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# Yangian of the Periplectic Lie Superalgebra

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**Abstract:** We study in detail the Yangian of the periplectic Lie superalgebra. For this Yangian we verify an analogue of the Poincaré–Birkhoff–Witt Theorem. Moreover we introduce a family of free generators of the centre of this Yangian.

1. Let  $\mathfrak{gl}_{M|N}$  be the general linear Lie superalgebra over the complex field  $\mathbb{C}$ . Here both  $M$  and  $N$  are positive integers. The Yangian of  $\mathfrak{gl}_{M|N}$  was introduced in [20] by extending the definition of the Yangian  $Y(\mathfrak{gl}_N)$  of the general linear Lie algebra  $\mathfrak{gl}_N$ , see for instance [24]. The Yangian of  $\mathfrak{gl}_{M|N}$  is a deformation of the universal enveloping algebra of the polynomial current Lie superalgebra  $\mathfrak{gl}_{M|N}[u]$  in the class of Hopf algebras, see [25] for further details.

Let  $\mathfrak{p}_N$  be the periplectic Lie superalgebra over  $\mathbb{C}$ . This is a subalgebra of  $\mathfrak{gl}_{N|N}$  preserving a non-degenerate odd symmetric bilinear form on the  $\mathbb{Z}_2$ -graded vector space  $\mathbb{C}^{N|N}$ . The Lie superalgebra  $\mathfrak{gl}_{N|N}$  acts on  $\mathbb{C}^{N|N}$  in the natural way. Equivalently, the Lie superalgebra  $\mathfrak{p}_N$  can be defined as a fixed point subalgebra of  $\mathfrak{gl}_{N|N}$  with respect to a certain involutive automorphism. This automorphism is denoted by  $\omega$ , see Section 3 for an explicit definition of  $\mathfrak{p}_N$  in these terms.

The corresponding Yangian  $Y(\mathfrak{p}_N)$  was briefly introduced in [21]. Note that the construction of  $Y(\mathfrak{p}_N)$  therein involved the *periplectic Brauer algebra* which was later studied by other authors, see for instance [8, 9, 14, 19].

According to the general scheme of [11], the Yangian of the Lie superalgebra  $\mathfrak{p}_N$  cannot be defined as a deformation of the universal enveloping algebra of the polynomial current Lie superalgebra  $\mathfrak{p}_N[u]$  in the class of Hopf algebras. This is because by [26] the only supersymmetric  $\mathfrak{p}_N$ -invariant element of  $\mathfrak{p}_N^{\otimes 2}$  is zero. Consequently, there is no natural Lie co-superalgebra structure on  $\mathfrak{p}_N[u]$ .

Remarkably, there is a natural Lie co-superalgebra structure on the *twisted* polynomial current Lie superalgebra  $\mathfrak{g} = \{g(u) \in \mathfrak{gl}_{N|N}[u] : \omega(g(u)) = g(-u)\}$ . The Yangian

$Y(\mathfrak{p}_N)$  is a deformation of the universal enveloping algebra  $U(\mathfrak{g})$  in the class of Hopf algebras. See [21] for further discussion of this fact.

The definition of the Yangian  $Y(\mathfrak{p}_N)$  is given in our Section 10. It is based on a new solution of the quantum Yang–Baxter equation constructed in [21]. This solution is a rational function of two variables  $u, v$  with values in the periplectic Brauer algebra. Unlike other rational solutions discovered before [21] this is *not* a function of only the difference  $u - v$  of the variables, see our Section 5.

Our Theorem 1 was stated in [21] without proof. We prove it here in Section 5. We also establish an analogue for  $Y(\mathfrak{p}_N)$  of the Poincaré–Birkhoff–Witt Theorem for  $U(\mathfrak{g})$ , see [17]. This analogue is Corollary 1 to our Theorem 2. The latter theorem was also stated in [21] without proof. We prove it here in Section 14.

Unlike its analogues [3] for the Yangians of the symplectic and orthogonal Lie algebras, Theorem 2 cannot be proved by using the method of [23]. The reason for it is explained in our Section 11. We use the Diamond Lemma of [6] instead.

The Lie superalgebra  $\mathfrak{p}_N$  is a subalgebra of  $\mathfrak{g}$ . It consists of all those elements of  $\mathfrak{g}$  which do not depend on the variable  $u$ . So the universal enveloping algebra  $U(\mathfrak{p}_N)$  is a Hopf subalgebra of  $U(\mathfrak{g})$ . We also have an embedding of  $U(\mathfrak{p}_N)$  to the Yangian  $Y(\mathfrak{p}_N)$  as a Hopf subalgebra, see Corollary 2 to our Theorem 2.

There also is a subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{gl}_N[u]$ . It consists of all  $g(u) \in \mathfrak{g}$  taking values only in the even part of  $\mathfrak{gl}_{N|N}$ . Hence we get a Hopf subalgebra of  $U(\mathfrak{g})$  isomorphic to  $U(\mathfrak{gl}_N[u])$ . We also have an embedding of  $Y(\mathfrak{gl}_N)$  to  $Y(\mathfrak{p}_N)$  as an associative subalgebra, see Corollary 3 to our Theorem 2. But this is not a Hopf algebra embedding.

Theorem 3 yields an explicit description of the centre of  $Y(\mathfrak{p}_N)$ , see Section 15. We use an analogue for  $Y(\mathfrak{p}_N)$  of the *quantum Berezinian* [20] for the Yangian of  $\mathfrak{gl}_{M|N}$ . To determine this analogue we construct a family of one-dimensional subquotient representations of  $Y(\mathfrak{p}_N)$  depending on a complex parameter, see Section 25 for an explanation. Theorem 3 itself is proved in Sections 16 to 24.

Irreducible finite-dimensional representations of the Lie superalgebra  $\mathfrak{p}_N$  were recently studied by many authors, see for instance [1, 2, 4, 7, 10, 12, 13]. It would be interesting to study irreducible finite-dimensional representations of  $Y(\mathfrak{p}_N)$ . For the Yangian of  $\mathfrak{gl}_{M|N}$  this was done in [28] and more recently in [15, 16, 27].

2. We shall use the following general conventions. Let  $A$  and  $B$  be associative  $\mathbb{Z}_2$ -graded algebras. Their tensor product  $A \otimes B$  is also an associative  $\mathbb{Z}_2$ -graded algebra such that for any homogeneous elements  $X, X' \in A$  and  $Y, Y' \in B$

$$(X \otimes Y)(X' \otimes Y') = XX' \otimes YY' (-1)^{\deg X' \deg Y},$$

$$\deg(X \otimes Y) = \deg X + \deg Y.$$

Furthermore, for any two  $\mathbb{Z}_2$ -graded modules  $U$  and  $V$  over  $A$  and  $B$  respectively, the vector space  $U \otimes V$  is a  $\mathbb{Z}_2$ -graded module over  $A \otimes B$  such that for any homogeneous elements  $X \in A, x \in U$  and  $Y \in B, y \in V$

$$(X \otimes Y)(x \otimes y) = Xx \otimes Yy (-1)^{\deg x \deg Y}, \tag{1}$$

$$\deg(x \otimes y) = \deg x + \deg y. \tag{2}$$

A homomorphism  $\alpha : A \rightarrow B$  is a linear map obeying  $\alpha(X X') = \alpha(X) \alpha(X')$  for all  $X, X' \in A$ . But an antihomomorphism  $\beta : A \rightarrow B$  is a linear map such that for all

homogeneous  $X, X' \in A$

$$\beta(XX') = \beta(X')\beta(X)(-1)^{\deg X \deg X'}. \tag{3}$$

Let  $n$  be any positive integer. If the algebra  $A$  is unital, let  $\iota_p$  be its embedding into the tensor product  $A^{\otimes n}$  as the  $p$ -th tensor factor:

$$\iota_p(X) = 1^{\otimes(p-1)} \otimes X \otimes 1^{\otimes(n-p)} \quad \text{for } p = 1, \dots, n.$$

We will also use various embeddings of  $A^{\otimes m}$  into  $A^{\otimes n}$  for  $m = 1, \dots, n$ . For any choice of  $m$  pairwise distinct indices  $p_1, \dots, p_m \in \{1, \dots, n\}$  and of an element  $X \in A^{\otimes m}$  of the form  $X = X^{(1)} \otimes \dots \otimes X^{(m)}$  we will denote

$$X_{p_1 \dots p_m} = \iota_{p_1}(X^{(1)}) \dots \iota_{p_m}(X^{(m)}) \in A^{\otimes n}.$$

We will then extend the notation  $X_{p_1 \dots p_m}$  to all elements  $X \in A^{\otimes m}$  by linearity.

**3.** Let the indices  $i, j, k, l$  run through  $\pm 1, \dots, \pm N$ . Put  $\bar{i} = 0$  if  $i > 0$  and  $\bar{i} = 1$  if  $i < 0$ . Now consider the  $\mathbb{Z}_2$ -graded vector space  $\mathbb{C}^{N|N}$ . Let  $e_i \in \mathbb{C}^{N|N}$  be an element of the standard basis. The  $\mathbb{Z}_2$ -grading on  $\mathbb{C}^{N|N}$  is defined by  $\deg e_i = \bar{i}$ . We will be using the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{C}^{N|N}$  defined by setting  $\langle e_i, e_j \rangle = \delta_{i, -j}$  for any indices  $i$  and  $j$ . This form is clearly symmetric.

Let  $E_{ij} \in \text{End } \mathbb{C}^{N|N}$  be the standard matrix unit, defined by  $E_{ij} e_k = \delta_{jk} e_i$ . The associative algebra  $\text{End } \mathbb{C}^{N|N}$  is  $\mathbb{Z}_2$ -graded so that  $\deg E_{ij} = \bar{i} + \bar{j}$ . Hence  $\mathbb{C}^{N|N}$  is a  $\mathbb{Z}_2$ -graded module over  $\text{End } \mathbb{C}^{N|N}$ . For any  $n$  we can also identify the tensor product  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$  with the algebra  $\text{End}((\mathbb{C}^{N|N})^{\otimes n})$  acting on the vector space  $(\mathbb{C}^{N|N})^{\otimes n}$  by repeatedly using the conventions (1) and (2).

We can define an antiautomorphism  $\tau$  of  $\text{End } \mathbb{C}^{N|N}$  by mapping

$$\tau : E_{ij} \mapsto E_{ji} (-1)^{\bar{i}\bar{j}+\bar{i}}.$$

Note that  $\tau$  is not involutive, while  $\tau^2$  is the *parity automorphism* of  $\text{End } \mathbb{C}^{N|N}$

$$E_{ij} \mapsto E_{ij} (-1)^{\bar{i}+\bar{j}}. \tag{4}$$

We can also define an involutive automorphism  $\pi$  of  $\text{End } \mathbb{C}^{N|N}$  by mapping

$$\pi : E_{ij} \mapsto E_{-i, -j}.$$

The compositions  $\tau\pi$  and  $\pi\tau$  again differ by the automorphism (4). Hence both  $\tau\pi$  and  $\pi\tau$  are involutive antiautomorphisms of  $\text{End } \mathbb{C}^{N|N}$ . Explicitly,

$$\tau\pi : E_{ij} \mapsto E_{-j, -i} (-1)^{\bar{i}\bar{j}+\bar{j}}.$$

Consider the Lie superalgebra  $\mathfrak{gl}_{N|N}$ . To avoid confusion, denote by  $e_{ij}$  the element of  $\mathfrak{gl}_{N|N}$  corresponding to  $E_{ij} \in \text{End } \mathbb{C}^{N|N}$ . Then  $\deg e_{ij} = \bar{i} + \bar{j}$  and

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}. \tag{5}$$

It follows that  $\pi$  and  $-\tau$  are automorphisms of the Lie superalgebra  $\mathfrak{gl}_{N|N}$ . Let  $\omega = -\tau\pi$  be their composition. The automorphism  $\omega$  is involutive. Further,

$$\langle Zx, y \rangle + \langle x, \omega(Z)y \rangle (-1)^{\deg x \deg Z} = 0$$

for any homogeneous  $x, y \in \mathbb{C}^{N|N}$  and  $Z \in \mathfrak{gl}_{N|N}$ . The Lie superalgebra  $\mathfrak{gl}_{N|N}$  acts on the vector space  $\mathbb{C}^{N|N}$  via the above identification with  $\text{End } \mathbb{C}^{N|N}$ .

The *periplectic Lie superalgebra*  $\mathfrak{p}_N$  is the fixed point subalgebra of  $\mathfrak{gl}_{N|N}$  relative to the automorphism  $\omega$ . It can also be defined as the subalgebra of  $\mathfrak{gl}_{N|N}$  preserving the form  $\langle , \rangle$  on  $\mathbb{C}^{N|N}$ . This subalgebra is spanned by the elements

$$f_{ij} = e_{ij} + \omega(e_{ij}) = e_{ij} - e_{-j, -i} (-1)^{\bar{i}\bar{j}+\bar{j}}$$
(6)

4. Take the element of the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$

$$P = \sum_{i,j} E_{ij} \otimes E_{ji} (-1)^{\bar{j}}$$
(7)

This element acts on the vector space  $(\mathbb{C}^{N|N})^{\otimes 2}$  so that

$$P : e_i \otimes e_j \mapsto e_j \otimes e_i (-1)^{\bar{i}\bar{j}}$$
(8)

Here we identify the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$  with the algebra  $\text{End } ((\mathbb{C}^{N|N})^{\otimes 2})$  using (1). Note that  $P^2 = 1$  and

$$(\tau \pi \otimes \tau \pi)(P) = -P.$$

Now let

$$Q = (-\tau \pi \otimes \text{id})(P) = (\text{id} \otimes \tau \pi)(P).$$
(9)

Explicitly,

$$Q = \sum_{i,j} E_{ij} \otimes E_{-i, -j} (-1)^{\bar{i}\bar{j}+\bar{i}+\bar{j}}$$
(10)

The image of the action of  $Q$  on  $(\mathbb{C}^{N|N})^{\otimes 2}$  is one-dimensional and is spanned by

$$\sum_i e_i \otimes e_{-i} (-1)^{\bar{i}}$$
(11)

Here we regard  $Q$  as an element of  $\text{End } ((\mathbb{C}^{N|N})^{\otimes 2})$  by once again identifying the latter algebra with  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$  using (1). Also note that

$$P Q = -Q, \quad Q P = Q \quad \text{and} \quad Q^2 = 0.$$
(12)

By [19, Theorem 4.5] the supercommutant of the image of  $\mathfrak{p}_N$  in  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$  for any  $n$  is generated by the elements  $P_{pq}$  and  $Q_{pq}$  with  $1 \leq p < q \leq n$ . Here we use the standard comultiplication on the universal enveloping algebra of  $\mathfrak{p}_N$ .

5. Consider a function of complex variables  $u, v$  with values in  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$

$$R(u, v) = 1 - \frac{P}{u - v} + \frac{Q}{u + v}.$$

By using (9) and the involutivity of  $\tau \pi$  we have the equalities

$$(\tau \pi \otimes 1)(R(u, v)) = R(u, -v), \tag{13}$$

$$(1 \otimes \tau \pi)(R(u, v)) = R(-u, v). \tag{14}$$

We also have

$$R(u, v) R(-u, -v) = 1 - \frac{1}{(u - v)^2}. \tag{15}$$

Indeed, due to the relation  $P^2 = 1$  and to (12)

$$\begin{aligned} & \left(1 - \frac{P}{u - v} + \frac{Q}{u + v}\right) \left(1 + \frac{P}{u - v} - \frac{Q}{u + v}\right) \\ &= 1 - \frac{P^2}{(u - v)^2} + \frac{P Q + Q P}{(u - v)(u + v)} - \frac{Q^2}{(u + v)^2} = 1 - \frac{1}{(u - v)^2}. \end{aligned}$$

**Theorem 1.** *The rational function  $R(u, v)$  satisfies the Yang–Baxter equation in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}(u, v, w)$*

$$R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v). \tag{16}$$

*Proof.* Using the definition of  $R(u, v)$  the equality in (16) will follow from the relations in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$  displayed below:

$$P_{12} P_{13} = P_{23} P_{12} = P_{13} P_{23}, \tag{17}$$

$$P_{13} P_{12} = P_{12} P_{23} = P_{23} P_{13}, \tag{18}$$

$$Q_{13} Q_{12} = -P_{23} Q_{12} = -Q_{13} P_{23}, \tag{19}$$

$$Q_{12} Q_{13} = -Q_{12} P_{23} = -P_{23} Q_{13}, \tag{20}$$

$$Q_{12} P_{13} = -Q_{12} Q_{23} = -P_{13} Q_{23}, \tag{21}$$

$$P_{13} Q_{12} = -Q_{23} Q_{12} = -Q_{23} P_{13}, \tag{22}$$

$$P_{12} Q_{13} = Q_{23} P_{12} = Q_{23} Q_{13}, \tag{23}$$

$$Q_{13} P_{12} = P_{12} Q_{23} = Q_{13} Q_{23}, \tag{24}$$

$$P_{12} P_{13} P_{23} = P_{23} P_{13} P_{12}, \tag{25}$$

$$P_{12} Q_{13} Q_{23} = Q_{23} Q_{13} P_{12}, \tag{26}$$

$$Q_{12} Q_{13} P_{23} = P_{23} Q_{13} Q_{12}, \tag{27}$$

$$Q_{12} P_{13} Q_{23} = 0, \tag{28}$$

$$Q_{23} P_{13} Q_{12} = 0, \tag{29}$$

$$P_{12} P_{13} Q_{23} = P_{23} Q_{13} P_{12} = Q_{12} P_{13} P_{23} = -Q_{12} Q_{13} Q_{23}, \tag{30}$$

$$Q_{23} P_{13} P_{12} = P_{12} Q_{13} P_{23} = P_{23} P_{13} Q_{12} = -Q_{23} Q_{13} Q_{12}. \tag{31}$$

To prove (16), the relations (17) and (18) are used along with the identity

$$\frac{1}{u - v} \frac{1}{u - w} - \frac{1}{u - v} \frac{1}{v - w} + \frac{1}{u - w} \frac{1}{v - w} = 0 \tag{32}$$

which is easy to verify. The relations (19) and (20) are used with the identity

$$\frac{1}{u+v} \frac{1}{u+w} + \frac{1}{u+v} \frac{1}{v-w} - \frac{1}{u+w} \frac{1}{v-w} = 0$$

which is obtained from (32) by changing the sign of  $u$ . The relations (21) and (22) are used with the identity

$$\frac{1}{u+v} \frac{1}{u-w} + \frac{1}{u+v} \frac{1}{v+w} - \frac{1}{u-w} \frac{1}{v+w} = 0$$

which is obtained from (32) by changing the sign of  $v$ . The relations (23) and (24) are used with the identity

$$\frac{1}{u-v} \frac{1}{u+w} - \frac{1}{u-v} \frac{1}{v+w} + \frac{1}{u+w} \frac{1}{v+w} = 0$$

which is obtained from (32) by changing the sign of  $w$ . Finally, the relations (30) and (31) are used along with another identity which is easy to verify:

$$\begin{aligned} & \frac{1}{u-v} \frac{1}{u-w} \frac{1}{v+w} - \frac{1}{u-v} \frac{1}{u+w} \frac{1}{v-w} \\ & + \frac{1}{u+v} \frac{1}{u-w} \frac{1}{v-w} - \frac{1}{u+v} \frac{1}{u+w} \frac{1}{v+w} = 0. \end{aligned}$$

Let us verify the relations (17) to (24). The relations (17) and (18) follow from the description (8) of the action of  $P$  on the vector space  $(\mathbb{C}^{N|N})^{\otimes 2}$ . Due to (9), the relations (19) and (20) can be obtained by applying the antiautomorphism  $\tau \pi$  of  $\text{End } \mathbb{C}^{N|N}$  to the relations (17) and (18) relative to the first tensor factor of  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$ . Similarly, the relations (21) and (22) can be now obtained by applying  $\tau \pi$  to the relations (17) and (18) relative to the second tensor factor of  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$ . The relations (23) and (24) can be obtained by applying  $\tau \pi$  to the relations (17) and (18) relative to the third tensor factor of  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$ .

Next let us verify the relations (25) to (31). The description (8) of the action of  $P$  on  $(\mathbb{C}^{N|N})^{\otimes 2}$  shows that each side of (25) equals  $P_{13}$ . The description (8) also implies (26) and (27). Further, by using the second equality in (21) we get

$$Q_{12} P_{13} Q_{23} = Q_{12} Q_{12} Q_{23} = 0$$

where we also used the last equality in (12). So we get the relation (28). Similarly, by using the second equality in (22) we get the relation (29):

$$Q_{23} P_{13} Q_{12} = Q_{23} Q_{12} Q_{12} = 0.$$

The description (8) also shows that the first three products in (30) are equal to each other. By using (17),(20) and (24) consecutively we get

$$Q_{12} P_{13} P_{23} = Q_{12} P_{23} P_{12} = -Q_{12} Q_{13} P_{12} = -Q_{12} Q_{13} Q_{23}.$$

Hence we get the last equality in (30). The proofs of the three equalities in (31) are very similar to those in (30) and are omitted here. □

6. We will first define the *extended Yangian* of the Lie superalgebra  $\mathfrak{p}_N$ . This is a complex associative unital algebra  $X(\mathfrak{p}_N)$  with a countable set of generators

$$T_{ij}^{(r)} \quad \text{where } r = 1, 2, \dots \quad \text{and } i, j = \pm 1, \dots, \pm N. \tag{33}$$

The algebra  $X(\mathfrak{p}_N)$  is  $\mathbb{Z}_2$ -graded so that  $\deg T_{ij}^{(r)} = \bar{i} + \bar{j}$  for all  $r$ . To write down defining relations for the generators (33) of  $X(\mathfrak{p}_N)$  we will use the formal power series in  $u^{-1}$  with coefficients from  $X(\mathfrak{p}_N)$

$$T_{ij}(u) = \delta_{ij} \cdot 1 + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \dots \tag{34}$$

Now combine all the series (34) into a single element

$$T(u) = \sum_{i,j} E_{ij} \otimes T_{ij}(u) \in (\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[u^{-1}]]. \tag{35}$$

For any  $n$  and any  $p = 1, \dots, n$  we will denote

$$T_p(u) = (\iota_p \otimes \text{id})(T(u)) \in (\text{End } \mathbb{C}^{N|N})^{\otimes n} \otimes X(\mathfrak{p}_N)[[u^{-1}]]. \tag{36}$$

By using this notation for  $n = 2$  the defining relations of  $X(\mathfrak{p}_N)$  are

$$(R(u, v) \otimes 1) T_1(u) T_2(v) = T_2(v) T_1(u) (R(u, v) \otimes 1). \tag{37}$$

Namely, after multiplying each side of (37) by  $u^2 - v^2$  it becomes an equality of formal Laurent series in  $u^{-1}, v^{-1}$  with coefficients in  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2} \otimes X(\mathfrak{p}_N)$ . By expanding the relation (37) in the basis of  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$  consisting of

$$E_{ij} \otimes E_{kl} (-1)^{(\bar{i} + \bar{j})(\bar{k} + \bar{l})} \quad \text{where } i, j, k, l = \pm 1, \dots, \pm N$$

while using the definitions (7) and (10), we get a collection of relations

$$\begin{aligned} [T_{ij}(u), T_{kl}(v)] &= \frac{T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u)}{u - v} (-1)^{\bar{i}\bar{k} + \bar{i}\bar{l} + \bar{k}\bar{l}} \\ &\quad - \delta_{-i,k} \sum_h \frac{T_{hj}(u) T_{-h,l}(v)}{u + v} (-1)^{\bar{h}\bar{l} + \bar{i}\bar{l} + \bar{i}} \\ &\quad + \delta_{j,-l} \sum_h \frac{T_{k,-h}(v) T_{ih}(u)}{u + v} (-1)^{\bar{h}\bar{k} + \bar{i}\bar{k} + \bar{i}\bar{l} + \bar{h}} \end{aligned} \tag{38}$$

where  $h = \pm 1, \dots, \pm N$  and the square brackets stand for the supercommutator.

By comparing (16) with (37), we obtain that for any  $t \in \mathbb{C}$  the assignment

$$(\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[u^{-1}]] \rightarrow (\text{End } \mathbb{C}^{N|N})^{\otimes 2}[[u^{-1}]] : T(u) \mapsto R(u, t)$$

defines a representation  $X(\mathfrak{p}_N) \rightarrow \text{End } \mathbb{C}^{N|N}$ . We will denote this representation by  $\rho_t$ . By the definitions (7) and (10), for any  $r \geq 0$

$$\rho_t : T_{ij}^{(r+1)} \mapsto (-E_{ji} + (-1)^r \tau \pi(E_{ji})) t^r (-1)^{\bar{j}}.$$



7. Due to (37) for any formal power series  $c(u)$  in  $u^{-1}$  with coefficients from  $\mathbb{C}$  and leading term 1, an automorphism of the algebra  $X(\mathfrak{p}_N)$  can be defined by

$$T_{ij}(u) \mapsto c(u) T_{ij}(u). \tag{39}$$

Now for all  $i$  and  $j$  define a formal power series  $T'_{ij}(u)$  in  $u^{-1}$  with coefficients in the algebra  $X(\mathfrak{p}_N)$  by using the element inverse to (35):

$$T(u)^{-1} = \sum_{i,j} E_{ij} \otimes T'_{ij}(u). \tag{40}$$

**Proposition 1.** *These mappings define two commuting antiautomorphisms of the algebra  $X(\mathfrak{p}_N)$ :*

$$T_{ij}(u) \mapsto T_{ij}(-u), \tag{41}$$

$$T'_{ij}(u) \mapsto T'_{ij}(u). \tag{42}$$

*Proof.* By using the notation (36) for  $n = 2$  we get the equalities

$$T_1(u) T_2(v) = \sum_{i,j,k,l} E_{ij} \otimes E_{kl} \otimes T_{ij}(u) T_{kl}(v) (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}, \tag{43}$$

$$T_2(-v) T_1(-u) = \sum_{i,j,k,l} E_{ij} \otimes E_{kl} \otimes T_{kl}(-v) T_{ij}(-u). \tag{44}$$

Due to (3) the antihomomorphism property of (41) follows from the relation

$$(R(u, v) \otimes 1) T_2(-v) T_1(-u) = T_1(-u) T_2(-v) (R(u, v) \otimes 1)$$

which is obtained from (37) by using (15). The antihomomorphism (41) is clearly involutive and therefore bijective.

Similarly to (44), by again using the notation (36) for  $n = 2$  we get

$$T_2(v)^{-1} T_1(u)^{-1} = \sum_{i,j,k,l} E_{ij} \otimes E_{kl} \otimes T'_{kl}(v) T'_{ij}(u).$$

By comparing this with (43), the antihomomorphism property of (42) follows from the relation

$$(R(u, v) \otimes 1) T_2(v)^{-1} T_1(u)^{-1} = T_1(u)^{-1} T_2(v)^{-1} (R(u, v) \otimes 1)$$

which is obtained by multiplying both sides of the defining relation (37) on the left and right by  $T_2(v)^{-1}$  and then by  $T_1(u)^{-1}$ .

The antihomomorphism (42) clearly commutes with (41). The bijectivity of the antihomomorphism (42) will follow from Proposition 5 below.  $\square$

Furthermore, let us introduce the element of  $(\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[u^{-1}]]$

$$\begin{aligned} T^*(u) &= (\tau \pi \otimes \text{id})(T(u)) \\ &= \sum_{i,j} E_{-j,-i} \otimes T_{ij}(u) (-1)^{\bar{i}\bar{j}+\bar{j}} = \sum_{i,j} E_{ij} \otimes T_{-j,-i}(u) (-1)^{\bar{i}\bar{j}+\bar{j}}. \end{aligned}$$

**Proposition 2.** *An automorphism of  $X(\mathfrak{p}_N)$  commuting with (41) is defined by*

$$T_{ij}(u) \mapsto T_{-j, -i}(u) (-1)^{\bar{i}\bar{j}+\bar{j}}. \tag{45}$$

*Proof.* The composition  $\tau \pi$  is an antiautomorphism of the  $\mathbb{Z}_2$ -graded associative algebra  $\text{End } \mathbb{C}^{N|N}$ . Therefore by applying the map  $\tau \pi \otimes \tau \pi \otimes \text{id}$  to both sides of (37) and then using (13) with (14) we get the relation

$$T_1^*(u) T_2^*(v) (R(-u, -v) \otimes 1) = (R(-u, -v) \otimes 1) T_2^*(v) T_1^*(u). \tag{46}$$

Here for  $p = 1, 2$  we denote

$$T_p^*(u) = \iota_p \otimes \text{id} (T^*(u)) \in (\text{End } \mathbb{C}^{N|N})^{\otimes 2} \otimes X(\mathfrak{p}_N) [[u^{-1}]].$$

Multiplying (46) on the left and right by  $R(u, v) \otimes 1$  and using (15) yields

$$(R(u, v) \otimes 1) T_1^*(u) T_2^*(v) = T_2^*(v) T_1^*(u) (R(u, v) \otimes 1)$$

which yields the homomorphism property of (45). This homomorphism is clearly involutive, hence bijective. It clearly commutes with (41) as well.  $\square$

8. Let us multiply the relation (37) by  $u + v$  and then set  $v = -u$ . We get

$$(Q \otimes 1) T_1(u) T_2(-u) = T_2(-u) T_1(u) (Q \otimes 1). \tag{47}$$

Since the image of the action of  $Q$  on  $(\mathbb{C}^{N|N})^{\otimes 2}$  is one-dimensional, either side of the last displayed relation must be equal to  $Q \otimes Z(u)$  where  $Z(u)$  is a power series in  $u^{-1}$  with coefficients from  $X(\mathfrak{p}_N)$ . It is immediate that every coefficient of the series  $Z(u)$  has  $\mathbb{Z}_2$ -degree 0 and that the leading term of  $Z(u)$  is 1. Hence

$$Z(u) = 1 + Z^{(1)}u^{-1} + Z^{(2)}u^{-2} + \dots$$

**Proposition 3.** *The elements  $Z^{(1)}, Z^{(2)} \dots$  of the algebra  $X(\mathfrak{p}_N)$  are central.*

*Proof.* We will work with the elements (36) where  $n = 3$ . Using (37) we get

$$\begin{aligned} &(R_{13}(u, v) R_{23}(-u, v) \otimes 1) T_1(u) T_2(-u) T_3(v) \\ &= T_3(v) T_1(u) T_2(-u) (R_{13}(u, v) R_{23}(-u, v) \otimes 1). \end{aligned} \tag{48}$$

Due to (8) and (15) we have the identity in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}(u, v)$

$$-R_{13}(u, -v) P_{12} R_{23}(-u, v) = -P_{12} \left( 1 - \frac{1}{(u+v)^2} \right).$$

Applying to this identity the antiautomorphism  $\tau \pi$  relative to the first tensor factor of  $(\text{End } \mathbb{C}^{N|N})^{\otimes 3}$  and then using (9) with (13) yields

$$Q_{12} R_{13}(u, v) R_{23}(-u, v) = Q_{12} \left( 1 - \frac{1}{(u+v)^2} \right).$$

Therefore by multiplying the relation (48) by  $Q_{12} \otimes 1$  on the left and using the above definition of the series  $Z(u)$  we obtain

$$(Q_{12} \otimes Z(u)) T_3(v) \left( 1 - \frac{1}{(u+v)^2} \right) = T_3(v) (Q_{12} \otimes Z(u)) \left( 1 - \frac{1}{(u+v)^2} \right).$$

The last relation implies that any generator  $T_{ij}^{(r)}$  of the algebra  $X(\mathfrak{p}_N)$  commutes with all coefficients of the series  $Z(u)$ .  $\square$

By the definition of  $Z(u)$  we have relations in  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2} \otimes X(\mathfrak{p}_N)[[u^{-1}]]$

$$\begin{aligned} (Q \otimes 1) T_1(u) T_2(-u) &= Q \otimes Z(u), \\ T_2(-u) T_1(u) (Q \otimes 1) &= Q \otimes Z(u). \end{aligned} \tag{49}$$

Applying  $\tau \pi \otimes \text{id} \otimes \text{id}$  to both sides of these relations and then using (9) gives

$$\begin{aligned} T_1^*(u) (-P \otimes 1) T_2(-u) &= -P \otimes Z(u), \\ T_2(-u) (-P \otimes 1) T_1^*(u) &= -P \otimes Z(u). \end{aligned}$$

Hence by (8)

$$\begin{aligned} T_1^*(u) T_1(-u) (-P \otimes 1) &= -P \otimes Z(u), \\ (-P \otimes 1) T_1(-u) T_1^*(u) &= -P \otimes Z(u). \end{aligned}$$

The latter two can be simpler written as relations in  $(\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[u^{-1}]]$

$$\begin{aligned} T^*(u) T(-u) &= 1 \otimes Z(u), \\ T(-u) T^*(u) &= 1 \otimes Z(u). \end{aligned} \tag{50}$$

Observe that the last two are also equivalent to each other due to Proposition 3. Explicitly, these last two relations can be rewritten respectively as

$$\sum_k T_{-k, -i}(u) T_{kj}(-u) (-1)^{\bar{j}\bar{k}+\bar{j}} = \delta_{ij} Z(u), \tag{51}$$

$$\sum_k T_{ik}(-u) T_{-j, -k}(u) (-1)^{\bar{i}\bar{k}+\bar{k}} = \delta_{ij} Z(u). \tag{52}$$

**Proposition 4.** *The antiautomorphism (42) maps  $Z(u) \mapsto Z(u)^{-1}$ .*

*Proof.* Due to Proposition 3 the relation (49) can be rewritten as

$$Q \otimes Z(u)^{-1} = (Q \otimes 1) T_2(-u)^{-1} T_1(u)^{-1}.$$

The right hand side of the last displayed relation can also be obtained by applying to the left hand side of (49) the antiautomorphism (42) relative to the tensor factor  $X(\mathfrak{p}_N)$ . Hence Proposition 4 follows from (49).  $\square$

**Proposition 5.** *The square of antiautomorphism (42) of  $X(\mathfrak{p}_N)$  is given by*

$$T_{ij}(u) \mapsto Z(u)^{-1} Z(-u) T_{ij}(u).$$

*Proof.* By (50)

$$T(u)^{-1} = Z(-u)^{-1} T^*(-u).$$

Thus for any indices  $i$  and  $j$  we have the relation

$$T'_{ij}(u) = Z(-u)^{-1} T_{-j, -i}(-u) (-1)^{\bar{i}\bar{j}+\bar{j}}. \tag{53}$$

By applying the antiautomorphism (42) to the right hand side of (53), and then using the same relation for  $-j, -i$  and  $-u$  instead of  $i, j$  and  $u$  we get the series

$$Z(u)^{-1} T_{ij}(u) Z(-u) = Z(u)^{-1} Z(-u) T_{ij}(u).$$

Here we also used Propositions 3 and 4.  $\square$

**Proposition 6.** *The automorphism (45) of  $X(\mathfrak{p}_N)$  maps  $Z(u) \mapsto Z(-u)$ .*

*Proof.* By setting  $j = i$  in (51) we get an explicit formula

$$Z(u) = \sum_k T_{-k, -i}(u) T_{ki}(-u) (-1)^{\bar{i}\bar{k}+\bar{i}}. \tag{54}$$

It follows from (54) that the automorphism (45) maps  $Z(u)$  to the sum

$$\sum_k T_{ik}(u) T_{-i, -k}(-u) (-1)^{\bar{i}\bar{k}+\bar{k}}.$$

On the other hand, by setting  $j = i$  in (52) we get another explicit formula

$$Z(u) = \sum_k T_{ik}(-u) T_{-i, -k}(u) (-1)^{\bar{i}\bar{k}+\bar{k}}. \tag{55}$$

Comparing the sum in (55) with the previous display completes our proof. □

**9.** There is a natural Hopf algebra structure on the extended Yangian  $X(\mathfrak{p}_N)$ . A coassociative comultiplication homomorphism  $\Delta : X(\mathfrak{p}_N) \rightarrow X(\mathfrak{p}_N) \otimes X(\mathfrak{p}_N)$  is defined by

$$\Delta : T_{ij}(u) \mapsto \sum_k T_{ik}(u) \otimes T_{kj}(u) (-1)^{(\bar{i}+\bar{k})(\bar{j}+\bar{k})} \tag{56}$$

where the tensor product is taken over the subalgebra  $\mathbb{C}[[u^{-1}]]$  of  $X(\mathfrak{p}_N)[[u^{-1}]]$ . The counit homomorphism  $\varepsilon : X(\mathfrak{p}_N) \rightarrow \mathbb{C}$  is defined by  $T_{ij}(u) \mapsto \delta_{ij}$ . Further, the antipodal mapping  $S : X(\mathfrak{p}_N) \rightarrow X(\mathfrak{p}_N)$  is just the antiautomorphism (42). Justification of these definitions is very similar to that in the case of the Yangian of the general linear Lie algebra  $\mathfrak{gl}_N$ , see [24, Section 4]. Here we omit the details.

**Proposition 7.** *For the formal power series  $Z(u)$  in  $u^{-1}$  we have*

$$\Delta : Z(u) \mapsto Z(u) \otimes Z(u) \text{ and } \varepsilon : Z(u) \mapsto 1. \tag{57}$$

*Proof.* By using the above formula (54) with the index  $k$  replaced by  $-h$ , and then using the definition (56) twice with  $k$  replaced respectively by  $-k$  and  $l$ , the comultiplication homomorphism  $\Delta$  maps  $Z(u)$  to the sum

$$\begin{aligned} & \sum_{h,k,l} (T_{h, -k}(u) T_{-h, l}(-u)) \otimes (T_{-k, -i}(u) T_{li}(-u)) (-1)^{\bar{h}\bar{l}+\bar{i}\bar{k}+\bar{k}\bar{l}+\bar{i}+\bar{k}} \\ &= \sum_{k,l} (\delta_{kl} Z(u)) \otimes (T_{-k, -i}(u) T_{li}(-u)) (-1)^{\bar{i}\bar{k}+\bar{i}+\bar{k}+\bar{l}} \\ &= \sum_k Z(u) \otimes (T_{-k, -i}(u) T_{ki}(-u)) (-1)^{\bar{i}\bar{k}+\bar{i}} = Z(u) \otimes Z(u) \end{aligned}$$

as needed. Here we also used the relation (51) with the indices  $k, l, -h$  instead of  $i, j, k$  respectively. Then we used the above formula (54) in its original form.

Due to the definition of the counit homomorphism  $\varepsilon$ , the second statement in (57) immediately follows from (54). □

Note that Proposition 4 also follows from (57), because the multiplication  $\mu : X(\mathfrak{p}_N) \otimes X(\mathfrak{p}_N) \rightarrow X(\mathfrak{p}_N)$  and the unit mapping  $\delta : \mathbb{C} \rightarrow X(\mathfrak{p}_N) : 1 \mapsto 1$  obey

$$\mu (S \otimes \text{id}) \Delta = \delta \varepsilon.$$

Indeed, by applying to the coefficients of the series  $Z(u)$  the homomorphisms at the two sides of the last displayed identity, we get the equality  $S(Z(u)) Z(u) = 1$ .

**10.** The *Yangian*  $Y(\mathfrak{p}_N)$  of the Lie superalgebra  $\mathfrak{p}_N$  is defined as the quotient of the  $\mathbb{Z}_2$ -graded algebra  $X(\mathfrak{p}_N)$  by the relations

$$Z^{(1)} = Z^{(2)} = \dots = 0.$$

Hence  $Y(\mathfrak{p}_N)$  is a complex associative unital algebra with the generators (33), subject to the relations (38) and  $Z(u) = 1$ . Here we still use the series (34). Explicit formulas for  $Z(u)$  in terms of these series are given by (54) and (55). The  $\mathbb{Z}_2$ -grading on  $Y(\mathfrak{p}_N)$  is still defined by setting  $\text{deg } T_{ij}^{(r)} = \bar{i} + \bar{j}$  for every  $r$ .

By Propositions 4 and 7 the Hopf algebra structure descends from  $X(\mathfrak{p}_N)$  to the quotient  $Y(\mathfrak{p}_N)$ . By Proposition 5 the antipodal map on  $Y(\mathfrak{p}_N)$  is involutive. In terms of the series (33), this map is still defined by (42).

By (53) the antipodal map on  $Y(\mathfrak{p}_N)$  coincides with the composition of the two commuting maps on  $Y(\mathfrak{p}_N)$  defined by (41) and (45). These two maps on  $Y(\mathfrak{p}_N)$  commute by Proposition 2. The map (41) on  $X(\mathfrak{p}_N)$  clearly descends to the quotient  $Y(\mathfrak{p}_N)$ , whereas (45) descends to  $Y(\mathfrak{p}_N)$  due to Proposition 6.

Consider the universal enveloping algebra  $U(\mathfrak{p}_N)$  of the Lie superalgebra  $\mathfrak{p}_N$ . This is an associative algebra generated by the elements  $f_{ij} \in \mathfrak{p}_N$ , see (6). The algebra  $U(\mathfrak{p}_N)$  is  $\mathbb{Z}_2$ -graded so that  $\text{deg } f_{ij} = \bar{i} + \bar{j}$ . The defining relations are

$$\begin{aligned} f_{-j,-i} &= -f_{ij} (-1)^{\bar{i}\bar{j}+\bar{j}}, \\ [f_{ij}, f_{kl}] &= \delta_{jk} f_{il} - \delta_{li} f_{kj} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \\ &\quad - \delta_{-i,k} f_{-j,l} (-1)^{\bar{i}\bar{j}+\bar{j}} - \delta_{j,-l} f_{i,-k} (-1)^{\bar{k}\bar{l}+\bar{l}}, \end{aligned}$$

see (5). But now the square brackets stand for the supercommutator in  $U(\mathfrak{p}_N)$ .

On the other hand, by setting  $Z(u) = 1$  in (51) and taking the coefficients at  $u^{-1}$  we obtain the relations in the algebra  $Y(\mathfrak{p}_N)$

$$T_{-j,-i}^{(1)} = T_{ij}^{(1)} (-1)^{\bar{i}\bar{j}+\bar{j}}.$$

Multiplying (38) by  $u^2 - v^2$  and then taking the coefficients at  $u v^{-1}$  yields

$$\begin{aligned} [T_{ij}^{(1)}, T_{kl}^{(1)}] &= (\delta_{kj} T_{il}^{(1)} - \delta_{il} T_{kj}^{(1)}) (-1)^{\bar{i}\bar{k}+\bar{i}\bar{l}+\bar{k}\bar{l}} \\ &\quad - \delta_{-i,k} T_{-j,l}^{(1)} (-1)^{\bar{i}\bar{l}+\bar{j}\bar{l}+\bar{i}} + \delta_{j,-l} T_{k,-i}^{(1)} (-1)^{\bar{i}\bar{l}+\bar{i}}. \end{aligned}$$

By comparing the last two displays with the above relations in  $U(\mathfrak{p}_N)$ , we deduce that a homomorphism  $U(\mathfrak{p}_N) \rightarrow Y(\mathfrak{p}_N)$  can be defined by mapping

$$-f_{ji} (-1)^{\bar{j}} \mapsto T_{ij}^{(1)}. \tag{58}$$

Our Theorem 2 will imply that this homomorphism is an embedding. Moreover it is an embedding of Hopf algebras, because for  $X(\mathfrak{p}_N)$  by our definitions

$$\Delta : T_{ij}^{(1)} \mapsto T_{ij}^{(1)} \otimes 1 + 1 \otimes T_{ij}^{(1)}, \quad \varepsilon : T_{ij}^{(1)} \mapsto 0, \quad S : T_{ij}^{(1)} \mapsto -T_{ij}^{(1)}.$$

**11.** For any formal power series  $c(u)$  in  $u^{-1}$  with coefficients from  $\mathbb{C}$  and leading term 1, the automorphism (39) of the associative algebra  $X(\mathfrak{p}_N)$  maps  $Z(u)$  to

$$c(u) c(-u) Z(u).$$

So (39) descends to  $Y(\mathfrak{p}_N)$  if and only if  $c(u) c(-u) = 1$ . The *special Yangian* of  $\mathfrak{p}_N$  is the fixed point subalgebra of  $Y(\mathfrak{p}_N)$  relative to all these automorphisms. We will denote this subalgebra of  $Y(\mathfrak{p}_N)$  by  $Z(\mathfrak{p}_N)$ . It will not be confused with the subalgebra of  $U(\mathfrak{p}_N)$  consisting of all elements invariant relative to the adoint action of  $\mathfrak{p}_N$ , since the latter subalgebra is just  $\mathbb{C}$  by [26, Proposition 3].

The special Yangian  $Z(\mathfrak{p}_N)$  is moreover a Hopf subalgebra of  $Y(\mathfrak{p}_N)$ . Indeed, the antipodal map on  $Y(\mathfrak{p}_N)$  preserves the subalgebra  $Z(\mathfrak{p}_N)$  by the definition of this subalgebra. Now let  $c(u)$  by any formal power series in  $u^{-1}$  with coefficients from  $\mathbb{C}$  and leading term 1, satisfying the condition  $c(u) c(-u) = 1$ . By (56) the comultiplication on  $Y(\mathfrak{p}_N)$  maps

$$\begin{aligned} c(u) T_{ij}(u) &\mapsto \sum_k (c(u) T_{ik}(u)) \otimes T_{kj}(u) (-1)^{(\bar{i}+\bar{k})(\bar{j}+\bar{k})} \\ &= \sum_k T_{ik}(u) \otimes (c(u) T_{kj}(u)) (-1)^{(\bar{i}+\bar{k})(\bar{j}+\bar{k})}. \end{aligned}$$

Hence the comultiplication intertwines the action of the automorphism (39) on  $Y(\mathfrak{p}_N)$  with its action on any one of the two tensor factors of  $Y(\mathfrak{p}_N) \otimes Y(\mathfrak{p}_N)$ . It follows that the image of the subalgebra  $Z(\mathfrak{p}_N)$  of  $Y(\mathfrak{p}_N)$  under comultiplication is contained in both  $Z(\mathfrak{p}_N) \otimes Y(\mathfrak{p}_N)$  and  $Y(\mathfrak{p}_N) \otimes Z(\mathfrak{p}_N)$ . Therefore this image is contained in  $Z(\mathfrak{p}_N) \otimes Z(\mathfrak{p}_N)$  as needed.

We will conclude this section with an important observation. For any  $t \in \mathbb{C}$  consider the representation  $\rho_t$  of  $X(\mathfrak{p}_N)$  defined at the end of Section 6. By using the formula (50) and the relations (13),(15) this representation maps  $Z(u)$  to

$$(\tau \pi \otimes \text{id})(R(u, t)) R(-u, t) = R(u, -t) R(-u, t) = 1 - \frac{1}{(u+t)^2}.$$

So the composition of  $\rho_t$  with the automorphism (39) of  $X(\mathfrak{p}_N)$  maps  $Z(u)$  to

$$c(u) c(-u) \left( 1 - \frac{1}{(u+t)^2} \right)$$

which may be equal to 1 for some  $c(u)$  if and only if  $t = 0$ . So the composition of  $\rho_t$  with (39) may factor to a representation of  $Y(\mathfrak{p}_N)$  if and only if  $t = 0$ . The choice of  $c(u)$  for that purpose is not unique then. For example, we can choose

$$c(u) = \frac{u}{u+1} \quad \text{or} \quad c(u) = \frac{u}{u-1}.$$

However, the composition of the tensor product  $\rho_t \otimes \rho_{-t}$  of representations of  $X(\mathfrak{p}_N)$  with an automorphism (39) may factor to a representation of  $Y(\mathfrak{p}_N)$  for any  $t \in \mathbb{C}$ . This is because due to (57), the latter composition maps  $Z(u)$  to

$$c(u) c(-u) \left( 1 - \frac{1}{(u+t)^2} \right) \left( 1 - \frac{1}{(u-t)^2} \right).$$

The choice of  $c(u)$  for  $\rho_t \otimes \rho_{-t}$  is still not unique. For example, we can choose

$$c(u) = \frac{(u+t)^2}{(u+t)^2 - 1} \quad \text{or} \quad c(u) = \frac{(u-t)^2}{(u-t)^2 - 1}.$$

**12.** There is a natural ascending  $\mathbb{Z}$ -filtration on the associative algebra  $Y(\mathfrak{p}_N)$ . It is defined by setting the degree of  $T_{ij}^{(r)}$  to  $r$  for each  $r \geq 1$  and all indices  $i, j$ . Let  $\text{gr } Y(\mathfrak{p}_N)$  be the  $\mathbb{Z}$ -graded algebra corresponding to this filtration. Denote by  $X_{ij}^{(r)}$  the image of the generator  $T_{ij}^{(r)}$  in the degree  $r$  component of  $\text{gr } Y(\mathfrak{p}_N)$ .

The  $\mathbb{Z}_2$ -grading on  $Y(\mathfrak{p}_N)$  descends to  $\text{gr } Y(\mathfrak{p}_N)$  so that  $\text{deg } X_{ij}^{(r)} = \bar{i} + \bar{j}$ . For any  $r, s \geq 1$  by taking the coefficients at  $u^{-r} v^{-s}$  in (38) we immediately obtain the supercommutation relation in the  $\mathbb{Z}_2$ -graded algebra  $\text{gr } Y(\mathfrak{p}_N)$

$$[X_{ij}^{(r)}, X_{kl}^{(s)}] = 0.$$

For any indices  $i, j$  introduce the series with coefficients in the algebra  $\text{gr } Y(\mathfrak{p}_N)$

$$X_{ij}(u) = \delta_{ij} \cdot 1 + X_{ij}^{(1)} u^{-1} + X_{ij}^{(2)} u^{-2} + \dots$$

Then the relations (51) with  $Z(u) = 1$  imply that for all  $i, j$  we have

$$\sum_k X_{-k,-i}(u) X_{kj}(-u) (-1)^{\bar{j}\bar{k}+\bar{j}} = \delta_{ij}. \tag{59}$$

The numerator in the first line of the defining relation (38) of  $X(\mathfrak{p}_N)$  vanishes at  $v = u$ . Further, if we multiply by  $u + v$  the expressions in the second and in the third line of (38) and then set  $v = -u$ , we will get

$$-\delta_{-i,k} \delta_{j,-l} Z(u) (-1)^{\bar{i}\bar{j}} \quad \text{and} \quad \delta_{j,-l} \delta_{-i,k} Z(u) (-1)^{\bar{l}\bar{l}+\bar{i}}$$

by using (51) and (52) respectively. The latter two expressions cancel each other. So by using (51) and (52) the defining relation (38) can be rewritten is an equality of formal power series in  $u^{-1}, v^{-1}$  with coefficients in  $X(\mathfrak{p}_N)$ .

**13.** We will also employ another ascending  $\mathbb{Z}$ -filtration on  $Y(\mathfrak{p}_N)$ . It is defined by setting the degree of  $T_{ij}^{(r)}$  to  $r - 1$  for any  $r \geq 1$ . The corresponding  $\mathbb{Z}$ -graded algebra will be denoted by  $\text{gr}' Y(\mathfrak{p}_N)$ . Let  $Y_{ij}^{(r)}$  be the image of  $T_{ij}^{(r)}$  in the degree  $r - 1$  component of  $\text{gr}' Y(\mathfrak{p}_N)$ .

The  $\mathbb{Z}_2$ -grading on  $Y(\mathfrak{p}_N)$  descends to  $\text{gr}' Y(\mathfrak{p}_N)$  so that  $\text{deg } Y_{ij}^{(r)} = \bar{i} + \bar{j}$ . For any  $r \geq 1$  by equating to  $\delta_{ij}$  the left hand side of (51) and then taking the coefficients at  $u^{-r}$  we obtain the relation in the algebra  $\text{gr}' Y(\mathfrak{p}_N)$

$$Y_{-j,-i}^{(r)} = -Y_{ij}^{(r)} (-1)^{\bar{i}\bar{j}+\bar{j}+r}. \tag{60}$$

For  $r, s \geq 1$  by taking coefficients at  $u^{-r} v^{-s}$  in (38) we obtain the relation

$$\begin{aligned} [Y_{ij}^{(r)}, Y_{kl}^{(s)}] &= (\delta_{kj} Y_{il}^{(r+s-1)} - \delta_{il} Y_{kj}^{(r+s-1)}) (-1)^{\bar{i}\bar{k}+\bar{i}\bar{l}+\bar{k}\bar{l}} \\ &\quad + \delta_{-i,k} Y_{-j,l}^{(r+s-1)} (-1)^{\bar{l}\bar{l}+\bar{j}\bar{l}+\bar{i}+r} - \delta_{j,-l} Y_{k,-i}^{(r+s-1)} (-1)^{\bar{l}\bar{l}+\bar{i}+r}. \end{aligned} \tag{61}$$

Here we used the equality to 1 of the coefficient at  $u^{-r} v^{-s}$  in the expansion of

$$\frac{v^{1-r-s} - u^{1-r-s}}{u - v}$$

as a polynomial in  $u^{-1}, v^{-1}$ . We also used the relation (60) obtained just above.

The structure of  $\mathbb{Z}_2$ -graded Hopf algebra on  $Y(\mathfrak{p}_N)$  descends to  $\text{gr}' Y(\mathfrak{p}_N)$ . Due to the definition (56) the comultiplication, the counit homomorphism and the antipodal antiautomorphism for  $\text{gr}' Y(\mathfrak{p}_N)$  are defined respectively by

$$Y_{ij}^{(r)} \mapsto Y_{ij}^{(r)} \otimes 1 + 1 \otimes Y_{ij}^{(r)}, \quad Y_{ij}^{(r)} \mapsto 0, \quad Y_{ij}^{(r)} \mapsto -Y_{ij}^{(r)}. \tag{62}$$

**14.** Now consider the twisted polynomial current Lie superalgebra  $\mathfrak{g}$  as defined in Section 1. This subalgebra of  $\mathfrak{gl}_{N|N}[u]$  is spanned by the elements

$$g_{ij}^{(r)} = e_{ij} u^r + \omega(e_{ij})(-u)^r = e_{ij} u^r - e_{-j, -i} u^r (-1)^{\bar{i}\bar{j}+\bar{j}+r}$$

where  $r = 0, 1, 2, \dots$  and  $i, j = \pm 1, \dots, \pm N$ . Here  $g_{ij}^{(0)} = f_{ij}$  by (6). For  $r, s \geq 0$

$$g_{-j, -i}^{(r)} = -g_{ij}^{(r)} (-1)^{\bar{i}\bar{j}+\bar{j}+r}, \tag{63}$$

$$\begin{aligned} [g_{ij}^{(r)}, g_{kl}^{(s)}] &= \delta_{jk} g_{il}^{(r+s)} - \delta_{li} g_{kj}^{(r+s)} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \\ &\quad - \delta_{-i, k} g_{-j, l}^{(r+s)} (-1)^{\bar{i}\bar{j}+\bar{j}+r} - \delta_{j, -l} g_{i, -k}^{(r+s)} (-1)^{\bar{k}\bar{l}+\bar{l}+s}. \end{aligned} \tag{64}$$

The universal enveloping algebra  $U(\mathfrak{g})$  is generated by the elements  $g_{ij}^{(r)}$  as an associative algebra. It is  $\mathbb{Z}_2$ -graded so that the degrees of these generators are equal to  $\bar{i} + \bar{j}$  respectively. The defining relations for these generators are (63) and (64) where the square brackets now denote the supercommutator.

By comparing (63), (64) with (60), (61) above we deduce that the mapping

$$-g_{ji}^{(r)} (-1)^{\bar{j}} \mapsto Y_{ij}^{(r+1)} \tag{65}$$

for  $r \geq 0$  defines a homomorphism  $U(\mathfrak{g}) \rightarrow \text{gr}' Y(\mathfrak{p}_N)$  of  $\mathbb{Z}_2$ -graded associative algebras. Moreover, this is a Hopf algebra homomorphism by the definitions (62).

Note that for any  $t \in \mathbb{C}$  the image of  $T_{ij}^{(r+1)} \in Y(\mathfrak{p}_N)$  under the representation  $\rho_t$  coincides with the image of the left hand side of (65) under the evaluation representation  $U(\mathfrak{gl}_{N|N}[u]) \mapsto \text{End } \mathbb{C}^{N|N}$  defined by mapping  $e_{ij} u^r \mapsto E_{ij} t^r$ .

**Theorem 2.** *The mapping (65) defines an isomorphism  $U(\mathfrak{g}) \rightarrow \text{gr}' Y(\mathfrak{p}_N)$  of  $\mathbb{Z}_2$ -graded Hopf algebras.*

*Proof.* The homomorphism  $U(\mathfrak{g}) \rightarrow \text{gr}' Y(\mathfrak{p}_N)$  defined by (65) above is clearly surjective. We have to prove that this homomorphism is injective as well.

Let  $\mathcal{I}$  be the set of all triples  $(i, j, r)$  with  $r \geq 0$ . Choose any total ordering  $<$  on this set. Also choose any subset  $\mathcal{J} \subset \mathcal{I}$  such that for  $i \neq -j$  it contains only one of the triples  $(i, j, r)$  and  $(-j, -i, r)$ , while for  $i = -j$  it contains  $(i, j, r)$  if and only if  $\bar{i} + r$  is odd. Due to (63) the left hand sides of (65) corresponding to the triples  $(i, j, r) \in \mathcal{J}$  form a basis of the vector space  $\mathfrak{g}$ .

Let  $\mathcal{T}$  be the set of all products of  $T_{ij}^{(r+1)}$  corresponding to  $(i, j, r) \in \mathcal{J}$  taken in non-decreasing orders, such that two adjacent triples can be the same  $(i, j, r)$  only if  $\bar{i} + \bar{j} = 0$  in  $\mathbb{Z}_2$ . We will prove that  $\mathcal{T}$  is linearly independent in  $Y(\mathfrak{p}_N)$ . This is equivalent to the linear independence in  $\text{gr}' Y(\mathfrak{p}_N)$  of the set of products obtained from  $\mathcal{T}$  by replacing



every factor  $T_{ij}^{(r+1)}$  by the corresponding  $Y_{ij}^{(r+1)}$ . Hence we will prove the injectivity of the homomorphism defined by (65).

Let  $\mathcal{S}$  be the set of all products of the generators  $T_{ij}^{(r+1)}$  of the algebra  $Y(\mathfrak{p}_N)$ . Let us call a pair of triples  $(i, j, r)$  and  $(k, l, s)$  *misordered* if  $(i, j, r) > (k, l, s)$ , or  $(i, j, r) = (k, l, s)$  and  $\bar{i} + \bar{j} = 1$  in  $\mathbb{Z}_2$ . For any product  $A \in \mathcal{S}$  let  $\mu(A)$  be the number of pairs of factors of  $A$  corresponding to misordered pairs of triples. The pairs of factors of  $A$  counted by  $\mu(A)$  need not to be adjacent. Let  $\nu(A)$  be the number of factors of  $A$  corresponding to the triples which are not in  $\mathcal{J}$ .

Let  $A$  and  $B$  be any two products from the set  $\mathcal{S}$ . Let  $a$  and  $b$  be the numbers of their factors respectively. Let  $c$  and  $d$  be respectively the sums of the degrees of their factors relative to the  $\mathbb{Z}$ -filtration on  $Y(\mathfrak{p}_N)$  from Section 13. We will write  $A < B$  if  $c < d$ , or  $c = d$  but  $a < b$ , or  $c = d$  and  $a = b$  but  $\nu(A) < \nu(B)$ , or  $A$  is a permutation of factors of  $B$  and  $\mu(A) < \mu(B)$ . The relation denoted by  $<$  is a partial ordering on the set  $\mathcal{S}$ , with every descending chain terminating. Notice that the condition  $A < B$  implies that  $XAY < XBY$  for any  $X, Y \in \mathcal{S}$ .

Let  $S(u)$  be the element of the algebra  $(\text{End } \mathbb{C}^{N|N}) \otimes Y(\mathfrak{p}_N)[[u^{-1}]]$  inverse to  $T^*(-u)$ . When determining the inverse element here we do not use any relations in  $Y(\mathfrak{p}_N)$ . The relation (50) with  $Z(u) = 1$  can be now written as  $S(u) = T(u)$ . For any  $i, j$  define a formal power series  $S_{ij}(u)$  in  $u^{-1}$  with coefficients in  $Y(\mathfrak{p}_N)$ ,

$$S(u) = \sum_{i,j} E_{ij} \otimes S_{ij}(u).$$

Define the elements  $S_{ij}^{(1)}, S_{ij}^{(2)}, \dots \in Y(\mathfrak{p}_N)$  as the coefficients of the series  $S_{ij}(u)$  at  $u^{-1}, u^{-2}, \dots$  respectively. For any  $r \geq 0$

$$S_{ij}^{(r+1)} = T_{-j,-i}^{(r+1)} (-1)^{\bar{i}\bar{j} + \bar{j} + r} + C \tag{66}$$

where  $C$  is a linear combination of products of two or more generators of  $Y(\mathfrak{p}_N)$  with the sums of the degrees of factors by the  $\mathbb{Z}$ -filtration from Section 13 being less than  $r$ . These products occur in the linear combination  $C$  only if  $r > 0$ .

Take any  $(i, j, r) \notin \mathcal{J}$ . If  $i \neq -j$ , then  $(-j, -i, r) \in \mathcal{J}$ . In this case we will replace any factor  $T_{ij}^{(r+1)}$  of  $B$  by the right hand side of (66). But if  $i = -j$  and  $\bar{i} + r$  is even, then we will replace  $T_{ij}^{(r+1)}$  by  $C/2$ . We will call either replacement a *reduction* of  $B$ . It does not change the value of  $B \in Y(\mathfrak{p}_N)$  and is compatible with  $<$  on  $\mathcal{S}$ , so that its result is a linear combination of products  $A < B$ .

For any triples  $(i, j, r)$  and  $(k, l, s)$  the relation (38) gives an equality in  $Y(\mathfrak{p}_N)$

$$T_{ij}^{(r+1)} T_{kl}^{(s+1)} = T_{kl}^{(s+1)} T_{ij}^{(r+1)} (-1)^{(\bar{i} + \bar{j})(\bar{k} + \bar{l})} + D \tag{67}$$

where  $D$  is a linear combination of single generators of  $Y(\mathfrak{p}_N)$  of degree  $r + s$  by the  $\mathbb{Z}$ -filtration from Section 13, and of products of two generators with the sums of the degrees of factors being equal to  $r + s - 1$ . Here we use the expansions

$$\frac{1}{u-v} = u^{-1} + u^{-2}v + \dots \quad \text{and} \quad \frac{1}{u+v} = u^{-1} - u^{-2}v + \dots$$

then equate the coefficients at  $u^{-r-1}v^{-s-1}$  of the series at the two sides of (38).

If  $(i, j, r) > (k, l, s)$  then we will replace any product of the corresponding two adjacent factors of  $B$  by the right hand side of (67). But if  $(i, j, r) = (k, l, s)$  and  $\bar{i} + \bar{j} = 1$  in  $\mathbb{Z}_2$  then we will replace the product of the two factors of  $B$  by  $D/2$ . The value of  $B \in Y(\mathfrak{p}_N)$  will not change then. Either replacement will be called a *reduction* of  $B$  too. It is compatible with the ordering  $<$  on  $S$  as well.

When proving the linear independence of the set  $\mathcal{T}$  in  $Y(\mathfrak{p}_N)$ , we may assume that  $(1, 1, r) \notin \mathcal{J}$  for every  $r \geq 0$ . The collection of relations  $S_{11}^{(r+1)} = T_{11}^{(r+1)}$  can be written as the equality  $S_{11}(u) = T_{11}(u)$ . Using only the relation (50) in  $(\text{End } \mathbb{C}^{N|N}) \otimes Y(\mathfrak{p}_N)[[u^{-1}]]$ , this equality gives  $Z(u) = 1$ . But the relation (50) follows from (37), see Section 8. So the relation  $Z(u) = 1$  follows from (37) and from the relations  $S_{11}^{(r+1)} = T_{11}^{(r+1)}$  for every  $r \geq 0$ . Under our assumption on the choice of  $\mathcal{J}$ , the latter relations are used to define reductions via (66).

Let now us multiply each side of the relation (37) by  $P \otimes 1$  on the left and on the right. Also exchange the variables  $u$  and  $v$ . The result is the relation

$$(R(-u, -v) \otimes 1) T_2(v) T_1(u) = T_1(u) T_2(v) (R(-u, -v) \otimes 1) \tag{68}$$

since by (12)

$$P R(v, u) \otimes 1) P = R(-u, -v). \tag{69}$$

But (68) can also be obtained by multiplying each side of (37) on the left and on the right by  $R(-u, -v) \otimes 1$  and then using (15). Inductively, this remark shows that the relations (67) with  $(i, j, r) \geq (k, l, s)$  imply all other relations (67). In the induction argument, we use the  $\mathbb{Z}$ -filtration on  $Y(\mathfrak{p}_N)$  from Section 12.

If  $(i, j) = (k, l)$  and  $\bar{i} + \bar{j} = 0$  in  $\mathbb{Z}_2$ , then the relation (38) simply means that the coefficients of the series  $T_{ij}(u)$  pairwise commute. Indeed, in this case the expressions in the second and the third lines of (38) vanish. Then (38) becomes

$$T_{ij}(u) T_{ij}(v) - T_{ij}(v) T_{ij}(u) = \frac{T_{ij}(u) T_{ij}(v) - T_{ij}(v) T_{ij}(u)}{u - v}.$$

Equivalently,

$$T_{ij}(u) T_{ij}(v) = T_{ij}(v) T_{ij}(u).$$

In particular, we can eliminate from the definition of  $Y(\mathfrak{p}_N)$  the relations (67) with  $(i, j, r) = (k, l, s)$  and  $\bar{i} + \bar{j} = 0$  in  $\mathbb{Z}_2$  as tautological relations. But to define reductions via (67), we used (67) only when  $(i, j, r) > (k, l, s)$  or  $(i, j, r) = (k, l, s)$  and  $\bar{i} + \bar{j} = 1$  in  $\mathbb{Z}_2$ . The above arguments show that the used relations imply all other relations (67). Thus our reductions involve all defining relations of  $Y(\mathfrak{p}_N)$ .

Let us now examine the *inclusion ambiguities* of our set of reductions. They correspond to misordered pairs of triples where one or both triples are not in  $\mathcal{J}$ . Let us show that they are resolvable relative to  $<$  in the sense of [6, Section 1].

Take any product  $T_{ij}^{(r+1)} T_{kl}^{(s+1)}$  where the pair of triples  $(i, j, r)$  and  $(k, l, s)$  is misordered. Suppose that  $(i, j, r) \notin \mathcal{J}$  but  $(k, l, s) \in \mathcal{J}$ . Then we can apply to the above product the reduction via (66) in the first factor, or alternatively the reduction via (67). The two results coincide by comparing (37) to the equality

$$(R(u, v) \otimes 1) S_1(u) T_2(v) = T_2(v) S_1(u) (R(u, v) \otimes 1). \tag{70}$$

Note that to prove the coincidence here we use equalities of linear combinations of products from  $\mathcal{S}$  which precede  $T_{ij}^{(r+1)} T_{kl}^{(s+1)}$  relative to the partial ordering.

To get the equality (70) we apply the map  $\tau \pi \otimes \text{id} \otimes \text{id}$  to both sides of (37) and then use (13). This yields the equality

$$T_1^*(u) (R(u, -v) \otimes 1) T_2(v) = T_2(v) (R(u, -v) \otimes 1) T_1^*(u).$$

By multiplying both sides of this equality by  $T_1^*(u)^{-1}$  on the left and on the right, changing  $u$  to  $-u$  and then using (15) we indeed get (70).

The case when  $(i, j, r) \in \mathcal{J}$  but  $(k, l, s) \notin \mathcal{J}$  can be treated in a similar way, but using instead of (70) the equality

$$(R(u, v) \otimes 1) T_1(u) S_2(v) = S_2(v) T_1(u) (R(u, v) \otimes 1). \tag{71}$$

To get this equality multiply each side of (37) by  $R(-u, -v) \otimes 1$  on the left and on the right, and then use (15). Apply the map  $\text{id} \otimes \tau \pi \otimes \text{id}$  to both sides of the resulting equality and then use (14). In this way we obtain the equality

$$T_1(u) (R(u, -v) \otimes 1) T_2^*(v) = T_2^*(v) (R(u, -v) \otimes 1) T_1(u).$$

Multiplying its both sides of this equality on the left and on the right by  $T_2^*(v)^{-1}$  and then changing  $v$  to  $-v$  yields (71).

The case when  $(i, j, r) \notin \mathcal{J}$  and  $(k, l, s) \in \mathcal{J}$  can also be treated in a similar way, but using instead of (70) the equality

$$(R(u, v) \otimes 1) S_1(u) S_2(v) = S_2(v) S_1(u) (R(u, v) \otimes 1).$$

To get this equality, multiply both sides of (46) on the left and on the right first by  $T_1^*(u)^{-1}$  and then by  $T_2^*(v)^{-1}$ . Then change  $u, v$  to  $-u, -v$  respectively.

Finally, let us examine the *overlap ambiguities* of our set of reductions. They correspond to the sequences of triples of length three, such that the pair of first two triples and the pair of the last two triples in the sequence are misordered. Let us show that they are also resolvable relative to  $<$  in the sense of [6, Section 1].

Take any product  $T_{ij}^{(r+1)} T_{kl}^{(s+1)} T_{gh}^{(t+1)}$  such that the pair of triples  $(i, j, r)$  and  $(k, l, s)$  is misordered, and so is the pair of triples  $(k, l, s)$  and  $(g, h, t)$ . We can apply to the product the reduction via (67) in the first two factors. Alternatively, in the last two factors we can apply the reduction via the equality obtained by replacing  $(i, j, r)$  and  $(k, l, s)$  in (67) respectively by  $(k, l, s)$  and  $(g, h, t)$ .

To see that the results of two reductions coincide, we can continue using (67). Thus for the first reduction we will use the equality

$$\begin{aligned} &(R_{23}(v, w) R_{13}(u, w) R_{12}(u, v) \otimes 1) T_1(u) T_2(v) T_3(w) \\ &= T_3(w) T_2(v) T_1(u) (R_{23}(v, w) R_{13}(u, w) R_{12}(u, v) \otimes 1) \end{aligned}$$

in the notation (36) with  $n = 3$ . Here  $w$  is another formal variable. In the same notation, for the second reduction we will use the equality

$$\begin{aligned} &(R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) \otimes 1) T_1(u) T_2(v) T_3(w) \\ &= T_3(w) T_2(v) T_1(u) (R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) \otimes 1). \end{aligned}$$

The two results will coincide due to (16). Note that to prove the coincidence here we will use equalities of linear combinations of products from  $\mathcal{S}$  which precede  $T_{ij}^{(r+1)} T_{kl}^{(s+1)}$  relative to the partial ordering.

So both the inclusion and the overlap ambiguities of our set of reductions are resolvable. But the elements of the set  $\mathcal{T}$  cannot be changed by the reductions. The linear independence of  $\mathcal{T}$  in  $Y(\mathfrak{p}_N)$  is now ensured by [6, Theorem 1.2].  $\square$

**Corollary 1.** *The elements of  $\text{gr } Y(\mathfrak{p}_N)$  corresponding to  $T_{ij}^{(r+1)}$  with the triples  $(i, j, r) \in \mathcal{J}$  are free generators of this supercommutative algebra.*

Our proof of Theorem 2 implies the Poincaré–Birkhoff–Witt Theorem for the algebra  $U(\mathfrak{g})$ , see [17, Theorem 5.15]. Thus we do not rely on the latter theorem.

**Corollary 2.** *The homomorphism  $U(\mathfrak{p}_N) \rightarrow Y(\mathfrak{p}_N)$  defined by the mapping (58) is an embedding of  $\mathbb{Z}_2$ -graded Hopf algebras.*

Let us link  $Y(\mathfrak{p}_N)$  to the Yangian  $Y(\mathfrak{gl}_N)$  of the general linear Lie algebra  $\mathfrak{gl}_N$ . The latter Yangian is a deformation of the universal enveloping algebra of the polynomial current Lie algebra  $\mathfrak{gl}_N[u]$  in the class of Hopf algebras. Comparing (38) with the relations in  $Y(\mathfrak{gl}_N)$  as given in [24, Section 1] immediately yields

**Corollary 3.** *The associative subalgebra of  $Y(\mathfrak{p}_N)$  generated by elements  $T_{ij}^{(r)}$  with  $i, j > 0$  is isomorphic to the algebra  $Y(\mathfrak{gl}_N)$ .*

Note that the associative subalgebra of  $Y(\mathfrak{p}_N)$  appearing in Corollary 3 is not a Hopf subalgebra, see the definition (56). Thus the isomorphism of Corollary 3 is not an isomorphism of Hopf algebras.

**15.** In this section we will explicitly describe the centre of the Yangian  $Y(\mathfrak{p}_N)$ . An element of an  $\mathbb{Z}_2$ -graded associative algebra is *central* if its supercommutator with every element of the algebra is zero.

First consider the subalgebra of the extended Yangian  $X(\mathfrak{p}_N)$  generated by the elements  $T_{ij}^{(r)}$  with  $i, j > 0$ . This subalgebra of  $X(\mathfrak{p}_N)$  is isomorphic to  $Y(\mathfrak{gl}_N)$ . Similarly to Corollary 3 this isomorphism property follows from Theorem 2 by (38). Consider the formal power series in  $u^{-1}$  with coefficients in this subalgebra

$$A(u) = \sum_{\sigma} (-1)^{\sigma} T_{1,\sigma(1)}(u) T_{2,\sigma(2)}(u + 1) \dots T_{N,\sigma(N)}(u + N - 1) \tag{72}$$

where  $\sigma$  ranges over the symmetric group  $\mathfrak{S}_N$  permuting the indices  $1, \dots, N$  and  $(-1)^{\sigma}$  stands for the sign of the permutation  $\sigma$ . In the sum (72) the power series in  $(u + 1)^{-1}, \dots, (u + N - 1)^{-1}$  should be reexpanded in  $u^{-1}$ . The series  $A(u)$  has the leading term 1. The coefficients of  $A(u)$  at  $u^{-1}, u^{-2}, \dots$  are free generators of the centre of that subalgebra of  $X(\mathfrak{p}_N)$  by [18, Theorem 2.13]. All these coefficients have  $\mathbb{Z}_2$ -degree zero, hence the supercommutator with any of them in the algebra  $X(\mathfrak{p}_N)$  is the usual commutator. By [18, Proposition 2.7]

$$A(u) = \sum_{\sigma} (-1)^{\sigma} T_{\sigma(1),1}(u + N - 1) \dots T_{\sigma(N-1),N-1}(u + 1) T_{\sigma(N),N}(u).$$

Now take the series in  $u^{-1}$  with coefficients in  $X(\mathfrak{p}_N)$  and the leading term 1

$$A(u) A(1 - N - u). \tag{73}$$

**Theorem 3.** *The coefficients of the series (73) belong to the centre of  $X(\mathfrak{p}_N)$ .*

We will prove this theorem in the next nine sections. We will now employ it to describe the centre of the algebra  $Y(\mathfrak{p}_N)$ . Let  $B^{(1)}, B^{(2)} \dots$  be the images in  $Y(\mathfrak{p}_N)$  of the coefficients of the series (73) at  $u^{-1}, u^{-2}, \dots$  respectively. These images are central in  $Y(\mathfrak{p}_N)$  by Theorem 3. They all are of  $\mathbb{Z}_2$ -degree zero.

**Corollary 4.** *The coefficients  $B^{(2)}, B^{(4)} \dots$  freely generate the centre of  $Y(\mathfrak{p}_N)$ .*

*Proof.* The elements  $B^{(2)}, B^{(4)} \dots$  of  $Y(\mathfrak{p}_N)$  are central by Theorem 3. It suffices to prove now that the images of these elements in the graded algebra  $\text{gr}' Y(\mathfrak{p}_N)$  are free generators of its centre. For  $r \geq 1$  the image of  $B^{(2r)}$  in  $\text{gr}' Y(\mathfrak{p}_N)$  equals

$$2 Y_{11}^{(2r)} + \dots + 2 Y_{NN}^{(2r)} \tag{74}$$

by the arguments from the beginning of the proof of [18, Theorem 2.13]. Due to our Theorem 2 the algebra  $\text{gr}' Y(\mathfrak{p}_N)$  is isomorphic to the universal enveloping algebra  $U(\mathfrak{g})$  via the mapping (65). Under this isomorphism the element (74) of  $\text{gr}' Y(\mathfrak{p}_N)$  with any  $r \geq 1$  corresponds to the element of  $U(\mathfrak{g})$

$$-2 g_{11}^{(2r-1)} - \dots - 2 g_{NN}^{(2r-1)} = -2 (e_{11} + e_{-1,-1} + \dots + e_{NN} + e_{-N,-N}) u^{2r-1}$$

from our Section 14. The latter elements freely generate of the centre of  $U(\mathfrak{g})$ .

Indeed, they are algebraically independent due to the Poincaré–Birkhoff–Witt Theorem for  $U(\mathfrak{g})$ . To show that they generate the centre of  $U(\mathfrak{g})$  consider the quotient of  $U(\mathfrak{g})$  by the ideal they generate. This quotient is isomorphic to the universal enveloping algebra  $U(\mathfrak{h})$  of the Lie superalgebra

$$\mathfrak{h} = \{h(u) \in \mathfrak{a}[u] : \omega(h(u)) = h(-u)\}$$

where  $\mathfrak{a}$  is the quotient of Lie superalgebra  $\mathfrak{gl}_{N|N}$  by the span of a central element

$$e_{11} + e_{-1,-1} + \dots + e_{NN} + e_{-N,-N}.$$

The automorphism  $\omega$  of  $\mathfrak{gl}_{N|N}$  maps this element to its negative and therefore descends to the quotient  $\mathfrak{a}$ . But the centre of the Lie superalgebra  $\mathfrak{a}$  is trivial. Hence the centre of  $U(\mathfrak{h})$  is also trivial by [23, Proposition 3.6]. □

To finish this section let us consider the images of the elements  $B^{(1)}, B^{(2)} \dots$  of  $Y(\mathfrak{p}_N)$  in the supercommutative algebra  $\text{gr} Y(\mathfrak{p}_N)$ . In the notation of Section 12 let  $X(u)$  the  $2N \times 2N$  matrix whose  $ij$  entry is the series

$$X_{ij}(u) (-1)^{\bar{i}\bar{j}+\bar{j}}.$$

We index the rows and columns of  $X(u)$  by the numbers  $1, \dots, N, -1, \dots, -N$ .

The matrix  $X(u)$  is invertible. Let  $X'_{ij}(u)$  be the  $ij$  entry of the inverse matrix. Then the relations (59) imply that for any indices  $i$  and  $j$

$$X_{ij}(-u) = X'_{-j,-i}(u).$$

Therefore by using the definition (72), the image in  $\text{gr} Y(\mathfrak{p}_N)[[u^{-1}]]$  of the series

$$B(u) = 1 + B^{(1)} u^{-1} + B^{(2)} u^{-2} + \dots$$

can be written as the product of the sum

$$\sum_{\sigma} (-1)^{\sigma} X_{1,\sigma(1)}(u) \dots X_{N,\sigma(N)}(u)$$

by the sum

$$\sum_{\sigma} (-1)^{\sigma} X'_{-\sigma(1),-1}(u) \dots X'_{-\sigma(N),-N}(u).$$

This product is the *Berezinian* or *superdeterminant* of the matrix  $X(u)$ , see [5]. Hence the series  $B(u)$  is an analogue for  $Y(\mathfrak{p}_N)$  of the quantum Berezinian [20] for the Yangian of the general linear Lie superalgebra  $\mathfrak{gl}_{M|N}$ , see [25] for details.

**16.** We will now begin our proof of Theorem 3. We will use the same method as in the proof of Proposition 3. However, the calculations will be more involved. Fix any  $n \geq 1$  and take the rational function of  $u$  with values in  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$

$$H(u) = \prod_{1 \leq p < q \leq n} R_{pq}(u+n-p, u+n-q)$$

where the factors are arranged from left to right by ordering their pairs of indices  $(p, q)$  lexicographically. The ordering of factors can be altered by employing (16), without changing the function. If  $n = 1$  then  $H(u) = 1$  by definition. For any  $n \geq 1$  by using the defining relation (37) repeatedly, we get the equality

$$(H(u) \otimes 1) T_1(u+n-1) \dots T_n(u) = T_n(u) \dots T_1(u+n-1) (H(u) \otimes 1)$$

of formal power series in  $u^{-1}$  with coefficients in  $(\text{End } \mathbb{C}^{N|N})^{\otimes n} \otimes X(\mathfrak{p}_N)$ .

**Lemma 1.** For any  $1 \leq p < q \leq n$  we have the equalities in  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}(u)$

$$P_{pq}H(u) = -H(u) \text{ and } Q_{pq}H(u) = 0.$$

*Proof.* For  $n = 1$  we have nothing to prove. Consider the case of  $n = 2$ . In this particular case  $H(u) = R(u+1, u)$  while

$$P R(u+1, u) = P \left( 1 - P + \frac{Q}{2u+1} \right) = P - 1 - \frac{Q}{2u+1} = -R(u+1, u)$$

by the equality  $P^2 = 1$  and by the first equality in (12). By the second and third equalities in (12) we also have

$$Q R(u+1, u) = Q \left( 1 - P + \frac{Q}{2u+1} \right) = Q - Q - \frac{0}{2u+1} = 0.$$

Now for any  $n > 2$  and for any given index  $p < n$  let  $q = p + 1$ . By (16) we can rearrange the factors of  $H(u)$  so that  $R_{p,p+1}(u+n-p, u+n-p-1)$  is the leftmost factor. Then the above calculations for  $n = 2$  imply the equalities of our lemma for any  $n$ . Hence our lemma holds whenever  $q = p + 1$ .

In turn, this implies the required equalities for every  $q > p$ . Indeed, the set of all elements  $P_{pq}$  and  $Q_{pq}$  with  $q > p$  is generated in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$  by those elements where  $q = p + 1$ . For example,  $Q_{13} = P_{12} Q_{23} P_{12}$  here.  $\square$

It follows from Lemma 1 that every value of the normalized function  $H(u)/n!$  is idempotent in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$ . Indeed, by using the lemma we get

$$H(u) H(u) = \prod_{1 \leq p < q \leq n} \left( 1 + \frac{1}{p - q} \right) \cdot H(u) = n! H(u).$$

Elements of  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}$  act on  $(\mathbb{C}^{N|N})^{\otimes n}$  by (1) and (2). Take any vector  $e_{i_1} \otimes \dots \otimes e_{i_n}$  with only positive indices  $i_1, \dots, i_n$ . Then all elements  $Q_{pq}$  with  $1 \leq p < q \leq n$  annihilate this vector. Hence

$$\begin{aligned} H(u)(e_{i_1} \otimes \dots \otimes e_{i_n}) &= \prod_{1 \leq p < q \leq n} \left( 1 + \frac{P_{pq}}{p - q} \right) \cdot (e_{i_1} \otimes \dots \otimes e_{i_n}) \\ &= \sum_{\sigma} (-1)^{\sigma} e_{i_{\sigma(1)}} \otimes \dots \otimes e_{i_{\sigma(n)}} \quad \text{where } i_1, \dots, i_n > 0 \end{aligned} \tag{75}$$

and  $\sigma$  ranges over  $\mathfrak{S}_n$ . The last equality here is well known [18, Proposition 2.3].

**Lemma 2.** For any  $1 \leq p < n$  we have an equality in  $(\text{End } \mathbb{C}^{N|N})^{\otimes n}(u)$

$$H(u) Q_{p,p+1} = (2u + 2n - 2p - 1) H(u) (1 + P_{p,p+1}).$$

*Proof.* For  $n = 1$  we have nothing to prove. Let  $n = 2$ . Then  $H(u) = R(u + 1, u)$  while by the relation (15) we have

$$R(u + 1, u) R(-u - 1, -u) = 0.$$

Multiplying this equality by  $2u + 1$  and using the definition of the second factor at the left hand side we get the equality

$$R(u + 1, u) ((2u + 1)(1 + P) - Q) = 0.$$

This implies our lemma for  $n = 2$  and  $p = 1$ . For any  $n > 2$  and  $1 \leq p < n$  by using (16) we can rearrange the factors of the product  $H(u)$  so that the factor

$$R_{p,p+1}(u + n - p, u + n - p - 1)$$

is the rightmost. Then the above calculation with  $u$  replaced by  $u + n - p - 1$  implies the lemma. □

It follows from Lemma 2 that for  $1 \leq p < n$  and for any indices  $i_1, \dots, i_n$

$$H(u) (1 + P_{p,p+1})(e_{i_1} \otimes \dots \otimes e_{i_n}) = 0 \quad \text{if } i_p + i_{p+1} \neq 0. \tag{76}$$

17. Let us continue our proof of Theorem 3. For any  $n \geq 1$  consider the rational function of  $u, v$  with values in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes(n+1)}$

$$F(u, v) = R_{1,n+1}(u + n - 1, v) \dots R_{n,n+1}(u, v).$$

By using (37) repeatedly we get the equality of formal power series in  $u^{-1}, v^{-1}$

$$\begin{aligned} (F(u, v) \otimes 1) T_1(u + n - 1) \dots T_n(u) T_{n+1}(v) \\ = T_{n+1}(v) T_1(u + n - 1) \dots T_n(u) (F(u, v) \otimes 1) \end{aligned} \tag{77}$$

which have their coefficients in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes(n+1)} \otimes X(\mathfrak{p}_N)$ .

**Lemma 3.** For any  $n \geq 1$  the rational function  $(H(u) \otimes 1) F(u, v)$  of  $u, v$  with values in  $(\text{End } \mathbb{C}^{N|N})^{\otimes(n+1)}$  is equal to

$$\left( 1 - \frac{P_{1,n+1} + \dots + P_{n,n+1}}{u - v} + \frac{Q_{1,n+1} + \dots + Q_{n,n+1}}{u + v + n - 1} \right) (H(u) \otimes 1)$$

plus terms divisible on the left by  $Q_{pq} \in (\text{End } \mathbb{C}^{N|N})^{\otimes(n+1)}$  with  $1 \leq p < q \leq n$ .

*Proof.* Using the relation (16) we can rewrite the product  $(H(u) \otimes 1) F(u, v)$  as

$$R_{n,n+1}(u, v) \dots R_{1,n+1}(u + n - 1, v) (H(u) \otimes 1). \tag{78}$$

We will prove by induction on  $n$  that the latter product is equal to the sum in the lemma. If  $n = 1$  then  $H(u) = 1$  while  $F(u, v) = R(u, v)$ . Then the required equality is just the definition of  $R(u, v)$  as given in Section 5.

Suppose that  $n > 1$ . Using (16) we obtain that  $H(u)$  is divisible on the left by the product of its factors  $R_{pq}(u + n - p, u + n - q)$  with  $2 \leq p < q \leq n$ , taken in the same order as in the definition of  $H(u)$ . All these factors commute with the factor  $R_{1,n+1}(u + n - 1, v)$  of (78). Using the induction assumption and the definition of  $H(u)$  for  $n - 1$  instead of  $n$ , the product (78) now equals

$$\begin{aligned} & \left( 1 - \frac{P_{2,n+1} + \dots + P_{n,n+1}}{u - v} + \frac{Q_{2,n+1} + \dots + Q_{n,n+1}}{u + v + n - 2} \right) \\ & \times \left( 1 - \frac{P_{1,n+1}}{u - v + n - 1} + \frac{Q_{1,n+1}}{u + v + n - 1} \right) (H(u) \otimes 1) \end{aligned}$$

plus terms divisible on the left by  $Q_{pq} \in (\text{End } \mathbb{C}^{N|N})^{\otimes(n+1)}$  with  $2 \leq p < q \leq n$ . We also used the relations (16). The product of two sums in the brackets equals

$$\begin{aligned} & 1 - \frac{P_{1,n+1}}{u - v + n - 1} + \frac{Q_{1,n+1}}{u + v + n - 1} \\ & - \frac{P_{2,n+1} + \dots + P_{n,n+1}}{u - v} + \frac{Q_{2,n+1} + \dots + Q_{n,n+1}}{u + v + n - 2} \\ & + \frac{(P_{2,n+1} + \dots + P_{n,n+1}) P_{1,n+1}}{(u - v)(u - v + n - 1)} - \frac{(Q_{2,n+1} + \dots + Q_{n,n+1}) P_{1,n+1}}{(u + v + n - 2)(u - v + n - 1)} \\ & - \frac{(P_{2,n+1} + \dots + P_{n,n+1}) Q_{1,n+1}}{(u - v)(u + v + n - 1)} + \frac{(Q_{2,n+1} + \dots + Q_{n,n+1}) Q_{1,n+1}}{(u + v + n - 2)(u + v + n - 1)}. \end{aligned}$$



Multiplying the first fraction in the third line by  $H(u) \otimes 1$  on the right gives

$$- \frac{(n-1) P_{1,n+1}}{(u-v)(u-v+n-1)} \tag{79}$$

times  $H(u) \otimes 1$ . Here we used the first equality from Lemma 1 and the relation

$$P_{p,n+1} P_{1,n+1} = P_{1,n+1} P_{1p} \quad \text{where } p = 2, \dots, n.$$

Multiplying the second fraction in the third line by  $H(u) \otimes 1$  on the right gives just zero due to the second equality from Lemma 1 and to the relations

$$Q_{p,n+1} P_{1,n+1} = P_{1,n+1} Q_{p1} = -P_{1,n+1} Q_{1p} \quad \text{where } p = 2, \dots, n.$$

The numerator of the first fraction in the fourth line can be written as the sum of

$$P_{p,n+1} Q_{1,n+1} = Q_{1p} P_{p,n+1} \quad \text{over } p = 2, \dots, n.$$

These summands evidently have the divisibility property stated in the lemma. Multiplying the second fraction in the fourth line by  $H(u) \otimes 1$  on the right gives

$$- \frac{Q_{2,n+1} + \dots + Q_{n,n+1}}{(u+v+n-2)(u+v+n-1)} \tag{80}$$

times  $H(u) \otimes 1$ . Here we used the first equality from Lemma 1 and the relation

$$Q_{p,n+1} Q_{1,n+1} = Q_{p,n+1} P_{1p} \quad \text{where } p = 2, \dots, n.$$

By adding to the first and the second lines the expressions (79) and (80) instead of the third and the fourth lines and by collecting similar terms we get the sum in brackets displayed in the lemma. Hence we make the induction step.  $\square$

Note that this proof implies the well known equality [22, Proposition 2.12]

$$\begin{aligned} & \prod_{1 \leq p < q \leq n} \left( 1 + \frac{P_{pq}}{p-q} \right) \cdot \left( 1 - \frac{P_{1,n+1}}{u-v+n-1} \right) \dots \left( 1 - \frac{P_{n,n+1}}{u-v} \right) \\ &= \prod_{1 \leq p < q \leq n} \left( 1 + \frac{P_{pq}}{p-q} \right) \cdot \left( 1 - \frac{P_{1,n+1} + \dots + P_{n,n+1}}{u-v} \right). \end{aligned} \tag{81}$$

**18.** Set  $n = N$ . Thus the above introduced functions  $H(u)$  and  $F(u, v)$  will now take values in  $(\text{End } \mathbb{C}^{N|N})^{\otimes N}$  and  $(\text{End } \mathbb{C}^{N|N})^{\otimes (N+1)}$  respectively. Elements of the latter algebra act on vectors in  $(\mathbb{C}^{N|N})^{\otimes (N+1)}$  by the conventions (1) and (2). From now on we will be denoting  $z = u - v$  and  $w = u + v + N - 1$  for short.

**Lemma 4.** *Applying  $(H(u) \otimes 1) F(u, v)$  to the vector  $e_1 \otimes \dots \otimes e_N \otimes e_1$  has the same effect as applying  $H(u) \otimes 1$  to the vector*

$$\frac{z-1}{z} e_1 \otimes \dots \otimes e_N \otimes e_1. \tag{82}$$

*Proof.* For  $1 \leq p < q \leq N + 1$  the elements  $Q_{pq} \in (\text{End } \mathbb{C}^{N|N})^{\otimes(N+1)}$  annihilate the vector  $e_1 \otimes \cdots \otimes e_N \otimes e_1$ . Hence applying  $(H(u) \otimes 1) F(u, v)$  to this vector has the same effect as applying

$$\begin{aligned} & \prod_{1 \leq p < q \leq N} \left( 1 + \frac{P_{pq}}{p - q} \right) \cdot \left( 1 - \frac{P_{1,N+1}}{u - v + N - 1} \right) \cdots \left( 1 - \frac{P_{N,N+1}}{u - v} \right) \\ &= \prod_{1 \leq p < q \leq N} \left( 1 + \frac{P_{pq}}{p - q} \right) \cdot \left( 1 - \frac{P_{1,N+1} + \cdots + P_{N,N+1}}{u - v} \right). \end{aligned}$$

Here we use the equality (81) with  $n = N$ . The summands  $P_{2,N+1}, \dots, P_{N,N+1}$  of the last numerator do not contribute to the image of  $e_1 \otimes \cdots \otimes e_N \otimes e_1$  due to the second equality in (75) with  $n = N$ . The summand  $P_{1,N+1}$  does not change this vector. Using the first equality in (75) with  $n = N$  completes the proof.  $\square$

**Lemma 5.** *Applying  $(H(u) \otimes 1) F(u, v)$  to the vector  $e_1 \otimes \cdots \otimes e_N \otimes e_{-1}$  has the same effect as applying  $H(u) \otimes 1$  to the linear combination*

$$\begin{aligned} & \frac{w + 1}{w} e_1 \otimes \cdots \otimes e_N \otimes e_{-1} \\ & - \frac{1}{z} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_k \\ & - \frac{1}{w} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_k. \end{aligned}$$

*Proof.* Applying  $F(u, v)$  to  $e_1 \otimes \cdots \otimes e_N \otimes e_{-1}$  has the same effect as applying

$$R_{1,N+1}(u + N - 1, v) \left( 1 - \frac{P_{2,N+1}}{u - v + N - 2} \right) \cdots \left( 1 - \frac{P_{N,N+1}}{u - v} \right).$$

Observe that for any  $n$  both equalities in (75) remain valid if one of the indices  $i_1, \dots, i_n$  is  $-1$  while the other  $n - 1$  indices are greater than 1. If  $N > 1$  then we will use this observation with  $n = N - 1$ . When applying  $(H(u) \otimes 1) F(u, v)$  to the vector  $e_1 \otimes \cdots \otimes e_N \otimes e_{-1}$  we can now replace the function  $F(u, v)$  by

$$R_{1,N+1}(u + N - 1, v) \left( 1 - \frac{P_{2,N+1} + \cdots + P_{N,N+1}}{u - v} \right).$$

If  $N > 1$  then we also use the equality (81) with  $n = N - 1$ . By simply opening the brackets, the last displayed product equals

$$\begin{aligned} & 1 - \frac{P_{1,N+1}}{u - v + N - 1} + \frac{Q_{1,N+1}}{u + v + N - 1} - \frac{P_{2,N+1} + \cdots + P_{N,N+1}}{u - v} \\ & + \frac{P_{1,N+1} (P_{2,N+1} + \cdots + P_{N,N+1})}{(u - v + N - 1)(u - v)} - \frac{Q_{1,N+1} (P_{2,N+1} + \cdots + P_{N,N+1})}{(u + v + N - 1)(u - v)}. \end{aligned}$$

The last two numerators are respectively equal to the sums

$$P_{12} P_{1,N+1} + \cdots + P_{1N} P_{1,N+1} \quad \text{and} \quad P_{2,N+1} Q_{12} + \cdots + P_{N,N+1} Q_{1N}.$$

The second of the two sums annihilates our vector. When applying the first sum to our vector we can replace each of the factors  $P_{12}, \dots, P_{1N}$  just by  $-1$ . Here we use (76) with  $n = N$ . Hence when applying  $(H(u) \otimes 1) F(u, v)$  to our vector we can replace the function  $F(u, v)$  by the sum

$$\begin{aligned} & 1 - \frac{P_{1,N+1}}{u - v + N - 1} + \frac{Q_{1,N+1}}{u + v + N - 1} - \frac{P_{2,N+1} + \dots + P_{N,N+1}}{u - v} \\ & - \frac{(N - 1) P_{1,N+1}}{(u - v + N - 1)(u - v)} \\ & = 1 - \frac{P_{1,N+1} + P_{2,N+1} + \dots + P_{N,N+1}}{z} + \frac{Q_{1,N+1}}{w}. \end{aligned}$$

Applying the expression in the last displayed line to  $e_1 \otimes \dots \otimes e_N \otimes e_{-1}$  gives

$$\begin{aligned} & e_1 \otimes \dots \otimes e_N \otimes e_{-1} - \frac{1}{z} \sum_{k=1}^N e_1 \otimes \dots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \dots \otimes e_N \otimes e_k \\ & - \frac{1}{w} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \dots \otimes e_N \otimes e_k + \frac{1}{w} \sum_{k=1}^N e_k \otimes e_2 \otimes \dots \otimes e_N \otimes e_{-k}. \end{aligned}$$

Due to (76) with  $n = N$  applying  $H(u) \otimes 1$  to the last sum over  $k = 1, \dots, N$  annihilates all summands but one where  $k = 1$ . This completes the proof.  $\square$

**19.** Introduce the rational function of  $u, v$  with values in  $(\text{End } \mathbb{C}^{N|N})^{\otimes(2N+1)}$

$$C(u, v) = F'(u, v) F''(1 - u - N, v)$$

where  $F'(u, v)$  and  $F''(u, v)$  are the images of  $F(u, v)$  by the two embeddings

$$(\text{End } \mathbb{C}^{N|N})^{\otimes(N+1)} \rightarrow (\text{End } \mathbb{C}^{N|N})^{\otimes(2N+1)}$$

defined as  $\iota_1 \otimes \dots \otimes \iota_N \otimes \iota_{2N+1}$  and as  $\iota_{N+1} \otimes \dots \otimes \iota_{2N} \otimes \iota_{2N+1}$  respectively. Using (77) twice we obtain that multiplying the formal power series in  $u^{-1}, v^{-1}$

$$T_1(u + N - 1) \dots T_N(u) T_{N+1}(-u) \dots T_{2N}(1 - N - u) T_{2N+1}(v)$$

by  $C(u, v) \otimes 1$  on the left gives the same result as multiplying by  $C(u, v) \otimes 1$  on the right the series

$$T_{2N+1}(v) T_1(u + N - 1) \dots T_N(u) T_{N+1}(-u) \dots T_{2N}(1 - N - u).$$

Both series have their coefficients in the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes(2N+1)} \otimes X(\mathfrak{p}_N)$ .

For any index  $j = \pm 1, \dots, \pm N$  denote

$$f_j = e_1 \otimes \dots \otimes e_N \otimes e_1 \otimes \dots \otimes e_N \otimes e_j. \tag{83}$$

Observe that replacing  $u$  by  $1 - N - u$  maps  $z = u - v$  to  $1 - N - u - v = -w$ . Therefore by using Lemma 4 with that replacement of the variable  $u$  and then in the original formulation we immediately get the following corollary to this lemma.

**Corollary 5.** Applying  $(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v)$  to the vector  $f_1$  has the same effect as applying  $H(u) \otimes H(1 - N - u) \otimes 1$  to the vector

$$\frac{(z - 1)(w + 1)}{z w} f_1. \tag{84}$$

The next is the counterpart of Corollary 5 for Lemma 5 instead of Lemma 4.

**Corollary 6.** Applying  $(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v)$  to  $f_{-1}$  has the same effect as applying  $H(u) \otimes H(1 - N - u) \otimes 1$  to the linear combination

$$\begin{aligned} & \frac{(z - 1)(w + 1)}{z w} f_{-1} \\ & - \frac{z - 1}{z^2} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\ & - \frac{z - 1}{z w} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\ & + \frac{z - 1}{z w} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_k \\ & + \frac{z - 1}{z^2} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_k. \end{aligned} \tag{85}$$

*Proof.* By Lemma 5 with the variable  $u$  replaced by  $1 - N - u$  we immediately obtain that applying  $(1 \otimes H(1 - N - u) \otimes 1) F''(1 - u - N, v)$  to the vector  $f_{-1}$  has the same effect as applying  $1 \otimes H(1 - N - u) \otimes 1$  to the linear combination

$$\begin{aligned} & \frac{z - 1}{z} f_{-1} \\ & + \frac{1}{w} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_k \\ & + \frac{1}{z} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_k. \end{aligned} \tag{86}$$

Further, by the original Lemma 5 applying  $(H(u) \otimes 1 \otimes 1) F'(u, v)$  to the first of three lines (86) has the same effect as applying  $H(u) \otimes 1 \otimes 1$  to the sum of first three lines in (85). Further, applying  $(H(u) \otimes 1 \otimes 1) F'(u, v)$  to the second and to the third lines of (86) has the same effect as applying the product

$$(H(u) \otimes 1 \otimes 1) \left( 1 - \frac{P_{1,2N+1}}{u - v + N - 1} \right) \cdots \left( 1 - \frac{P_{N,2N+1}}{u - v} \right).$$

Applying the last displayed product to the second and third lines of (86) gives the same result as applying the product

$$(H(u) \otimes 1 \otimes 1) \left( 1 - \frac{P_{1,2N+1} + \cdots + P_{N,2N+1}}{u - v} \right). \tag{87}$$

Here we used (75) and (81) with  $n = N$ . When applying (87) to the summands with the index  $k$  in the second and third lines of (86), all summands of the numerator in (87) vanish except for  $P_{k, 2N+1}$ . The latter does not change the summands with the index  $k$  in the second and third lines of (86). So applying  $(H(u) \otimes 1 \otimes 1) F'(u, v)$  to the second and the third lines of (86) gives the same as applying  $H(u) \otimes 1 \otimes 1$  respectively to the fourth and the fifth lines of (85).  $\square$

**20.** So far we used the equality in Lemma 2 only in the cases when both sides of the equality annihilate a given vector. In the next section we will use that equality in more general cases. We will employ the following corollary to Lemma 2. Denote

$$d_k = e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \quad \text{for } k = 1, \dots, N.$$

**Corollary 7.** For any  $k > 1$  applying  $H(u)$  to the vector  $e_{-k} \otimes e_2 \otimes \cdots \otimes e_N$  has the same effect as applying  $H(u)$  to the linear combination

$$a(u) d_k \otimes e_{-1} (-1)^{N+k} - b(u) d_1 \otimes e_{-k} (-1)^N$$

where

$$a(u) = \frac{2}{2u + N + 1} \quad \text{and} \quad b(u) = \frac{2u + N - 1}{2u + N + 1}.$$

For any  $k > 1$  applying  $H(1 - N - u)$  to  $e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N$  has the same effect as applying  $H(1 - N - u)$  to the linear combination

$$a(u) e_{-k} \otimes d_1 - b(u) e_{-1} \otimes d_k (-1)^k.$$

*Proof.* For any index  $i$  denote by  $x_i(u)$  the result of applying  $H(u)$  to the vector

$$e_2 \otimes \cdots \otimes e_{k-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_i \otimes e_{-i}. \tag{88}$$

Due to (76) with  $n = N$  applying  $H(u)$  to  $e_{-k} \otimes e_2 \otimes \cdots \otimes e_N$  gives the vector  $(-1)^k x_{-k}(u)$ . Let us apply both sides of the equality in Lemma 2 with  $n = N$  and  $p = N - 1$  to the vectors (88) with  $i = 1, \dots, N$ . We obtain the equations

$$x_1(u) - x_{-1}(u) + \cdots + x_N(u) - x_{-N}(u) = (2u + 1) (x_i(u) + x_{-i}(u)). \tag{89}$$

But for  $i = 2, \dots, k - 1, k + 1, \dots, N$  we get  $x_i(u) = 0$  by (76) with  $n = N$ . So by (89) the vectors  $x_{-i}(u)$  with  $i = 2, \dots, k - 1, k + 1, \dots, N$  are equal to each other. Using these observations, it is easy to deduce from the equations (89) that

$$x_{-k}(u) = a(u) x_1(u) - b(u) x_k(u).$$

Using (76) with  $n = N$  again we now obtain the first statement of our corollary.

Changing the notation, let  $x_i(u)$  be the result of applying  $H(1 - N - u)$  to

$$e_{-i} \otimes e_i \otimes e_2 \otimes \cdots \otimes e_{k-1} \otimes e_{k+1} \otimes \cdots \otimes e_N. \tag{90}$$

Due to (76) applying  $H(1 - N - u)$  to  $e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N$  gives the vector  $(-1)^k x_{-1}(u)$ . Let us replace  $u$  by  $1 - N - u$  in the equality in Lemma 2 and apply both sides of the resulting equality with  $n = N$  and  $p = 1$  to the vectors (90) with  $i = 1, \dots, N$ . In this way we obtain the same equations (89) although for the changed vectors. For  $i = 2, \dots, k - 1, k + 1, \dots, N$  we still have  $x_i(u) = 0$  by (76). So the equations (89) easily imply that

$$x_{-1}(u) = a(u) x_k(u) - b(u) x_1(u).$$

Using (76) with  $n = N$  again we get the second statement of the corollary.  $\square$

**21.** Let us relate two vectors in  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}(u, v)$  by the symbol  $\sim$  to each other if their difference vanishes under the action of  $H(u) \otimes H(1 - N - u) \otimes 1$ , or lies in the image of the action of  $Q_{N, N+1}$ . Extend the relation  $\sim$  transitively. Recall the notations  $z = u - v$  and  $w = u + v + N - 1$  adopted in Section 18.

**Proposition 8.** *For any index  $j = \pm 1, \dots, \pm N$  we have the relation*

$$C(u, v) f_j \sim \frac{(z - 1)(w + 1)}{z w} f_j.$$

*Proof.* Proposition 8 follows from its particular cases of  $j = 1, -1$ . Indeed, the sums in (7) and (10) are invariant under any permutation of the indices  $1, \dots, N$  with the corresponding permutation of  $-1, \dots, -N$  at the same time. So  $H(u)$  and  $C(u, v)$  are also invariant under any such permutation. While the vector

$$e_1 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \tag{91}$$

changes under the permutation, its image by the action of  $H(u) \otimes H(1 - N - u)$  does not change thanks to (75). Finally, observe that the product of  $C(u, v)$  by  $H(u) \otimes H(1 - N - u) \otimes 1$  on the left is also divisible by this factor on the right. This remark follows by setting  $n = N$  in the beginning of the proof of Lemma 3.

For  $j = 1$  our proposition immediately follows from Corollary 5. For  $j = -1$  we will derive the proposition from Corollary 6. We will show that the summands with the index  $k$  in the second and fifth lines of (85) cancel each other by  $\sim$  and so do the summands with the index  $k$  in the third and fourth lines of (85).

Consider the second and the fifth lines of (85). For  $k = 1, \dots, N$  by (76)

$$\begin{aligned} & -e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\ & + e_1 \otimes \cdots \otimes e_N \otimes e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_k \\ & \sim -d_k \otimes e_{-1} \otimes e_1 \otimes d_1 \otimes e_k (-1)^{N+k} + d_k \otimes e_k \otimes e_{-k} \otimes d_1 \otimes e_k (-1)^{N+k} \\ & \sim d_k \otimes x \otimes d_1 \otimes e_k (-1)^{N+k} \end{aligned}$$

where  $x$  is the vector (11). So the last displayed line lies in the image of  $Q_{N, N+1}$ .

Consider the third and the fourth lines of (85). For their summands with the index  $k = 1$  by using (76) we obtain the relations

$$\begin{aligned} & -e_{-1} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_1 \\ & + e_1 \otimes \cdots \otimes e_N \otimes e_{-1} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \\ & \sim -d_1 \otimes e_{-1} \otimes e_1 \otimes d_1 \otimes e_1 (-1)^{N+1} + d_1 \otimes e_1 \otimes e_{-1} \otimes d_1 \otimes e_1 (-1)^{N+1} \\ & \sim d_1 \otimes x \otimes d_1 \otimes e_1 (-1)^{N+1} \end{aligned}$$

where  $x$  is as above. Thus the last displayed line lies in the image of  $Q_{N, N+1}$ .

So far in the proof of our proposition we used the equality in Lemma 2 only in the cases when both sides of the equality annihilate a given vector. Now we will also use Corollary 7 that stands outside of these cases. Consider the summands in the third and

fourth lines of (85) with any index  $k > 1$ . We get the relations

$$\begin{aligned}
 & -e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\
 & + e_1 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_k \\
 & \sim -a(u) d_k \otimes e_{-1} \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k (-1)^{N+k} \\
 & + b(u) d_1 \otimes e_{-k} \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k (-1)^N \\
 & + a(u) e_1 \otimes \cdots \otimes e_N \otimes e_{-k} \otimes d_1 \otimes e_k \\
 & - b(u) e_1 \otimes \cdots \otimes e_N \otimes e_{-1} \otimes d_k \otimes e_k (-1)^k \\
 & \sim -a(u) d_k \otimes e_{-1} \otimes e_1 \otimes d_1 \otimes e_k (-1)^{N+k} \\
 & - b(u) d_1 \otimes e_{-k} \otimes e_k \otimes d_k \otimes e_k (-1)^{N+k} \\
 & + a(u) d_k \otimes e_k \otimes e_{-k} \otimes d_1 \otimes e_k (-1)^{N+k} \\
 & + b(u) d_1 \otimes e_1 \otimes e_{-1} \otimes d_k \otimes e_k (-1)^{N+k} \\
 & \sim a(u) d_k \otimes x \otimes d_1 \otimes e_k (-1)^{N+k} + b(u) d_1 \otimes x \otimes d_k \otimes e_k (-1)^{N+k}
 \end{aligned}$$

where  $x$  is as above. So the last displayed line lies in the image of  $Q_{N,N+1}$ . □

**22.** Let us employ Proposition 8 in the proof of Theorem 3. Consider the algebra

$$(\text{End } \mathbb{C}^{N|N})^{\otimes(2N+1)} \otimes X(\mathfrak{p}_N). \tag{92}$$

Elements of this algebra can be applied to vectors of  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$  via  $2N + 1$  tensor factors  $\text{End}(\mathbb{C}^{N|N})$ . Equip the vector space  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$  with the basis of tensor products of the vectors  $e_i \in \mathbb{C}^{N|N}$ . This basis includes the vectors (83).

We have an equality of formal power series in  $u^{-1}, v^{-1}$  with coefficients in (92)

$$\begin{aligned}
 & (H(u) \otimes H(1 - N - u) \otimes 1 \otimes 1) (C(u, v) \otimes 1) \\
 & \times T_1(u + N - 1) \dots T_N(u) T_{N+1}(-u) \dots T_{2N}(1 - N - u) T_{2N+1}(v) \\
 & = (H(u) \otimes H(1 - N - u) \otimes 1 \otimes 1) T_{2N+1}(v) \\
 & \times T_1(u + N - 1) \dots T_N(u) T_{N+1}(-u) \dots T_{2N}(1 - N - u) (C(u, v) \otimes 1),
 \end{aligned}$$

see the beginning of Section 19. For any indices  $i, j$  apply the right hand side of this equality to the basis vector  $f_j$  and take the coefficient of the result at the basis vector  $f_i$ . By Proposition 8 and by the second formula for  $A(u)$  as given in Section 15 this coefficient is equal to

$$\frac{(z - 1)(w + 1)}{z w} T_{ij}(v) A(u) A(1 - N - u). \tag{93}$$

Indeed, due to (37) the product of

$$T_1(u + N - 1) \dots T_N(u) T_{N+1}(-u) \dots T_{2N}(1 - N - u)$$

by  $H(u) \otimes H(1 - N - u) \otimes 1 \otimes 1$  on the left is also divisible by the same factor on the right, see the beginning of Section 16. Further, due to (47) the product of  $T_N(u) T_{N+1}(-u)$  by  $Q_{N,N+1} \otimes 1$  on the right is divisible by the same factor on the left. Here we use the relation  $Q_{N+1,N} = -Q_{N,N+1}$  which follows from (12). But the coefficient at  $f_i$  of

any vector from the image of the action of  $(H(u) \otimes H(1 - N - u) \otimes 1) Q_{N,N+1}$  on  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$  is zero.

Next we shall prove that by applying the left hand side of the above displayed equality to the vector  $f_j$  and then taking the coefficient of the result at  $f_i$  we get

$$\frac{(z - 1)(w + 1)}{z w} A(u) A(1 - N - u) T_{ij}(v). \tag{94}$$

Thus the product (93) equals (94). These equalities for all  $i, j$  imply Theorem 3. Considering the left hand side of the above displayed equality will be easier than the right hand side, thanks to the obvious advantages of Lemma 1 over Lemma 2.

**23.** Let us now use the right action of the  $\mathbb{Z}_2$ -graded algebra  $\text{End}(\mathbb{C}^{N|N})$  on the  $\mathbb{Z}_2$ -graded vector space  $\mathbb{C}^{N|N}$  instead of the left action which we used before. For the matrix units  $E_{ij} \in \text{End} \mathbb{C}^{N|N}$  we have  $e_k E_{ij} = \delta_{ik} e_j$ . For any positive integer  $n$  we can then define the right action of the algebra  $(\text{End} \mathbb{C}^{N|N})^{\otimes n}$  on the vector space  $(\mathbb{C}^{N|N})^{\otimes n}$  by the conventions similar to (1) and (2).

Notice that the right action of the element  $P \in (\text{End} \mathbb{C}^{N|N})^{\otimes 2}$  defined by (7) still maps  $e_i \otimes e_j \mapsto e_j \otimes e_i (-1)^{i j}$ . The image of the right action on  $(\mathbb{C}^{N|N})^{\otimes 2}$  of the element  $Q$  in (10) is still one-dimensional, but is spanned by the vector

$$\sum_j e_j \otimes e_{-j}. \tag{95}$$

By using Lemmas 1 and 3 with  $n = N$  we immediately get the next two lemmas.

**Lemma 6.** *Applying  $(H(u) \otimes 1) F(u, v)$  to  $e_1 \otimes \dots \otimes e_N \otimes e_1$  from the right has the same effect as applying  $H(u) \otimes 1$  from the right to the vector (82).*

**Lemma 7.** *Applying  $(H(u) \otimes 1) F(u, v)$  to  $e_1 \otimes \dots \otimes e_N \otimes e_{-1}$  from the right has the same effect as applying  $H(u) \otimes 1$  from the right to the linear combination*

$$\begin{aligned} & \frac{w + 1}{w} e_1 \otimes \dots \otimes e_N \otimes e_{-1} \\ & - \frac{1}{z} \sum_{k=1}^N e_1 \otimes \dots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \dots \otimes e_N \otimes e_k \\ & + \frac{1}{w} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \dots \otimes e_N \otimes e_k. \end{aligned}$$

By using Lemma 6 first in its original formulation and then with the variable  $u$  replaced by  $1 - N - u$  we immediately get the following corollary to this lemma.

**Corollary 8.** *Applying  $(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v)$  to  $f_1$  from the right has the same effect as applying  $H(u) \otimes H(1 - N - u) \otimes 1$  from the right to (84).*

The next is the counterpart of Corollary 8 for Lemma 7 instead of Lemma 6.



**Corollary 9.** Applying  $(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v)$  to  $f_{-1}$  from the right has the same effect as applying  $H(u) \otimes H(1 - N - u) \otimes 1$  from the right to

$$\begin{aligned} & \frac{(z - 1)(w + 1)}{z w} f_{-1} \\ & + \frac{w + 1}{w^2} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_k \\ & - \frac{w + 1}{z w} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_N \otimes e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_k \\ & - \frac{w + 1}{z w} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\ & + \frac{w + 1}{w^2} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k. \end{aligned} \tag{96}$$

*Proof.* Using Lemma 7 we obtain that applying  $(H(u) \otimes 1 \otimes 1) F'(u, v)$  to  $f_{-1}$  from the right has the same effect as applying  $H(u) \otimes 1 \otimes 1$  from the right to

$$\begin{aligned} & \frac{w + 1}{w} f_{-1} \\ & - \frac{1}{z} \sum_{k=1}^N e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k \\ & + \frac{1}{w} \sum_{k=1}^N e_{-k} \otimes e_2 \otimes \cdots \otimes e_N \otimes e_1 \otimes \cdots \otimes e_N \otimes e_k. \end{aligned} \tag{97}$$

By Lemma 7 with the variable  $u$  replaced by  $1 - N - u$ , applying the product  $(1 \otimes H(1 - N - u) \otimes 1) F''(1 - N - u, v)$  to the first of three lines (97) from the right has the same effect as applying  $1 \otimes H(1 - N - u) \otimes 1$  from the right to the sum of first three lines in (96).

Applying  $(1 \otimes H(1 - N - u) \otimes 1) F''(1 - N - u, v)$  to the second and to the third lines of (97) from the right has the same effect as applying from the right

$$\left(1 + \frac{P_{2N, 2N+1}}{u + v + N - 1}\right) \cdots \left(1 + \frac{P_{N+1, 2N+1}}{u + v}\right) (1 \otimes H(1 - N - u) \otimes 1),$$

see the beginning of the proof of Lemma 3 for  $n = N$ . By Lemma 1 applying the last displayed product to the second and the third lines of (97) from the right gives the same result as applying to them from the right the product

$$\left(1 + \frac{P_{N+1, 2N+1} + \cdots + P_{2N, 2N+1}}{u + v + N - 1}\right) (1 \otimes H(1 - N - u) \otimes 1). \tag{98}$$

When applying (98) to the summands with the index  $k$  in the second and third lines of (97), all summands of the numerator in (98) vanish except  $P_{N+k, 2N+1}$ . The latter does not change the summands with the index  $k$  in the second and third lines of (97). Hence applying  $(1 \otimes H(1 - N - u) \otimes 1) F''(1 - N - u, v)$  to the second and the third lines of (97) from the right gives the same as applying  $1 \otimes H(1 - N - u) \otimes 1$  respectively to the fourth and the fifth lines of (96).  $\square$

**24.** Changing the notation of Section 21 let us now relate by the symbol  $\sim$  two vectors in  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}(u, v)$  to each other if either their difference vanishes under the right action of  $H(u) \otimes H(1 - N - u) \otimes 1$ , or it lies in the image of the right action of  $Q_{1,2N}$ . Extend the new relation  $\sim$  transitively. Denote

$$D(u, v) = R_{N,2N+1}(u, v) \dots R_{1,2N+1}(u + N - 1, v) \\ \times R_{2N,2N+1}(1 - u - N, v) \dots R_{N+1,2N+1}(-u, v).$$

By choosing  $n = N$  in the beginning of the proof Lemma 3 we obtain the equality

$$(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v) \\ = D(u, v) (H(u) \otimes H(1 - N - u) \otimes 1). \tag{99}$$

**Proposition 9.** For  $i = \pm 1, \dots, \pm N$  and for the right action on the vector  $f_i$

$$f_i D(u, v) \sim \frac{(z - 1)(w + 1)}{z w} f_i.$$

*Proof.* Proposition 9 follows from its particular cases of  $i = 1, -1$  by the same remark as already used in the beginning of the proof of Proposition 8. For  $i = 1$  our proposition follows from Corollary 8, see the equality (99) above. For  $i = -1$  we will derive Proposition 9 from Corollary 9. We will prove that the summands with the index  $k$  in the second and fifth lines of (96) cancel each other by  $\sim$  and that so do the summands with the index  $k$  in the third and fourth lines of (96).

Consider the second and the fifth lines of (96). For  $k = 1, \dots, N$  by Lemma 1

$$e_1 \otimes \dots \otimes e_N \otimes e_1 \otimes \dots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \dots \otimes e_N \otimes e_k \\ + e_{-k} \otimes e_2 \otimes \dots \otimes e_N \otimes e_1 \otimes \dots \otimes e_N \otimes e_k \\ \sim e_1 \otimes d_1 \otimes d_k \otimes e_{-k} \otimes e_k (-1)^{N+k} + e_{-k} \otimes d_1 \otimes d_k \otimes e_k \otimes e_k (-1)^{N+k} \\ \sim \sum_j e_j \otimes d_1 \otimes d_k \otimes e_{-j} \otimes e_k (-1)^{N+k}.$$

So the last displayed line lies in the image of the right action of  $Q_{1,2N}$ . Here we use the notation introduced just before stating Corollary 7.

Now consider the third and fourth lines of (85). For  $k = 1, \dots, N$  by Lemma 1

$$e_1 \otimes \dots \otimes e_N \otimes e_{-k} \otimes e_2 \otimes \dots \otimes e_N \otimes e_k \\ + e_1 \otimes \dots \otimes e_{k-1} \otimes e_{-1} \otimes e_{k+1} \otimes \dots \otimes e_N \otimes e_1 \otimes \dots \otimes e_N \otimes e_k \\ \sim e_k \otimes d_k \otimes d_1 \otimes e_{-k} \otimes e_k (-1)^{N+k} + e_{-1} \otimes d_k \otimes d_1 \otimes e_1 \otimes e_k (-1)^{N+k} \\ \sim \sum_j e_j \otimes d_k \otimes d_1 \otimes e_{-j} \otimes e_k (-1)^{N+k}.$$

So the last displayed line also lies in the image of the right action of  $Q_{1,2N}$ . □

The vector space  $(\mathbb{C}^{N|N})^{\otimes 2}$  has a basis of the vectors  $e_i \otimes e_j$ . Applying any  $X \in (\text{End } \mathbb{C}^{N|N})^{\otimes 2}$  to a basis vector from the left and taking the coefficient of the result at another basis vector is not always the same as applying  $X$  to the latter vector from the right and taking the coefficient of the result at the former vector. But these two coefficients will be the same for  $X = P$  and for  $X = Q$ .

Indeed, both actions of  $P$  map  $e_i \otimes e_j \mapsto e_j \otimes e_i (-1)^{i\bar{j}}$ . Further, applying  $Q$  to the vector  $e_j \otimes e_{-j}$  from the left yields the sum (11). Applying  $Q$  to the vector  $e_i \otimes e_{-i}$  in (11) from the right yields the sum (95) multiplied by  $(-1)^{\bar{i}}$ .

It follows that applying the product  $(H(u) \otimes H(1 - N - u) \otimes 1) C(u, v)$  to any basis vector of  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$  from the left and taking the coefficient of the result at another basis vector is the same as applying this product to the latter vector from the right and taking the coefficient of the result at the former vector.

Let us now consider the product at the left hand side of the equality displayed in Section 22. To complete our proof Theorem 3 we have to show that for any indices  $i, j$  applying this product to the basis vector  $f_j$  and taking the coefficient of the result at  $f_i$  gives (94). But this follows from Proposition 9 due to the above remark about the left and right actions on the basis vectors of  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$ .

Here we use second formula for  $A(u)$  in Section 15 and the equality (99). We also use the equality displayed just before stating Lemma 1 with  $n = N$ . And we also use the following observation. Due to (47) the product

$$(Q_{1,2N} \otimes 1) T_1(u + N - 1) T_{2N}(1 - N - u)$$

is also divisible by  $Q_{1,2N} \otimes 1$  on the right. But the coefficient at  $f_j$  of any vector from the image of the right action of  $Q_{1,2N} (H(u) \otimes H(1 - N - u) \otimes 1)$  on  $(\mathbb{C}^{N|N})^{\otimes(2N+1)}$  is zero. Our proof of Theorem 3 is now complete.

**25.** In this section we will provide an interpretation of Proposition 8. We will employ the usual left action of the algebra  $\text{End } \mathbb{C}^{N|N}$  on the vector space  $\mathbb{C}^{N|N}$ . Proposition 9 admits a similar interpretation. But then one has to use the right action of the algebra  $\text{End } \mathbb{C}^{N|N}$  on the vector space  $\mathbb{C}^{N|N}$  instead of left action.

For any  $t \in \mathbb{C}$  consider the representation  $\rho_t \otimes \rho_{-t}$  of the algebra  $X(\mathfrak{p}_N)$ . Using the comultiplication (56) this representation is determined by mapping

$$\begin{aligned} (\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[u^{-1}]] &\rightarrow (\text{End } \mathbb{C}^{N|N})^{\otimes 3}[[u^{-1}]] : \\ T(u) &\mapsto R_{12}(u, t) R_{13}(u, -t). \end{aligned}$$

See the end of our Section 6 for the definition of the representation  $\rho_t$  of  $X(\mathfrak{p}_N)$ .

Multiplying the relation (16) by  $v + w$  and then setting  $v = -w = t$  yields

$$R_{12}(u, t) R_{13}(u, -t) Q_{23} = Q_{23} R_{13}(u, -t) R_{12}(u, t).$$

The latter relation shows that that under the representation  $\rho_t \otimes \rho_{-t}$  the action of the algebra  $X(\mathfrak{p}_N)$  preserves the subspace  $\text{im } Q \subset (\mathbb{C}^{N|N})^{\otimes 2}$ .

Suppose that  $-2t \neq 1, 2, \dots, 2N - 3$  if  $N > 1$ . Then the rational functions  $H(u)$  and  $H(1 - N - u)$  have no poles at  $u = t$ . For  $N = 1$  we get  $H(u) = 1$  and the complex number  $t$  is still arbitrary. For any  $N \geq 1$  let  $W \in (\text{End } \mathbb{C}^{N|N})^{\otimes 2N}$  be the value of the function  $H(u) \otimes H(1 - N - u)$  at  $u = t$ .

Now consider the representation of  $X(\mathfrak{p}_N)$  on the vector space  $(\mathbb{C}^{N|N})^{\otimes 2N}$

$$\rho = \rho_{1-N-t} \otimes \dots \otimes \rho_{-t} \otimes \rho_t \otimes \dots \otimes \rho_{t+N-1}.$$

For this particular choice of a representation  $\rho$  of  $X(\mathfrak{p}_N)$  we have

$$\begin{aligned} \text{id} \otimes \rho : (\text{End } \mathbb{C}^{N|N}) \otimes X(\mathfrak{p}_N)[[v^{-1}]] &\rightarrow (\text{End } \mathbb{C}^{N|N})^{\otimes 2N+1} [[v^{-1}]] : \\ T(-v) &\mapsto R_{12}(-v, 1 - N - t) \dots R_{1,N+1}(-v, -t) \times \\ &R_{1,N+2}(-v, t) \dots R_{1,2N+1}(-v, t + N - 1) \\ &= P_{12} \dots P_{2N,2N+1} R_{2N+1,1}(-v, 1 - N - t) \dots R_{2N+1,N}(-v, -t) \\ &\quad \times R_{2N+1,N+1}(-v, t) \dots R_{2N+1,2N}(-v, t + N - 1) P_{2N,2N+1} \dots P_{12} \\ &= P_{12} \dots P_{2N,2N+1} C(t, v) P_{2N,2N+1} \dots P_{12}. \end{aligned}$$

The last equality is obtained by using (69). The equality (99) now shows that the action of the algebra  $X(\mathfrak{p}_N)$  by  $\rho$  preserves the subspace  $\ker W \subset (\mathbb{C}^{N|N})^{\otimes 2N}$ . This action also preserves the subspace  $\text{im } Q_{N,N+1} \subset (\mathbb{C}^{N|N})^{\otimes 2N}$  by the above remark. Proposition 8 implies that this action preserves the subspace spanned by  $\ker W$ ,  $\text{im } Q_{N,N+1}$  and the vector (91). Moreover under the action of the algebra  $X(\mathfrak{p}_N)$  on the one-dimensional subquotient of  $(\mathbb{C}^{N|N})^{\otimes 2N}$  spanned by (91)

$$T_{ij}(-v) \mapsto \delta_{ij} \frac{(t - v - 1)(t + v + N)}{(t - v)(t + v + N - 1)}.$$

Let us choose  $c(u)$  in (39) so that  $c(-v)$  is the reciprocal of the last displayed fraction. Then  $c(u) T_{ij}(u)$  acts on our one-dimensional subquotient just as  $\delta_{ij}$ . For this  $c(u)$  the composition of the representation  $\rho$  with the automorphism (39) of  $X(\mathfrak{p}_N)$  maps  $Z(u) \mapsto 1$ , see Section 11. Hence the composite representation of  $X(\mathfrak{p}_N)$  on  $(\mathbb{C}^{N|N})^{\otimes 2N}$  factors through  $Y(\mathfrak{p}_N)$ .

**26.** There is a natural  $\mathbb{Z}$ -grading on the associative algebra  $Y(\mathfrak{p}_N)$ . It is defined by setting the degree of the generator  $T_{ij}^{(r)}$  to 0 if the indices  $i, j$  are of the same sign, to 1 if  $i > 0 > j$  and to  $-1$  if  $i < 0 < j$ . Therefore twice the  $\mathbb{Z}$ -degree of  $T_{ij}^{(r)}$  is the eigenvalue of the element  $E_{ij} \in \text{End } \mathbb{C}^{N|N}$  under the adjoint action of

$$E = E_{11} - E_{-1,-1} + \dots + E_{NN} - E_{-N,-N}.$$

In the algebra  $(\text{End } \mathbb{C}^{N|N})^{\otimes 2}$  the elements (7), (10) commute with the element  $E \otimes 1 + 1 \otimes E$ . So the defining relations of  $Y(\mathfrak{p}_N)$  are homogeneous, see (38) and (54) with  $Z(u) = 1$ . Hence our  $\mathbb{Z}$ -grading is well defined. We shall use it to prove that  $B(u) \mapsto B(u)^{-1}$  under the antipodal map (42) on  $Y(\mathfrak{p}_N)$ .

Let  $I$  be the subspace of  $Y(\mathfrak{p}_N)$  consisting of all elements of  $\mathbb{Z}$ -degree 0. This subspace is a subalgebra. Using the relations (38) each element of  $Y(\mathfrak{p}_N)$  can be written as a linear combination of monomials in the generators  $T_{ij}^{(r)}$  where all factors of degree 1 are on the left and all factors of degree  $-1$  are on the right. If such a monomial belongs to  $I \setminus \{0\}$  and has a factor of degree 1, it also has a factor of degree  $-1$ . So the monomials from  $I \setminus \{0\}$  span a two-sided ideal of  $I$ .

Denote this ideal by  $J$ . Take any monomial in the generators  $T_{ij}^{(r)}$  of degree 0. By using the relations (38) and (54) with  $Z(u) = 1$ , this monomial can be written as a linear combination of monomials in the generators  $T_{ij}^{(r)}$  with  $i, j > 0$  only and of some monomials from  $J$ . So the quotient algebra  $I/J$  can be identified with the subalgebra of

$Y(\mathfrak{p}_N)$  in Corollary 3. Let  $A$  be the latter subalgebra and  $\alpha : I \rightarrow A$  be the canonical homomorphism coming from this identification.

The homomorphism  $\alpha$  maps the series  $B(u)$  to itself, because all coefficients of this series belong to  $A$ . Thus  $\alpha$  maps the series  $B(u)^{-1}$  to itself too. Consider the image of the series  $B(u)$  by the antipodal map (42) on  $Y(\mathfrak{p}_N)$ . This image is a formal power series in  $u^{-1}$  with coefficients in the centre of  $Y(\mathfrak{p}_N)$ . We will show that by applying  $\alpha$  to this image we get the series  $B(u)^{-1}$  as well. Due to Corollary 4 this fact will imply that  $B(u) \mapsto B(u)^{-1}$  under the map (42).

Let  $G(u)$  the  $2N \times 2N$  matrix whose  $ij$  entry is the series

$$G_{ij}(u) = T_{ij}(u) (-1)^{\bar{i}\bar{j}+\bar{j}}.$$

We index the rows and columns of  $G(u)$  by the numbers  $1, \dots, N, -1, \dots, -N$ . The matrix  $G(u)$  is invertible. Let  $G'_{ij}(u)$  be the  $ij$  entry of the inverse matrix. Then by the definition (40) for any indices  $i$  and  $j$  we have the equality

$$G'_{ij}(u) = T'_{ij}(u) (-1)^{\bar{i}\bar{j}+\bar{j}}.$$

The matrix  $G(u)$  splits to four blocks of size  $N \times N$  corresponding to different signs of the indices  $i$  and  $j$ . By using for instance [5, Lemma 3.2] we obtain that for  $i, j > 0$  the homomorphism  $\alpha$  maps  $G'_{ij}(u)$  to the  $ij$  entry of the inverse of the block of  $G(u)$  corresponding to the positive matrix indices. But for  $i, j > 0$  we have  $G_{ij}(u) = T_{ij}(u)$  and  $G'_{ij}(u) = T'_{ij}(u)$ . Therefore by the definition (72) the homomorphism  $\alpha$  maps the image of  $B(u)$  by (42) to  $B(u)^{-1}$  as needed. Here we use [18, Proposition 2.19] and our Corollary 3, also see the end of our Section 9.

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#### Declarations

**Conflict of interest** The author did not receive support from any organization for this work, and has no Conflict of interest to declare that are relevant to the content of this article. All research data supporting this work are available within this article.

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