } \infty \notin S, \\ \min_{\mathbf{b}\in\mathbb{Z}^n}|\mathbf{a}\boldsymbol{\alpha} + \mathbf{b} + \boldsymbol{\gamma}|_{\boldsymbol{\tau}} & \text{if } \infty \in S \end{cases} \le \frac{\delta}{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}}\right\}.$$

Then by Theorem 5.1,

$$\mu_{S,n}\left(\bigcup_{\delta>0}\limsup_{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\to\infty} B_{\phi(\mathbf{a},\mathbf{b})}(\delta,\boldsymbol{\tau})\right)=\mu_{S,n}(\mathbb{Z}_S^n).$$

Note that δ above plays a role of C in Theorem 5.1, and C depends on $\boldsymbol{\alpha}$. This is why we need to take union over δ. Our goal is to turn the union into intersection using Lemma 4.1.

First, by continuity of measure,

$$\lim_{\delta\to+\infty}\mu_{S,n}\left(\limsup_{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\to\infty} B_{\phi(\mathbf{a},\mathbf{b})}(\delta,\boldsymbol{\tau})\right)=\mu_{S,n}(\mathbb{Z}_S^n).$$

By Lemma 4.1, for any $\delta_1,\delta_2>0$ we have

$$\mu_{S,n}\left(\limsup_{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\to\infty} B_{\phi(\mathbf{a},\mathbf{b})}(\delta_1,\boldsymbol{\tau})\right)=\mu_{S,n}\left(\limsup_{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\to\infty} B_{\phi(\mathbf{a},\mathbf{b})}(\delta_2,\boldsymbol{\tau})\right).$$

Hence, we get that

$$\mu_{S,n}\left(\bigcap_{\delta>0}\limsup_{|\phi(\mathbf{a},\mathbf{b})|_{\boldsymbol{\eta}}\to\infty} B_{\phi(\mathbf{a},\mathbf{b})}(\delta,\boldsymbol{\tau})\right)=\mu_{S,n}(\mathbb{Z}_S^n) \tag{46}$$

as required.

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