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Hamiltonians for the quantised Volterra hierarchy

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Abstract

This paper builds upon our recent work, published in Carpentier *et al* (2022 *Lett. Math. Phys.* **112** 94), where we established that the integrable Volterra lattice on a free associative algebra and the whole hierarchy of its symmetries admit a quantisation dependent on a parameter ω. We also uncovered an intriguing aspect: all odd-degree symmetries of the hierarchy admit an alternative, non-deformation quantisation, resulting in a non-commutative algebra for any choice of the quantisation parameter ω. In this study, we demonstrate that each equation within the quantum Volterra hierarchy can be expressed in the Heisenberg form. We provide explicit expressions for all quantum Hamiltonians and establish their commutativity. In the classical limit, these quantum Hamiltonians yield explicit expressions for the classical ones of the commutative Volterra hierarchy. Furthermore, we present Heisenberg equations and their Hamiltonians in the case of non-deformation quantisation. Finally, we discuss commuting first integrals, central elements of the quantum algebra, and the integrability problem for periodic reductions of the Volterra lattice in the context of both quantisations.

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1. Introduction

In this paper, we further develop the quantisation theory of the Volterra lattice [1], based on the notion of quantisation ideals [2]. In our previous work [1], we proved that the non-Abelian Volterra lattice, along with the entire hierarchy of its commuting symmetries, admits a quantisation with quadratic commutation relations between dynamical variables that depend on a complex parameter ω. The algebra generated by the dynamical variables becomes commutative in the specialisation $\omega = 1$. This quantisation can be viewed as a finite deformation of a commutative algebra, a deformation that is consistent with all equations of the hierarchy. Slightly abusing terminology, we shall call it *deformation quantisation*, although it does not use to any Poisson structure of the Volterra lattice and the noncommutative multiplication is presented in an explicit form in contrast to the well known theory of deformation quantisation [3, 4]. In addition, we also showed that all odd-degree symmetries of the Volterra hierarchy admit a non-deformation quantisation whose multiplication law is noncommutative for any choice of the quantisation parameter ω [1]. While the deformation quantisation for the Volterra lattice is known in the literature [5], the non-deformation quantisation appeared in [2] for the first time. In physics, a quantum description of fermions can be regarded as non-deformation quantisation, since the 'classical' limit of the fermion dynamical variables is represented by a $\mathbb{Z}_2$ graded (Grassmann) algebra with commutative and anti-commutative variables. The 'classical' limit of the Volterra non-deformation quantum algebra is not commutative and is not graded. It is a new type of non-commutative associative algebras whose representation theory has not yet been developed.

Traditionally, commuting quantum integrals are obtained in the frame of the quantum inverse scattering method using a lax representation with ultra-local l operator [5–7]. in this method, the quantum commuting operators, including the hamiltonian of the system, can be obtained recursively from the logarithm of the trace of the monodromy matrix (the transfer matrix) using it as a generating function. the coefficients in the expansion of this generating function commute thanks to the existence of a quantum r-matrix compatible with the ultra-local l operators. the goal of this paper is to present the hamiltonian operators in explicit form and show that they are formally self-adjoint, commute with each other, and yield heisenberg equations for every member of the integrable hierarchy of the commuting symmetries. we show it without making use of the quantum lax structure or the corresponding transfer matrix. we prove this result both for the conventional and the non-deformation quantisations.

The notion of quantisation ideals for dynamical systems defined on free algebras was proposed in [2]. Let $\mathfrak{A}$ be a free associative algebra with a finite or infinite number of multiplicative generators. The dynamical system defines a derivation $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$. A quantisation is a canonical projection of the dynamical system on $\mathfrak{A}$ to a system defined on a quotient algebra $\mathfrak{A}_{\mathfrak{I}} = \mathfrak{A} \diagup \mathfrak{I}$ over a two-sided ideal $\mathfrak{I} \subset \mathfrak{A}$ satisfying the following properties:

(i) the ideal $\mathfrak{I}$ is ∂_t–stable, that is, $\partial_t(\mathfrak{I}) \subset \mathfrak{I}$;
(ii) the quotient algebra $\mathfrak{A}_{\mathfrak{I}}$ admits an additive basis of normally ordered monomials.

An ideal satisfying the above two conditions is called a *quantisation ideal*, and $\mathfrak{A}_{\mathfrak{I}}$ is called a *quantum algebra*.

In [1], we applied this approach to the integrable nonabelian Volterra lattice

$$\partial_{t_1}(u_n) = K^{(1)}(u_{n+1}, u_n, u_{n-1}), \quad K^{(1)} = u_{n+1}u_n - u_n u_{n-1}, \qquad n \in \mathbb{Z} \tag{1}$$

and its hierarchy of symmetries

$$\partial_{t_\ell}(u_n) = K^{(\ell)}(u_{n+\ell}, \ldots, u_{n-\ell}), \qquad \ell \in \mathbb{N}, \quad n \in \mathbb{Z}, \tag{2}$$

where $K^{(\ell)}(u_{n+\ell}, \ldots, u_{n-\ell})$ are homogeneous polynomials of degree $\ell + 1$ (explicitly given in section 2.1). The second member of the hierarchy

$$\partial_{t_2}(u_n) = K^{(2)} = u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 - u_n^2 u_{n-1} - u_n u_{n-1}^2 - u_n u_{n-1} u_{n-2} \tag{3}$$

is a cubic polynomial and we refer to it as the cubic symmetry of (1). In this case the free algebra $\mathfrak{A} = \mathbb{C}[\omega]\langle u_n \,;\, n \in \mathbb{Z}\rangle$ is generated by infinite number of non-commutative variables u_n. We proved that the Volterra lattice (1) and its whole hierarchy of symmetries admit a quantisation with the quantization ideal

$$\mathfrak{I}_a = \langle \{u_n u_{n+1} - \omega u_{n+1} u_n \,;\, n \in \mathbb{Z}\} \cup \{u_n u_m - u_m u_n \,;\, |n-m| > 1,\ n, m \in \mathbb{Z}\,\}\rangle, \tag{4}$$

leading to the commutation relations

$$u_n u_{n+1} = \omega u_{n+1} u_n, \qquad u_n u_m = u_m u_n \ \text{ if } \ |n-m| \geqslant 2, \quad n, m \in \mathbb{Z} \tag{5}$$

in the quotient algebra $\mathfrak{A}_{\mathfrak{I}_a}$, where $\omega \in \mathbb{C}^*$ is a quantisation parameter. Moreover, we showed that the cubic symmetry of the Volterra lattice, equation (3), and all odd degree members of the Volterra hierarchy also admit a non-deformation quantisation with the quantisation ideal

$$\mathfrak{I}_b = \langle \left\{u_n u_{n+1} - (-1)^n \omega u_{n+1} u_n \,;\, n \in \mathbb{Z}\right\} \cup \{u_n u_m + u_m u_n \,;\, |n-m| > 1,\ n, m \in \mathbb{Z}\}\rangle \tag{6}$$

with commutation relations

$$u_n u_{n+1} = (-1)^n \omega u_{n+1} u_n, \qquad u_n u_m + u_m u_n = 0 \ \text{ if } \ |n-m| \geqslant 2, \quad n, m \in \mathbb{Z} \tag{7}$$

in the quotient algebra $\mathfrak{A}_{\mathfrak{I}_b}$.

In the quantum theory, real valued dynamical variables are replaced by self-adjoint operators, with respect to a Hermitian conjugation †. The ideals $\mathfrak{I}_a$ and $\mathfrak{I}_b$ and corresponding commutation relations are stable with respect to the Hermitian conjugation † (defined in section 3), assuming the variables u_n are self-adjoint and $\omega = e^{i\hbar}$. Here $\hbar$ is an arbitrary real parameter, an analogue of the Plank constant, and $i = \sqrt{-1}$. For the quantised equations of the Volterra hierarchy, we introduce the factors $e^{\frac{1}{2}i\ell\hbar}$ to make the right-hand side of the equations self-adjoint, that is,

$$\partial_{t_\ell}(u_n) = q^\ell K^{(\ell)}(u_{n+\ell}, \ldots, u_{n-\ell}), \qquad q = e^{\frac{1}{2}i\hbar}, \quad \ell = 1, 2, \ldots, \quad n \in \mathbb{Z}. \tag{8}$$

In this paper we show that the infinite sequence of quantum Hamiltonians H_ℓ for the quantised Volterra hierarchy defined on the quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}$ is given by

$$H_\ell = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{N}^\ell} \frac{\omega^\ell - 1}{\omega^{\nu(\alpha,0)} - 1} P_\alpha^{\mathfrak{I}_a}(\omega)\, u_{\alpha+k},$$

where the set

$$\mathcal{N}^\ell=\left\{\alpha=(\alpha_1,\alpha_2\cdots,\alpha_{\ell-1},0)\in\mathbb{Z}^\ell\middle|\alpha_i=\sum_{s=i}^{\ell-1}\theta_s,\ \theta_s\in\{0,1\};i=1,\cdots,\ell-1\right\},$$

and $\nu(\alpha,i)$ denotes the number of i's in the ℓ-tuple α. For $\alpha\in\mathcal{N}^\ell$, the polynomials $P_\alpha^{\mathfrak{I}_a}(\omega)$ are given by products of Gaussian binomials

$$P_\alpha^{\mathfrak{I}_a}(\omega)=\binom{\nu(\alpha,\alpha_1)+\nu(\alpha,\alpha_1-1)-1}{\nu(\alpha,\alpha_1)}_\omega\cdots\binom{\nu(\alpha,1)+\nu(\alpha,0)-1}{\nu(\alpha,1)}_\omega.$$

We prove that the Hamiltonians H_ℓ are self-adjoint $H_\ell^\dagger=H_\ell$ and commute with each other $[H_\ell,H_k]=0,\ k,\ell\in\mathbb{N}$. Furthermore, the dynamical equations of the quantum hierarchy (8) can be written in the Heisenberg form [8]:

$$\partial_{t_\ell}(u_n)=\frac{i}{2\sin\left(\frac{1}{2}\ell\hbar\right)}[H_\ell,u_n],\qquad n\in\mathbb{Z},\ \ell\in\mathbb{N}.$$

In the classical limit $\hbar\to0$ we obtain the Volterra hierarchy in the Hamiltonian form $\partial_{t_\ell}(u_n)=\{u_n,\tilde{H}_\ell\}$ and explicit expressions for all Hamiltonians $\tilde{H}_\ell=\lim_{\hbar\to0}\ell^{-1}H_\ell$ (see section 4).

In the case of the non-deformation quantisation (6) we have also found explicit expressions for self-adjoint commuting quantum Hamiltonians and present the quantum hierarchy with even times (ℓ being even in (8)) in the Heisenberg form. These results are stated in theorem 7.

The problem of quantisation of the Volterra lattice has a long history. In 1992, using the quantum version of the inverse spectral transform method, Volkov proposed quantum commutation relations between the dynamical variables [6] (see also [7]). These commutation relations are hardly suitable for the derivation of the Heisenberg equations and the study of the corresponding quantum algebra structure. In the paper by Inoue and Hikami [5], the commutation relations (5), as well as the first four Hamiltonians of the quantum Volterra hierarchy were found using ultra-local Lax representation and the R–matrix technique. Our alternative approach does not rely on the existence of an ultra-local Lax representation, R–matrix or Hamiltonian structures. It enables us to explicitly present all quantum Hamiltonians for the Volterra hierarchy in the case of the deformation quantisation (5). Moreover, we are able to explicitly find the Hamiltonians and Heisenberg equations for the non-deformation quantisation (7), defined for all odd-degree members of the Volterra hierarchy. Both results are new and rather surprising.

2. The nonabelian Volterra hierarchy and its quantisations

In this section, we derive the explicit expressions for the quantised Volterra hierarchy under the quantisation ideal $\mathfrak{I}_a$ defined by (4), making use of Gaussian binomial coefficients. When $\omega=1$, this also reduces to the hierarchy of symmetries for the classical (abelian) Volterra chain. We first give a brief description of the nonabelian Volterra hierarchy and introduce some basic notations required for this paper.

2.1. The nonabelian Volterra hierarchy

Let $\mathfrak{A}=\mathbb{C}\langle u_n\,;n\in\mathbb{Z}\rangle$ be the free associative algebra of polynomials generated by an infinite number of non-commuting variables u_n. There is a natural automorphism $\mathcal{S}:\mathfrak{A}\mapsto\mathfrak{A}$, which we call the *shift operator*, defined by

$$\mathcal{S}: a(u_k,\ldots,u_r) \mapsto a(u_{k+1},\ldots,u_{r+1}), \quad \mathcal{S}: \alpha \mapsto \alpha, \qquad a(u_k,\ldots,u_r) \in \mathfrak{A}, \;\; \alpha \in \mathbb{C}.$$

Thus $\mathfrak{A}$ is a difference algebra. A derivation $\mathcal{D}$ of the algebra $\mathfrak{A}$ is a $\mathbb{C}$–linear map satisfying Leibniz's rule

$$\mathcal{D}(\alpha a + \beta b) = \alpha \mathcal{D}(a) + \beta \mathcal{D}(b), \qquad \mathcal{D}(a \cdot b) = \mathcal{D}(a) \cdot b + a \cdot \mathcal{D}(b), \qquad a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C}.$$

It is uniquely defined by its action on the generators and $\mathcal{D}(\alpha) = 0,\ \alpha \in \mathbb{C}$.

A derivation $\mathcal{D}$ is called evolutionary if it commutes with the shift operator $\mathcal{S}$. An evolutionary derivation is completely characterised by its action on the generator u (we often write u instead of u_0), that is,

$$\mathcal{D}(u) = a \quad \text{and} \quad \mathcal{D}(u_k) = \mathcal{S}^k(a), \qquad a \in \mathfrak{A}.$$

We adopt the notation $\mathcal{D}_a$ for the unique evolutionary derivation of $\mathfrak{A}$ such that $\mathcal{D}_a(u) = a$. Evolutionary derivations form a Lie subalgebra of the Lie algebra of derivations of $\mathfrak{A}$, and the characteristic of a commutator $[\mathcal{D}_a, \mathcal{D}_b] = \mathcal{D}_c$ is given by $c = \mathcal{D}_a(b) - \mathcal{D}_b(a)$. This expression induces a Lie bracket on the difference algebra $\mathfrak{A}$.

Assuming that the generators u_k depend on $t \in \mathbb{C}$ we then identify the evolutionary derivation $\mathcal{D}_a$ with an infinite system of Equations

$$\partial_t(u_n) = \mathcal{D}_a(u_n) = \mathcal{S}^n(a). \qquad n \in \mathbb{Z}.$$

From now on we will think of the system of evolutionary equations and the evolutionary derivation as the same object.

The Volterra lattice (1) defines an evolutionary derivation $\partial_{t_1}: \mathfrak{A} \mapsto \mathfrak{A}$. The differential-difference system (3) defines another evolutionary derivation ∂_{t_2}. Evolutionary derivations commuting with ∂_{t_1} are called (generalised) symmetries of the Volterra lattice. It can be straightforwardly verified that $[\partial_{t_1}, \partial_{t_2}] = 0$ and thus equation (3) is a symmetry of the Volterra lattice.

It is well known that the Volterra lattice has an infinite hierarchy of commuting symmetries. They can be found using Lax representations both in commutative [9] and noncommutative [10] cases, or using recursion operators [11, 12]. Remarkably, the symmetries of the Volterra lattice (1) can be explicitly presented in terms of a family of nonabelian homogeneous difference polynomials [12], which was inspired by the family of polynomials discovered in the commutative case (see [13, 14]).

Let us assume that the generators u_k of the free associative algebra $\mathfrak{A}$ depend on an infinite set of 'times' $t_1, t_2, \ldots$. It follows from [12] that the hierarchy of commuting symmetries of the nonabelian Volterra lattice (1) can be written in the following explicit form

$$\partial_{t_\ell}(u) = \mathcal{S}\left(X^{(\ell)}\right) u - u \mathcal{S}^{-1}\left(X^{(\ell)}\right), \qquad \ell \in \mathbb{N}, \tag{9}$$

where the (noncommutative) polynomials $X^{(\ell)}$ are given by

$$X^{(\ell)} = \sum_{0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_\ell \leqslant \ell - 1} \left(\overrightarrow{\prod_{j=1}^{\ell}} u_{\lambda_j + 1 - j} \right). \tag{10}$$

Here $\prod_{j=1}^{\to\ell}$ denotes the order of the values j, from 1 to ℓ in the product of the noncommutative generators u_{λ_j+1-j}. For example, we have $X^{(1)} = u$ and

$$X^{(2)} = u_1u + u^2 + uu_{-1}; \tag{11}$$

$$X^{(3)} = u_2u_1u + u_1^2u + uu_1u + u_1u^2 + u^3 + uu_{-1}u + u_1uu_{-1} + u^2u_{-1} + uu_{-1}^2 + uu_{-1}u_{-2}; \tag{12}$$

$$\begin{aligned} X^{(4)} = {} & u_3u_2u_1u + u_2^2u_1u + u_2u_1^2u + u_1u_2u_1u + u_2u_1u^2 + uu_2u_1u + u_2u_1uu_{-1} + u_1^2u^2 + u_1uu_1u \\ & + uu_1^2u + u_1^3u + uu_1u^2 + u_1u^3 + u^2u_1u + u_1^2uu_{-1} + u^4 + uu_1uu_{-1} + u_1uu_{-1}u + u_1u^2u_{-1} \\ & + uu_{-1}^2u + uu_{-1}uu_{-1} + uu_{-1}u^2 + uu_{-1}u_1u + u^2u_{-1}^2 + u^2u_{-1}u + u^3u_{-1} + u_1uu_{-1}^2 + uu_{-1}^3 \\ & + u_1uu_{-1}u_{-2} + uu_{-1}u_{-2}u + u^2u_{-1}u_{-2} + uu_{-1}u_{-2}u_{-1} + uu_{-1}^2u_{-2} + uu_{-1}u_{-2}^2 + uu_{-1}u_{-2}u_{-3}. \end{aligned} \tag{13}$$

Clearly, we get the Volterra equation (1) when $\ell = 1$ and the system (3) when $\ell = 2$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \in \mathbb{Z}^k$ be a k-component vector. For each $\alpha \in \mathbb{Z}^k$, we define the k-degree monomial $u_\alpha = u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_k}$. We denote the degree of α by $|\alpha| = k$. We say that a monomial u_α is normally ordered if $\alpha_i > \alpha_{i+1}$ for all $1 \leqslant i \leqslant k-1$. Conventionally, we write $(\alpha_1+1, \alpha_2+1, \cdots, \alpha_k+1)$ as $\alpha+1$. Thus we have $\mathcal{S}^i u_\alpha = u_{\alpha+i}$ for $i \in \mathbb{Z}$. The multiplicity of u_i in the monomial u_α is denoted by $\nu(\alpha, i)$. Similarly, we denote by $\nu(\alpha, \geqslant i)$ the number of $k \geqslant i$ such that u_k appears in u_α, counted with multiplicities. We say that two monomials u_α and u_β are *similar* written as $\alpha \sim \beta$ if $\nu(\alpha, i) = \nu(\beta, i)$ for all $i \in \mathbb{Z}$.

We define two sets of distinguished monomials, namely, *admissible* and *nonincreasing* monomials. For $k \geqslant 1$, let

$$\mathcal{A}^k = \left\{\alpha \in \mathbb{Z}^k \middle| 1-j \leqslant \alpha_j \leqslant k-j, j = 1, \cdots, k;\ \alpha_{i+1}+1 \geqslant \alpha_i,\ i = 1, \cdots, k-1\right\}; \tag{14}$$

$$\mathcal{Z}_{\geqslant}^k = \left\{\alpha \in \mathbb{Z}^k \middle| \alpha_{i+1}+1 \geqslant \alpha_i \geqslant \alpha_{i+1},\ i = 1, \ldots, k-1\right\}. \tag{15}$$

A k-degree monomial u_α is admissible if $\alpha \in \mathcal{A}^k$ and is nonincreasing if $\alpha \in \mathcal{Z}_{\geqslant}^k$.

Using these notations, the expression $X^{(k)}$ given by (10) can be written as

$$X^{(k)} = \sum_{\alpha \in \mathcal{A}^k} u_\alpha. \tag{16}$$

In what follows, we use this form to present $X^{(k)}$ in normal ordering under the quantisation ideals of the Volterra hierarchy and to derive Hamiltonians for their quantised equations.

2.2. *The quantised Volterra hierarchies in normal ordering*

Assume that $\mathfrak{I} \subset \mathfrak{A}$ is a two-sided ideal generated by the infinite set of polynomials $\mathfrak{f}_{i,j}$:

$$\mathfrak{I} = \langle \mathfrak{f}_{i,j}\,;\, i < j,\ i,j \in \mathbb{Z}\rangle, \qquad \mathfrak{f}_{i,j} = u_iu_j - \omega_{i,j}u_ju_i, \tag{17}$$

where $\omega_{i,j} \in \mathbb{C}^*$ are arbitrary non-zero complex parameters. Specifying the nonzero constants $\omega_{i,j}$ leads to either $\mathfrak{I}_a$ defined by (4) or $\mathfrak{I}_b$ defined by (6).

Given such an ideal $\mathfrak{I}$, we denote the projection on the quotient algebra $\mathfrak{A}_\mathfrak{I}$ by $\pi_\mathfrak{I} : \mathfrak{A} \to \mathfrak{A}_\mathfrak{I}$. The algebra $\mathfrak{A}_\mathfrak{I}$ has an additive basis of *standard normally ordered monomials*

$$u_{i_1}u_{i_2}\cdots u_{i_n}; \qquad i_1 \geqslant i_2 \geqslant \cdots \geqslant i_n,\ i_k \in \mathbb{Z},\ n \in \mathbb{N}.$$

The canonical projection $\pi_{\mathfrak{I}}:\mathfrak{A}\to\mathfrak{A}_{\mathfrak{I}}$ acts on the polynomial $X^{(k)}$ given by (16) as follows:

$$\pi_{\mathfrak{I}}\left(X^{(k)}\right)=\sum_{\alpha\in\mathcal{A}^k\cap\mathcal{Z}^k_{\geqslant}}P^{\mathfrak{I}}_{\alpha}(\omega)\,u_\alpha,\tag{18}$$

where $P^{\mathfrak{I}}_{\alpha}(\omega)$ is the unique polynomial in $\mathbb{Z}[\omega]$ such that for $\alpha\in\mathcal{A}^k\cap\mathcal{Z}^k_{\geqslant}$,

$$P^{\mathfrak{I}}_{\alpha}(\omega)\,u_\alpha=\pi_{\mathfrak{I}}\left(\sum_{\beta\in\mathcal{A}^k,\beta\sim\alpha}u_\beta\right).\tag{19}$$

We often write it as $P_\alpha(\omega)$ if there is no ambiguity regarding the choice of ideal.

We now study the polynomials $P_\alpha(\omega)$ for the quantisation ideals $\mathfrak{I}_a$ (4) and $\mathfrak{I}_b$ (6). For example, we have

$$\begin{aligned}
&\pi_{\mathfrak{I}_a}\left(X^{(1)}\right)=X^{(1)}=u;\qquad \pi_{\mathfrak{I}_a}\left(X^{(2)}\right)=X^{(2)}=u_1u+u^2+uu_{-1};\\
&\pi_{\mathfrak{I}_a}\left(X^{(3)}\right)=u_2u_1u+u_1^2u+(1+\omega)u_1u^2+u^3+(1+\omega)u^2u_{-1}+u_1uu_{-1}+uu_{-1}^2+uu_{-1}u_{-2};\\
&\pi_{\mathfrak{I}_a}\left(X^{(4)}\right)=u_3u_2u_1u+u_2^2u_1u+u_1^3u+u^4+u_1^2uu_{-1}+u_2u_1uu_{-1}+u_1uu_{-1}^2+u_1uu_{-1}u_{-2}\\
&\qquad+uu_{-1}^3+uu_{-1}u_{-2}^2+uu_{-1}u_{-2}u_{-3}+(1+\omega)\left(u_2u_1^2u+u_2u_1u^2+u^2u_{-1}u_{-2}+uu_{-1}^2u_{-2}\right)\\
&\qquad+(1+\omega)^2u_1u^2u_{-1}+\left(1+\omega+\omega^2\right)\left(u_1^2u^2+u_1u^3+u^2u_{-1}^2+u^3u_{-1}\right)
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&\pi_{\mathfrak{I}_b}\left(X^{(1)}\right)=X^{(1)}=u;\qquad \pi_{\mathfrak{I}_b}\left(X^{(2)}\right)=X^{(2)}=u_1u+u^2+uu_{-1};\\
&\pi_{\mathfrak{I}_b}\left(X^{(3)}\right)=u_2u_1u+u_1^2u+(1+\omega)u_1u^2+u^3+(1-\omega)u^2u_{-1}+u_1uu_{-1}+uu_{-1}^2+uu_{-1}u_{-2};\\
&\pi_{\mathfrak{I}_b}\left(X^{(4)}\right)=u_3u_2u_1u+u_2^2u_1u+u_2u_1uu_{-1}+u_1^3u+u_1^2uu_{-1}+u^4+u_1uu_{-1}^2+u_1uu_{-1}u_{-2}\\
&\qquad+uu_{-1}^3+uu_{-1}u_{-2}^2+uu_{-1}u_{-2}u_{-3}+(1-\omega)\left(u_2u_1^2u+u_2u_1u^2\right)+\left(1+\omega^2\right)u_1u^2u_{-1}\\
&\qquad+(1+\omega)\left(u^2u_{-1}u_{-2}+uu_{-1}^2u_{-2}\right)+\left(1+\omega+\omega^2\right)\left(u_1^2u^2+u_1u^3\right)\\
&\qquad+\left(1-\omega+\omega^2\right)\left(u^2u_{-1}^2+u^3u_{-1}\right).
\end{aligned}\tag{21}$$

This defines the polynomials $P_\alpha(\omega)$ for all α that are admissible, nonincreasing and of degree 1 to 4. For example, $P^{\mathfrak{I}_a}_{(0,0,-1)}(\omega)=1+\omega$ and $P^{\mathfrak{I}_b}_{(0,0,-1)}(\omega)=1-\omega$.

For the quantisation ideal $\mathfrak{I}_a$, these polynomials can be computed explicitly using the Gaussian binomial coefficients:

$$\binom{m}{r}_\omega=\frac{(1-\omega^m)\left(1-\omega^{m-1}\right)\cdots\left(1-\omega^{m-r+1}\right)}{(1-\omega)(1-\omega^2)\cdots(1-\omega^r)},$$

where m and r are non-negative integers. If $r>m$, this equals zero. When $r=0$, its value is 1.

Proposition 1. *For $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathcal{A}^k \cap \mathcal{Z}^k_{\geqslant}$, let $\kappa_i = \nu(\alpha, i)$, where $\alpha_k \leqslant i \leqslant \alpha_1$. Then*

$$P^{\mathfrak{I}_a}_{\alpha}(\omega) = \binom{\kappa_{\alpha_1} + \kappa_{\alpha_1 - 1} - 1}{\kappa_{\alpha_1}}_{\omega} \cdots \binom{\kappa_2 + \kappa_1 - 1}{\kappa_2}_{\omega} \binom{\kappa_1 + \kappa_0 - 1}{\kappa_1}_{\omega} \times \binom{\kappa_0 + \kappa_{-1} - 1}{\kappa_{-1}}_{\omega} \cdots \binom{\kappa_{\alpha_k + 1} + \kappa_{\alpha_k} - 1}{\kappa_{\alpha_k}}_{\omega}. \tag{22}$$

Proof. An admissible monomial similar to α is equivalent to the following data:

(i). for each integer i such that $0 \leqslant i \leqslant \alpha_1 - 1$, a monomial similar to $u_{i+1}^{\kappa_{i+1}} u_i^{\kappa_i}$ ending on the right by u_i,

(ii). for each integer i such that $0 \geqslant i \geqslant \alpha_k + 1$, a monomial similar to $u_i^{\kappa_i} u_{i-1}^{\kappa_{i-1}}$ starting on the left by u_i.

This is true since we have $u_n u_m = u_m u_n$ for $|n - m| > 1$ in the quantised algebra $\mathfrak{A}_{\mathfrak{I}_a}$. We now need to compute the sums of monomials in (i) for fixed i. Let us denote by $(n|m)$ the monomial $u_{i+1}^n u_i^m$ and by $Q_{(n|m)}(\omega)$ the coefficient in front of $u_{i+1}^n u_i^m$ when summing all monomials in (i). If $m = 1$ then we have $Q_{(n|1)}(\omega) = 1$ since the only monomial similar to $(n|1)$ and ending by u_i is itself. We also have $Q_{(0|m)}(\omega) = 1$. Otherwise, such admissible monomials start either with u_i or u_{i+1} that gives the induction formula $Q_{(n|m)} = Q_{(n-1|m)} + \omega^n Q_{(n|m-1)}$. This can be integrated into

$$Q_{(n|m)}(\omega) = \binom{n + m - 1}{n}_{\omega}.$$

Using a mirror argument, one sees that the sum of all monomials in (ii) is equal to

$$\binom{\kappa_i + \kappa_{i-1} - 1}{\kappa_{i-1}}_{\omega} u_i^{\kappa_i} u_{i-1}^{\kappa_{i-1}},$$

which concludes the proof. □

It follows from this proposition that

$$P^{\mathfrak{I}_a}_{\alpha}(\omega) + \omega^{\nu(\alpha,0)} P^{\mathfrak{I}_a}_{\alpha - 1}(\omega) = P^{\mathfrak{I}_a}_{\alpha - 1}(\omega) + \omega^{\nu(\alpha,1)} P^{\mathfrak{I}_a}_{\alpha}(\omega), \quad \alpha \in \mathcal{Z}^k_{\geqslant}, \tag{23}$$

which was alternatively proved based on combinatoric counting in [1] when we showed that the ideal $\mathfrak{I}_a$ defined by (4) is preserved by the symmetry flows (9), for all $\ell \in \mathbb{N}$.

Using proposition 1, one can directly compute the canonical projections under the quantisation ideal $\mathfrak{I}_a$ without first writing down $X^{(l)}$. For example,

$$P^{\mathfrak{I}_a}_{(1,0,0,-1)}(\omega) = \binom{2}{1}_{\omega} \binom{2}{1}_{\omega} = (1 + \omega)^2,$$

which is the coefficient of $u_1 u^2 u_{-1}$ in $\pi_{\mathfrak{I}_a}(X^{(4)})$ as shown in (20).

For the quantisation ideal $\mathfrak{I}_b$, we have not been able to obtain such neat formula since an admissible monomial is not the product of canonical projections of monomials $u_i^{\kappa_i} u_{i-1}^{\kappa_{i-1}}$ due to the relation $u_n u_m + u_m u_n = 0$ for $|n - m| > 1$ in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$. In [1], we proved the following important identity:

$$P^{\mathfrak{I}_b}_{\alpha}(\omega) + (-1)^{\nu(\alpha, \geqslant 0)} \omega^{\nu(\alpha,0)} P^{\mathfrak{I}_b}_{\alpha - 1}(-\omega) = P^{\mathfrak{I}_b}_{\alpha - 1}(-\omega) + (-1)^{\nu(\alpha, \geqslant 2)} \omega^{\nu(\alpha,1)} P^{\mathfrak{I}_b}_{\alpha}(\omega), \quad \alpha \in \mathcal{Z}^{2k}_{\geqslant}. \tag{24}$$

In order to write down the quantised equations in normal ordering (18), we investigate the set $\mathcal{A}^k \cap \mathcal{Z}^k_{\geqslant}$. We define a subset of $\mathcal{A}^k \cap \mathcal{Z}^k_{\geqslant}$ (cf (14) and (15)), denoted by $\mathcal{N}^k$:

$$\mathcal{N}^k = \left\{\alpha \in \mathcal{Z}^k_{\geqslant} \cap \mathcal{A}^k \middle| \alpha_k = 0\right\}, \qquad k \in \mathbb{N}, \tag{25}$$

which is useful to write explicitly the Hamiltonians for the quantised Volterra hierarchy next section. For any fixed $k \in \mathbb{N}$, all its elements can be constructed following the same manner of Pascal's triangle. For example, $\mathcal{N}^1 = \{(0)\}$, $\mathcal{N}^2 = \{(1,0),(0,0)\}$ and

$$\mathcal{N}^3 = \{(2,1,0),(1,1,0),(1,0,0),(0,0,0)\}; \tag{26}$$

$$\begin{aligned}\mathcal{N}^4 = \{&(3,2,1,0),(2,2,1,0),(2,1,1,0),(2,1,0,0),(1,1,1,0),\\ &(1,1,0,0),(1,0,0,0),(0,0,0,0)\}.\end{aligned} \tag{27}$$

In fact, the set $\mathcal{N}^k$ is in bijection with the set

$$\mathcal{U}^k = \left\{(\theta_1,\ldots,\theta_k) \middle| \theta_k = 0,\ \theta_s \in \{0,1\},\ s = 1,\cdots,k-1\right\}.$$

Elements of $\mathcal{U}^k$ are sequences of zeros and ones of length k with the last element $\theta_k = 0$. The set $\mathcal{U}^k$ has 2^{k-1} elements. The bijection with $\mathcal{N}^k$ is given by the invertible linear transformation

$$(\alpha_1,\ldots\alpha_k) = (\theta_1,\ldots,\theta_k)\mathcal{B}, \quad \mathcal{B}_{ij} = \begin{cases} 0 & \text{if} \quad i<j \\ 1 & \text{if} \quad i \geqslant j \end{cases}, \tag{28}$$

or simply $\alpha_m = \sum_{n=m}^{k} \theta_n$. Thus, we can rewrite the set $\mathcal{N}^k$ given by (25) as

$$\mathcal{N}^k = \left\{\alpha = (\alpha_1,\alpha_2\cdots,\alpha_{k-1},0) \in \mathbb{Z}^k \middle| \ \alpha_i = \sum_{s=i}^{k-1} \theta_s,\ \theta_s \in \{0,1\}; i = 1,\cdots,k-1\right\}. \tag{29}$$

Proposition 2. *The cardinality of set* $\mathcal{N}^{k+1}$ *is* 2^k, *and set* $\mathcal{A}^{k+1} \cap \mathcal{Z}^{k+1}_{\geqslant}$ *has a cardinality of* $(k+2)2^{k-1}$, $0 \leqslant k \in \mathbb{Z}$.

Proof. Due to the bijection (28), the first part of the statement is obvious. To prove the second half, we define a subset of $\mathcal{N}^{k+1}$ as $\mathcal{N}^{k+1}_{(j)} = \left\{\alpha \in \mathcal{N}^{k+1} \middle| 0 \leqslant \alpha_1 = j \leqslant k\right\}$, whose cardinal number is $\binom{k}{j}$. Note that $\mathcal{N}^{k+1} = \cup_{j=0}^{k} \mathcal{N}^{k+1}_{(j)}$ and there is no intersection among any subsets with different j. The set $\mathcal{A}^{k+1} \cap \mathcal{Z}^{k+1}_{\geqslant}$ can be obtained from the subset $\mathcal{N}^{k+1}_{(j)}$: for any $\alpha \in \mathcal{N}^{k+1}_{(j)}$, we can generate j more distinct elements in the set, namely, $\mathcal{S}^{-l}\alpha \in \mathcal{A}^{k+1} \cap \mathcal{Z}^{k+1}_{\geqslant} \setminus \mathcal{N}^{k+1}$ for $1 \leqslant l \leqslant \alpha_1$. Thus its cardinality is

$$\sum_{j=0}^{k} (j+1)\binom{k}{j} = \sum_{j=0}^{k} \binom{k}{j} + \sum_{j=1}^{k} j\binom{k}{j} = 2^k + k2^{k-1} = (k+2)2^{k-1}$$

as stated in the proposition. □

Combining proposition 1 and the construction of the set $\mathcal{A}^{k+1} \cap \mathcal{Z}^{k+1}_{\geqslant}$ described in proposition 2, we are able to explicitly write down the expressions of $X^{(k)}$ in the quantum algebras:

Theorem 3. *Let $\mathfrak{I}$ be either $\mathfrak{I}_a$ or $\mathfrak{I}_b$. Then*

$$\pi_{\mathfrak{I}}\left(X^{(k)}\right) = \sum_{\alpha\in\mathcal{N}^k}\sum_{j=0}^{\alpha_1} P^{\mathfrak{I}}_{\alpha-\alpha_1+j}(\omega)\, u_{\alpha-\alpha_1+j}. \tag{30}$$

Using this theorem and (9), we can explicitly write down the quantum Volterra hierarchy. In the quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}$, for $k\in\mathbb{N}$, we have

$$\begin{aligned}\partial_{t_k}(u) &= \sum_{\alpha\in\mathcal{N}^k}\sum_{j=0}^{\alpha_1} P^{\mathfrak{I}_a}_{\alpha-\alpha_1+j}(\omega)\ \pi_{\mathfrak{I}_a}\left(u_{\alpha-\alpha_1+j+1}u - u u_{\alpha-\alpha_1+j-1}\right)\\ &= \sum_{\alpha\in\mathcal{N}^k}\sum_{j=0}^{\alpha_1} P^{\mathfrak{I}_a}_{\alpha-\alpha_1+j}(\omega)\left(\omega^{\nu(\alpha-\alpha_1+j,-2)}u_{\overline{\alpha-\alpha_1+j+1,0}} - \omega^{\nu(\alpha-\alpha_1+j,2)}u_{\overline{0,\alpha-\alpha_1+j-1}}\right),\end{aligned} \tag{31}$$

where the notation $u_{\overline{\beta}}$ stands for the standard normally ordered monomial which is similar to u_β.

As an example, we work out the case when $k=3$. The elements in the set $\mathcal{N}^3$ are listed in (26). According to (31), we have

$$\begin{aligned}\partial_{t_3}(u) &= P^{\mathfrak{I}_a}_{(2,1,0)}(\omega)\left(u_3u_2u_1u - \omega u_1u^2u_{-1}\right) + P^{\mathfrak{I}_a}_{(1,0,-1)}(\omega)\left(u_2u_1u^2 - u^2u_{-1}u_{-2}\right)\\ &\quad + P^{\mathfrak{I}_a}_{(0,-1,-2)}(\omega)\left(\omega u_1u^2u_{-1} - uu_{-1}u_{-2}u_{-3}\right) + P^{\mathfrak{I}_a}_{(1,1,0)}(\omega)\left(u_2^2u_1u - u^3u_{-1}\right)\\ &\quad + P^{\mathfrak{I}_a}_{(0,0,-1)}(\omega)\left(u_1^2u^2 - uu_{-1}^2u_{-2}\right) + P^{\mathfrak{I}_a}_{(1,0,0)}(\omega)\left(u_2u_1^2u - u^2u_{-1}^2\right)\\ &\quad + P^{\mathfrak{I}_a}_{(0,-1,-1)}(\omega)\left(u_1u^3 - uu_{-1}u_{-2}^2\right) + P^{\mathfrak{I}_a}_{(0,0,0)}(\omega)\left(u_1^3u - uu_{-1}^3\right)\\ &= u_3u_2u_1u + u_2u_1u^2 - u^2u_{-1}u_{-2} - uu_{-1}u_{-2}u_{-3} + u_2^2u_1u - u^3u_{-1}\\ &\quad + (1+\omega)(u_1^2u^2 - uu_{-1}^2u_{-2}) + (1+\omega)(u_2u_1^2u - u^2u_{-1}^2) + u_1u^3 - uu_{-1}u_{-2}^2 + u_1^3u - uu_{-1}^3,\end{aligned}$$

where we compute $P^{\mathfrak{I}_a}_{\alpha}(\omega)$ using (22) in proposition 1.

Although theorem 3 is valid for the quantisation ideal $\mathfrak{I}_b$, to compute the quantum Volterra hierarchy in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$ is much harder since we do not have the similar result for $\mathfrak{I}_b$ as the one in proposition 1 for $\mathfrak{I}_a$.

When $\omega=1$, Gaussian binomial coefficients become the ordinary binomial coefficients. Proposition 1 gives the formulas $P_\alpha(1)$ for the commutative polynomials $X^{(k)}$ given by (16). It follows from (31) that the explicit formula for the whole hierarchy of symmetries of the classical (commutative) Volterra lattice is

$$\partial_{t_k}(u) = \sum_{\alpha\in\mathcal{N}^k}\sum_{j=0}^{\alpha_1} P_{\alpha-\alpha_1+j}(1)\left(u_{\alpha-\alpha_1+j+1} - u_{\alpha-\alpha_1+j-1}\right)u, \quad k\in\mathbb{N}. \tag{32}$$

3. Quantum Hamiltonians for the quantised Volterra hierarchies

In the quantum theory we replace real valued commutative variables by self adjoint operators with respect to some Hermitian conjugation $\dagger$. The Hermitian conjugation $\dagger$ in algebra $\mathfrak{A}$ is defined by the following rules

$$u_n^\dagger = u_n,\quad \alpha^\dagger = \bar{\alpha},\quad (a+b)^\dagger = a^\dagger + b^\dagger,\quad (ab)^\dagger = b^\dagger a^\dagger,\qquad u_n, a, b\in\mathfrak{A},\ \alpha\in\mathbb{C},$$

where $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$. The Hermitian conjugation $\dagger$ can be extended to the quantum algebras $\mathfrak{A}_{\mathfrak{I}_a}$ and $\mathfrak{A}_{\mathfrak{I}_b}$ by letting $\omega^\dagger = \omega^{-1}$. The quantisation ideals $\mathfrak{I}_a$ (4) and $\mathfrak{I}_b$ (6) are $\dagger$–stable. We introduce the square root $q = e^{\frac{1}{2}i\hbar}$ of $\omega = e^{i\hbar}$ with $\hbar \in \mathbb{R}$ a real constant (an analog of the Plank constant). The quantised Volterra hierarchy in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}$ is presented in the form [1]

$$u_{t_1} = q\left(u_1 u - u u_{-1}\right), \qquad u_{t_\ell} = q^\ell \left(\mathcal{S}\left(X^{(\ell)}\right) u - u \mathcal{S}^{-1}\left(X^{(\ell)}\right)\right), \quad \ell \in \mathbb{N}. \tag{33}$$

As a consequence of the results shown later in this paper, all these derivations are self-adjoint, which justifies the rescaling by q^ℓ.

In the paper [1], we presented the Volterra lattice and its first symmetry in Heisenberg form

$$\begin{aligned} \partial_{t_1}(u_n) &= \frac{1}{q^{-1} - q}\left[H_1, u_n\right], \quad H_1 = \sum_{k\in\mathbb{Z}} u_k; \\ \partial_{t_2}(u_n) &= \frac{1}{q^{-2} - q^2}\left[H_2, u_n\right], \quad H_2 = \sum_{k\in\mathbb{Z}} \left(u_k^2 + u_{k+1}u_k + u_k u_{k+1}\right), \end{aligned} \tag{34}$$

where H_1 and H_2 are self-adjoint, algebraically independent and commuting Hamiltonians in $\mathfrak{A}_{\mathfrak{I}_a}$.

In the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$ with commutation relations (7) we can also write the equation (3) in Heisenberg form

$$\partial_{t_2}(u_n) = \frac{1}{q^{-2} - q^2}\left[H_2, u_n\right]. \tag{35}$$

Note that in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$ we have $H_2 = H_1^2$ and $H_2^\dagger = H_2$. In this section, we derive the explicit expressions for the self-adjoint, algebraically independent and commuting Hamiltonians of the Volterra hierarchy in both quantised algebras $\mathfrak{A}_{\mathfrak{I}_a}$ and $\mathfrak{A}_{\mathfrak{I}_b}$.

3.1. Quantum Hamiltonians H_n in $\mathfrak{A}_{\mathfrak{I}_a}$

In section 2.2, we give the definition of the sets $\mathcal{A}^k$, $\mathcal{Z}_\geqslant^k$ and $\mathcal{N}^k$, cf (14), (15) and (25) (or equivalently (29)), whose elements α are associated to the k-degree monomials u_α for $k \in \mathbb{N}$. We now define another set related to them, namely,

$$\mathcal{M}_j^k = \left\{\alpha \in \mathcal{Z}_\geqslant^k \,\middle|\, \exists\, i \in \{1, 2, \cdots, k\} \text{ such that } \alpha_i = j\right\}.$$

Clearly, we have

$$\mathcal{M}_0^k = \mathcal{Z}_\geqslant^k \cap \mathcal{A}^k \quad \text{and} \quad \mathcal{N}^k \subset \mathcal{M}_0^k.$$

Note that the definition of the polynomials $P_\alpha(\omega)$ for $\alpha \in \mathcal{M}_0^k$ in (19), which can be extended to the set $\mathcal{Z}_\geqslant^k$ by the convention $P_\alpha(\omega) = 0$ if $\alpha \notin \mathcal{M}_0^k$.

We first prove several lemmas. In the proofs of these lemmas, we drop the up-index and simply write $P_\alpha(\omega)$ for $P_\alpha^{\mathfrak{I}_a}(\omega)$.

Lemma 4. *Let $\alpha \in \mathcal{Z}_\geqslant^\ell$ be such that $\nu(\alpha, 1) = \nu(\alpha, -1)$. Then $P_{\alpha+1}^{\mathfrak{I}_a}(\omega) = P_{\alpha-1}^{\mathfrak{I}_a}(\omega)$ and u commutes with u_α in $\mathfrak{A}_{\mathfrak{I}_a}$, i.e. $\pi_{\mathfrak{I}_a}([u, u_\alpha]) = 0$.*

Proof. It is obvious that u commutes with such u_α in $\mathfrak{A}_{\mathfrak{I}_a}$ due to the commutation relations (5). We now prove the rest of the statement by considering two cases. If $\nu(\alpha, 1) = \nu(\alpha, -1) = 0$, that is, α contains neither -1 nor 1, this leads to $P_{\alpha+1}(\omega) = P_{\alpha-1}(\omega) = 0$ since neither $\alpha + 1$

nor $\alpha-1$ is admissible (requiring to contain 0). If $\nu(\alpha,1)=\nu(\alpha,-1)\neq 0$, we also have that $\nu(\alpha,0)\neq 0$ since $\alpha\in\mathcal{Z}^{\ell}_{\geqslant}$. From (23) it follows that

$$P_{\alpha+1}(\omega)=\frac{\omega^{\nu(\alpha,-1)}-1}{\omega^{\nu(\alpha,0)}-1}P_{\alpha}(\omega);\quad P_{\alpha-1}(\omega)=\frac{\omega^{\nu(\alpha,1)}-1}{\omega^{\nu(\alpha,0)}-1}P_{\alpha}(\omega) \tag{36}$$

implying $P_{\alpha+1}(\omega)=P_{\alpha-1}(\omega)$. □

Lemma 5. *Let $Y^{(\ell)}=\mathcal{S}(X^{(\ell)})u-u\mathcal{S}^{-1}(X^{(\ell)})$. Then in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}$ we have*

$$Y^{(\ell)}=\sum_{\alpha\in\mathcal{N}^{\ell}}\sum_{k\in\mathbb{Z}}\frac{P_{\alpha}(\omega)}{\omega^{\nu(\alpha,0)}-1}[u,u_{\alpha+k}]. \tag{37}$$

Proof. It follows from (19) that

$$\pi_{\mathfrak{I}_a}\left(\mathcal{S}\left(X^{(\ell)}\right)\right)=\sum_{\alpha\in\mathcal{M}_0^{\ell}}P_{\alpha}(\omega)\,u_{\alpha+1}.$$

Thus we have

$$\begin{aligned}\pi_{\mathfrak{I}_a}\left(\mathcal{S}\left(X^{(\ell)}\right)u\right)&=\sum_{\alpha\in\mathcal{M}_0^{\ell}}P_{\alpha}(\omega)\,\pi_{\mathfrak{I}_a}(u_{\alpha+1}u)\\&=\sum_{\alpha\in\mathcal{M}^{\ell}_{0,(0,-2)}}P_{\alpha}(\omega)\,\pi_{\mathfrak{I}_a}(u_{\alpha+1}u)+\sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{0,(0,-2)}}P_{\alpha}(\omega)\,\pi_{\mathfrak{I}_a}(u_{\alpha+1}u),\end{aligned} \tag{38}$$

where we use the notations $\mathcal{M}^{\ell}_{k,(i,j)}=\{\alpha\in\mathcal{M}^{\ell}_k|\nu(\alpha,i)=\nu(\alpha,j)\}$ and $\overline{\mathcal{M}}^{\ell}_{k,(i,j)}=\mathcal{M}^{\ell}_k\setminus\mathcal{M}^{\ell}_{k,(i,j)}$. In the same way, we have

$$\pi_{\mathfrak{I}_a}\left(u\mathcal{S}^{-1}\left(X^{(\ell)}\right)\right)=\sum_{\alpha\in\mathcal{M}^{\ell}_{0,(0,2)}}P_{\alpha}(\omega)\,\pi_{\mathfrak{I}_a}(uu_{\alpha-1})+\sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{0,(0,2)}}P_{\alpha}(\omega)\,\pi_{\mathfrak{I}_a}(uu_{\alpha-1}).$$

We claim that

$$\sum_{\alpha\in\mathcal{M}^{\ell}_{0,(0,2)}}P_{\alpha}(\omega)\,u_{\alpha-1}=\sum_{\alpha\in\mathcal{M}^{\ell}_{0,(0,-2)}}P_{\alpha}(\omega)\,u_{\alpha+1}$$

and that both sides commute with u in $\mathfrak{A}_{\mathfrak{I}_a}$. This is equivalent to

$$\sum_{\alpha\in\mathcal{M}^{\ell}_{-1,(-1,1)}}P_{\alpha+1}(\omega)\,u_{\alpha}=\sum_{\alpha\in\mathcal{M}^{\ell}_{1,(-1,1)}}P_{\alpha-1}(\omega)\,u_{\alpha},$$

which is indeed true using lemma 4.

We then rewrite the rest of sums in (38) as

$$\begin{aligned}\sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{0,(0,-2)}} P_\alpha(\omega)\,u_{\alpha+1}u &= \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{0,(0,-2)}} \frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)-\nu(\alpha,-2)}-1}[u,u_{\alpha+1}]\\ &= \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{1,(1,-1)}} \frac{P_{\alpha-1}(\omega)}{\omega^{\nu(\alpha,1)-\nu(\alpha,-1)}-1}[u,u_\alpha]\\ &= \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{1,(1,-1)}} \frac{\omega^{\nu(\alpha,-1)}P_{\alpha-1}(\omega)}{\omega^{\nu(\alpha,1)}-\omega^{\nu(\alpha,-1)}}[u,u_\alpha].\end{aligned}$$

Similarly, we are able to show that

$$\sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{(0,2)}} P_\alpha(\omega)\,uu_{\alpha-1} = \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{-1,(-1,1)}} \frac{\omega^{\nu(\alpha,1)}P_{\alpha+1}(\omega)}{\omega^{\nu(\alpha,1)}-\omega^{\nu(\alpha,-1)}}[u,u_\alpha].$$

Taking the difference yields

$$Y^{(\ell)} = \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{1,(-1,1)}} \frac{\omega^{\nu(\alpha,-1)}P_{\alpha-1}(\omega)}{\omega^{\nu(\alpha,1)}-\omega^{\nu(\alpha,-1)}}[u,u_\alpha] - \sum_{\alpha\in\overline{\mathcal{M}}^{\ell}_{-1,(-1,1)}} \frac{\omega^{\nu(\alpha,1)}P_{\alpha+1}(\omega)}{\omega^{\nu(\alpha,1)}-\omega^{\nu(\alpha,-1)}}[u,u_\alpha]. \quad (39)$$

We simplify it by splitting into different cases. When $\nu(\alpha,1)\nu(\alpha,-1)\neq 0$, α belongs to both sums in (39). Using (36), the difference of fractions simplifies into

$$\frac{\omega^{\nu(\alpha,-1)}P_{\alpha-1}(\omega)-\omega^{\nu(\alpha,1)}P_{\alpha+1}(\omega)}{\omega^{\nu(\alpha,1)}-\omega^{\nu(\alpha,-1)}} = \frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}u_\alpha.$$

When $\nu(\alpha,-1)=0$ but $\nu(\alpha,0)\nu(\alpha,1)\neq 0$, α only appears in the first sum in (39) and using (36) in that case one can write

$$\frac{P_{\alpha-1}(\omega)}{\omega^{\nu(\alpha,1)}-1} = \frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}.$$

Similarly, when $\nu(\alpha,1)=0$ but $\nu(\alpha,0)\nu(\alpha,-1)\neq 0$, we have

$$\frac{P_{\alpha+1}(\omega)}{\omega^{\nu(\alpha,-1)}-1} = \frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}.$$

Finally, if $\nu(\alpha,-1)=\nu(\alpha,0)=0$ but $\nu(\alpha,1)\neq 0$, then α appears in the first sum only in (39) and we can rewrite the term as $\frac{P_\beta(\omega)}{\omega^{\nu(\beta,0)}-1}u_{\beta+1}$, where β is an element of $\mathcal{N}^\ell$, that is, to say $\beta_\ell=0$.

In the mirror case where $\nu(\alpha,1)=\nu(\alpha,0)=0$ but $\nu(\alpha,-1)\neq 0$, then α appears in the second sum in (39) and we can rewrite the term as $\frac{P_\gamma(\omega)}{\omega^{\nu(\gamma,0)}-1}u_{\gamma-1}$, where $\gamma\in\mathcal{M}_0^k$ is such that $\gamma_1=0$.

Thus we have so far proved that

$$Y^\ell=\sum_{\alpha\in\overline{\mathcal{M}}^\ell_{0,(1,-1)}}\frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}[u,u_\alpha]+\sum_{\beta\in\mathcal{N}^\ell}\frac{P_\beta(\omega)}{\omega^{\nu(\beta,0)}-1}[u,u_{\beta+1}]$$
$$+\sum_{\gamma\in\mathcal{M}_0^\ell,\gamma_1=0}\frac{P_\gamma(\omega)}{\omega^{\nu(\gamma,0)}-1}[u,u_{\gamma-1}].$$

Since elements u_α for $\alpha\in\mathcal{M}^\ell_{0,(1,-1)}$ commute with u in $\mathfrak{A}_{\mathfrak{I}_a}$ following from Lemma 4, one can add them to the first sum in the above formula and it becomes

$$Y^\ell=\sum_{\alpha\in\mathcal{M}_0^\ell}\frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}[u,u_\alpha]+\sum_{\beta\in\mathcal{N}^\ell}\frac{P_\beta(\omega)}{\omega^{\nu(\beta,0)}-1}[u,u_{\beta+1}]$$
$$+\sum_{\gamma\in\mathcal{M}_0^\ell,\gamma_1=0}\frac{P_\gamma(\omega)}{\omega^{\nu(\gamma,0)}-1}[u,u_{\gamma-1}]. \qquad (40)$$

Recursively applying (23), we get $\frac{P_{\alpha+m}(\omega)}{\omega^{\nu(\alpha+m,0)}-1}=\frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}$ when $\alpha\in\mathcal{M}_0^\ell$ and $\alpha+m\in\mathcal{M}_0^\ell$ for some $m\in\mathbb{Z}$. Hence, we can rewrite (40) as

$$Y^\ell=\sum_{\alpha\in\mathcal{N}^\ell}\sum_{k=-\alpha_1-1}^{1}\frac{P_\alpha(\omega)}{\omega^{\nu(\alpha,0)}-1}[u,u_{\alpha+k}].$$

Finally, adding extra shifts of u_α does not change the sum as these commute with u, and thus we complete the proof of the statement. □

Theorem 6. *The quantum Volterra hierarchy (33) in the algebra $\mathfrak{A}_{\mathfrak{I}_a}$ is presented in the Heisenberg form:*

$$\partial_{t_\ell}(u_n)=\frac{i}{2\sin\left(\frac{1}{2}\ell\hbar\right)}[H_\ell,u_n] \qquad (41)$$

where the Hamiltonians

$$H_\ell=\sum_{\alpha\in\mathcal{N}^\ell}\sum_{k\in\mathbb{Z}}P_\alpha^{\mathfrak{I}_a}(\omega)\frac{\omega^\ell-1}{\omega^{\nu(\alpha,0)}-1}u_{\alpha+k},\quad \omega=q^2=e^{i\hbar},\qquad \ell\in\mathbb{N} \qquad (42)$$

are self-adjoint and commute with each other.

Proof. The first part of the statement follows immediately from the definition of the quantised hierarchy (33) and lemma 5. We now show all Hamiltonians H_ℓ are self-adjoint. First note that, for any nonnegative integers a and b,

$$\binom{a+b-1}{a}_{\omega^{-1}}=\omega^{a(1-b)}\binom{a+b-1}{a}_{\omega}.$$

For $\alpha \in \mathcal{N}^{\ell}$, we have $\ell = \kappa_{\alpha_1} + \kappa_{\alpha_1 - 1} + \cdots + \kappa_1 + \kappa_0$ and $\pi_{\mathfrak{J}_a}\left(u_{\alpha}{}^{\dagger}\right) = \omega^{\left(\kappa_{\alpha_1}\kappa_{\alpha_1-1} + \cdots + \kappa_2\kappa_1 + \kappa_1\kappa_0\right)} u_{\alpha}$. Thus it follows from proposition 1 that

$$P_{\alpha}^{\mathfrak{J}_a}\left(\omega^{-1}\right) = \omega^{\ell - \kappa_0 - \left(\kappa_{\alpha_1}\kappa_{\alpha_1-1} + \cdots + \kappa_2\kappa_1 + \kappa_1\kappa_0\right)} P_{\alpha}^{\mathfrak{J}_a}\left(\omega\right), \quad \alpha \in \mathcal{N}^{\ell}$$

and hence

$$H_{\ell}^{\dagger} = \sum_{\alpha \in \mathcal{N}^{\ell}} \frac{\omega^{-\ell} - 1}{\omega^{-\nu(\alpha,0)} - 1} P_{\alpha}^{\mathfrak{J}_a}\left(\omega^{-1}\right) \sum_{k \in \mathbb{Z}} \mathcal{S}^k \pi_{\mathfrak{J}_a}\left(u_{\alpha}{}^{\dagger}\right) = H_{\ell}.$$

Finally, we show that the Hamiltonians commute with each other. Let ℓ_1 and ℓ_2 be two positive integers and define $Q = [H_{\ell_1}, H_{\ell_2}]$. Then Q is of the form

$$Q = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{N}^{\ell_1 + \ell_2}} T_{\alpha}\left(\omega\right) u_{\alpha + k}$$

for some fractions $T_{\alpha}(\omega)$. We know from [1] that $[\partial_{t_{\ell_1}}, \partial_{t_{\ell_2}}] = 0$. Hence for any $f \in \mathfrak{A}_{\mathfrak{J}_a}$, we have

$$[H_{\ell_1}, [H_{\ell_2}, f]] - [H_{\ell_2}, [H_{\ell_1}, f]] = [f, [H_{\ell_1}, H_{\ell_2}]] = [f, Q] = 0.$$

Thus, if $Q \neq 0$, then every monomial of Q belongs to the center of $\mathfrak{A}_{J_a}$, which is impossible (see the proof of proposition 8).

□

We apply theorem 6 to find the Hamiltonians for lower numbers ℓ. When $\ell = 1$, we know $\mathcal{N}^1 = \{(0)\}$ leading to

$$H_1 = \sum_{k \in \mathbb{Z}} u_k.$$

When $\ell = 2$, there are two elements in $\mathcal{N}^2$, namely, $(1,0)$ and $(0,0)$. It follows from (20) (or using proposition 1) that $P_{(1,0)}(\omega) = 1$ and $P_{(0,0)}(\omega) = 1$. Thus

$$H_2 = \sum_{k \in \mathbb{Z}} u_k^2 + (\omega + 1) \sum_{k \in \mathbb{Z}} u_{k+1} u_k.$$

These are the same as those given by (34).

When $\ell = 3$, the set $\mathcal{N}^3$ is given by (26) and we have

$$P_{(2,1,0)}\left(\omega\right) = 1, \quad P_{(1,1,0)}\left(\omega\right) = 1, \quad P_{(1,0,0)}\left(\omega\right) = 1 + \omega, \quad P_{(0,0,0)}\left(\omega\right) = 1.$$

Hence

$$H_3 = \sum_{k \in \mathbb{Z}} u_k^3 + \left(\omega^2 + \omega + 1\right) \sum_{k \in \mathbb{Z}} \left(u_{k+2} u_{k+1} u_k + u_{k+1}^2 u_k + u_{k+1} u_k^2\right).$$

For the quintic symmetry of the Volterra equation in $\mathfrak{A}_{\mathfrak{J}_a}$, that is, $\ell = 4$, the cardinality of $\mathcal{N}^4$ is 8 and whose elements is given in (27). Following (20) or using proposition 1, we get

$$P_{(3,2,1,0)}\left(\omega\right) = P_{(2,2,1,0)}\left(\omega\right) = P_{(1,1,1,0)}\left(\omega\right) = P_{(0,0,0,0)}\left(\omega\right) = 1;$$
$$P_{(2,1,1,0)}\left(\omega\right) = P_{(2,1,0,0)}\left(\omega\right) = 1 + \omega; \quad P_{(1,1,0,0)}\left(\omega\right) = P_{(1,0,0,0)} = 1 + \omega + \omega^2.$$

Thus using theorem 6 we obtain

$$H_4 = \sum_{k\in\mathbb{Z}} u_k^4 + \left(\omega^2+1\right)(\omega+1)^2 \sum_{k\in\mathbb{Z}} u_{k+2}u_{k+1}^2u_k + \left(1+\omega+\omega^2\right)\left(\omega^2+1\right)\sum_{k\in\mathbb{Z}} u_{k+1}^2u_k^2$$
$$+\left(\omega^2+1\right)(\omega+1)\sum_{k\in\mathbb{Z}}\left(u_{k+3}u_{k+2}u_{k+1}u_k + u_{k+2}^2u_{k+1}u_k + u_{k+2}u_{k+1}u_k^2 + u_{k+1}^3u_k + u_{k+1}u_k^3\right).$$

3.2. Quantum Hamiltonians in $\mathfrak{A}_{\mathfrak{I}_b}$

We have shown in [1] that the derivations with even index ℓ in the nonabelian Volterra hierarchy stabilise the ideal $\mathfrak{I}_b$. In this quantisation we can also find the Hamiltonians following the lines of the proofs in the previous section, except that we now use the identity (24) instead of (23). Because of the similarity we will omit the proof and simply state the result with examples.

Theorem 7. *The quantum Volterra hierarchy (33) in the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$ is presented in the Heisenberg form:*

$$\partial_{t_{2\ell}}(u_n) = \frac{i}{2\sin(\ell\hbar)}\left[\hat{H}_{2\ell}, u_n\right],\quad \hat{H}_{2\ell} = \sum_{\alpha\in\mathcal{N}^{2\ell}}\sum_{k\in\mathbb{Z}} \frac{\omega^{2\ell}-1}{\left((-1)^k\omega\right)^{\nu(\alpha,0)}-1} P_\alpha^{\mathfrak{I}_b}\left((-1)^k\omega\right)u_{\alpha+k}, \tag{43}$$

where $\omega = q^2 = e^{i\hbar}, \hbar\in\mathbb{R}$ and $\ell\in\mathbb{N}$. Moreover, all Hamiltonians are self-adjoint and commute with each other.

We apply theorem 7 to write down the quantum Hamiltonians for the cubic and quintic members of the Volterra hierarchy in $\mathfrak{A}_{\mathfrak{I}_b}$.

Example 1. The cubic symmetry of the Volterra equation corresponds to $\ell = 1$ in theorem 7. We have $\mathcal{N}^2 = \{(1,0),(0,0)\}$. It follows from (21) that $P_{(1,0)}^{\mathfrak{I}_b}(\omega) = 1$ and $P_{(0,0)}^{\mathfrak{I}_b}(\omega) = 1$. Thus

$$\hat{H}_2 = \sum_{k\in\mathbb{Z}} u_k^2 + \sum_{k\in\mathbb{Z}}\left(1+(-1)^k\omega\right)u_{k+1}u_k,$$

which is the same as those given by (35).

Example 2. When $\ell = 2$ in theorem 7, the set $\mathcal{N}^4$ is given in (27). From (21) we get

$$P_{(3,2,1,0)}(\omega) = P_{(2,2,1,0)}(\omega) = P_{(1,1,1,0)}(\omega) = P_{(0,0,0,0)}(\omega) = 1;$$
$$P_{(2,1,1,0)}(\omega) = P_{(2,1,0,0)}(\omega) = 1-\omega;\quad P_{((1,1,0,0)}(\omega) = P_{(1,0,0,0)} = 1+\omega+\omega^2.$$

Thus, using theorem 7 we obtain

$$\hat{H}_4 = \sum_{k\in\mathbb{Z}} u_k^4 - \left(\omega^4-1\right)\sum_{k\in\mathbb{Z}} u_{k+2}u_{k+1}^2u_k + \sum_{k\in\mathbb{Z}}\left(\omega^2+1\right)\left(\omega^2+(-1)^k\omega+1\right)u_{k+1}^2u_k^2$$
$$+\sum_{k\in\mathbb{Z}}\left(\omega^2+1\right)\left(1+(-1)^k\omega\right)\left(u_{k+3}u_{k+2}u_{k+1}u_k + u_{k+2}^2u_{k+1}u_k + u_{k+1}^3u_k + u_{k+1}u_k^3 + u_{k+3}u_{k+2}u_{k+1}^2\right).$$

3.3. Periodic quantum Volterra system

The infinite Volterra hierarchy admits a periodic reduction $u_{n+M} = u_n$ for any integer period M. The periodic reduction can be obtained by taking a quotient of the algebra $\mathfrak{A}$ over the ideal $\mathcal{I}_M = \langle \{u_{n+M} - u_n\}_{n\in\mathbb{Z}} \rangle$. The ideal $\mathcal{I}_M$ is obviously ∂_{t_ℓ}–stable. We denote the quotient algebra $\mathfrak{A}_M = \mathfrak{A}/\mathcal{I}_M \simeq \mathbb{C}\langle u_1,\dots,u_M\rangle$. The M–periodic Volterra system and its symmetries are the sets of M equations of the form (1) and (2), where the index $n \in \mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$.

The quantisation ideal $\mathfrak{I}_a$ is $\mathcal{S}$–stable, that is, $\mathcal{S}(\mathfrak{I}_a) = \mathfrak{I}_a$. Thus the quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}$ admits a periodic reduction for any $M>2$ and $\mathfrak{A}^a_M = \mathfrak{A}_{\mathfrak{I}_a}/\mathcal{I}_M$ is isomorphic to $\mathbb{C}\langle u_1,\dots,u_M\rangle/\mathfrak{I}^M_a$, where

$$\mathfrak{I}^M_a = \langle u_M u_1 - \omega u_1 u_M,\ u_n u_{n+1} - \omega u_{n+1} u_n,\ \ u_n u_m - u_m u_n\ ;\ 1 \leqslant n < m \leqslant M,\ 1 < m-n < M-1\rangle.$$

In contrast to the infinite case, the algebra $\mathfrak{A}^a_M$ has a nontrivial center $Z(\mathfrak{A}^a_M)$.

Proposition 8 ([15]). *If M is odd then $Z(\mathfrak{A}^a_M) = \mathbb{C}[C]$, where $C = u_M u_{M-1}\cdots u_1$. If M is even, then $Z(\mathfrak{A}^a_M) = \mathbb{C}[C_1, C_2]$ where $C_1 = u_{M-1}u_{M-3}\cdots u_1$ and $C_2 = u_M u_{M-2}\cdots u_2$.*

Proof. [15] We consider the monomials $u_M^{i_M}\cdots u_2^{i_2}u_1^{i_1}$, where the powers i_n are nonnegative integers, as a basis for the algebra $\mathfrak{A}^a_M$. Since the commutation relations in $\mathfrak{A}^a_M$ are homogeneous, the center is generated by monomials. A monomial $u_M^{i_M}\cdots u_2^{i_2}u_1^{i_1}$ belongs to the center if and only if

$$0 = \left[u_M^{i_M}\cdots u_2^{i_2}u_1^{i_1}, u_n\right] = \left(\omega^{i_{n-1}} - \omega^{i_{n+1}}\right) u_M^{i_M}\cdots u_n^{i_n+1} u_2^{i_2}u_1^{i_1} \quad \text{for all } n \in \mathbb{Z}_M.$$

Therefore $i_{n+2} = i_n$ for $n \in \mathbb{Z}_M$, which yields the claim due to the M–periodicity of the indices. □

The central elements are first integrals (constants of motion) of the corresponding quantum periodic Volterra system. In the periodic case equations of the quantum Volterra hierarchy can also be written in Heisenberg form (41) with the commuting self-adjoint Hamiltonians (42) being the finite sums

$$H_\ell = \sum_{\alpha\in\mathcal{N}^\ell}\sum_{k\in\mathbb{Z}_M} P_\alpha(\omega)\,\frac{\omega^\ell - 1}{\omega^{\nu(\alpha,0)} - 1} u_{\alpha+k}, \qquad \ell \in \mathbb{N}. \tag{44}$$

In contrast to the infinite dimensional case, only $k = \lfloor \frac{M-1}{2} \rfloor$ first Hamiltonians are algebraically independent first integrals and the rest are polynomials in the first integrals $H_1,\dots,H_k$ and central elements of the algebra $\mathfrak{A}^a_M$. Thus, periodic reductions of the Heisenberg equations (41) are integrable quantum systems, since

$$\#\text{commuting Hamiltonians } = \frac{1}{2}\left(M - \#\text{generators of the center}\right).$$

The cases $M=3$ and $M=4$ are superintegrable [1]. The system of three (resp. four) equations admits two (resp. three) algebraically independent quantum first integrals.

For example, in the case $M=3$, the center of algebra $\mathfrak{A}^a_3$ is $C = u_3u_2u_1$. There is only one commuting Hamiltonian, namely, $H_1 = u_1+u_2+u_3$. The Hamiltonians H_k, $k \geqslant 2$ are polynomials in C and H_1:

$$H_2 = H_1^2, \quad H_3 = H_1^3 + 3\omega^2 C, \quad H_4 = H_1^4 + 4\omega^2(1+\omega)CH_1,\ \dots\ .$$

In the case $M=4$, the independent first integrals are

$$C_1=u_3u_1,\quad C_2=u_4u_2,\quad H_1=u_1+u_2+u_3+u_4,$$

where C_1,C_2 are central elements of $\mathfrak{A}_4^a$. The Hamiltonians $H_k,\ k\geqslant 2$ are polynomials in these first integrals, namely,

$$H_2=H_1^2-2(C_1+C_2),\quad H_3=H_1^3-3(C_1+C_2)H_1,$$
$$H_4=H_1^4-4(C_1+C_2)H_1^2+4\left(1+\omega^2\right)C_1C_2+2\left(C_1^2+C_2^2\right),\ \dots\ .$$

In the case $M=5$, there are two commuting Hamiltonians, namely, H_1 and H_2, and we have

$$H_3=\frac{3}{2}H_2H_1-\frac{1}{2}H_1^3,\quad H_4=H_1^2H_2+\frac{1}{2}H_2^2-\frac{1}{2}H_1^4,\ \dots\ .$$

In the case $M=6$, the independent first integrals are C_1,C_2 as well as H_1, H_2, and

$$H_3=\frac{3}{2}H_2H_1-\frac{1}{2}H_1^3+3(C_1+C_2),\quad H_4=H_1^2H_2+\frac{1}{2}H_2^2-\frac{1}{2}H_1^4+4(C_1+C_2)H_1,\ \dots\ .$$

The quantisation ideal $\mathfrak{I}_b$ is $\mathcal{S}^2$–stable. Thus the quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}$ admits a periodic reduction for any *even* $M=2N>2$. The algebra $\mathfrak{A}_M^b=\mathfrak{A}_{\mathfrak{I}_b}\diagup\mathcal{I}_M$ is isomorphic to $\mathbb{C}\langle u_1,\dots,u_M\rangle\diagup\mathfrak{I}_b^M$, where

$$\mathfrak{I}_b^M=\langle u_Mu_1-\omega u_1u_M,\ u_nu_{n+1}-(-1)^n\omega u_{n+1}u_n,\ \ u_nu_m+u_mu_n\ ;$$
$$1\leqslant n<m\leqslant M,\ 1<m-n<M-1\rangle.$$

Proposition 9 ([15]). *Let* $M=2N$.

(i) *If N is odd, then* $Z(\mathfrak{A}_M^b)=\mathbb{C}[C_1,C_2]$ *where*

$$C_1=u_{M-1}u_{M-3}\cdots u_1,\qquad C_2=u_Mu_{M-2}\cdots u_2.$$

(ii) *If N is even, then the center of* $\mathfrak{A}_M^b$ *is generated by the elements*

$$\hat C_1=u_{M-1}^2u_{M-3}^2\cdots u_1^2,\quad \hat C_2=u_M^2u_{M-2}^2\cdots u_2^2,\quad \hat C=u_Mu_{M-1}\cdots u_2u_1,$$

where the generators $\hat C_1,\hat C_2$ *and* $\hat C$ *are algebraically dependent* $\hat C_1\hat C_2=\omega^{M-2}\hat C^2$.

Proof. The proof is similar to the proposition 8. Details of the proof can be found in [15] □

Example 3. In the case $M=4$, the center of algebra $\mathfrak{A}_4^b$ is generated by $\hat C_1=u_3^2u_1^2$, $\hat C_2=u_4^2u_2^2$ and $\hat C=u_4u_3u_2u_1$. The Hamiltonian of the cubic member of the periodic Volterra hierarchy is

$$\hat H_2=\sum_{k=1}^4u_k^2+(1-\omega)(u_2u_1+u_4u_3)+(1+\omega)(u_3u_2+u_1u_4)=H_1^2,$$

s where $H_1=u_1+u_2+u_3+u_4$. The elements $B_1=u_3u_1$ and $B_2=u_4u_2$ commute with each other, with $\hat H_2$ and anti-commute with H_1, that is,

$$[B_1,B_2]=0,\quad \left[B_1,\hat H_2\right]=0,\quad \left[B_2,\hat H_2\right]=0,\ \ [B_1,H_1]_+=B_1H_1+H_1B_1=0,\ \ [B_2,H_1]_+=0.$$

Moreover, the generators of the center $Z(\mathfrak{A}_4^b)$ can be represented as

$$\hat{C} = \omega^{-1} B_2 B_1 = u_4 u_3 u_2 u_1, \quad \hat{C}_1 = -B_1^2 = u_3^2 u_1^2, \quad \hat{C}_2 = -B_2^2 = u_4^2 u_2^2.$$

In algebra $\mathfrak{A}_4^b$, we have the Hamiltonian $\hat{H}_4 = \hat{H}_2^2 - 2\hat{C}_1 - 2\hat{C}_2$.

Since the elements B_1 and B_2 commute with the Hamiltonian $\hat{H}_2$, and are not central, they can be regarded as Hamiltonians of commuting quantum symmetries

$$\partial_{\tau_1}(u_n) = [B_1, u_n] = 2u_3 u_1 u_n, \qquad \partial_{\tau_2}(u_n) = [B_2, u_n] = 2u_4 u_2 u_n, \qquad n \in \mathbb{Z}_4,$$

which is not possible in the commutative case.

The results in the above example can be generalised to the case when $M = 2N$ and N is even. In $\mathfrak{A}_M^b$ the elements $B_1 = u_{M-1} u_{M-3} \cdots u_1$ and $B_2 = u_M u_{M-2} \cdots u_2$ satisfy the commutation relations

$$[B_1, u_k]_+ = [B_2, u_k]_+ = 0, \qquad k \in \mathbb{Z}_M,$$
$$[B_1, B_2] = \left[B_1, \hat{H}_{2\ell}\right] = \left[B_2, \hat{H}_{2\ell}\right] = 0, \qquad \ell \in \mathbb{N},$$

and the generators of the center $Z(\mathfrak{A}_M^b)$ can be represented as

$$\hat{C}_1 = (-1)^{\frac{N(N-1)}{2}} B_1^2, \qquad \hat{C}_2 = (-1)^{\frac{N(N-1)}{2}} B_2^2, \qquad \hat{C} = (-1)^{\frac{(N-1)(N-2)}{2}} \omega^{1-N} B_2 B_1,$$

where $\hat{C}_1, \hat{C}_2$ and $\hat{C}$ are same as in the second statement of proposition 9.

4. Summary and discussion

In this paper, we present explicit expressions for the infinite hierarchy of quantum Hamiltonians corresponding to both quantisations of the Volterra hierarchy, namely, quantisation ideals $\mathfrak{I}_a$ and $\mathfrak{I}_b$, and show that they are self-adjoint and commute with each other. Moreover, the dynamical equations of the quantum hierarchy can be written in the Heisenberg form using these Hamiltonians. The proofs mainly rely on the explicit expressions of the Volterra hierarchy on a free associative algebra.

The quantum algebra $\mathfrak{A}_{\mathfrak{I}_a}[\omega]$ can be regarded as a deformation of the commutative algebra $\tilde{\mathfrak{A}} = \mathfrak{A}_{\mathfrak{I}_a}[1] = \mathbb{C}[u_n; n \in \mathbb{Z}]$. It is well known that taking the classical limit $\hbar \to 0$, and thus $\omega = e^{i\hbar} \to 1$, one can equip $\tilde{\mathfrak{A}}$ with a Poisson algebra structure and turn the Heisenberg equations into the corresponding Hamiltonian ones [8, 16, 17]. Let us denote by $\tilde{a} \in \tilde{\mathfrak{A}}$ the limit of $a \in \mathfrak{A}_{\mathfrak{I}_a}[\omega]$, that is, $\tilde{a} = \lim_{\hbar \to 0} a$. It is clear that for any $a, b \in \mathfrak{A}_{\mathfrak{I}_a}[\omega]$ the commutator $[a, b] \in (\omega - 1)\mathfrak{A}_{\mathfrak{I}_a}[\omega]$ and thus we can define the bracket

$$\left\{\tilde{a}, \tilde{b}\right\} = \lim_{\hbar \to 0} \frac{1}{e^{\frac{1}{2}i\hbar} - e^{-\frac{1}{2}i\hbar}} [a, b]. \tag{45}$$

The bracket (45) is a Poisson bracket on $\tilde{\mathfrak{A}}$. Indeed, it is $\mathbb{C}$–bilinear and skew symmetric satisfying the Jacobi and Leibniz identities. Thus, we have

$$\{u_m, u_n\} = (\delta_{m,n-1} - \delta_{m,n+1}) u_m u_n. \tag{46}$$

It follows from theorem 6 that

$$\partial_{t_\ell}(u_n) = \lim_{\hbar \to 0} \frac{1}{e^{-\frac{1}{2}i\ell\hbar} - e^{\frac{1}{2}i\ell\hbar}} [H_\ell, u_n] = \left\{u_n, \tilde{H}_\ell\right\},$$

where

$$\tilde{H}_\ell = \lim_{\hbar\to 0}\frac{1}{\ell}H_\ell = \sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{N}^\ell}\frac{P_\alpha(1)}{\nu(\alpha,0)}u_{\alpha+k}.$$

Hence the densities $\tilde{h}_\ell$ of the local conservation laws for the Volterra hierarchy are given by $\tilde{H}_\ell = \sum_{k\in\mathbb{Z}}\mathcal{S}^k(\tilde{h}_\ell)$ with

$$\tilde{h}_\ell = \sum_{\alpha\in\mathcal{N}^\ell}\frac{P_\alpha(1)}{\nu(\alpha,0)}u_\alpha = \sum_{\alpha\in\mathcal{N}^\ell}\frac{1}{\kappa_{\alpha_0}}\binom{\kappa_{\alpha_1}+\kappa_{\alpha_1-1}-1}{\kappa_{\alpha_1}}\cdots\binom{\kappa_2+\kappa_1-1}{\kappa_2}\binom{\kappa_1+\kappa_0-1}{\kappa_1}u_\alpha,$$

where we recall that $\kappa_i = \nu(\alpha,i)$ is the number of i's in α. Indeed, we have

$$\tilde{h}_1 = u,\quad \tilde{h}_2 = \frac{u^2}{2}+u_1u,\quad \tilde{h}_3 = \frac{u^3}{3}+u_1u^2+u_1^2u+u_2u_1u,$$
$$\tilde{h}_4 = u_3u_2u_1u+u_2^2u_1u+u_1^3u+\frac{u^4}{4}+2u_2u_1^2u+u_2u_1u^2+\frac{3}{2}u_1^2u^2+u_1u^3,\ \cdots$$

As a byproduct of our results, we obtained explicit expressions for all local classical Hamiltonians $\tilde{h}_\ell$ of the classical commutative Volterra hierarchy. Traditional approaches [5], based on the Lax representation of the Volterra lattice or the transfer matrix approach, enable one to find a *generating function* for the Hamiltonians, but not their explicit expressions.

The quantum algebra $\mathfrak{A}_{\mathfrak{I}_b}[\omega]$ can be regarded as a deformation of the noncommutative algebra $\breve{\mathfrak{A}} = \mathfrak{A}_{\mathfrak{I}_b}[1]$. A Poisson algebra structure and Hamiltonian description of equations associated with deformations of noncommutative algebras have been recently developed in [15].

In the classical theory of integrable systems with commutative variables and systems on free associative algebra, there are numerous powerful tools and useful concepts, including Lax representations, Darboux transformations, recursion operators, and master symmetries [12, 18–20]. Their connections with the concept of quantisation ideals have not been explored yet. Developing this aspect of the theory will enable us to take advantage of a wide range of results in integrable systems and to advance the theory based on quantisation ideals.

Data availability statement

No new data were created or analysed in this study.

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