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Commutative Poisson algebras from deformations of noncommutative algebras

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Abstract
It is well-known that a formal deformation of a commutative algebra $\mathcal{A}$ leads to a Poisson bracket on $\mathcal{A}$ and that the classical limit of a derivation on the deformation leads to a derivation on $\mathcal{A}$, which is Hamiltonian with respect to the Poisson bracket. In this paper we present a generalization of it for formal deformations of an arbitrary noncommutative algebra $\mathcal{A}$. The deformation leads in this case to a Poisson algebra structure on $\Pi(\mathcal{A}) := Z(\mathcal{A}) \times (\mathcal{A}/Z(\mathcal{A}))$ and to the structure of a $\Pi(\mathcal{A})$-Poisson module on $\mathcal{A}$. The limiting derivations are then still derivations of $\mathcal{A}$, but with the Hamiltonian belong to $\Pi(\mathcal{A})$, rather than to $\mathcal{A}$. We illustrate our construction with several cases of formal deformations, coming from known quantum algebras, such as the ones associated with the nonabelian Volterra chains, Kontsevich integrable map, the quantum plane and the quantized Grassmann algebra.

Keywords Poisson algebra · Poisson module · Deformations of noncommutative algebras · Deformation quantisation · Heisenberg derivations · Hamiltonian derivations

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1 Introduction

By a well-known procedure, usually referred to as "taking the classical limit", quantum systems become classical systems, equipped with a Hamiltonian structure (symplectic or Poisson). Dirac [8] observed that Heisenberg's noncommutative multiplication of operators in a quantum algebra $\mathcal{A}_\hbar$ is a deformation of the commutative multiplication of functions on phase space. As the deformation parameter, the Planck constant $\hbar$, tends to zero, the algebra $\mathcal{A}_\hbar$ tends to the commutative algebra $\mathcal{A}$ of functions of phase space and the commutator of operators $\hat{a}, \hat{b} \in \mathcal{A}_\hbar$ converges to the Poisson bracket

$$\{a, b\} = \lim_{\hbar \to 0} \frac{i}{\hbar}[\hat{a}, \hat{b}]$$

of the corresponding functions $a, b \in \mathcal{A}$ on phase space. Heisenberg's equation with a quantum Hamiltonian $\hat{H} \in \mathcal{A}_\hbar$ tends to the Hamiltonian equation

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] \quad \xrightarrow{\hbar \to 0} \quad \frac{da}{dt} = \{H, a\}$$

and defines the Hamiltonian derivation $\partial_H := \{H, \cdot\} : \mathcal{A} \to \mathcal{A}$.

A novel approach to quantum algebras has recently be introduced by the first author [14]. These algebras, which will here be simply called *quantum algebras*, admit by definition a basis of ordered monomials (see Definition 3.1). They appear naturally in the study of nonabelian systems [3–5] and by taking the classical limit one obtains the underlying classical Hamiltonian system, under the assumption that the limiting algebra is a *commutative* algebra.

An example of where the method of taking the classical limit fails is given by the quantum algebra [14]

$$\mathcal{A}_q := \frac{\mathbb{C}(q)\langle x_i \rangle_{i \in \mathbb{Z}}}{\langle x_{i+1}x_i - (-1)^i q x_i x_{i+1}, x_i x_j + x_j x_i \rangle_{|i-j| \neq 1}}\ , \tag{1.1}$$

where $\mathbb{C}(q)\langle x_i \rangle_{i \in \mathbb{Z}}$ is the free algebra on $\dots, x_{-1}, x_0, x_1, \dots$. It is clear that for no value of the deformation parameter q this algebra becomes commutative, or even graded commutative. On $\mathcal{A}_q$ there is a well-defined differential-difference equation

$$\partial_2 x_\ell = \frac{1}{q^2 - 1}\,[\mathfrak{H}_2, x_\ell] = x_\ell x_{\ell+1}^2 - x_{\ell-1}^2 x_\ell + x_\ell^2 x_{\ell+1} - x_{\ell-2}x_{\ell-1}x_\ell + x_\ell x_{\ell+1}x_{\ell+2} - x_{\ell-1}x_\ell^2$$

which is the first member of an infinite hierarchy of odd-degree Volterra type systems. All equations of the hierarchy can be presented for $m = 1, 2, \dots$ in the Heisenberg form [5]

$$\partial_{2m} x_\ell = \frac{1}{q^{2m} - 1}\,[\mathfrak{H}_{2m}, x_\ell]\ .$$

They have nontrivial limits when q goes to any $2m$-th root of unity. The problem is that the Hamiltonian structure of these limits cannot be obtained by taking the conventional classical limit. It was one of the motivations to undertake this study. We solved this problem, and in Sect. 5.3 we use this example to illustrate our method.

There were several attempts to develop a Hamiltonian description of differential equations on associative algebras which are not commutative. In the case of a free associative algebra $\mathcal{A}$, a Lie ("Poisson") bracket can be defined on the quotient space $\mathcal{A}^\natural = \mathcal{A}/[\mathcal{A},\mathcal{A}]$ of the algebra $\mathcal{A}$ over the linear space $[\mathcal{A},\mathcal{A}]$ spanned by all commutators in $\mathcal{A}$, and elements of $\mathcal{A}^\natural$ define Hamiltonian derivations of $\mathcal{A}$ [15]. The linear space $\mathcal{A}^\natural$ does not admit a structure of a Poisson algebra since a compatible multiplication is missing. Farkas and Letzter demonstrated that for any prime Poisson algebra $\mathcal{A}$, which is *not commutative*, the Poisson bracket must be the commutator in $\mathcal{A}$, up to an appropriate scalar factor [9]. Consequently, it became widely acknowledged that the definition of a Poisson algebra in the noncommutative case is too restrictive. Several modifications of the definition, inspired by noncommutative Poisson and differential geometries, as well as Hamiltonian description of noncommutative differential equations were explored in [6, 7, 17].

In this paper we propose another approach, where the structure that we put on $\mathcal{A}$ is that of a Poisson module over a commutative Poisson algebra $\Pi(\mathcal{A})$ which we also construct from the deformation. Specifically, suppose that $(\mathcal{A}[[\nu]],\star)$ is a (formal) deformation of an associative algebra $\mathcal{A}$, whose center is denoted $Z(\mathcal{A})$. The (noncommutative) Poisson algebra $(\mathcal{A}[[\nu]],[\cdot\,,\cdot]_\star)$ admits $\mathcal{H}_\nu := Z(\mathcal{A})+\nu\mathcal{A}[[\nu]]$ as a Poisson subalgebra, and on it we can define a rescaled Lie bracket $[\cdot\,,\cdot]_\nu := \frac{1}{\nu}[\cdot\,,\cdot]_\star$, which makes $(\mathcal{H}_\nu,[\cdot\,,\cdot]_\nu)$ into a Poisson algebra. The latter admits $\nu\mathcal{H}_\nu$ as a Poisson ideal, so that $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ inherits the structure of a Poisson algebra, i.e., a multiplication and a Poisson bracket. Since $\mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq Z(\mathcal{A})\times(\mathcal{A}/Z(\mathcal{A}))$, in a natural way, $\Pi(\mathcal{A}) := Z(\mathcal{A})\times(\mathcal{A}/Z(\mathcal{A}))$ is a Poisson algebra. It is in fact a *commutative* Poisson algebra. The commutative multiplication and the Poisson bracket on $\Pi(\mathcal{A})$ will be denoted by $\cdot$ and $\{\cdot\,,\cdot\}$. Like any Poisson bracket on a commutative algebra, the bracket is completely specified on generators and can be computed for arbitrary elements by using derivatives (see (4.1)), which is one of the main virtues of Poisson brackets on commutative algebras.

In order to construct the $\Pi(\mathcal{A})$-Poisson module structure on $\mathcal{A}$, we first consider the (noncommutative) Poisson algebra $(\mathcal{H}_\nu/\nu^2\mathcal{A}[[\nu]],[\cdot\,,\cdot]_\nu)$ as a Poisson module over itself, and then show that $\nu\mathcal{A}[[\nu]]/\nu^2\mathcal{A}[[\nu]] \simeq \mathcal{A}$ is a Poisson submodule. Now $\Pi(\mathcal{A})$ can be identified with the quotient of $\mathcal{H}_\nu/\nu^2\mathcal{A}[[\nu]]$ with respect to the Poisson ideal $\nu Z(\mathcal{A})$ and by reduction $\mathcal{A}$ becomes a Poisson module over $\Pi(\mathcal{A})$, with actions denoted by $\cdot$ and $\{\cdot;\cdot\}$. Moreover we show that for any $\mathbf{H}\in\Pi(\mathcal{A})$, $\{\mathbf{H}\,;\cdot\}$ is a derivation of $\mathcal{A}$, making it into a *Hamiltonian* derivation of $\mathcal{A}$. Since $\Pi(\mathcal{A})$ is commutative, the Poisson module structure can again easily be computed from the formulas, given in terms of generators for $\Pi(\mathcal{A})$ and of $\mathcal{A}$.

The construction of $\Pi(\mathcal{A})$ and of the Poisson module structure on $\mathcal{A}$ will be illustrated at length in several examples (Sect. 4), associated with quantum algebras; its applications to nonabelian systems, related to these quantum algebras will be illustrated in Sect. 5. We will in this introduction only describe shortly one example, which is worked out in detail in Sects. 4.2 and 5.6.

For concreteness, we illustrate our approach in a simple example, related to the integrable Kontsevich equation [18]. Recall that the (two-dimensional) quantum torus is defined as the quantum algebra

$$\mathcal{A}_q := \mathbb{T}_q[x,y] = \frac{\mathbb{C}(q)\langle x,y,x^{-1},y^{-1}\rangle}{\langle yx-qxy\rangle} . \tag{1.2}$$

It is a localization of the quantum plane $\frac{\mathbb{C}(q)\langle x,y\rangle}{\langle yx-qxy\rangle}$. On $\mathcal{A}_q$ there is a hierarchy of commuting derivations, given for $n=1,2,\dots$ by

$$\partial_n x = \frac{1}{1-q^n}[\mathfrak{H}^{(n)},x] , \quad \partial_n y = \frac{1}{1-q^n}[\mathfrak{H}^{(n)},y] . \tag{1.3}$$

The quantum Hamiltonians $\mathfrak{H}^{(n)}$ are given for $n=1,2$ by

$$\begin{aligned}\mathfrak{H}^{(1)} &= qx^{-1}y^{-1}+qy^{-1}+y+qx+x^{-1} ,\\ \mathfrak{H}^{(2)} &= \left(\mathfrak{H}^{(1)}\right)^2-(1+q)\mathfrak{H}^{(1)}-4q .\end{aligned}$$

The coefficients in the right-hand side of Eqs. (1.3) are Laurent polynomials in the variable q. Consequently, the derivation ∂_n possesses a well-defined limit as $q\to\xi$, where ξ represents a primitive n-th root of unity. By setting $q=\xi+\nu$, we can regard $\mathcal{A}_q$ as an algebra of power series $\mathcal{A}[[\nu]]$, where $\mathcal{A}=\mathcal{A}_\xi$. In the algebra $\mathcal{A}_\xi$ we have $yx=\xi xy$, implying that the center of $\mathcal{A}$ is generated by x^n and y^n. Moreover, $X:=\left(x^n,\overline{0}\right)$, $Y:=\left(y^n,\overline{0}\right)$ and $W_{i,j}:=\left(0,\overline{x^iy^j}\right)$, where $0\leqslant i,j<n,\ i+j\neq 0$ generate $\Pi(\mathcal{A})$, while the $\Pi(\mathcal{A})$–module $\mathcal{A}$ is generated by $1,\ x$ and y. Table 1 describes in this case the Poisson structure $\{\cdot\,,\cdot\}$ on $\Pi(\mathcal{A})$ in terms of these generators. The $\cdot$ action and Lie action $\{\cdot\,;\cdot\}$ of $\Pi(\mathcal{A})$ on $\mathcal{A}$ are given in Table 2. These tables can be used to describe the Hamiltonian structure of the limiting derivation of ∂_n on $\mathcal{A}$.

$\{\cdot\,,\cdot\}$	X	Y	$W_{k,\ell}$
X	0	$-\xi^{-1}n^2XY$	$-\xi^{-1}n\ell XW_{k,\ell}$
Y	$\xi^{-1}n^2XY$	0	$\xi^{-1}nkYW_{k,\ell}$
$W_{i,j}$	$\xi^{-1}njXW_{i,j}$	$-\xi^{-1}niYW_{i,j}$	$\left(\xi^{jk}-\xi^{i\ell}\right)W_{i+k,j+\ell}$

Table 1 Poisson brackets between the generators of the Poisson algebra $\Pi(\mathcal{A})$

Let us consider the case $n=2,\ q(\nu)=-1+\nu$ and $\mathcal{A}=\mathbb{C}\langle x,y,x^{-1},y^{-1}\rangle/\langle xy+yx\rangle$. Then

$$\frac{\mathfrak{H}^{(2)}}{q(\nu)^2-1} = \frac{1}{\nu}\left(H_0^{(2)}+\nu H_1^{(2)}\right) \pmod{\mathcal{H}_\nu} ,$$

$\cdot$	x	y
X	x^{n+1}	$x^n y$
Y	xy^n	y^3
$W_{i,j}$	0	0

$\{\cdot\,;\cdot\}$	x	y
X	0	$-\xi^{-1}nx^n y$
Y	$\xi^{-1}nxy^n$	0
$W_{i,j}$	$\left(\xi^j - 1\right) x^{i+1} y^j$	$\left(1 - \xi^i\right) x^i y^{j+1}$

Table 2 Multiplication table and Lie brackets for the generators of the $\Pi(\mathcal{A})$ module $\mathcal{A}$

where

$$H_0^{(2)} = -\frac{1}{2}\left(x^2 + y^2 + x^{-2} + y^{-2} - x^{-2}y^{-2}\right) ,$$
$$H_1^{(2)} = -\frac{1}{2}\left(x - y - xy - xy^{-1} - x^{-1}y + x^{-1}y^{-2} - x^{-2}y^{-1}\right) .$$

Let $\mathbf{H}^{(2)} := (H_0^{(2)}, H_1^{(2)})$. Then, writing U, V and W, respectively, as a shorthand for $W_{1,0}$, $W_{0,1}$ and $W_{1,1}$,

$$\mathbf{H}^{(2)} = -\frac{1}{2}\Big(X + Y + X^{-1} + Y^{-1} - X^{-1}Y^{-1} + U - V - W - Y^{-1}W$$
$$+X^{-1}Y^{-1}U - X^{-1}Y^{-1}V\Big) .$$

Using Table 2, we obtain the Hamiltonian equation on the algebra $\mathbb{C}\langle x, y, x^{-1}, y^{-1}\rangle/\langle xy + yx\rangle$

$$\begin{aligned}\partial_{\mathbf{H}^{(2)}} x = \left\{\mathbf{H}^{(2)}\,; x\right\} &= \frac{\partial \mathbf{H}^{(2)}}{\partial Y}\cdot(-2xy^2) + \frac{\partial \mathbf{H}^{(2)}}{\partial V}\cdot(-2xy^2) + \frac{\partial \mathbf{H}^{(2)}}{\partial W}\cdot(-2x^2y)\\ &= (1 - Y^{-2} + X^{-1}Y^{-2})\cdot xy^2 - (1 + X^{-1}Y^{-1})\cdot xy - (1 + Y^{-1} + X^{-1})\cdot x^2 y\\ &= xy^2 - xy^{-2} + x^{-1}y^{-2} - xy - x^{-1}y^{-1} - x^2y - x^2y^{-1} - y ,\end{aligned}$$

and similarly for $\partial_{\mathbf{H}^{(2)}} y$ (see the detailed calculations in Sect. 5.6).

The structure of the paper is as follows. We show in Sect. 2 how deformations of an associative algebra $\mathcal{A}$ lead to a Poisson algebra $\Pi(\mathcal{A})$ and to a $\Pi(\mathcal{A})$-Poisson module structure on $\mathcal{A}$. We show that the construction is functorial and show its relevance constructing Hamiltonian derivation on $\mathcal{A}$ from Heisenberg derivations on $\mathcal{A}[[\nu]]$. We show in Sect. 3 how a quantized algebra, depending on one or several parameters, can be viewed naturally as a formal deformation of some associative algebra. Examples of quantized algebras, the corresponding deformations, Poisson algebras and Poisson modules will be given in Sect. 4, together with an application. Examples of nonabelian systems related to these quantized algebras will be given in 5; we illustrate how the Hamiltonian structure of their limit derivations is obtained by our methods.

Lastly, in the paper, we do not explicitly consider the complex or Hermitian structure of the Hamiltonians and the equations. While it can be straightforwardly incorporated in each case, doing so might compromise the clarity and readability of the expressions.

The results of this paper were presented at several seminars and conferences, including the international conference "Geometry and Integrability" at Skoltech and the Higher School of Economics, Moscow, in September 2023, and in the preprint arXiv:2402.16191. We are grateful to N. Reshetikhin for the invitation to give a talk on this subject at his seminar at BIMSA, Beijing, China, in April 2024. On that occasion we learned that with his collaborators he was studying some of the problems solved in this paper using a different approach, based on the notion of a hybrid quantum system, that they introduced. In particular, they independently discovered the derivations on a noncommutative algebra, which are Hamiltonian derivations in our approach and hybrid derivation in theirs. Their results are now available in the preprint [13]. We are also grateful to N. Reshetikhin for pointing out the reference [16], which contains useful examples of nonflat Azumaya algebras, that would naturally fit in our list of examples.

2 Poisson algebras and Poisson modules from deformations

In this section we show how any deformation of a (not necessarily commutative) associative algebra $\mathcal{A}$ leads in a natural way to a commutative Poisson algebra $\Pi(\mathcal{A})$ and a $\Pi(\mathcal{A})$-Poisson module structure on $\mathcal{A}$.

2.1 Poisson algebras and deformations

We first recall the definition of a (not necessarily commutative) Poisson algebra (see, for example, [9]). Let $\mathcal{A}$ be any (unitary) associative algebra over a commutative ring R. For $a, b \in \mathcal{A}$ their product $a \cdot b$ in $\mathcal{A}$ will simply be denoted by ab and their commutator $ab - ba$ by $[a, b]$.

Definition 2.1 A skew-symmetric R-bilinear map $\{\cdot\,,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be a *Poisson bracket* on $\mathcal{A}$ when it satisfies the Jacobi and Leibniz identities: for all $a, b, c \in \mathcal{A}$,

$$(1)\ \{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0\ , \quad \text{(Jacobi identity)},$$
$$(2)\ \{a, bc\} = \{a, b\}\, c + b\, \{a, c\}\ , \quad \text{(Leibniz identity)}.$$

$(\mathcal{A}, \{\cdot\,,\cdot\})$ or $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ is then said to be a *Poisson algebra (over R)*. When $\mathcal{A}$ is commutative one says that the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$ is *commutative*.

Example 2.2 Any associative algebra $\mathcal{A}$ has a natural Poisson bracket, given by the commutator $\{a, b\} := [a, b]$. Indeed, it is well-known that $[\cdot\,,\cdot]$ is a Lie bracket on $\mathcal{A}$ and one easily checks that the Leibniz identity also holds, so $(\mathcal{A}, [\cdot\,,\cdot])$ is a Poisson algebra. This Poisson bracket is trivial if and only if $\mathcal{A}$ is commutative.

Example 2.3 When $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a Poisson algebra and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ which is also a Lie subalgebra of $(\mathcal{A}, \{\cdot\,,\cdot\})$, then $(\mathcal{B}, \{\cdot\,,\cdot\})$ is also a Poisson algebra; we say that it is a *Poisson subalgebra* of $\mathcal{A}$. Similarly, if $\mathcal{I}$ is an ideal of $\mathcal{A}$ which is also a Lie ideal of $(\mathcal{A}, \{\cdot\,,\cdot\})$ then $\mathcal{I}$ is a *Poisson ideal* of $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ is a Poisson algebra; we say that it is a *quotient Poisson algebra* of $\mathcal{A}$. The inclusion $\mathcal{B} \to \mathcal{A}$ and the projection $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ are then *morphisms of Poisson algebras*, that is they are algebra as well as Lie algebra morphisms.

These examples will be used in what follows to construct, by Poisson reduction, commutative Poisson algebras from deformations of a (not necessarily commutative) associative algebra, a notion which we first recall (see, for example, [11, Ch. 13] for the commutative case). Let $\mathcal{A}$ be any associative algebra over R and consider $\mathcal{A}[[\nu]]$, the $R[[\nu]]$-module of formal power series in some formal variable ν with the elements of $\mathcal{A}$ as coefficients. By definition, any element A of $\mathcal{A}[[\nu]]$ can be written in a unique way as $A = a_0 + \nu a_1 + \nu^2 a_2 + \cdots$, where $a_i \in \mathcal{A}$ for all i.

Definition 2.4 Suppose that $\mathcal{A}[[\nu]]$ is equipped with the structure of an associative algebra over $R[[\nu]]$, with product denoted by $\star$. Then $(\mathcal{A}[[\nu]], \star)$ (or more simply $\mathcal{A}[[\nu]]$) is said to be a *(formal) deformation* of $\mathcal{A}$ if for any $a, b \in \mathcal{A}$, $a\star b = ab + \mathcal{O}(\nu)$, i.e., $a\star b - ab \in \nu\mathcal{A}[[\nu]]$.

Said differently, the latter condition states that under the natural identification of $\mathcal{A}$ with $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]]$ the canonical projection $p : (\mathcal{A}[[\nu]], \star) \to (\mathcal{A}, \cdot)$ is a morphism of algebras. One naturally views p as evaluation at $\nu = 0$.

Example 2.5 A first classical example is the Moyal product. It defines a nontrivial deformation of the algebra of functions on any symplectic manifold (M, ω). If we denote by Γ the inverse to ω, then Γ is a Poisson structure on M and the Moyal product of $f, g \in C^\infty(M)$ is given by

$$f\star g = m \circ e^{\nu\Gamma/2}(f \otimes g)\ ,$$

where m denotes the usual multiplication in $C^\infty(M)$.

Example 2.6 A second classical example is the standard deformation of the algebra $\operatorname{Sym}\mathfrak{g}$ of polynomial functions on a Lie algebra $\mathfrak{g}$. If we denote by $T^\bullet\mathfrak{g}$ the tensor algebra of $\mathfrak{g}$ and by $\mathcal{I}$ the two-sided ideal of $T^\bullet\mathfrak{g}[[\nu]]$ generated by all $x \otimes y - y \otimes x - \nu[x, y]$ with $x, y \in \mathfrak{g}$, then by the Poincaré–Birkhoff–Witt Theorem $T^\bullet\mathfrak{g}[[\nu]]/\mathcal{I} \simeq \operatorname{Sym}\mathfrak{g}[[\nu]]$ making $\operatorname{Sym}\mathfrak{g}[[\nu]]$ with the transported product $\star$ into a deformation of $\operatorname{Sym}\mathfrak{g}$.

See [1, 2] for more information on these examples and for the relevance of deformation theory to quantization. Notice that in both of these examples the associative algebra $\mathcal{A}$, which is deformed, is commutative.

The commutator in $(\mathcal{A}[[\nu]], \star)$ is denoted by $[\cdot\,,\cdot]_\star$: $[A, B]_\star := A\star B - B\star A$ for $A, B \in \mathcal{A}[[\nu]]$. We also introduce for all $i \in \mathbb{Z}_{>0}$, R-bilinear maps $(\cdot, \cdot)_i, \{\cdot\,,\cdot\}_i$:

$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by setting, for all $a, b \in \mathcal{A} \subset \mathcal{A}[[\nu]]$,

$$a\star b = ab + \nu(a,b)_1 + \nu^2(a,b)_2 + \cdots\,, \tag{2.1}$$

$$[a,b]_\star = [a,b] + \nu\,\{a,b\}_1 + \nu^2\,\{a,b\}_2 + \cdots\,, \tag{2.2}$$

where the values of the leading terms follow from $p(a\star b) = ab$ and $p([a,b]_\star) = ab - ba = [a,b]$. Of course, $\{a,b\}_i = (a,b)_i - (b,a)_i$ for all i and all $a, b \in \mathcal{A}$.

2.2 Commutative Poisson algebras from deformations

It is well-known that when the associative algebra $\mathcal{A}$ is commutative, $\mathcal{A}$ inherits from any deformation $(\mathcal{A}[[\nu]], \star)$ of $\mathcal{A}$ a Poisson bracket (see, for example, [1, 11]). Recall that this Poisson bracket is classically defined for $a, b \in \mathcal{A} \subset \mathcal{A}[[\nu]]$ by

$$\{a,b\} = \lim_{\nu\to 0} \frac{a\star b - b\star a}{\nu}\,, \tag{2.3}$$

and that the fact that it is a Poisson bracket follows from the associativity of $\star$. In order to anticipate the construction of the Poisson bracket in the noncommutative case, we first reformulate the construction of the Poisson bracket (2.3) in a different, more abstract way. In view of Example 2.2, $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\star)$ is a Poisson algebra over $R[[\nu]]$. Since $\mathcal{A}$ is commutative, the commutator $[\cdot\,,\cdot]_\star$ takes values in $\nu\mathcal{A}[[\nu]]$ and we can also consider on $\mathcal{A}[[\nu]]$ its rescaling, defined for $A, B \in \mathcal{A}[[\nu]]$ by

$$[A,B]_\nu := \frac{1}{\nu}\,[A,B]_\star = \frac{A\star B - B\star A}{\nu} \in \mathcal{A}[[\nu]]\,. \tag{2.4}$$

It is clear that this rescaling does not affect the Leibniz and Jacobi identities, so that $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\nu)$ is also a Poisson algebra over the ring $R[[\nu]]$. Since

$$[\nu\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]]_\nu = [\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]]_\star \subset \nu\mathcal{A}[[\nu]]\,,$$

the (associative) ideal $\nu\mathcal{A}[[\nu]]$ of $\mathcal{A}[[\nu]]$ is also a Lie ideal of $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\nu)$, hence is a Poisson ideal of it. The quotient $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]]$ is therefore a Poisson algebra. Under the natural identification of $\mathcal{A}$ with $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]]$, we recover from (2.4) the Poisson bracket (2.3) on $\mathcal{A}$.

Example 2.7 In the case of Example 2.5, the Poisson structure that one obtains is Γ. In the case of Example 2.6 one obtains the canonical Lie–Poisson structure on $\operatorname{Sym}\mathfrak{g}$ (see [11, Ch. 7]).

We now consider the more general case in which $\mathcal{A}$ is not necessarily commutative. The center of $\mathcal{A}$ is denoted $Z(\mathcal{A})$. We suppose that $\mathcal{A}[[\nu]]$ is a deformation of $\mathcal{A}$ and consider again the Poisson algebra $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\star)$. In this case we cannot define the rescaled bracket on $\mathcal{A}[[\nu]]$ as in (2.4) because $[\cdot\,,\cdot]_\star$ does not take values in $\nu\mathcal{A}[[\nu]]$ (in general). Let us define

$$\mathcal{H}_\nu := Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]],$$

the $R[[\nu]]$-submodule of $\mathcal{A}[[\nu]]$ generated by $Z(\mathcal{A})$ and $\nu\mathcal{A}$. It consists of those elements of $\mathcal{A}[[\nu]]$ whose ν-independent term belongs to the center $Z(\mathcal{A})$ of $\mathcal{A}$.

Lemma 2.8 *$\mathcal{H}_\nu$ is a Poisson subalgebra of $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\star)$. Moreover, the commutator $[\cdot\,,\cdot]_\star$, restricted to $\mathcal{H}_\nu$ takes values in $\nu\mathcal{H}_\nu$, so that $(\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu)$ is also a Poisson algebra.*

Proof Since $Z(\mathcal{A})$ is a subalgebra of $\mathcal{A}$, $\mathcal{H}_\nu$ is a subalgebra of $\mathcal{A}[[\nu]]$. But $\mathcal{H}_\nu$ is also a Lie subalgebra of $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\star)$ because

$$[\mathcal{H}_\nu, \mathcal{H}_\nu]_\star = [Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]], Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]]]_\star \subset \nu\mathcal{A}[[\nu]] \subset \mathcal{H}_\nu\ ,$$

where we have used that $[Z(\mathcal{A}), Z(\mathcal{A})]_\star \subset \nu\mathcal{A}[[\nu]]$. It follows that $\mathcal{H}_\nu$ is a Poisson subalgebra of $(\mathcal{A}[[\nu]], [\cdot\,,\cdot]_\star)$, which is the first statement, and that the bracket $[\cdot\,,\cdot]_\star$, restricted to $\mathcal{H}_\nu$ takes values in $\nu\mathcal{A}[[\nu]]$. In order to prove the second statement, we need to show that the restriction of $[\cdot\,,\cdot]_\star$ to $\mathcal{H}_\nu$ actually takes values in $\nu\mathcal{H}_\nu$, i.e., that when $A, B \in \mathcal{H}_\nu$ then $[A, B]_\star \in \nu\mathcal{H}_\nu$. Writing $A = a + \nu a_1 + \mathcal{O}(\nu^2)$ and $B = b + \nu b_1 + \mathcal{O}(\nu^2)$ we have, using that a and b belong to the center of $\mathcal{A}$ and using (2.2), that

$$[A, B]_\star = [a, b]_\star + \nu\,[a, b_1] + \nu\,[a_1, b] + \mathcal{O}\left(\nu^2\right) = \nu\,\{a, b\}_1 + \mathcal{O}\left(\nu^2\right)\ , \tag{2.5}$$

so we need to show that $\{a, b\}_1 \in Z(\mathcal{A})$ for any $a, b \in Z(\mathcal{A})$. Let c be any element of $\mathcal{A}$. In view of the Jacobi identity for $[\cdot\,,\cdot]_\star$ (which follows from the associativity of $\star$),

$$[[a, b]_\star\,, c]_\star + [[b, c]_\star\,, a]_\star + [[c, a]_\star\,, b]_\star = 0\ . \tag{2.6}$$

Using (2.5) and that $a \in Z(\mathcal{A})$, the first term reads

$$[[a, b]_\star\,, c]_\star = \nu\left[\{a, b\}_1\,, c\right]_\star + \mathcal{O}\left(\nu^2\right) = \nu(\{a, b\}_1\,c - c\,\{a, b\}_1) + \mathcal{O}\left(\nu^2\right)\ .$$

Now since $a, b \in Z(\mathcal{A})$,

$$[[b, c]_\star\,, a]_\star = \nu\left[\{b, c\}_1\,, a\right]_\star + \mathcal{O}\left(\nu^2\right) = \nu^2\,\{\{b, c\}_1\,, a\}_1 + \mathcal{O}\left(\nu^2\right) = \mathcal{O}\left(\nu^2\right)\ ,$$

and similarly $[[c, a]_\star\,, b]_\star = \mathcal{O}(\nu^2)$. Substituted in (2.6) we get $\{a, b\}_1\,c - c\,\{a, b\}_1 = 0$ for all $c \in \mathcal{A}$, which shows that $\{a, b\}_1 \in Z(\mathcal{A})$. □

We are now ready to construct the commutative Poisson algebra which is naturally associated with a deformation.

Proposition 2.9 *Let $(\mathcal{A}[[\nu]], \star)$ be a deformation of an associative algebra $\mathcal{A}$.*

(1) *$\nu\mathcal{H}_\nu$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu)$.*
(2) *The quotient algebra $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ is a commutative Poisson algebra.*

Proof $\nu\mathcal{H}_\nu \subset \mathcal{H}_\nu$, which is an ideal of $\mathcal{H}_\nu$, is also a Lie ideal for the bracket $[\cdot\,,\cdot]_\nu$ on $\mathcal{H}_\nu$ because

$$[\nu\mathcal{H}_\nu, \mathcal{H}_\nu]_\nu = \nu\,[\mathcal{H}_\nu, \mathcal{H}_\nu]_\nu \subset \nu\mathcal{H}_\nu\ .$$

It follows that $\nu\mathcal{H}_\nu$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu)$ and that the quotient algebra $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ is a Poisson algebra. According to Lemma 2.8, $A{\star}B - B{\star}A = [A, B]_\star \in \nu\mathcal{H}_\nu$ when $A, B \in \mathcal{H}_\nu$, which shows that the Poisson algebra $(\mathcal{H}_\nu/\nu\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu)$ is commutative. □

By construction, $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ is a Poisson algebra over $R[[\nu]]$, in particular it is a Poisson algebra over R. Moreover, we will write elements of $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ as pairs $(a, \overline{b}) := (a, b + Z(\mathcal{A}))$ with $a \in Z(\mathcal{A})$ and $b \in \mathcal{A}$, and identify $\mathcal{H}_\nu/\nu\mathcal{H}_\nu$ with

$$\Pi(\mathcal{A}) := Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}\ ,$$

which we call the *Poisson algebra associated to the deformation* $(\mathcal{A}[[h]], \star)$ of $\mathcal{A}$. Notice that the quotient $\mathcal{A}/Z(\mathcal{A})$ is not a quotient of algebras ($Z(\mathcal{A})$ is in general not an ideal of $\mathcal{A}$) but of R-modules. Under this identification and notation, the canonical projection $p_\Pi : \mathcal{H}_\nu \to \Pi(\mathcal{A})$ is given by $A = a_0 + \nu a_1 + \nu^2 a_2 + \ldots \mapsto (a_0, \overline{a_1})$. The associative product and Poisson bracket on $\Pi(\mathcal{A})$ are denoted by $\cdot$ and $\{\cdot\,,\cdot\}$, respectively. If we denote the unit of $\mathcal{A}$ by 1, then $(1, \overline{0})$ is the unit of $\Pi(\mathcal{A})$. By construction, we have the following corollary of Proposition 2.9:

Proposition 2.10 *The projection $p_\Pi : (\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu) \to (\Pi(\mathcal{A}), \{\cdot\,,\cdot\})$ is a surjective morphism of Poisson algebras.* □

Explicit formulas for $\cdot$ and $\{\cdot\,,\cdot\}$ are given in the following proposition:

Proposition 2.11 *Let $(a, \overline{a_1})$, $(b, \overline{b_1}) \in \Pi(\mathcal{A})$. Then*

$$(a, \overline{a_1}) \cdot (b, \overline{b_1}) = \left(ab, \overline{ab_1 + a_1 b + (a, b)_1}\right)\ , \tag{2.7}$$

and

$$\left\{(a, \overline{a_1}), (b, \overline{b_1})\right\} = \left(\{a, b\}_1\,, \overline{\{a, b\}_2 + \{a_1, b\}_1 + \{a, b_1\}_1 + [a_1, b_1]}\right)\ . \tag{2.8}$$

Proof Since p_Π is surjective, we can write $(a, \overline{a_1}) = p_\Pi(A)$ and $(b, \overline{b_1}) = p_\Pi(B)$, where $A = a + \nu a_1 + \nu^2 a_2 + \cdots$ and $B = b + \nu b_1 + \nu^2 b_2 + \cdots$ belong to $\mathcal{H}_\nu$. Then, using Proposition 2.10,

$$(a, \overline{a_1}) \cdot (b, \overline{b_1}) \;=\; p_\Pi(A) \cdot p_\Pi(B) = p_\Pi(A{\star}B) = p_\Pi((a + \nu a_1){\star}(b + \nu b_1))$$

$$\overset{(2.1)}{=} p_\Pi(ab + \nu(ab_1 + a_1 b + (a,b)_1)) = \left(ab, \overline{ab_1 + a_1 b + (a,b)_1}\right),$$

and

$$\begin{aligned}\left\{(a, \overline{a_1}), (b, \overline{b_1})\right\} &= p_\Pi\left[a + \nu a_1 + \nu^2 a_2, b + \nu b_1 + \nu^2 b_2\right]_\nu \\ &\overset{(2.2)}{=} p_\Pi(\{a,b\}_1 + \nu(\{a,b\}_2 + \{a_1,b\}_1 + \{a,b_1\}_1 + [a_1,b_1])) \\ &= \left(\{a,b\}_1, \overline{\{a,b\}_2 + \{a_1,b\}_1 + \{a,b_1\}_1 + [a_1,b_1]}\right).\end{aligned}$$

□

Remark 2.12 When $\mathcal{A}$ is commutative, $Z(\mathcal{A}) = \mathcal{A}$ and $\mathcal{H}_\nu = \mathcal{A}[[\nu]]$ so that, as Poisson algebras,

$$\Pi(\mathcal{A}) \simeq \mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq \mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]] \simeq \mathcal{A},$$

and we recover the (commutative) Poisson algebra constructed in the commutative case, with Poisson bracket $\{\cdot\,,\cdot\} = \{\cdot\,,\cdot\}_1$.

Example 2.13 Let $\mathcal{A}$ be any associative algebra over R and consider the trivial deformation of $\mathcal{A}$: the product $\star$ on $\mathcal{A}[[\nu]]$ is the $R[[\nu]]$-linear extension of the product on $\mathcal{A}$. Then $(a,b)_i = \{a,b\}_i = 0$ for $a, b \in \mathcal{A}$ and $i \geqslant 1$, since $a \star b = ab$ for all $a, b \in \mathcal{A}$. Hence, the formulas (2.7) and (2.8) for the product and Poisson bracket on $\Pi(\mathcal{A})$ become

$$(a, \overline{a_1}) \cdot \left(b, \overline{b_1}\right) = \left(ab, \overline{ab_1 + a_1 b}\right), \quad \text{and} \quad \left\{(a, \overline{a_1}), \left(b, \overline{b_1}\right)\right\} = \left(0, \overline{[a_1, b_1]}\right). \tag{2.9}$$

It follows that for any associative algebra $\mathcal{A}$, (2.9) defines the structure of a commutative Poisson algebra on $\Pi(\mathcal{A})$. The Poisson bracket in (2.9) is trivial if and only if $\mathcal{A}$ is two-step nilpotent, $[[\mathcal{A}, \mathcal{A}], \mathcal{A}] = 0$.

In order to define the $\Pi(\mathcal{A})$-Poisson module structure on $\mathcal{A}$, we will need a slightly larger Poisson algebra, which is non necessarily commutative, given by the following proposition. Its proof is very similar to the proof of Proposition 2.9.

Proposition 2.14 *Let $(\mathcal{A}[[\nu]], \star)$ be a deformation of an associative algebra $\mathcal{A}$.*

(1) *$\nu^2\mathcal{A}[[\nu]]$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot\,,\cdot]_\nu)$.*
(2) *The quotient algebra $\mathcal{H}_\nu/\nu^2\mathcal{A}[[\nu]] \simeq Z(\mathcal{A}) \times \mathcal{A}$ is a Poisson algebra.* □

By a similar computation as in the proof of Proposition 2.11, the product and the Poisson bracket of $(a, a_1), (b, b_1) \in Z(\mathcal{A}) \times \mathcal{A}$, again denoted by $\cdot$ and $\{\cdot\,,\cdot\}$, take the following form:

$$\begin{aligned}(a, a_1) \cdot (b, b_1) &= (ab, ab_1 + a_1 b + (a,b)_1), \\ \{(a, a_1), (b, b_1)\} &= (\{a,b\}_1, \{a,b\}_2 + \{a_1,b\}_1 + \{a,b_1\}_1 + [a_1,b_1]).\end{aligned} \tag{2.10}$$

It is clear that, alternatively, the commutative Poisson algebra $\Pi(\mathcal{A})$ can be constructed as a quotient of the Poisson algebra $(Z(\mathcal{A}) \times \mathcal{A}, \{\cdot\,,\cdot\})$ by considering the Poisson ideal $\{0\} \times Z(\mathcal{A})$ of $Z(\mathcal{A}) \times \mathcal{A}$.

The Poisson algebra $Z(\mathcal{A}) \times \mathcal{A}$ is not commutative when $\mathcal{A}$ is not commutative. According to [9], the Poisson bracket of any prime Poisson algebra that is not commutative is a multiple of the commutator $a \cdot b - b \cdot a$, where $\cdot$ denotes the (associative) product of the Poisson algebra. This result does not apply to $Z(\mathcal{A}) \times \mathcal{A}$ because it is not prime. In fact, in this case the Poisson bracket is not a multiple of the commutator.

2.3 Poisson modules from deformations

We first recall the definition of a Poisson module over a (not necessarily commutative) Poisson algebra.

Definition 2.15 Let $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ be a Poisson algebra over R and let $\mathcal{M}$ be an R-module. Then $\mathcal{M}$ is said to be a $\mathcal{A}$-*Poisson module* (or Poisson module *over* $\mathcal{A}$ or *over* $(\mathcal{A}, \{\cdot\,,\cdot\})$) when $\mathcal{M}$ is both a $(\mathcal{A}, \cdot)$-bimodule and a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module, satisfying the following derivation properties: for all $a, b \in \mathcal{A}$ and $m \in \mathcal{M}$,

$$\{a\,;b \cdot m\} = \{a, b\} \cdot m + b \cdot \{a\,;m\}\ , \tag{2.11}$$

$$\{a\,;m \cdot b\} = m \cdot \{a, b\} + \{a\,;m\} \cdot b\ , \tag{2.12}$$

$$\{a \cdot b\,;m\} = a \cdot \{b\,;m\} + \{a\,;m\} \cdot b\ . \tag{2.13}$$

In the above formulas, the three actions of $\mathcal{A}$ on $\mathcal{M}$ have been written $a \cdot m$, $m \cdot a$ and $\{a\,;m\}$ for $a \in \mathcal{A}$ and $m \in \mathcal{M}$. In this notation, the fact that $\mathcal{M}$ is a $\mathcal{A}$-bimodule (respectively, a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module), takes the form

$$a \cdot (b \cdot m) = (a \cdot b) \cdot m\ , \qquad (m \cdot a) \cdot b = m \cdot (a \cdot b)\ , \qquad a \cdot (m \cdot b) = (a \cdot m) \cdot b\ , \tag{2.14}$$

$$\{\{a, b\}\,;m\} = \{a\,;\{b\,;m\}\} - \{b\,;\{a\,;m\}\}\ , \tag{2.15}$$

for $a, b \in \mathcal{A}$ and $m \in \mathcal{M}$. When $\mathcal{A}$ is commutative and the left and right actions of $\mathcal{A}$ on $\mathcal{M}$ coincide, (2.11) and (2.12) are equivalent, just like the first two conditions in (2.14). The properties (2.14) and (2.15) are similar to the associativity and Jacobi identity in $\mathcal{A}$, while the properties (2.11)–(2.13) are similar to the Leibniz identity in $\mathcal{A}$. In the example that follows, they are exactly these properties.

Example 2.16 It is well-known that every associative algebra is a bimodule over itself and that every Lie algebra is a Lie module over itself, in both cases in a natural way. When $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a Poisson algebra this leads to a natural $\mathcal{A}$-bimodule structure on $\mathcal{A}$, given by left and right multiplication, as well as a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module structure, given by taking the Poisson bracket. Then $\{\cdot\,;\cdot\} = \{\cdot\,,\cdot\}$ and each one of the properties (2.11)–(2.13) is equivalent to the Leibniz identity in $(\mathcal{A}, \{\cdot\,,\cdot\})$. It follows that every Poisson algebra is in a natural way a Poisson module over itself.

Proposition 2.17 *Let* $(\mathcal{A}[[\nu]], \star)$ *be a deformation of an associative algebra* $\mathcal{A}$. *Consider the Poisson algebra* $(Z(\mathcal{A}) \times \mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ *(Proposition 2.14).*

(1) $\mathcal{A}$ *is a Poisson module over* $(Z(\mathcal{A}) \times \mathcal{A}, \cdot, \{\cdot\,,\cdot\})$.
(2) $\mathcal{A}$ *is a Poisson module over* $(\Pi(\mathcal{A}), \cdot, \{\cdot\,,\cdot\})$, *with actions given for* $a \in Z(\mathcal{A})$ *and* $a_1, b \in \mathcal{A}$ *by*

$$(a, \overline{a_1}) \cdot b = ab\ , \quad b \cdot (a, \overline{a_1}) = ba = ab\ , \quad \{(a, \overline{a_1})\,; b\} = \{a, b\}_1 + [a_1, b]\ . \tag{2.16}$$

Proof It is clear from Example 2.16 that the Poisson algebra $(Z(\mathcal{A}) \times \mathcal{A}, \{\cdot\,,\cdot\})$ is a Poisson module over itself. The formulas for the product and bracket are given by (2.10). Let $(a, a_1), (0, b) \in Z(\mathcal{A}) \times \mathcal{A}$. Then (2.10) specializes to

$$\begin{aligned} (a, a_1) \cdot (0, b) = (0, ab)\ , \quad (0, b) \cdot (a, a_1) = (0, ba)\ , \\ \{(a, a_1), (0, b)\} = (0, \{a, b\}_1 + [a_1, b])\ , \end{aligned} \tag{2.17}$$

so that $\cdot$ and $\{\cdot\,,\cdot\}$ can be restricted to $\{0\} \times \mathcal{A}$, making the latter into a Poisson module over $(Z(\mathcal{A}) \times \mathcal{A}, \{\cdot\,,\cdot\})$. Under the identification $\{0\} \times \mathcal{A} \simeq \mathcal{A}$ we get (1). We use these formulas to prove (2), where we need to show that the restriction to the Poisson ideal $\{0\} \times Z(\mathcal{A})$ of both actions of $Z(\mathcal{A}) \times \mathcal{A}$ on $\mathcal{A}$ is trivial. Let $a \in Z(\mathcal{A})$ and $b \in \mathcal{A}$. Then (2.17) becomes

$$(0, a) \cdot (0, b) = (0, 0) = (0, b) \cdot (0, a)\ , \text{ and } \{(0, a), (0, b)\} = (0, [a, b]) = (0, 0)\ , \tag{2.18}$$

where we have used that $a \in Z(\mathcal{A})$. In terms of the notation that we have introduced for the elements of $\Pi(\mathcal{A})$, upon identifying $\{0\} \times \mathcal{A}$ with $\mathcal{A}$ and writing $\{\cdot\,;\cdot\}$ for the induced Lie action, we get (2.16) from (2.17). □

We stress that the Poisson module $\mathcal{A}$ that we have constructed is a Poisson module over the *commutative* Poisson algebra $\Pi(\mathcal{A})$. Notice also that the left and right actions of $Z(\mathcal{A}) \times \mathcal{A}$, and hence of $\Pi(\mathcal{A})$, on $\mathcal{A}$ are the same (see (2.16) and (2.18)).

Example 2.18 As we have already pointed out, when $\mathcal{A}$ is commutative, $\Pi(\mathcal{A}) \simeq \mathcal{A}$ as a Poisson algebra, where both algebras have the rescaled commutator $[\cdot\,,\cdot]_\nu$ as Poisson bracket. Under this identification, the $\mathcal{A}$-Poisson module structure of $\mathcal{A}$ constructed in Proposition 2.17 is precisely the canonical Poisson module structure of $\mathcal{A}$ as a Poisson module over itself (cfr. Example 2.16).

2.4 Hamiltonian derivations from deformations

We now show how any element $\mathbf{H}$ of $\Pi(\mathcal{A})$ leads to a *derivation* $\partial_{\mathbf{H}}$ on $\mathcal{A}$, i.e., a linear map $\mathcal{A} \to \mathcal{A}$ satisfying the Leibniz identity. We call them *Hamiltonian derivations* in analogy with the terminology used in the classical case, i.e., when $\mathcal{A}$ is commutative, so that $\Pi(\mathcal{A}) \simeq \mathcal{A}$. For $a \in \mathcal{A}$, we define $\partial_{\mathbf{H}} a := \{\mathbf{H}\,; a\}$. We show in the following proposition that $\partial_{\mathbf{H}}$ is a derivation of $\mathcal{A}$.

Proposition 2.19 *The map* $\partial_{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$ *is a derivation of* $\mathcal{A}$.

Proof Writing $\mathbf{H} = (H_0, \overline{H_1})$, we find from (2.16) that

$$\{\mathbf{H}\,; a\} = \{H_0, a\}_1 + [H_1, a] \ . \tag{2.19}$$

It is clear that $\partial_{\mathbf{H}}$ is a linear map, so we only need to establish that $\{\mathbf{H}\,; \cdot\}$ satisfies the Leibniz identity. Now $[H_1, \cdot]$ satisfies the Leibniz identity, so according to (2.19) we only need to check that for any $a, b \in \mathcal{A}$, $\{H_0, ab\}_1 = a\{H_0, b\}_1 + \{H_0, a\}_1 b$. This follows by comparing the following two expressions for $[H_0, ab]_\star$:

$$\begin{aligned}
[H_0, ab]_\star &= [H_0, ab] + \nu\{H_0, ab\}_1 + \mathcal{O}\left(\nu^2\right) \\
&= a[H_0, b] + [H_0, a]b + \nu\{H_0, ab\}_1 + \mathcal{O}\left(\nu^2\right) \ , \\
[H_0, ab]_\star &= a[H_0, b]_\star + [H_0, a]_\star b = a[H_0, b] + a\nu\{H_0, b\}_1 \\
&\quad + [H_0, a]b + \nu\{H_0, a\}_1 b + \mathcal{O}\left(\nu^2\right) \ .
\end{aligned}$$

□

The Hamiltonian derivation $\partial_{\mathbf{H}}$ on $\mathcal{A}$ should not be confused with the Hamiltonian derivation $\partial'_{\mathbf{H}}$ on $\Pi(\mathcal{A})$, which is defined for $\mathbf{F} \in \Pi(\mathcal{A})$ by $\partial'_{\mathbf{H}}\mathbf{F} := \{\mathbf{H}, \mathbf{F}\}$, where we recall that the latter Poisson bracket is explicitly given by (2.8). When $\mathcal{A}$ is commutative, $\partial_{\mathbf{H}}$ and $\partial'_{\mathbf{H}}$ are both derivations of $\mathcal{A}$ (under the obvious identifications) and they coincide. When $\mathcal{A}$ is not commutative, these derivations are defined on different algebras and though they are related, none of the two determines the other one.

The following proposition generalizes a well-known property from the commutative case:

Proposition 2.20 *Suppose that* $\mathbf{F} = (F_0, \overline{F_1}), \mathbf{G} = (G_0, \overline{G_1}) \in \Pi(\mathcal{A})$. *Then* $[\partial_{\mathbf{F}}, \partial_{\mathbf{G}}] = \partial_{\{\mathbf{F},\mathbf{G}\}}$. *In particular, if* $\mathbf{F}$ *and* $\mathbf{G}$ *are in involution,* $\{\mathbf{F}, \mathbf{G}\} = 0$, *their associated derivations* $\partial_{\mathbf{F}}$ *and* $\partial_{\mathbf{G}}$ *of* $\mathcal{A}$ *commute and both* F_0 *and* G_0 *are first integrals of them.*

Proof For the first statement, we need to prove that for any $a \in \mathcal{A}$,

$$\partial_{\mathbf{F}}(\partial_{\mathbf{G}} a) - \partial_{\mathbf{G}}(\partial_{\mathbf{F}} a) = \partial_{\{\mathbf{F},\mathbf{G}\}} a \ ,$$

i.e., that

$$\{\mathbf{F}\,; \{\mathbf{G}\,; a\}\} - \{\mathbf{G}\,; \{\mathbf{F}\,; a\}\} = \{\{\mathbf{F}, \mathbf{G}\}\,; a\} \ .$$

This is precisely the property that $\mathcal{A}$ is a Lie module over $(\Pi(\mathcal{A}), \{\cdot\,, \cdot\})$ (see (2.15)). For the last statement, it remains to be shown that G_0 is a first integral of $\partial_{\mathbf{F}}$, i.e., that $\partial_{\mathbf{F}} G_0 = 0$. According to (2.19), $\partial_{\mathbf{F}} G_0 = \{\mathbf{F}\,; G_0\} = \{F_0, G_0\}_1 + [F_1, G_0]$ and both terms in the sum are zero, the first one because $\mathbf{F}$ and $\mathbf{G}$ are in involution (see (2.8)) and the second one because $G_0 \in Z(\mathcal{A})$. □

The proposition also shows that the commutator of two Hamiltonian derivations of $\mathcal{A}$ is a Hamiltonian derivation of $\mathcal{A}$ and the Poisson algebra $\Pi(\mathcal{A})$ provides a tool to compute a corresponding Hamiltonian. Notice that, contrarily to the classical case where $\Pi(\mathcal{A}) \simeq \mathcal{A}$, in the general case Hamiltonians are pairs of $Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}$, while first integrals are elements of $\mathcal{A}$, so they live on different spaces.

The derivation $\partial_{\mathbf{H}}$ of $\mathcal{A}$, given by (2.19), is naturally obtained as the limit of a Heisenberg derivation of $\mathcal{A}[[\nu]]$, an observation which has been our original motivation for the construction of the Poisson algebra $\Pi(\mathcal{A})$ and the Poisson module $\mathcal{A}$ over it. Indeed, let $H \in \mathcal{H}_\nu$ be any element with $H = H_0 + \nu H_1 + \nu^2 H_2 + \cdots$, and denote as before $\mathbf{H} = (H_0, \overline{H_1})$. The *Heisenberg derivation* associated with H, which we denote by δ_H, is by definition the derivation of $\mathcal{A}[[\nu]]$, given for $a \in \mathcal{A} \subset \mathcal{A}[[\nu]]$ by

$$\delta_H a = \frac{1}{\nu}\,[H, a]_\star \ .$$

We get, using (2.2) and (2.19),

$$\delta_H a = \frac{1}{\nu}\,[H_0 + \nu H_1 + \cdots, a]_\star = \{H_0, a\}_1 + [H_1, a] + \mathcal{O}(\nu) = \partial_{\mathbf{H}} a + \mathcal{O}(\nu)\ ,$$

so that in the limit $\nu \to 0$ we get indeed the Hamiltonian derivation $\partial_{\mathbf{H}}$ of $\mathcal{A}$.

Example 2.21 When $\mathcal{A}$ is commutative one recovers the well-known fact that the limit of the Heisenberg derivation associated with H_0, where $\mathbf{H} = \left(H_0, \overline{H_1}\right) = (H_0, \overline{0})$, is the Hamiltonian derivation associated with H_0 by means of the Poisson structure, associated with the deformation since $\partial_{\mathbf{H}} a = \{H_0, a\}_1 + [H_1, a] = \{H_0, a\}$, since in the commutative case the commutator vanishes and $\{\cdot\,, \cdot\}_1$ is the Poisson bracket (see Remark 2.12).

2.5 The reduced Poisson algebra

Let $(\mathcal{A}, \star)$ be a deformation of an associative algebra $\mathcal{A}$ and recall that $\Pi(\mathcal{A}) = Z(\mathcal{A}) \times (\mathcal{A}/Z(\mathcal{A}))$ is the associated Poisson algebra. Explicit formulas for the product and bracket on $\Pi(\mathcal{A})$ are given in (2.7) and (2.8). It is clear from these formulas that $Z(\mathcal{A}) \times \left\{\overline{0}\right\}$ is in general neither a subalgebra nor a Lie subalgebra of $\Pi(\mathcal{A})$. Yet, we show that, by reduction, $Z(\mathcal{A})$ is also a (commutative) Poisson algebra. In fact, $\{0\} \times (\mathcal{A}/Z(\mathcal{A}))$ is both an ideal and a Lie ideal of $\Pi(\mathcal{A})$, since according to (2.7) and (2.8),

$$(0, \overline{a_1}) \cdot \left(b, \overline{b_1}\right) = \left(0, \overline{a_1 b}\right)\ , \quad \left\{(0, \overline{a_1}), \left(b, \overline{b_1}\right)\right\} = \left(0, \overline{\{a_1, b\}_1 + [a_1, b_1]}\right)\ .$$

It follows that $\{0\} \times (\mathcal{A}/Z(\mathcal{A}))$ is a Poisson ideal of $\Pi(\mathcal{A})$, so that $\Pi(\mathcal{A})/(\{0\} \times (\mathcal{A}/Z(\mathcal{A}))) \simeq Z(\mathcal{A})$ is a Poisson algebra, where the latter isomorphism is according to (2.7) not just an isomorphism of modules but of (associative) algebras. The induced Poisson structure on $Z(\mathcal{A})$ is given for $a, b \in Z(\mathcal{A})$ by $\{a, b\}_1$ since according to

(2.8),

$$\left\{\left(a,\overline{0}\right),\left(b,\overline{0}\right)\right\}=\left(\{a,b\}_1,\overline{\{a,b\}_2}\right)\ .$$

We call $(Z(\mathcal{A}),\{\cdot\,,\cdot\}_1)$ the *reduced Poisson algebra (associated to the deformation)*. Notice that it follows that $\{\cdot\,,\cdot\}_1$ satisfies the Jacobi identity when restricted to $Z(\mathcal{A})$, though in general $\{\cdot\,,\cdot\}_1$ does not satisfy the Jacobi identity on $\mathcal{A}$, since in general $\overline{\{a,b\}_2}$ is nonzero, for $a,b\in Z(\mathcal{A})$; see Sect. 4.6 for a counterexample.

Example 2.22 According to Example 2.13, the reduced Poisson structure associated with a trivial deformation is trivial.

2.6 Functoriality

In deformation theory, the natural notion of isomorphism is the one of equivalence. We will show in this subsection that equivalent deformations of an algebra lead to isomorphic Poisson algebras. We first recall the definition of equivalence of deformations (see, for example, [11, Ch. 13]).

Definition 2.23 Two deformations $(\mathcal{A}[[\nu]],\star)$ and $(\mathcal{A}[[\nu]],\star')$ of an associative algebra $\mathcal{A}$ are said to be *equivalent* if there exists a morphism of $R[[\nu]]$-algebras $F:(\mathcal{A}[[\nu]],\star)\to(\mathcal{A}[[\nu]],\star')$ such that $F(a)=a+\mathcal{O}(\nu)$ for all $a\in\mathcal{A}$. Then F is called an *equivalence (of deformations)*.

Notice that F is automatically an isomorphism and that F can be viewed as a deformation of the identity map on $\mathcal{A}[[\nu]]$. More precisely, expanding $F(a)$ for all $a\in\mathcal{A}$ as a formal power series in ν,

$$F(a)=a+\nu F_1(a)+\nu^2F_2(a)+\cdots\ ,$$

we get R-linear maps $F_i:\mathcal{A}\to\mathcal{A}$ and we can write

$$F=\mathrm{Id}_{\mathcal{A}[[\nu]]}+\nu F_1+\nu^2F_2+\cdots \tag{2.20}$$

where, by a slight abuse of notation, F_i stands for the $R[[\nu]]$-linear extension of F_i to a morphism $F_i:\mathcal{A}[[\nu]]\to\mathcal{A}[[\nu]]$. Formula (2.20) is convenient for computations. One computes, for example, easily from it that $F^{-1}=\mathrm{Id}_{\mathcal{A}[[\nu]]}-\nu F_1+\nu^2(F_1^2-F_2)+\cdots$.

The following proposition shows that, under some condition which is automatically satisfied for equivalences, an algebra homomorphism between deformations of two algebras induces a Poisson morphism between the two associated Poisson algebras.

Proposition 2.24 *Let $\mathcal{A}$ and $\mathcal{A}'$ be two associative algebras with deformations $(\mathcal{A}[[\nu]],\star)$ and $(\mathcal{A}'[[\nu]],\star')$, and let $F=F_0+\nu F_1+\nu^2F_2+\cdots:\mathcal{A}[[\nu]]\to\mathcal{A}'[[\nu]]$ be a morphism of $R[[\nu]]$-algebras. If $F_0(Z(\mathcal{A}))\subset Z(\mathcal{A}')$, then F restricts to a morphism of Poisson algebras $F:(\mathcal{H}_\nu,[\cdot\,,\cdot]_\nu)\to(\mathcal{H}'_\nu,[\cdot\,,\cdot]'_\nu)$ and induces a map $F_\Pi:\Pi(\mathcal{A})\to\Pi(\mathcal{A}')$. Moreover, F_Π and the map induced by it between the reduced Poisson algebras $(Z(\mathcal{A}),\{\cdot\,,\cdot\}_1)$ and $(Z(\mathcal{A}'),\{\cdot\,,\cdot\}'_1)$ are morphisms of Poisson algebras.*

Proof Suppose that $F_0(Z(\mathcal{A})) \subset Z(\mathcal{A}')$. Then

$$F(\mathcal{H}_\nu) = (F_0 + \nu F_1 + \cdots)(Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]]) \subset Z(\mathcal{A}') + \nu\mathcal{A}'[[\nu]] = \mathcal{H}'_\nu \; .$$

It follows that F can be restricted to a morphism $F : \mathcal{H}_\nu \to \mathcal{H}'_\nu$. Notice that since F is a morphism of $R[[\nu]]$-algebras it also preserves the commutator, $F([A, B]_\star) = [F(A), F(B)]'_\star$, and hence also the rescaled bracket, $F([A, B]_\nu) = [F(A), F(B)]'_\nu$, so that $F : (\mathcal{H}_\nu, [\cdot\, , \cdot]_\nu) \to (\mathcal{H}'_\nu, [\cdot\, , \cdot]'_\nu)$ is a morphism of Poisson algebras. Consider the following diagram of Poisson algebras:

$$\begin{array}{ccc}
(\mathcal{H}_\nu, [\cdot\, , \cdot]_\nu) & \xrightarrow{\;\;F\;\;} & (\mathcal{H}'_\nu, [\cdot\, , \cdot]'_\nu) \\
{\scriptstyle p_\Pi}\big\downarrow & & \big\downarrow{\scriptstyle p_{\Pi'}} \\
(\Pi(\mathcal{A}), \{\cdot\, , \cdot\}) & \xrightarrow[\;\;F_\Pi\;\;]{} & (\Pi(\mathcal{A}'), \{\cdot\, , \cdot\}')
\end{array}$$

As we just showed, F is a morphism of Poisson algebras. According to Proposition 2.10, p_Π and $p_{\Pi'}$ are also morphisms of Poisson algebras. Let us now consider F_Π; by surjectivity of p_Π, if a map F_Π making the diagram commutative exists, it is unique. In fact, one establishes quite easily a formula for F_Π: for $(a, \overline{a_1}) \in \Pi(\mathcal{A})$,

$$\begin{aligned}
F_\Pi(a, \overline{a_1}) &= F_\Pi p_\Pi(a + \nu a_1 + \cdots) = p_{\Pi'} F(a + \nu a_1 + \cdots) \\
&= p_{\Pi'}(F_0(a) + \nu(F_0(a_1) + F_1(a))) = \Big(F_0(a), \overline{F_0(a_1) + F_1(a)}\Big) \; .
\end{aligned} \tag{2.21}$$

This gives a formula for F_Π and the above computation shows that it makes the diagram commutative. The surjectivity of p_Π and the commutativity of the diagram imply that F_Π is a morphism of Poisson algebras. For example, to check that F_Π is a morphism of Lie algebras, it suffices to check that $F_\Pi\, \{p_\Pi A, p_\Pi B\} = \{F_\Pi p_\Pi A, F_\Pi p_\Pi B\}'$ for all $A, B \in \mathcal{A}[[\nu]]$, which follows easily from the cited properties:

$$\begin{aligned}
F_\Pi\, \{p_\Pi A, p_\Pi B\} &= F_\Pi p_\Pi\, [A, B]_\nu = p_{\Pi'} F\, [A, B]_\nu = p_{\Pi'}\, [F(A), F(B)]'_\nu \\
&= \{p_{\Pi'} F(A), p_{\Pi'} F(B)\}' = \{F_\Pi p_\Pi A, F_\Pi p_\Pi B\}' \; .
\end{aligned}$$

To finish to proof, we consider the corresponding reduced Poisson algebras (see Sect. 2.5). Consider the following diagram of Poisson algebras:

$$\begin{array}{ccc}
(\Pi(\mathcal{A}), \{\cdot\, , \cdot\}) & \xrightarrow{\;\;F_\Pi\;\;} & (\Pi(\mathcal{A}'), \{\cdot\, , \cdot\}') \\
\big\downarrow & & \big\downarrow \\
(Z(\mathcal{A}), \{\cdot\, , \cdot\}_1) & \xrightarrow[\;\;F_0\;\;]{} & (Z(\mathcal{A}'), \{\cdot\, , \cdot\}'_1)
\end{array}$$

The vertical arrows in this diagram are the reduction maps. The lower arrow has been labeled F_0, as it is just the restriction of F_0 to the center of $\mathcal{A}$; it is obvious from

(2.21) that F_0 makes the diagram commutative. Since F_Π and the reduction maps are Poisson maps, the latter moreover being surjective, we may conclude as above that F_0 is a Poisson map. □

When F_0 is surjective, the condition $F_0(Z(\mathcal{A})) \subset Z(\mathcal{A}')$ is automatically satisfied. In particular, the proposition can be applied to equivalences of deformations and we get the following result:

Corollary 2.25 *Let* $(\mathcal{A}[[\nu]], \star)$ *and* $(\mathcal{A}[[\nu]], \star')$ *be equivalent deformations of an associative algebra* $\mathcal{A}$*. Then the Poisson algebras associated to these deformations are isomorphic,* $(\Pi(\mathcal{A}), \{\cdot\,,\cdot\}) \simeq (\Pi(\mathcal{A}), \{\cdot\,,\cdot\}')$*; the reduced Poisson algebras are also isomorphic,* $(Z(\mathcal{A}), \{\cdot\,,\cdot\}_1) \simeq (Z(\mathcal{A}), \{\cdot\,,\cdot\}'_1)$. □

The converse is not true in general, as the Poisson algebras and reduced Poisson algebras only "see" the first two terms of the deformation; replacing in a nontrivial deformation ν by ν^k leads to a nontrivial deformation for which the first $k-1$ deformation terms are zero, hence leading to a trivial Poisson algebra $\Pi(\mathcal{A})$ when $k > 2$.

We show that under the hypothesis of Corollary 2.25, the Poisson module structure on $\mathcal{A}$ which is induced by the two deformations is the same, i.e., that for any $(a, \overline{a_1}) \in \Pi(\mathcal{A})$ and b in $\mathcal{A}$, one has $F_\Pi(a, \overline{a_1}) \cdot' b = (a, \overline{a_1}) \cdot b$ and $\{F_\Pi(a, \overline{a_1})\,; b\}' = \{(a, \overline{a_1})\,; b\}$, where $\cdot'$ denotes the action of $\Pi(\mathcal{A})$ on $\mathcal{A}$ coming from the deformation $(\mathcal{A}[[\nu]], \star')$. Now $F_\Pi\,(a, \overline{a_1}) = \left(a, \overline{F_1(a) + a_1}\right)$, as follows from (2.21), so that according to (2.16) we need to show that

$$\left(a, \overline{a_1 + F_1(a)}\right) \cdot' b = (a, \overline{a_1}) \cdot b\,, \quad \text{and} \quad \left\{\left(a, \overline{a_1 + F_1(a)}\right)\,; b\right\}' = \{(a, \overline{a_1})\,; b\}\,, \tag{2.22}$$

for any $a \in Z(\mathcal{A})$ and $a_1, b \in \mathcal{A}$. The first equality in (2.22) holds because both sides evaluate to ab (see (2.16)). Proving the second equality amounts in view of the formulas (2.16) to showing that $\{a, b\}_1 = \{a, b\}'_1 + [F_1(a), b]$ for all $a \in Z(\mathcal{A})$ and $b \in \mathcal{A}$. To prove the latter, we expand both sides of $[F(a), F(b)]'_\star = F\,[a, b]_\star$ using (2.21). First,

$$\begin{aligned}
[F(a), F(b)]'_\star &= [a + \nu F_1(a), b + \nu F_1(b)]'_\star + \mathcal{O}\left(\nu^2\right) \\
&= [a, b] + \nu(\{a, b\}'_1 + [a, F_1(b)] + [F_1(a), b]) + \mathcal{O}\left(\nu^2\right) \\
&= \nu(\{a, b\}'_1 + [F_1(a), b]) + \mathcal{O}\left(\nu^2\right)\,,
\end{aligned}$$

where we have used twice that $a \in Z(\mathcal{A})$. Similarly,

$$\begin{aligned}
F\,[a, b]_\star = [a, b]_\star + \nu F_1\,[a, b]_\star + \mathcal{O}\left(\nu^2\right) = [a, b] + \nu(\{a, b\}_1 + F_1\,[a, b]) \\
+ \mathcal{O}\left(\nu^2\right) = \nu\,\{a, b\}_1 + \mathcal{O}\left(\nu^2\right)\,.
\end{aligned}$$

Comparing these two results yields the required equality, proving the second equality in (2.22).

3 Free quotients of free algebras

The associative algebra underlying a quantum system usually depends on a parameter, the Planck constant, which is small and the algebra becomes commutative when the parameter is set to 0. These algebras can (under some assumptions) naturally be viewed as deformations (in the sense of Definition 2.4) of that commutative algebra and one speaks of *deformation quantization*. We are interested here in the more general case in which the limiting algebra is not necessarily commutative. Moreover, in view of the examples which we will treat, we may be interested in other values of the parameter, such as roots of unity, and we may want to deal with several parameters. We will describe in this section a natural and quite general setup in which we can view these more general algebras as deformations, so that we can apply the techniques and results of the previous sections to them, as we will do in Sects. 4 and 5. We denote by $\mathbb{K}$ any commutative field of characteristic 0, which in the examples in the next sections will be taken equal to $\mathbb{C}$.

3.1 Quantization ideals and quantum algebras

We first recall the notion of quantization ideal and of quantum algebra which were first introduced in [14]. Let $x_1, x_2, \ldots$ be a (possible infinite, but at most countable) collection of independent variables. We denote $\mathfrak{A} = \mathbb{K}\langle x_1,\ x_2,\ \ldots\rangle$ the free associative (unital) $\mathbb{K}$-algebra on these variables. Elements of $\mathfrak{A}$ are finite $\mathbb{K}$-linear combinations of words in $x_1, x_2, \ldots,$ and the product of two words is their concatenation. Assume that $\mathfrak{A}$ is equipped with a derivation $\partial : \mathfrak{A} \to \mathfrak{A}$.

Definition 3.1 A two-sided ideal $\mathcal{I}$ of $\mathfrak{A}$ is said to be a *quantization ideal* for $(\mathfrak{A}, \partial)$ if it satisfies the following two properties:

(1) The ideal $\mathcal{I}$ is ∂-stable: $\partial(\mathcal{I}) \subset \mathcal{I}$;
(2) The quotient $\mathfrak{A}/\mathcal{I}$ admits a basis $\mathcal{B}$ of normally ordered monomials[1] in $x_1,\ x_2,\ \ldots$.

The quotient algebra $\mathfrak{A}/\mathcal{I}$ is then said to be a *quantum algebra* (over $\mathbb{K}$).

The first condition implies that ∂ descends to a derivation of $\mathfrak{A}/\mathcal{I}$; we will come back to this in Sect. 5. In the absence of a derivation, we can still speak of a quantization ideal and of a quantum algebra by considering the trivial (zero) derivation of $\mathfrak{A}$; then (1) is automatically satisfied.

The generators of the quantization ideals that define quantum algebras often depend on one or several parameters $\mathbf{q} = (q_1, q_2, \ldots)$, which can be thought of as being

[1] Our convention is that a monomial in the generators $x_1,\ x_2,\ x_3, \ldots$ is normally ordered when it is of the form $x_{m_1}^{n_1} x_{m_2}^{n_2} \ldots x_{m_s}^{n_s}$ where $m_1, m_2, \cdots, m_s$ is strictly increasing, all n_i are (usually positive) integers and $s \geqslant 0$. Notice that it is not required that all such elements are in the basis $\mathcal{B}$, i.e., that $\mathcal{B}$ is a PBW basis.

elements of the field $\mathbb{K}$. Moreover, in all relevant examples this dependency is rational. It is then natural to think of the family of quantum $\mathbb{K}$-algebras $\mathfrak{A}/\mathcal{I}$, depending on $\mathbf{q}$, as being the quantum $\mathbb{K}(\mathbf{q})$-algebra $\mathcal{A}_{\mathbf{q}} := \mathfrak{A}(\mathbf{q})/\mathcal{I}_{\mathbf{q}}$, where $\mathfrak{A}(\mathbf{q})$ is a shorthand for $\mathbb{K}(\mathbf{q})\langle x_1,\ x_2,\ \dots\rangle$ and $\mathcal{I}_{\mathbf{q}}$ stands for the ideal of $\mathfrak{A}(\mathbf{q})$, with the elements of $\mathcal{I}$ being considered as elements of $\mathfrak{A}(\mathbf{q})$. The basis $\mathcal{B}$ of normally ordered monomials is then a basis for $\mathcal{A}_{\mathbf{q}}$ as a $\mathbb{K}(\mathbf{q})$-module and will be simply referred to as a *monomial basis* for $\mathcal{A}_{\mathbf{q}}$. The product and commutator of elements A, B of $\mathcal{A}_{\mathbf{q}}$ will be denoted AB and $[A, B]$, respectively.

Example 3.2 Let $\mathfrak{A} = \mathbb{K}\langle x_1,\ x_2,\ \dots, x_N\rangle$ be a free associative algebra as above and consider the ideal $\mathcal{I}_{\mathbf{q}}$ of $\mathfrak{A}(\mathbf{q})$, generated by

$$\left\{x_i x_j - q_{i,j} x_j x_i \mid 1 \leqslant j < i \leqslant N\right\}\ ,$$

where $\mathbf{q} = (q_{i,j})_{1\leqslant j<i\leqslant N}$ are the parameters. Then it is clear that a monomial basis $\mathcal{B}$ for $\mathcal{A}_{\mathbf{q}} = \mathfrak{A}(\mathbf{q})/\mathcal{I}_{\mathbf{q}}$ is given by

$$\mathcal{B} = \left\{x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N} \mid i_1,\ i_2,\ \dots, i_N \in \mathbb{N}\right\}\ , \tag{3.1}$$

where we have used the same notation x_i for the generators of $\mathfrak{A}$ as for their images in $\mathcal{A}_{\mathbf{q}}$. Notice that in $\mathcal{A}_{\mathbf{q}}$ we have $x_i x_j = q_{i,j} x_j x_i$ for $i > j$, which can be written equivalently as $\left[x_i, x_j\right] = (q_{i,j} - 1)x_j x_i$, in particular the commutator in $\mathcal{A}_{\mathbf{q}}$ of any two generators x_i is a multiple of their product; more generally, the commutator in $\mathcal{A}_{\mathbf{q}}$ of two monomials in $x_1, \dots, x_N$ is a multiple of their product. A basis of the form (3.1) is called a *PBW basis*.

Example 3.3 With $\mathfrak{A}$ as in the previous example, consider now the ideal $\mathcal{I}_{\mathbf{q}}$ of $\mathfrak{A}(\mathbf{q})$, generated by

$$\left\{x_i x_j - x_j x_i - q_{i,j} \mid 1 \leqslant j < i \leqslant N\right\}\ ,$$

where $\mathbf{q} = (q_{i,j})_{1\leqslant j<i\leqslant N}$ are again the parameters. Then $\mathcal{B}$, as given by (3.1), is again a PBW basis for $\mathcal{A}_{\mathbf{q}}$. However, in this case the commutator in $\mathcal{A}_{\mathbf{q}}$ of two monomials in $x_1, \dots, x_N$ is in general not a multiple of their product. Both examples can of course be combined to produce more general quantization ideals and quantum algebras with a PBW basis. See [12] for a very large class of ideals whose associated quotient algebra has a basis of normally ordered monomials, making it into a quantum algebra.

3.2 Free quotients and deformations

We now show how a quantum algebra which depends on one or several parameters can be turned into a (formal) deformation of an algebra (Definition 2.4). This will allow us to apply the results and methods of Sect. 2.

For clarity, and in view of the examples which we will treat, let us assume that there is only one parameter, $\mathbf{q} = q$; see Remark 3.5 below for the case of several parameters. We would like to specialize q to a value q_0, in such a way that $\mathcal{B}$ is still a basis after specialization. Notice that since $\mathcal{B}$ is a basis for $\mathcal{A}_q = \mathfrak{A}(q)/\mathcal{I}_q$, we can write for any $j > i$ the product $x_j x_i$ as a finite linear combination of elements of $\mathcal{B}$, with as coefficients rational fractions in q. When none of these fractions has a pole at q_0, we will say that q_0 is a *regular* value of $\mathcal{B}$.

When q_0 is a regular value of $\mathcal{B}$, the expression for $x_j x_i$ in terms of $\mathcal{B}$ can be evaluated at q_0. This shows that $\mathcal{B}$ remains a (monomial) basis after specialization. In fact, these expressions for $x_j x_i$ with $j > i$ can be taken as generators of $\mathcal{I}_q$; they will be used as such in what follows. Suppose that q_0 is a regular value of $\mathcal{B}$ and set $q(\nu) = q_0 + c_1\nu + c_2\nu^2 + \cdots$, any polynomial or formal power series in ν with $c_1 \neq 0$; in practice we will take $q(\nu) = q_0 + \nu$, see Remark 3.6 below. Elements of $\mathbb{K}(\mathbf{q})$ which do not have a pole at q_0 are then formal power series in ν. We consider in $\mathfrak{A}[[\nu]] := \mathbb{K}[[\nu]]\langle x_1,\ x_2,\ \dots\rangle$ the closed ideal $\mathcal{I}_{q(\nu)}$ generated by the generators of $\mathcal{I}_q$ in which every occurrence of q has been replaced by $q(\nu)$, and denote $\mathcal{A}_\nu := \mathfrak{A}[[\nu]]/\mathcal{I}_{q(\nu)}$. We consider in particular the $\mathbb{K}$-algebra $\mathcal{A} := \mathcal{A}_{q_0} := \mathfrak{A}/\mathcal{I}_{q_0}$. Then $\mathcal{B}$ is a $\mathbb{K}$-basis of $\mathcal{A}$ and is a $\mathbb{K}[[\nu]]$-basis of $\mathcal{A}_\nu$. Notice that since q_0 is a regular value of $\mathcal{B}$, the elements of $\mathcal{I}_{q(\nu)}$ are formal power series in ν.

Proposition 3.4 *The algebra $\mathcal{A}_\nu$ is isomorphic, as a $\mathbb{K}[[\nu]]$-module, to $\mathcal{A}[[\nu]]$.*

Proof We use the basis $\mathcal{B}$ construct a surjective morphism $\mathfrak{A}[[\nu]] \to \mathcal{A}[[\nu]]$ with kernel $\mathcal{I}_{q(\nu)}$. Let $A = \sum_{i=0}^{\infty} a_i\nu^i$ be any element of $\mathfrak{A}[[\nu]]$, where $a_i \in \mathfrak{A}$ for all i. Since $\mathcal{B}$ is a basis of $\mathcal{A}_\nu$, each coefficient a_i can be written uniquely as $a_i = \sum_j \alpha_{i,j}\nu^j + r_i$, where the sum is finite, all $\alpha_{i,j}$ belong to $\operatorname{Span}_{\mathbb{K}}\mathcal{B}$ and $r_i \in \mathcal{I}_{q(\nu)}$. Summing up, we can write A uniquely as

$$A = \sum_{i=0}^{\infty} a_i\nu^i = \sum_{i,j}\alpha_{i,j}\nu^{i+j} + \sum_{i=0}^{\infty} r_i\nu^i = \sum_{k=0}^{\infty}\beta_k\nu^k + \sum_{i=0}^{\infty} r_i\nu^i\ , \tag{3.2}$$

where the first term belongs to $\mathcal{A}[[\nu]]$, with all β_k belonging to $\operatorname{Span}_{\mathbb{K}}\mathcal{B}$, while the second term belongs to $\mathcal{I}_{q(\nu)}$, since all r_i, which are formal power series in ν, belong to it (recall that the ideal $\mathcal{I}_{q(\nu)}$ is closed). The assignment $A \mapsto \sum_{k=0}^{\infty}\beta_k\nu^k$ defines a surjective $\mathbb{K}[[\nu]]$-linear map $\mathfrak{A}[[\nu]] \to \mathcal{A}[[\nu]]$. Since the decomposition (3.2) of A is unique, its kernel is $\mathcal{I}_{q(\nu)}$. It follows that $\mathfrak{A}[[\nu]]/\mathcal{I}_{q(\nu)}$ and $\mathcal{A}[[\nu]]$ are isomorphic, as was to be shown. □

In view of the proposition, using the monomial basis $\mathcal{B}$ of $\mathcal{A}_\nu$ we may identify $\mathcal{A}_\nu$ with $\mathcal{A}[[\nu]]$ as $\mathbb{K}[[\nu]]$-modules and the product in $\mathcal{A}_\nu$ yields a product $\star$ on $\mathcal{A}[[\nu]]$, making $(\mathcal{A}[[\nu]], \star)$ into a (formal) deformation of $\mathcal{A}$. Under this identification we may write any $A \in \mathcal{A}_\nu$ uniquely as

$$A = \mathbf{n}_0(A) + \nu\mathbf{n}_1(A) + \nu^2\mathbf{n}_2(A) + \cdots\ . \tag{3.3}$$

Here, every $\mathbf{n}_i(A)$ belongs to $\operatorname{Span}_{\mathbb{K}}\mathcal{B}$ and the maps $\mathbf{n}_i$ are linear maps $\mathcal{A}_\nu \to \mathcal{A}$.

Remark 3.5 The adaptation to the case of several parameters is straightforward: one puts $\mathbf{q}(\nu) = \mathbf{q}_0 + \nu\mathbf{q}_1 + \mathcal{O}\left(\nu^2\right)$, where $\mathbf{q}_0$ is a regular value for $(q_1, \dots, q_\ell)$ and $\mathbf{q}_1$ is any nonzero vector. *Regular* means as in the single parameter case that $\mathbf{q}_0$ is a regular value for all the rational coefficients that appear when writing $x_j x_i$ for $j > i$ in terms of the $\mathbb{K}(\mathbf{q})$-basis $\mathcal{B}$ of $\mathcal{A}_\mathbf{q}$. The algebra $\mathcal{A}_\nu$ is constructed as before and from there on everything proceeds as in the one-parameter case.

Remark 3.6 In the examples we will always set $q(\nu) = q_0 + \nu$. In fact, the higher-order terms in ν do not play a role in the construction of the Poisson structure; also, in the one-parameter case the value of the coefficient of ν is not very important since it amounts to rescaling ν, so we will always pick it equal to 1.

4 Examples and applications

In this section we show on a few different families of examples how to determine explicitly the Poisson algebra, the reduced Poisson algebra and the Poisson module, associated with a (formal) deformation; all our examples are obtained from quantum algebras, as explained in the previous section. As a first application, we use the (reduced) Poisson algebra to show that two particular deformations of some algebra which is not commutative are not equivalent. As a second application, we will show in the next section how the Poisson module is used for obtaining a Hamiltonian formulation of nonabelian systems.

4.1 Computing the Poisson brackets

We first explain a few general facts on the computation of the Poisson brackets $\{\cdot\,,\cdot\}$ and $\{\cdot\,,\cdot\}_1$ on $\Pi(\mathcal{A})$ and on $Z(\mathcal{A})$, respectively, and on the Lie action $\{\cdot\,;\cdot\}$ of $\Pi(\mathcal{A})$ on $\mathcal{A}$. Recall that these are associated with a deformation $(\mathcal{A}[[\nu]], \star)$ of some (not necessarily commutative) algebra $\mathcal{A}$ and that the deformation itself is associated with a quantum algebra $\mathcal{A}_\mathbf{q}$ and a regular value $\mathbf{q}_0$ of $\mathbf{q}$; in the examples that follow there is a single parameter, $\mathbf{q} = q$. Also, in these examples we always take as base ring R the field of complex numbers $\mathbb{C}$ so that q_0 is just a complex number.

In each example that we consider, the Poisson brackets $\{\cdot\,,\cdot\}$ and $\{\cdot\,,\cdot\}_1$ will be computed explicitly for generators of $\Pi(\mathcal{A})$ and of $Z(\mathcal{A})$, respectively; also, the Lie action $\{\cdot\,;\cdot\}$ will be computed for generators of $\Pi(\mathcal{A})$ and of $\mathcal{A}$. In view of the Leibniz identity, this yields the Poisson bracket for any pair of elements of $\Pi(\mathcal{A})$; see section 5.1 for the case of the Lie action. In the case of the Poisson brackets this can be done by using a formula that generalizes the classical formula for computing Poisson brackets on $\mathbb{C}^M$ in terms of the Poisson brackets between the coordinate functions on $\mathbb{C}^M$. If, say, $X_1, X_2, \ldots, X_M$ are generators of a commutative Poisson algebra and $P = P(X_1, X_2, \ldots, X_M)$ and $Q = Q(X_1, X_2, \ldots, X_M)$ are two elements expressed in terms of these generators, then

$$\{P, Q\} = \sum_{i,j=1}^{M} \frac{\partial P}{\partial X_i}\frac{\partial Q}{\partial X_j}\left\{X_i, X_j\right\} . \tag{4.1}$$

For a more formal statement and a proof, see [11, Prop. 1.9]; notice that the cited proposition says in particular that the right hand side of (4.1) is independent of the way in which P and Q are expressed in terms of the generators. The skew-symmetric matrix $\left(\{X_i, X_j\}\right)_{1\leqslant i,j\leqslant M}$, which in view of (4.1) completely determines the Poisson bracket, is called the *Poisson matrix of* $\{\cdot,\cdot\}$ *with respect to the generators* $X_1,\ldots,X_M$.

We will need to determine the center $Z(\mathcal{A})$ of $\mathcal{A}$ and find generators for it. In general this can be very complicated, but in most examples that follow we have for every $1\leqslant j<i\leqslant N$ a relation in $\mathcal{A}$ of the form $x_ix_j=q_{i,j}x_jx_i$, with $q_{i,j}\in\mathbb{C}^*$. As we already pointed out in Example 3.2, on the one hand this implies that we have a monomial basis of $\mathcal{A}$ whose elements are of the form $x_1^{i_1}x_2^{i_2}\ldots x_N^{i_N}$, and on the other hand that the commutator of two such monomials is a multiple (element of $\mathbb{C}$) of their product. This in turn implies that when an element of the center is written in the monomial basis, each one of its terms belongs to the center. The center of $\mathcal{A}$ is therefore generated by monomials in $x_1,\ldots,x_N$. The same argument shows that for such commutation relations the center of $\mathcal{A}_{\mathbf{q}}$ is generated by monomials (the multiple is then an element of $\mathbb{C}(\mathbf{q})$).

In order to write explicit formulas for the Poisson brackets and for the Lie action, we will need to find algebra generators for $\Pi(\mathcal{A})=Z(\mathcal{A})\times(\mathcal{A}/Z(\mathcal{A}))$, where we recall that the associative product in $\Pi(\mathcal{A})$ is denoted by $\cdot$ and is given by (2.7). Such generators can be chosen in the union of $Z(\mathcal{A})\times\left\{\overline{0}\right\}$ and $\{0\}\times\mathcal{A}/Z(\mathcal{A})$ since $\Pi(\mathcal{A})$ is the direct sum of these subspaces. For such generators, (2.7) simplifies to

$$(a,\overline{0})\cdot(b,\overline{0})=(ab,\overline{(a,b)_1})\ ,\tag{4.2}$$

$$(0,\overline{a_1})\cdot(0,\overline{b_1})=\left(0,\overline{0}\right)\ ,\tag{4.3}$$

$$(a,\overline{0})\cdot(0,\overline{b_1})=\left(0,\overline{ab_1}\right)\ ,\tag{4.4}$$

where we recall that $(a,b)_1$ is the coefficient in ν of $a\star b$ (see (2.1)). Notice that when $(a,b)_1\in Z(\mathcal{A})$, (4.2) simplifies further to

$$(a,\overline{0})\cdot(b,\overline{0})=\left(ab,\overline{0}\right)\ .\tag{4.5}$$

This happens, for example, when the commutator of a and b in $\mathcal{A}_\nu$ is a constant (element of $R[[\nu]]$), or is a multiple of their product ab; for the latter, see again Example 3.2. In general, a generating set of $\Pi(\mathcal{A})$ can be constructed using the following proposition:

Proposition 4.1 *Suppose that* $z_1,\ldots,z_k$ *are generators of* $Z(\mathcal{A})$ *as a (unital) algebra and that* $\overline{t_1},\ldots,\overline{t_\ell}$ *are generators for* $\mathcal{A}/Z(\mathcal{A})$ *as a* $Z(\mathcal{A})$*-module. Denote* $Z_i:=(z_i,\overline{0})$ *and* $T_j:=(0,\overline{t_j})$ *for* $i=1,\ldots,k$ *and* $j=1,\ldots,\ell$*. Then* $Z_1\ldots,Z_k,T_1,\ldots,T_\ell$ *are algebra generators of* $(\Pi(\mathcal{A}),\cdot)$.

Proof Let $(Z,\overline{A})\in\Pi(\mathcal{A})$. Since $z_1,\ldots,z_k$ are generators of $Z(\mathcal{A})$, there exists a polynomial P such that $P(z_1,\ldots,z_k)=Z$. In view of (4.2)–(4.4), $P(Z_1,\ldots,Z_k)=(Z,\overline{T})$ where $T\in\mathcal{A}$. Since $\mathcal{A}/Z(\mathcal{A})$ is generated by $\overline{t_1},\ldots,\overline{t_\ell}$ as a $Z(\mathcal{A})$-module, there exist $\alpha_1, \ldots, \alpha_\ell \in Z(\mathcal{A})$ such that $\overline{A - T} = \sum_{j=1}^{\ell} \alpha_j \overline{t_j}$. Writing each of these α_j as $\alpha_j = P_j(z_1, \ldots, z_k)$, where each P_j is a polynomial, we get

$$(Z, \overline{A}) = P(Z_1, \ldots, Z_k) + (0, \overline{A - T}) = P(Z_1, \ldots, Z_k) + \sum_{j=1}^{\ell} P_j(Z_1, \ldots, Z_k) \cdot T_j \;,$$

which expresses explicitly $(Z, \overline{A})$ in terms of the elements $Z_1, \ldots, Z_k$ and $T_1, \ldots, T_\ell$, which shows that the latter are generators of $\Pi(\mathcal{A})$. □

Using the algebra generators $Z_1, \ldots, Z_k$ and $T_1, \ldots, T_\ell$ of $\Pi(\mathcal{A})$, we get a surjective algebra homomorphism $\mathbb{C}[Z_1, \ldots, Z_k, T_1, \ldots, T_\ell] \to \Pi(\mathcal{A})$ and we can describe $(\Pi(\mathcal{A}), \cdot)$ as the quotient $\mathbb{C}[Z_1, \ldots, Z_k, T_1, \ldots, T_\ell]/\mathcal{J}$, where $\mathcal{J}$ is the kernel of the homomorphism; we will explicitly compute this kernel in our examples, providing thereby the algebra structure of $\Pi(\mathcal{A})$.

The Poisson brackets of the chosen generators can then be explicitly computed from the following formulas, which are a specialization of (2.8):

$$\left\{(a, \overline{0}), (b, \overline{0})\right\} = \left(\{a, b\}_1 \,, \overline{\{a, b\}_2}\right) \;, \tag{4.6}$$

$$\left\{(0, \overline{a_1}), (0, \overline{b_1})\right\} = \left(0, \overline{[a_1, b_1]}\right) \;, \tag{4.7}$$

$$\left\{(a, \overline{0}), (0, \overline{b_1})\right\} = \left(0, \overline{\{a, b_1\}_1}\right) \;, \tag{4.8}$$

where $a, b \in Z(\mathcal{A})$ and $a_1, b_1 \in \mathcal{A}$, and so we only need to compute $\{a, b\}_1$, $\{a, b\}_2$, $[a_1, b_1]$ and $\{a, b_1\}_1$ for such elements to determine these Poisson brackets. Recall from (2.2) that $\{a, b\}_1$ and $\{a, b\}_2$ are the coefficients in ν and ν^2 of $a \star b$. As before, when $\{a, b\}_2 \in Z(\mathcal{A})$, (4.6) simplifies further to

$$\left\{(a, \overline{0}), (b, \overline{0})\right\} = \left(\{a, b\}_1 \,, \overline{0}\right) \;. \tag{4.9}$$

Similarly, the Lie action of $\Pi(\mathcal{A})$ on $\mathcal{A}$ is given for $a \in Z(\mathcal{A})$ and $a_1, b \in \mathcal{A}$ by the following formulas, which are a specialization of (2.16):

$$\left\{(a, \overline{0})\,; b\right\} = \{a, b\}_1 \quad \text{and} \quad \{(0, \overline{a_1})\,; b\} = [a_1, b] \;. \tag{4.10}$$

In view of the following proposition, we will also be interested in the center $Z(\mathcal{A}_{\mathbf{q}})$ of the quantum algebra $\mathcal{A}_{\mathbf{q}}$.

Proposition 4.2 *Suppose that $X = X_0 + \nu X_1 + \cdots$ is a central element of $\mathcal{A}[[\nu]]$. Then $\left(X_0, \overline{X_1}\right)$ is a Casimir of $(\Pi(\mathcal{A}), \{\cdot\,, \cdot\})$ and belongs to the Lie annihilator of $\mathcal{A}$.*

Proof Let $\left(Y_0, \overline{Y_1}\right)$ be any element of $\Pi(\mathcal{A})$ and denote $Y = Y_0 + \nu Y_1 \in \mathcal{A}[[\nu]]$. According to Proposition 2.10,

$$\left\{\left(X_0, \overline{X_1}\right), \left(Y_0, \overline{Y_1}\right)\right\} = p_\Pi[X_0 + \nu X_1, Y_0 + \nu Y_1]_\nu = p_\Pi\,[X, Y]_\nu = 0 \;,$$

since X belongs to the center of $\mathcal{A}[[\nu]]$. This shows that $\left(X_0, \overline{X_1}\right)$ Poisson commutes with all elements of $\Pi(\mathcal{A})$, i.e., is a *Casimir* of $\Pi(\mathcal{A})$. Similarly, for $a \in \mathcal{A}$ one has $\left\{\left(X_0, \overline{X_1}\right) ; a\right\} = 0$ since $[X, a]_\star = 0$ for any $a \in \mathcal{A} \subset \mathcal{A}[[\nu]]$. □

Let $A = A(\mathbf{q}) \in Z(\mathcal{A}_\mathbf{q})$, which we may assume to depend polynomially on $\mathbf{q}$. For any regular value $\mathbf{q}_0$ of $\mathbf{q}$, and any nonzero $\mathbf{q}_1$, expand $A(\mathbf{q}(\nu)) = A(\mathbf{q}_0 + \nu\mathbf{q}_1) = X_0 + \nu X_1 + \mathcal{O}(\nu)$, which is a central element of $\mathcal{A}[[\nu]]$, hence leads in view of the proposition to the Casimir $\left(X_0, \overline{X_1}\right)$ of $\Pi(\mathcal{A})$. In general, not all Casimirs of $\Pi(\mathcal{A})$ are obtained in this way.

4.2 The quantum plane

As a first example, we consider the (complex) quantum plane, which is defined as being the noncommutative algebra

$$\mathcal{A}_q := \mathbb{C}_q[x, y] = \frac{\mathbb{C}(q)\langle x, y\rangle}{\langle yx - qxy\rangle} . \tag{4.11}$$

As a basis $\mathcal{B}$ for this quantum algebra, we take the normally ordered monomials $x^m y^n$, $m, n \in \mathbb{N}$. It is a PBW basis and any $q_0 \in \mathbb{C}$ is a regular value of it. In $\mathcal{A}_q$ we have $yx = qxy$, or equivalently $[x, y] = (1-q)xy$. As we already pointed out in Sect. 4.1, this implies that the center of $\mathcal{A}_q$ is generated by monomials, from which it is clear that the center of $\mathcal{A}_q$ consists of constants only, $Z(\mathcal{A}_q) = \mathbb{C}(q)$.

The evaluation of $\mathcal{A}_q$ at $q = 1$ is the polynomial algebra $\mathbb{C}[x, y]$, which is commutative, so let us consider its evaluation at $q = -1$ to illustrate how to obtain the Poisson and reduced Poisson algebras associated with the deformation; see below for other values of q. As explained in Sect. 3.2, we set $q(\nu) = -1 + \nu$ and consider

$$\mathcal{A}_\nu = \frac{\mathbb{C}[[\nu]]\langle x, y\rangle}{\langle yx - (\nu - 1)xy\rangle} \simeq \mathcal{A}[[\nu]] \ , \ \text{where } \mathcal{A} := \frac{\mathbb{C}\langle x, y\rangle}{\langle yx + xy\rangle} .$$

The product $\star$ on $\mathcal{A}[[\nu]]$ is induced by the above isomorphism and is completely specified by $y\star x = (\nu-1)x\star y$. In $\mathcal{A}$ we have $yx = -xy$, or equivalently $[x, y] = 2xy$, so that $Z(\mathcal{A})$ is also generated by monomials. Since x and y anticommute in $\mathcal{A}$, the center $Z(\mathcal{A})$ consists of all polynomials that are even in x and in y, hence is generated by x^2 and y^2. Then $\mathcal{A}/Z(\mathcal{A})$ is generated as a $Z(\mathcal{A})$-module by $\overline{x}$, $\overline{y}$ and $\overline{xy}$. According to Proposition 4.1, the following 5 elements are algebra generators of $\Pi(\mathcal{A})$:

$$X = \left(x^2, \overline{0}\right) \ , \ Y = \left(y^2, \overline{0}\right) \ , \ U = (0, \overline{x}) \ , \ V = (0, \overline{y}) \ , \ W = (0, \overline{xy}) \ ,$$

with product $\cdot$ given by (4.3)–(4.5). Consider the algebra homomorphism $\mathbb{C}[X, Y, U, V, W] \to \Pi(\mathcal{A})$. We show that its kernel $\mathcal{J}$ is the ideal, generated by $U^2,\ V^2,\ W^2,\ U \cdot V,\ V \cdot W,\ U \cdot W$. To do this, first notice that the kernel clearly contains these elements, while the elements $X^i \cdot Y^j \cdot U^{\epsilon_1} V^{\epsilon_2} \cdot W^{\epsilon_3}$ with at most one ϵ_i equal to 1 and the others equal to 0, are all linearly independent. Indeed, they are given by

$$X^i \cdot Y^j = \left(x^{2i} y^{2j}, \overline{0}\right) , \quad X^i \cdot Y^j \cdot U^{\epsilon_1} \cdot V^{\epsilon_2} \cdot W^{\epsilon_3} = \left(0, \overline{x^{2i+\epsilon_1+\epsilon_3} y^{2j+\epsilon_2+\epsilon_3}}\right) ,$$

where exactly one of the ϵ_i is equal to 1. Therefore, as an algebra, $(\Pi(\mathcal{A}), \cdot) \simeq \mathbb{C}[X, Y, U, V, W]/\mathcal{J}$, while $Z(\mathcal{A}) \simeq \mathbb{C}[X, Y]$. The Poisson brackets $\{\cdot\,,\cdot\}$ between the generators $X, Y, \ldots, W$ are given in Table 3. They are computed using (4.6)–(4.8). Let us show for example that $\{X, Y\} = \left(4x^2y^2, \overline{0}\right) = 4X \cdot Y$. For the first equality, one needs to verify in view of (4.9) that $\left\{x^2, y^2\right\}_1 = 4x^2y^2$. To do this, we first compute, in $\mathcal{A}_\nu$,

$$\left[x^2, y^2\right]_\star = \left(1 - q(\nu)^4\right) x^2y^2 = \left(1 - (\nu - 1)^4\right) x^2y^2 = 4\nu x^2y^2 + \mathcal{O}\left(\nu^2\right) ,$$

from which it follows that $\left[x^2, y^2\right]_\star = 4\nu x^2y^2 + \mathcal{O}\left(\nu^2\right)$, as we needed to show. For the second equality, it suffices to notice that the product $X \cdot Y$ is given by (4.5) because $\left(x^2, y^2\right)_1 = 0$. Similarly, $\{U, V\} = 2W$ since $[x, y] = 2xy$. As a last example, $\{X, W\} = \left(0, 2\overline{x^3y}\right) = X \cdot W$ because $\left\{x^2, xy\right\}_1 = 2x^3y$, which follows from

$$\left[x^2, xy\right]_\star = \left(1 - q(\nu)^2\right) x^3y = \left(1 - (\nu - 1)^2\right) x^3y = 2\nu x^3y + \mathcal{O}\left(\nu^2\right) .$$

It is clear from the table that the reduced Poisson algebra $(Z(\mathcal{A}), \{\cdot\,,\cdot\}_1)$ can be described as the polynomial algebra $\mathbb{C}[X, Y]$, with Poisson bracket $\{X, Y\}_1 = 4XY$. The action and Lie action of the generators of $\Pi(\mathcal{A})$ on the generators x and y of $\mathcal{A}$ is given in Table 4. The entries of the rightmost table are computed from (4.10). For example, $\{X\,; y\} = \left\{\left(x^2, \overline{0}\right)\,; y\right\} = \left\{x^2, y\right\}_1 = 2x^2y$, where the last equality is obtained from

$$\left[x^2, y\right] = (1 - q(\nu)^2)x^2y = \left(1 - (1 - \nu)^2\right) x^2y = 2\nu x^2y + \mathcal{O}(\nu^2) .$$

Also, $\{U\,; y\} = \{(0, \overline{x})\,; y\} = [x, y] = 2xy$, since in $\mathcal{A}$, x and y anticommute.

$\{\cdot\,,\cdot\}$	X	Y	U	V	W
X	0	$4X \cdot Y$	0	$2X \cdot V$	$2X \cdot W$
Y	$-4X \cdot Y$	0	$-2U \cdot Y$	0	$-2Y \cdot W$
U	0	$2U \cdot Y$	0	$2W$	$2X \cdot V$
V	$-2X \cdot V$	0	$-2W$	0	$-2Y \cdot U$
W	$-2X \cdot W$	$2Y \cdot W$	$-2X \cdot V$	$2Y \cdot U$	0

Table 3 Poisson brackets between the generators of the Poisson algebra $\Pi(\mathcal{A})$ in the case of quantum plane with $q = -1$

$\cdot$	x	y
X	x^3	x^2y
Y	xy^2	y^3
U	0	0
V	0	0
W	0	0

$\{\cdot\,;\cdot\}$	x	y
X	0	$2x^2y$
Y	$-2xy^2$	0
U	0	$2xy$
V	$-2xy$	0
W	$-2x^2y$	$2xy^2$

Table 4 Multiplication table and Lie brackets for the generators of the $\Pi(\mathcal{A})$ module $\mathcal{A}$ in the case of quantum plane with $q = -1$

We now consider arbitrary values of q. Since $\left[x^i y^j, x\right] = (q^j - 1)x^{i+1}y^j$ in $\mathcal{A}_q$, the center of $\mathcal{A}$ will be $\mathbb{C}$, unless q is a root of unity. Let us therefore consider a primitive n-th root of unity ξ (where $n > 1$). Setting $q(\nu) = \xi + \nu$, the above considerations and computations for $n = 2$ are easily generalized. First, since we know that $Z(\mathcal{A})$ is generated by monomials, it is easily checked as above that x^n and y^n generate $Z(\mathcal{A})$ and from it that $X := \left(x^n, \overline{0}\right)$, $Y := \left(y^n, \overline{0}\right)$ and $W_{i,j} := \left(0, \overline{x^i y^j}\right)$, where $0 \leqslant i, j < n$, $i + j \neq 0$ generate $\Pi(\mathcal{A})$. As an algebra, $(\Pi(\mathcal{A}), \cdot) \simeq \mathbb{C}[X, Y, W_{i,j}]/\langle W_{i,j} W_{k,\ell}\rangle$, where the indices i, j, k, ℓ are in the range $0, \ldots, n-1$, with $i + j \neq 0$ and $k + \ell \neq 0$. The Poisson brackets of these generators are computed as above and are given in Table 1 in the introduction.

Notice that $W_{i+k,j+\ell}$ is for large values of i, j, k, ℓ not one of the chosen generators of $\Pi(\mathcal{A})$ but can easily be rewritten in terms of these generators using $W_{\alpha n+\beta,\alpha' n+\beta'} = X^\alpha \cdot Y^{\alpha'} \cdot W_{\beta,\beta'}$. The reduced Poisson algebra $(Z(\mathcal{A}), \{\cdot\,,\cdot\}_1)$ can be described as the polynomial algebra $\mathbb{C}[X, Y]$, with Poisson bracket $\{X, Y\}_1 = -\xi^{-1}n^2 XY$. The action and Lie action of $\Pi(\mathcal{A})$ on $\mathcal{A}$ are given in Table 2.

Remark 4.3 The quantum plane (4.11) is closely related to the (two-dimensional) quantum torus, which is defined as the noncommutative algebra

$$\mathcal{A}_q := \mathbb{T}_q[x, y] = \frac{\mathbb{C}(q)\langle x, y, x^{-1}, y^{-1}\rangle}{\langle yx - qxy\rangle}\,. \tag{4.12}$$

The above considerations and computations are easily adapted to this case. Still considering the case of q being an n-th root of unity, $Z(\mathcal{A})$ is now generated by x^n, x^{-n}, y^n and y^{-n} and $Z(\mathcal{A}) \simeq \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$, where $X^{-1} := \left(x^{-n}, \overline{0}\right)$ and $Y^{-1} := \left(y^{-n}, \overline{0}\right)$ are two extra generators for $\Pi(\mathcal{A})$. The ideal $\mathcal{J}$ has two extra generators $XX^{-1} - 1$ and $YY^{-1} - 1$. The above tables containing the Poisson brackets of $\Pi(\mathcal{A})$ and the actions of $\Pi(\mathcal{A})$ on $\mathcal{A}$ in the case of the quantum plane still contain all information for the case of the quantum torus because by the Leibniz identity, $\left\{X^{-1}, \cdot\right\} = -X^{-2}\{X, \cdot\}$ and $\left\{X^{-1}\,;\cdot\right\} = -X^{-2} \cdot \{X\,;\cdot\}$, and similarly for the brackets with Y^{-1}. In more formal terms, for any value of $q \in \mathbb{C}^*$, the Poisson algebra

$\Pi(\mathcal{A})$ for the quantum torus is the localization of the Poisson algebra $\Pi(\mathcal{A})$ of the quantum plane, with respect to the multiplicative system of $\mathcal{A}$ generated by X and Y (see [11, Section 2.4.2]). The algebra $\mathcal{A}$, which is a Poisson module over $\Pi(\mathcal{A})$, then becomes a Poisson module over this localization of $\Pi(\mathcal{A})$.

4.3 A quantum algebra related to the Volterra chain

We next consider a more elaborate example which is related to the N-periodic non-abelian Volterra chain (see Sect. 5.2). The quantum algebra, which is a particular case of Example 3.2, is generated by $x_1, \ldots, x_N$, which all commute except the neighboring pairs $x_{i+1}x_i = qx_ix_{i+1}$ for $i = 1, \ldots, N$; in this formula the index i of x_i is taken modulo N, so that $x_1x_N = qx_Nx_1$. For $1 \leqslant i, j \leqslant N$ let us denote their N-periodic distance by $d_N(i, j)$; so $d_N(i, j) = \min\{|i-j|, N-|i-j|\}$. Then $x_ix_j = x_jx_i$ when $d_N(i, j) \neq 1$. We therefore consider

$$\mathcal{A}_q := \frac{\mathbb{C}(q)\langle x_1, \ldots, x_N\rangle}{\mathcal{I}_q}\ , \text{ where } \mathcal{I}_q = \langle x_{i+1}x_i - qx_ix_{i+1}, x_ix_j - x_jx_i\rangle_{d_N(i,j)\neq 1}\ . \tag{4.13}$$

An automorphism of order N of $\mathcal{A}_q$ is defined by $\mathcal{S}(x_i) := x_{i+1}$ for all i. We use as a basis $\mathcal{B}$ of $\mathcal{A}_q$ the normally ordered monomials $x_1^{i_1} \ldots x_N^{i_N}$ with $i_1, \ldots, i_N \in \mathbb{N}$. Notice that $q_0 = 0$ is not a regular value of $\mathcal{B}$, since $x_Nx_1 = q^{-1}x_1x_N$, but all other values are regular values. Again, $\mathcal{B}$ is a PBW basis. As we explained in Sect. 4.1, the center of $\mathcal{A}_q$ is in this case generated by monomials. We use this fact to show that $Z(\mathcal{A}_q)$ is generated by $x_1x_2 \ldots x_N$ when N is odd, while it is generated by $x_1x_3 \ldots x_{N-1}$ and $x_2x_4 \ldots x_N$ when N is even. A monomial $x_1^{i_1}x_2^{i_2} \ldots x_N^{i_N}$ of $\mathcal{A}_q$ belongs to the center if and only if its commutator (in $\mathcal{A}_q$) with any x_ℓ vanishes. From

$$0 = \left[x_1^{i_1} \ldots x_N^{i_N}, x_\ell\right] = (q^{i_{\ell+1}-i_{\ell-1}} - 1)x_\ell x_1^{i_1} \ldots x_N^{i_N} \tag{4.14}$$

it follows that if $x_1^{i_1} \ldots x_N^{i_N}$ belongs to the center of $\mathcal{A}_q$, then $i_k = i_{k+2}$ for $k = 1, \ldots, N$, which yields the claim (recall that the indices of x are N-periodic, so that also $i_{N+1} = i_1$ and $i_0 = i_N$). It is also clear from (4.14) that for values of q that are not roots of unity, the center of the corresponding algebra $\mathcal{A}$ will contain no other elements and hence $Z(\mathcal{A}) \times \left\{\overline{0}\right\} \subset \Pi(\mathcal{A})$ consists of Casimirs only (Proposition 4.2) and the only nontrivial Poisson brackets in $\Pi(\mathcal{A})$ are given by commutators (see (4.6)–(4.8)). We will therefore consider the evaluation of q at roots of unity only. Also, in view of the difference between N even and odd, we will first consider in detail the case that N is odd and spell out afterward how to adapt the results in case N is even.

Let $N > 2$ be odd and let ξ denote a primitive n-th root of unity, $\xi^n = 1$, where $n > 1$; see Remark 4.4 below for the case of $n = 1$. We set $q(\nu) = \xi + \nu$ and consider

$$\mathcal{A}_\nu = \frac{\mathbb{C}[[\nu]]\langle x_1, \ldots, x_N\rangle}{\mathcal{I}_{q(\nu)}} \simeq \mathcal{A}[[\nu]]\ , \text{ where } \mathcal{A} := \frac{\mathbb{C}\langle x_1, \ldots, x_N\rangle}{\mathcal{I}_\xi}\ .$$

It is clear that the center of $\mathcal{A}$ contains $x_1^n, \dots, x_N^n$ and $x_1 x_2 \dots x_N$. We claim that $Z(\mathcal{A})$ is generated by these elements. To show this, we look for monomials $x_1^{i_1} x_2^{i_2} \dots x_N^{i_N}$ in the center of $\mathcal{A}$, with $0 \leqslant i_1, i_2, \dots, i_N < n$; as above, it follows however from (4.14) with $q = \xi$ that then all i_k must be equal (recall that N is assumed odd), and so the monomial is of the form $x_1^k x_2^k \dots x_N^k$, for some k. This shows our claim. It is then also clear that $\mathcal{A}$ is generated as a $Z(\mathcal{A})$-module by the monomials $x_1^{i_1} x_2^{i_2} \dots x_N^{i_N}$, where $0 \leqslant i_1, \dots, i_N < n$ are not all zero and at least one of them is zero. In view of Proposition 4.1, $\Pi(\mathcal{A})$ is generated by

$$X_1 := (x_1^n, \bar{0}) , \dots, X_N := (x_N^n, \bar{0}) , \ X := (x_1 x_2 \dots x_N, \bar{0}) \text{ and}$$
$$W_{i_1,\dots,i_N} := \left(0, \overline{x_1^{i_1} x_2^{i_2} \dots x_N^{i_N}}\right) ,$$

where $0 \leqslant i_1, \dots, i_N < n$ are not all zero and at least one of them is zero. Since $\Pi(\mathcal{A})$ is generated by these elements, we have a surjective morphism $\mathbb{C}[X_i, X, W_{i_1,\dots,i_N}] \to \Pi(\mathcal{A})$, with kernel

$$\mathcal{J} = \langle X_1 \cdot X_2 \cdots X_N - X^n, W_{i_1,\dots,i_N} \cdot X - W_{i_1+1,\dots,i_N+1}, W_{i_1,\dots,i_N} \cdot W_{j_1,\dots,j_N} \rangle , \tag{4.15}$$

where the indices $i_1, \dots, i_N$ and $j_1, \dots, j_N$ are as above. It is understood that when one of the indices of the term $W_{i_1+1,\dots,i_N+1}$ in (4.15) is at least n, it is rewritten in terms of the generators, for example, $W_{n,i_2,\dots,i_N} = X_1 W_{0,i_2,\dots,i_N}$ and $W_{n,0,0,\dots,0} = 0$. It follows that, as algebras, $\Pi(\mathcal{A}) \simeq \mathbb{C}[X_i, X, W_{i_1,\dots,i_N}]/\mathcal{J}$ and $Z(\mathcal{A}) \simeq \mathbb{C}[X_i, X]/\langle X_1 X_2 \dots X_N - X^n \rangle$.

$\{\cdot\,,\cdot\}$	X_ℓ	W_J
X_k	$n^2(\delta_{k,\ell+1} - \delta_{k,\ell-1})\xi^{-1} X_k \cdot X_\ell$	$(j_{k-1} - j_{k+1})\xi^{-1} n X_k \cdot W_J$
W_I	$(i_{\ell+1} - i_{\ell-1})\xi^{-1} n X_\ell \cdot W_I$	$\left(\xi^{\eta(I,J)} - \xi^{\eta(J,I)}\right) W_{I+J}$

Table 5 Poisson brackets between the generators of the Poisson algebra $\Pi(\mathcal{A})$ (Volterra chain)

In terms of these generators, the Poisson bracket $\{\cdot\,,\cdot\}$ of $\Pi(\mathcal{A})$ is given in Table 5, in which $I = (i_1, \dots, i_N)$ and $J = (j_1, \dots, j_N)$. In this table, the indices of i and j are again taken modulo N, so that $i_{N+1} = i_1$ and $i_0 = i_N$, and similarly for j; $I + J$ simply stands for the componentwise sum of I and J. Notice that in view of the automorphism $\mathcal{S}$ we may assume that $1 < k < N$, or that $1 < \ell < N$, when verifying the entries of the table. Also, $\eta(I, J)$ is a shorthand for $\sum_{s=1}^{N-1} i_{s+1} j_s - i_N j_1$ and is computed from

$$(x_1^{i_1} \dots x_N^{i_N})(x_1^{j_1} \dots x_N^{j_N}) = \xi^{\eta(I,J)} x_1^{i_1+j_1} \dots x_N^{i_N+j_N} ,$$

using the relations in $\mathcal{A}$ which say that all variables x_i commute, except the neighboring pairs $x_{i+1} x_i = \xi x_i x_{i+1}$ for all i mod N. As before, when W_{I+J} is not one of the chosen generators of $\Pi(\mathcal{A})$ it is easily written as a product of such generators. We did not put the generator X in the table because it is a Casimir, so the brackets of X with any other generator (or element of $\Pi(\mathcal{A})$) is zero; it is a Casimir because $x_1x_2\dots x_N$ is a central element of $\mathcal{A}_q$, see Proposition 4.2. The reduced Poisson algebra $(Z(\mathcal{A}), \{\cdot\,,\cdot\}_1)$ is easily read off from the upper-left corner of Table 5. It is a so-called *diagonal* Poisson structure (see [11, Example 8.14]).

For future use, we also compute the Poisson module structure on $\mathcal{A}$, still assuming N odd. It is clear that the action of $\Pi(\mathcal{A})$ on $\mathcal{A}$ is given by $X_k\cdot x_\ell = x_k^n x_\ell$, $W_I\cdot x_\ell = 0$ for all I, k, ℓ. According to (2.16) and since X is in the annihilator of $\mathcal{A}$ (see Proposition 4.2), $\{X\,;x_\ell\} = 0$. With $k, \ell = 1,\dots,N$ and $I = (i_1,\dots,i_N)$ $(0 \leqslant i_1,\dots,i_N < n)$,

$$\{X_k\,;x_\ell\} \overset{(2.16)}{=} \left\{x_k^n, x_\ell\right\}_1 = (\delta_{k,\ell+1} - \delta_{k,\ell-1})\xi^{-1} n x_k^n x_\ell\ , \tag{4.16}$$

and

$$\{W_I\,;x_\ell\} = \left[x_1^{i_1}x_2^{i_2}\dots x_N^{i_N}, x_\ell\right] = \alpha(I,\ell)x_1^{i_1}\dots x_\ell^{i_\ell+1}\dots x_N^{i_N}\ , \tag{4.17}$$

where

$$\alpha(I,\ell) = \alpha(i_1,\dots,i_N,\ell) = \begin{cases} \xi^{i_2-i_N} - 1 & \ell = 1\ , \\ \xi^{i_{\ell+1}} - \xi^{i_{\ell-1}} & 1 < \ell < N\ , \\ 1 - \xi^{i_{N-1}-i_1} & \ell = N\ . \end{cases}$$

These results were obtained from (4.10) using the commutation rules in $\mathcal{A}_\nu \simeq \mathcal{A}[[\nu]]$. For example, $\left[x_k^n, x_\ell\right]_\star = 0$ when $d_N(k,\ell) \neq 1$, since then x_k and x_ℓ commute. Also, if $k \neq N$ then $\left[x_k^n, x_{k+1}\right] = (1-q^n)x_k^n x_{k+1} = (1-(\xi+\nu)^n)x_k^n x_{k+1} = -\xi^{-1}n\nu x_k^n x_{k+1} + \mathcal{O}\left(\nu^2\right)$, so that $\left[x_k^n, x_{k+1}\right]_\star = -\xi^{-1}n\nu x_k^n x_{k+1} + \mathcal{O}\left(\nu^2\right)$ and hence $\{X_k\,;x_{k+1}\} = \left\{x_k^n, x_{k+1}\right\}_1 = -\xi^{-1}n\nu x_k^n x_{k+1}$; this result is also valid for $k = N$ since $x_N^n x_1 = x_1 x_N^n$ hence belongs to $\mathcal{B}$. The formulas in (4.17) are just commutators in $\mathcal{A}$, so they follow immediately from the relations in $\mathcal{A}$, which are given by $x_{i+1}x_i = \xi x_i x_{i+1}$, and $x_i x_j = x_j x_i$ when $d_N(i,j) \neq 1$.

We now consider the minor adaptations to be done in case N is even, $N > 2$. As we have already shown, $Z(\mathcal{A}_q)$ is generated by $x_1x_3\dots x_{N-1}$ and $x_2x_4\dots x_N$. By the same proof as above, $Z(\mathcal{A})$ is generated by

$$X_i := (x_i^n, \bar{0})\ ,\ \ Y_1 := (x_1x_3\dots x_{N-1}, \bar{0})\ ,\ \ Y_2 := (x_2x_4\dots x_N, \bar{0})\ ,$$

where $i = 1,\dots,N$, and

$$W_{i_1,\dots,i_N} := \left(0, \overline{x_1^{i_1}x_2^{i_2}\dots x_N^{i_N}}\right)\ , \qquad 0 \leqslant i_1,\dots,i_N < n\ ,$$

where not all indices i_k are zero, where at least one index i_k with k even is zero, as well as at least one index i_k with k odd. As an algebra, $\Pi(\mathcal{A}) \simeq \mathbb{C}[X_i, Y_1, Y_2, W_{i_1,\dots,i_N}]/\mathcal{J}'$, where

$$\begin{aligned}\mathcal{J}' = \langle X_1\cdot X_3\cdots X_{N-1} - Y_1^n, X_2\cdot X_4\cdots X_N - Y_2^n, W_{i_1,\dots,i_N}\cdot Y_1 - W_{i_1+1,i_2,\dots,i_N},\\ W_{i_1,\dots,i_N}\cdot Y_2 - W_{i_1,i_2+1,\dots,i_{N+1}}, W_{i_1,\dots,i_N}\cdot W_{j_1,\dots,j_N}\rangle\ ,\end{aligned}$$

and $Z(\mathcal{A}) \simeq \mathbb{C}[X_i, Y_1, Y_2]/\langle X_1X_3\dots X_{N-1} - Y_1^n, X_2X_4\dots X_N - Y_2^n\rangle$. The table of Poisson brackets for the generators is again given in Table 5, where we now leave out the zero rows and columns corresponding to the Casimirs Y_1 and Y_2. Also, the formulas for the actions of $\Pi(\mathcal{A})$ on $\mathcal{A}$ are the same. So, finally the cases N even and odd are formally not very different.

Remark 4.4 When $n = 1$, $\xi = 1$ and $\mathcal{A}$ is commutative, $\mathcal{A} = Z(\mathcal{A})$, so that we are in the classical case. Then $\mathcal{A}/Z(\mathcal{A})$ is trivial and $\mathcal{A}$ is generated by $X_1, \dots, X_N$, which now take the simple form $X_i = (x_i, \overline{0})$, i.e., $X_i = x_i$ under the natural identification of $\Pi(\mathcal{A})$ with $\mathcal{A}$. The above computation of the Poisson brackets is still valid and leads to the following nonzero Poisson brackets between the x_i:

$$\left\{x_i, x_j\right\} = (\delta_{i,j+1} - \delta_{i,j-1})x_ix_j \ . \tag{4.18}$$

It is the standard Poisson structure of the (periodic) Volterra chain.

Remark 4.5 The infinite case ($N = \infty$) is not very different and is in a sense simpler. The formula for $\eta(I, J)$ in Table 5 simplifies to $\eta(I, J) = \sum i_{s+1} j_s$ and (4.17) becomes

$$\{W_I\,;x_\ell\} = (\xi^{i_{\ell+1}} - \xi^{i_{\ell-1}})x_1^{i_1}\dots x_\ell^{i_\ell+1}\dots x_N^{i_N} \ .$$

The Poisson algebra $\Pi(\mathcal{A})$ does not have any Casimirs; it can be extended to a larger Poisson algebra for which the infinite products $\prod_{i \text{ odd}} x_i$ and $\prod_{i \text{ even}} x_i$ are Casimirs.

4.4 Another quantum algebra related to the Volterra chain

We now describe the Poisson algebra corresponding to another quantum algebra, also related to the periodic Volterra chain. The algebra is given by

$$\mathcal{A}_q := \frac{\mathbb{C}(q)\langle x_1, \dots, x_N\rangle}{\langle x_{i+1}x_i - (-1)^i q x_i x_{i+1}, x_ix_j + x_jx_i\rangle_{d_N(i,j)\neq 1}} \ ,$$

with N even, $N = 2M$. As before, the indices are periodic modulo N and $d_N(i, j)$ denotes the periodic distance between i and j. As a basis of $\mathcal{A}_q$ we take the monomials $x_1^{i_1}\dots x_N^{i_N}$, with $i_1, \dots, i_N \in \mathbb{N}$. It is a PBW basis and again only $q_0 = 0$ is not a regular value of $\mathcal{B}$. Notice that since N is even we can speak unambiguously of even and odd indices. A $\mathbb{C}$-algebra automorphism $\mathcal{S}_q$ of $\mathcal{A}_q$ is defined by $\mathcal{S}_q(x_i) = x_{i+1}$ and $\mathcal{S}_q(q) = -q$. In view of the commutation relations in $\mathcal{A}_q$, the center of $\mathcal{A}_q$ is again generated by monomials. Since

$$\left[x_1^{i_1}\dots x_N^{i_N}, x_\ell\right] = \left((-1)^{\sum_{k=1}^N i_k - i_\ell - i_{\ell+(-1)^\ell}} q^{i_{\ell+1}-i_{\ell-1}} - 1\right) x_\ell x_1^{i_1}\dots x_N^{i_N}, \tag{4.19}$$

$x_1^{i_1}\dots x_N^{i_N}$ belongs to $Z(\mathcal{A}_q)$ if and only if (1) all exponents that correspond to even indices are equal, (2) all exponents that correspond to odd indices are equal and (3) the sum $\sum_{k=1}^N i_k - i_\ell - i_{\ell+(-1)^\ell}$ is even for all ℓ. Since this sum is of the form

$\sum_{k=1}^{N} i_k - i_\ell - i_{\ell\pm 1}$, it contains $M-1$ terms i_k with k even and $M-1$ terms i_k with k odd; therefore, when M is odd, condition (3) is satisfied as soon as conditions (1) and (2) are satisfied, while when M is even, (3) imposes furthermore that all the exponents have the same parity. It follows that

$$Z(\mathcal{A}_q) \text{ is generated by } \begin{cases} \prod_{i \text{ odd}} x_i^2 \ , \quad \prod_{i \text{ even}} x_i^2 \ , \ \prod_i x_i \ , & M \text{ even,} \\ \prod_{i \text{ odd}} x_i \ , \quad \prod_{i \text{ even}} x_i \ , & M \text{ odd.} \end{cases}$$

We will consider the evaluation of $\mathcal{A}_q$ at $q = 1$ only; notice that in view of the automorphism $\mathcal{S}_q$ the Poisson algebras obtained for $q = 1$ and $q = -1$ are isomorphic. When $q = 1$ the commutation relations in $\mathcal{A}$ take the simple form

$$x_{i+1}x_i = (-1)^i x_i x_{i+1} \ , \quad x_j x_i = -x_i x_j \ , \ d_N(i, j) \neq 1 \ , \tag{4.20}$$

so all variables anticommute, except for half of the neighbors that commute. From these relations and from the fact that $Z(\mathcal{A})$ is generated by monomials, it follows that

$$Z(\mathcal{A}) \text{ is generated by } \begin{cases} x_k^2 \ , x_{2j}x_{2j+1} \ , & M \text{ even,} \\ x_k^2 \ , x_{2j}x_{2j+1} \ , \ \prod_{i \text{ odd}} x_i \ , \ \prod_{i \text{ even}} x_i \ , & M \text{ odd,} \end{cases}$$

where $k = 1, \ldots, N$ and $j = 1, \ldots, M$.

From these generators we construct, using Proposition 4.1, generators for $\Pi(\mathcal{A})$. Let us denote $X_k := \left(x_k^2, \overline{0}\right)$ for $k = 1, \ldots, N$, $Y_j := \left(x_{2j}x_{2j+1}, \overline{0}\right)$ for $j = 1, \ldots, M$, $Y := \left(\prod_{i \text{ odd}} x_i, \overline{0}\right)$, $Z := \left(\prod_{i \text{ even}} x_i, \overline{0}\right)$ and $W_{i_1,\ldots,i_N} := \left(0, \overline{x_1^{i_1} \ldots x_N^{i_N}}\right)$. Then $\Pi(\mathcal{A})$ is generated by $X_1, \ldots, X_N, Y_1, \ldots, Y_M$ and

$$\begin{cases} W_{i_1,\ldots,i_N}, \ i_k \in \{0, 1\}, \text{not all } i_k = 0, \ i_{2j}i_{2j+1} = 0 \ , & M \text{ even,} \\ Y, \ Z, \ W_{i_1,\ldots,i_N}, \ i_k \in \{0, 1\}, \text{not all } i_k = 0, \ i_{2j}i_{2j+1} = 0, \text{ not all } \begin{bmatrix} \text{even} \\ \text{odd} \end{bmatrix} i_k = 1 \ , & M \text{ odd.} \end{cases}$$

When M is even, $(\Pi(\mathcal{A}), \cdot) \simeq \mathbb{C}[X_i, Y_j, W_{i_1,\ldots,i_N}]/\mathcal{J}$, where

$$\begin{aligned} \mathcal{J} := \langle & Y_j^2 - X_{2j} \cdot X_{2j+1}, Y_j \cdot W_{\ldots,i_{2j-1},1,0,i_{2j+2},\ldots} - X_{2j} \cdot W_{\ldots,i_{2j-1},0,1,i_{2j+2},\ldots}, \\ & Y_j \cdot W_{\ldots,i_{2j-1},0,1,i_{2j+2},\ldots} - X_{2j+1} \cdot W_{\ldots,i_{2j-1},1,0,i_{2j+2},\ldots}, W_{i_1,\ldots,i_N} \cdot W_{j_1,\ldots,j_N} \rangle \ , \end{aligned}$$

and similarly when M is odd. Setting $q(\nu) = 1 + \nu$, we can now determine the Poisson brackets between the above generators. Since Y and Z are constructed from central elements of $\mathcal{A}_q$, they are Casimirs of $(\Pi(\mathcal{A}), \{\cdot\,, \cdot\})$ (Proposition 4.2), so they have zero brackets with all elements of $\Pi(\mathcal{A})$. We therefore do not add these generators to the table of Poisson brackets and can provide a table which is valid for both M

$\{\cdot,\cdot\}$	X_ℓ	Y_j	W_J
X_k	$4\left(\delta_{k,\ell+1}-\delta_{k,\ell-1}\right)X_k\cdot X_\ell$	$2\left(\delta_{j,\left[\frac{k-1}{2}\right]}-\delta_{j,\left[\frac{k+1}{2}\right]}\right)X_k\cdot Y_j$	$2\left(j_{k-1}-j_{k+1}\right)X_k\cdot W_J$
Y_i	$2\left(\delta_{i,\left[\frac{\ell+1}{2}\right]}-\delta_{i,\left[\frac{\ell-1}{2}\right]}\right)X_i\cdot Y_\ell$	$\left(\delta_{i,j+1}-\delta_{i,j-1}\right)Y_i\cdot Y_j$	$\epsilon(J,i)Y_i\cdot W_J$
W_I	$2\left(i_{\ell+1}-i_{\ell-1}\right)X_\ell\cdot W_I$	$-\epsilon(I,j)Y_jW_I$	$\left((-1)^{\eta(I,J)}-(-1)^{\eta(J,I)}\right)W_{I+J}$

Table 6 Poisson brackets between the generators of the Poisson algebra $\Pi(\mathcal{A})$ (Volterra chain with alternative quantisation)

even and M odd; in the table, we use again the abbreviations $I=(i_1,\ldots,i_N)$ and $J=(j_1,\ldots,j_N)$, and $1\leqslant k,\ell\leqslant N$ while $1\leqslant i,j\leqslant M$.

In this table, $\epsilon(I,\ell):=i_{2\ell-1}+i_{2\ell}-i_{2\ell+1}-i_{2\ell+2}$, and the exponent $\eta(I,J)$, which can be reduced modulo 2, is defined by the equality

$$(x_1^{i_1}\ldots x_N^{i_N})(x_1^{j_1}\ldots x_N^{j_N})=(-1)^{\eta(I,J)}\;x_1^{i_1+j_1}\ldots x_N^{i_N+j_N}\;,$$

where the product in the left-hand side is the product in $\mathcal{A}$. An explicit formula is computed from it using the commutation relations (4.20) and is given by $\eta(I,J)=\sum_{s=1}^{M}(i_{2s}+i_{2s+1})\sum_{t=1}^{2s-1}j_t$, where we have set $i_{N+1}:=0$. Notice that again $W_{i_1+j_1,\ldots,i_N+j_N}$ may not belong to the chosen generators of $\Pi(\mathcal{A})$ (for example, when one of the indices becomes 2), but is then easily written as a product of such generators.

For future use, we also give the $\Pi(\mathcal{A})$-Poisson module structure of $\mathcal{A}$ in terms of the generators $X_1,\ldots,X_N,Y_1,\ldots,Y_M$ of $\Pi(\mathcal{A})$ (to which the Casimirs Y and Z, which belong to the annihilator of $\mathcal{A}$ (see Proposition 4.2), need to be added when M is odd) and the generators $x_1,\ldots,x_N$ of $\mathcal{A}$. Let $1\leqslant k,\ell\leqslant N$, $1\leqslant i\leqslant M$ and $I=(i_1,\ldots,i_N)$. Then $X_k\cdot x_\ell=x_k^2x_\ell$, $Y_i\cdot x_\ell=x_{2i}x_{2i+1}x_\ell$ and $W_I\cdot x_\ell=0$, while

$$\{X_k\,;x_\ell\}=\left\{x_k^2,x_\ell\right\}_1=2(\delta_{k,\ell+1}-\delta_{k,\ell-1})x_k^2x_\ell\;, \tag{4.21}$$

$$\{Y_i\,;x_\ell\}=\{x_{2i}x_{2i+1},x_\ell\}_1=\left(\delta_{i,\left[\frac{\ell+1}{2}\right]}-\delta_{i,\left[\frac{\ell-1}{2}\right]}\right)x_{2i}x_{2i+1}x_\ell\;, \tag{4.22}$$

and

$$\begin{aligned}\{W_I\,;x_\ell\}&=\left[x_1^{i_1}\ldots x_N^{i_N},x_\ell\right]\\&=\begin{cases}\left((-1)^{i_{\ell+2}+\cdots+i_N}-(-1)^{i_1+\cdots+i_{\ell-1}}\right)x_1^{i_1}\ldots x_\ell^{i_\ell+1}\ldots x_N^{i_N} & \ell\text{ even},\\ \left((-1)^{i_{\ell+1}+\cdots+i_N}-(-1)^{i_1+\cdots+i_{\ell-2}}\right)x_1^{i_1}\ldots x_\ell^{i_\ell+1}\ldots x_N^{i_N} & \ell\text{ odd}.\end{cases}\end{aligned} \tag{4.23}$$

It is understood that when, for example, $\ell=N$ then the sums $i_{\ell+1}+\cdots+i_N$ and $i_{\ell+2}+\cdots+i_N$ in (4.23) have no terms, hence are equal to 0.

Remark 4.6 A similar remark as Remark 4.5 applies here: the formulas for the infinite case are the same, only the formula for $\eta(I, J)$ needs to be adapted, and the Casimirs from the periodic case become infinite products to be dealt with appropriately.

4.5 A quantization of the Grassmann algebra

We now consider the deformation of the (complex two-dimensional) Grassmann algebra, given by the following commutation relations:

$$
\begin{aligned}
&[x, p] = \nu\ , \qquad \psi^2 = \phi^2 = 0\ ,\\
&\psi\phi + \phi\psi = \nu\ ,\ [p, \psi] = [x, \psi] = [p, \phi] = [x, \phi] = 0\ .
\end{aligned}
$$

We will only be interested in the evaluation for $\nu = 0$, so we can consider right away the algebra

$$
\mathcal{A}_\nu = \frac{\mathbb{C}[[\nu]]\langle p, x, \psi, \phi\rangle}{\mathcal{I}_\nu} \simeq \mathcal{A}[[\nu]]\ ,\ \text{where } \mathcal{A} := \frac{\mathbb{C}\langle p, x, \psi, \phi\rangle}{\mathcal{I}_0}\ ,
$$

and where $\mathcal{I}_\nu$ stands for the closed ideal of $\mathbb{C}[[\nu]]$, generated by $[x, p] - \nu,\ \psi^2,\ \phi^2$, $\psi\phi + \phi\psi - \nu,\ [p, \psi],\ [x, \psi], [p, \phi],\ [x, \phi]$, and $\mathcal{I}_0$ stands for its evaluation at $\nu = 0$. As a monomial basis for $\mathcal{A}_\nu$ we take the monomials $p^i x^j \psi^k \phi^\ell$ with $i, j \in \mathbb{N}$ and $k, \ell \in \{0, 1\}$. In view of the latter restrictions on k and ℓ it is not a PBW basis. Since in $\mathcal{A}$ the commutator of two elements is proportional to their product, its center is generated by monomials, from which it is clear that $Z(\mathcal{A})$ is generated by $p,\ x$ and $\psi\phi$ and that $\mathcal{A}/Z(\mathcal{A})$ is generated as a $Z(\mathcal{A})$-module by $\overline{\psi}$ and $\overline{\phi}$. According to Proposition 4.1, it follows that $\Pi(\mathcal{A})$ is generated by

$$
P := \left(p, \overline{0}\right)\ ,\quad X := \left(x, \overline{0}\right)\ ,\quad W := \left(\psi\phi, \overline{0}\right)\ ,\quad \Psi := \left(0, \overline{\psi}\right)\ ,\quad \Phi := \left(0, \overline{\phi}\right)\ .
$$

To determine the algebra structure of $\Pi(\mathcal{A})$, we consider the surjective morphism $\mathbb{C}[P, X, W, \Psi, \Phi] \to \Pi(\mathcal{A})$, defined by these generators. Let $\mathcal{J} := \langle \Psi^2, \Phi^2, W^2, \Psi\cdot \Phi, \Psi\cdot W, \Phi\cdot W\rangle$. It is clear that $\mathcal{J}$ is contained in the kernel of this morphism; to show that it coincides with the kernel, it suffices to notice that

$$
P^i \cdot X^j = (p^i x^j, \overline{0})\ , \qquad P^i \cdot X^j \cdot W = (p^i x^j \psi\phi, \overline{0})\ , \tag{4.24}
$$

$$
P^i \cdot X^j \cdot \Psi = (0, \overline{p^i x^j \psi})\ , \qquad P^i \cdot X^j \cdot \Phi = (0, \overline{p^i x^j \phi})\ , \tag{4.25}
$$

which shows that the family $\{P^i \cdot X^j,\ P^i \cdot X^j \cdot W,\ P^i \cdot X^j \cdot \Psi,\ P^i \cdot X^j \cdot \Phi \mid i, j \in \mathbb{N}\}$ is linearly independent. To compute the two products in (4.24), we have used that $(p^i, x^j)_1 = 0$ and $(p^i x^j, \psi\phi)_1 = 0$, for $i, j \in \mathbb{N}$. It follows that $(\Pi(\mathcal{A}), \cdot) \simeq \mathbb{C}[P, X, W, \Psi, \Phi]/\mathcal{J}$. In order to compute the Poisson brackets between the generators, we use the following nontrivial brackets: $\{\phi, \psi\}_1 = 1$, $\{x, p\}_1 = 1$, $\{\psi\phi, \psi\}_1 = \psi$ and $\{\psi\phi, \phi\}_1 = -\phi$. From these formulas and (4.6)–(4.8), the Poisson brackets between the generators of $\Pi(\mathcal{A})$ are easily computed. They are given in Table 7.

$\{\cdot,\cdot\}$	P	X	W	Ψ	Φ
P	0	-1	0	0	0
X	1	0	0	0	0
W	0	0	0	Ψ	$-\Phi$
Ψ	0	0	$-\Psi$	0	0
Φ	0	0	Φ	0	0

Table 7 Poisson brackets between the generators of the Poisson algebra $\Pi(\mathcal{A})$ in the case of the Grassmann algebra

For example,

$$\{P, X\} = \left\{(p, \overline{0}), (x, \overline{0})\right\} = (\{p, x\}_1\,, \overline{0}) = (-1, \overline{0}) = -1\ ,$$

and

$$\{W, \Psi\} = \left\{(\psi\phi, 0), (0, \overline{\psi})\right\} = \left(0, \overline{\{\psi\phi, \psi\}_1}\right) = \left(0, \overline{\psi}\right) = \Psi\ .$$

It follows that the reduced Poisson algebra $Z(\mathcal{A})$ is isomorphic to the polynomial algebra $\mathbb{C}[P, X, W]$ with W as Casimir and $\{X, P\} = 1$. The Poisson module structure on $\mathcal{A}$ is computed similarly and is given in Table 8.

$\cdot$	p	x	ψ	ϕ
P	p^2	px	$p\psi$	$p\phi$
X	px	x^2	$x\psi$	$x\phi$
W	$p\psi\phi$	$x\psi\phi$	0	0
Ψ	0	0	0	0
Φ	0	0	0	0

$\{\cdot\,;\cdot\}$	p	x	ψ	ϕ
P	0	-1	0	0
X	1	0	0	0
W	0	0	ψ	$-\phi$
Ψ	0	0	0	$2\psi\phi$
Φ	0	0	$2\psi\phi$	0

Table 8 Multiplication table and Lie brackets for the generators of the $\Pi(\mathcal{A})$ module $\mathcal{A}$ in the case of the Grassmann algebra

4.6 The algebra $M_q(2)$

The entries of the 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which preserve the quantum plane $\mathbb{C}_q[x, y]$ satisfy the following relations (see [10, Ch. 4]):

$$ba = qab\ , \qquad db = qbd\ , \qquad bc = cb\ ,$$
$$dc = qcd\ , \qquad ca = qac\ , \qquad ad - da = (q^{-1} - q)bc\ .$$

We are therefore led to consider the following quotient algebra:

$$\mathcal{A}_q = \frac{\mathbb{C}(q)\langle a, b, c, d\rangle}{\mathcal{I}_q} \ ,$$

where $\mathcal{I}_q$ is the ideal of $\mathbb{C}(q)$ generated by the following polynomials:

$$ba - qab \ , db - qbd \ , bc - cb \ , dc - qcd \ , ca - qac \ , ad - da - (q^{-1} - q)bc \ . \tag{4.26}$$

It is well-known (and easy to check) that $ad - q^{-1}bc = da - qbc$ is a central element of $\mathcal{A}_q$. As a monomial basis for $\mathcal{A}_q$ we choose the monomials $a^i b^j c^k d^\ell$, with i, j, k and ℓ in $\mathbb{N}$. It is a PBW basis and $q_0 = 0$ is the only nonregular value of $\mathcal{B}$.

We will first consider the evaluation of $\mathcal{A}_q$ to $q = 1$, which is commutative. We do it to show how the classical case of deformations of commutative algebras is treated as a special case of our methods. We set $q(\nu) = 1 + \nu$ and consider

$$\frac{\mathbb{C}[[\nu]]\langle a, b, c, d\rangle}{\mathcal{I}_{q(\nu)}} \simeq \mathcal{A}[[\nu]] \ , \ \text{where } \mathcal{A} = \frac{\mathbb{C}\langle a, b, c, d\rangle}{\mathcal{I}_1} \simeq \mathbb{C}[a, b, c, d] \ .$$

In this case, $Z(\mathcal{A}) = \mathcal{A}$ and we may identify $(\Pi(\mathcal{A}), \cdot)$ with the polynomial algebra $\mathcal{A}$. Under this identification the Poisson algebra $(\Pi(\mathcal{A}), [\cdot\,,\cdot]_\nu)$ and the reduced Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\}_1)$ coincide and only the brackets $\{\cdot\,,\cdot\}_1$ between the elements a, b, c, d need to be computed. These brackets are given in Table 9.

$\{\cdot\,,\cdot\} = \{\cdot\,,\cdot\}_1 = \{\cdot\,;\cdot\}$	a	b	c	d
a	0	$-ab$	$-ac$	$-2bc$
b	ab	0	0	$-bd$
c	ac	0	0	$-cd$
d	$2bc$	bd	cd	0

Table 9 Poisson brackets for the generators of $\mathbb{C}[a, b, c, d]$ corresponding to the deformation of $M_q(2),\ q \to 1$

For example, $\{a, d\} = \{a, d\}_1 = -2bc$ since

$$[a, d] = ad - da = \left(q^{-1} - q\right) bc = ((1 - \nu) - (1 + \nu))bc + \mathcal{O}\left(\nu^2\right) = -2\nu bc + \mathcal{O}\left(\nu^2\right) \ .$$

This Poisson structure has rank 2. Since $ad - q^{-1}bc$ is a central element of $M_q(2)$, $ad - bc$ is a Casimir. Since b/c as also a (rational) Casimir, the Poisson structure can be described as a Nambu–Poisson structure (see [11, Ch. 8.3]).

We now consider the case $q = -1$. We set $q(\nu) = \nu - 1$ and consider

$$\frac{\mathbb{C}[[\nu]]\langle a, b, c, d\rangle}{\mathcal{I}_{q(\nu)}} \simeq \mathcal{A}[[\nu]] \ , \ \text{where } \mathcal{A} = \frac{\mathbb{C}\langle a, b, c, d\rangle}{\mathcal{I}_{-1}} \ .$$

Notice that $\mathcal{A}$ is not commutative since the following relations hold in $\mathcal{A}$:

$$\begin{array}{lll} ba = -ab \ , & db = -bd \ , & bc = cb \ , \\ dc = -cd \ , & ca = -ac \ , & ad = da \ . \end{array} \tag{4.27}$$

It is clear from these relations that $Z(\mathcal{A})$ contains the following 6 elements:

$$a^2, \ b^2, \ c^2, \ d^2, \ ad, \ bc \ . \tag{4.28}$$

We show that these elements generate $Z(\mathcal{A})$. To do this, first notice that $Z(\mathcal{A})$ is generated by monomials, since the commutator of any two monomials is a multiple of their product, as follows from (4.27). Given a monomial in $Z(\mathcal{A})$ we may strip off any even power of a, b, c and d, which leaves us with a monomial of degree 1 at most in each of these variables. Among the 15 possible monomials of this type, it is easily checked that only ad, bc and their product $abcd$ belong to the center. This shows that the six elements (4.28) generate $Z(\mathcal{A})$. It is then clear that $\mathcal{A}/Z(\mathcal{A})$ is generated as a $Z(\mathcal{A})$-module by the following 8 elements: $\overline{a}, \ldots, \overline{d}, \overline{ab}, \overline{ac}, \overline{bd}, \overline{cd}$. According to Proposition 4.1, it follows that the following elements generate $(\Pi(\mathcal{A}), \cdot)$:

$$\begin{gathered} U_1 = \left(a^2, \overline{0}\right) \ , U_2 = \left(b^2, \overline{0}\right) \ , U_3 = \left(c^2, \overline{0}\right) \ , U_4 = \left(d^2, \overline{0}\right) \ , U_5 = \left(ad, \overline{0}\right) \ , U_6 = \left(bc, \overline{0}\right) \ , \\ V_1 = (0, \overline{a}) \ , \ V_2 = \left(0, \overline{b}\right) \ , \ V_3 = (0, \overline{c}) \ , \ V_4 = \left(0, \overline{d}\right) \ , \\ W_1 = \left(0, \overline{ab}\right) \ , \ W_2 = (0, \overline{ac}) \ , \ W_3 = \left(0, \overline{bd}\right) \ , \ W_4 = \left(0, \overline{cd}\right) \ . \end{gathered} \tag{4.29}$$

We now have all elements needed to determine the algebra structure of $(\Pi(\mathcal{A}), \cdot)$, to compute the Poisson brackets of the 14 generators $U_1, \ldots, W_4$ of $\Pi(\mathcal{A})$ and to compute the Lie action of these generators on the four generators a, b, c, d of $\mathcal{A}$. This can be done as in the previous examples by setting $q(\nu) = \nu - 1$ and computing the first terms of the commutators $[\cdot\,,\cdot]_\star$ and $[\cdot\,,\cdot]$ and using (4.6)–(4.8) and (4.10). We will only describe here the reduced Poisson algebra $Z(\mathcal{A})$. To do this, let us write $U_i = (u_i, \overline{0})$ for $i = 1, \ldots, 6$, so that $Z(\mathcal{A})$ is generated by $u_1, \ldots, u_6$: as an associative algebra, it is clear that $Z(\mathcal{A}) \simeq \mathbb{C}[u_1, \ldots, u_6]/\langle u_5^2 - u_1u_4, u_6^2 - u_2u_3\rangle$. The reduced brackets, which are the brackets $\{\cdot\,,\cdot\}_1$ of $(Z(\mathcal{A}), \{\cdot\,,\cdot\}_1)$, are given in Table 10.

For example, $\{u_1, u_5\}_1 = -4u_1u_6$ since $\left\{a^2, ad\right\}_1 = -4a^2bc$, as follows from

$$\begin{aligned} \left[a^2, ad\right] &= a^3d - ada^2 = a^3d - a(ad + 2\nu bc)a + \mathcal{O}\left(\nu^2\right) \\ &= a^3d - a^2(ad + 2\nu bc) - 2\nu a^2bc + \mathcal{O}\left(\nu^2\right) = -4\nu a^2bc + \mathcal{O}\left(\nu^2\right) \ . \end{aligned}$$

$\{\cdot,\cdot\}_1$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0	$4u_1u_2$	$4u_1u_3$	$-8u_5u_6$	$-4u_1u_6$	$4u_1u_6$
u_2	$-4u_1u_2$	0	0	$4u_2u_4$	0	0
u_3	$-4u_1u_3$	0	0	$4u_3u_4$	0	0
u_4	$8u_5u_6$	$-4u_2u_4$	$-4u_3u_4$	0	$4u_4u_6$	$-4u_4u_6$
u_5	$4u_1u_6$	0	0	$-4u_4u_6$	0	0
u_6	$-4u_1u_6$	0	0	$4u_4u_6$	0	0

Table 10 Reduced brackets for the generators of $Z(M_{-1}(2))$

Since $ad - q^{-1}bc$ is a central element of $\mathcal{A}_q$, $(ad+bc, \overline{bc/2}) = (ad+bc, \overline{0}) = U_5 + U_6$ is a Casimir of $\Pi(\mathcal{A})$. This is reflected in Table 10 and it reduces the number of Poisson brackets to be computed, since $\{\cdot, U_5\} = -\{\cdot, U_6\}$, so that $\{\cdot, u_5\}_1 = -\{\cdot, u_6\}_1$.

On $\mathbb{C}^6$, with coordinates $u_1, \dots, u_6$, Table 10 gives the Poisson matrix of a Poisson structure of rank 2, with Casimirs $C_1 = u_5^2 - u_1u_4$, $C_2 = u_6^2 - u_2u_3$, $C_3 = u_2/u_3$ and $C_4 = u_5 + u_6$. The Poisson structure on $Z(\mathcal{A})$ can therefore also be described as the Nambu–Poisson structure on $\mathbb{C}^6$ with respect to these Casimirs, restricted to the subvariety defined by $C_1 = C_2 = 0$.

One considers similarly the Poisson algebra and Poisson module structure for any primitive n-th root of unity. When $n > 2$, the bracket $\overline{\{a, b\}_2}$ is not zero for some $a, b \in Z(\mathcal{A})$, as already pointed out in Sect. 2.5. See [16] for other examples that are nonflat in that sense.

4.7 Nonequivalent deformations

As it turns out, we have encountered in the above examples three isomorphic algebras $\mathcal{A}$ and three deformations of it. Indeed, taking $N = 4$ and $q_0 = -1$ in (4.13), $\mathcal{I}_{q_0} = \langle x_{i+1}x_i + x_ix_{i+1}, x_ix_j - x_jx_i\rangle_{d_N(i,j)\neq 1}$, so that $\mathcal{A} := \mathbb{C}\langle x_1, \dots, x_4\rangle/\mathcal{I}_{q_0}$ is defined by

$$x_2x_1 = -x_1x_2\ , \qquad x_3x_2 = -x_2x_3\ , \qquad x_4x_2 = x_2x_4\ ,$$
$$x_4x_3 = -x_3x_4\ , \qquad x_4x_1 = -x_1x_4\ , \qquad x_3x_1 = x_1x_3\ .$$

These relations are the same relations as (4.27) under the correspondence $a \leftrightarrow x_1$, $b \leftrightarrow x_2$, $c \leftrightarrow x_4$ and $d \leftrightarrow x_3$. We will use the reduced Poisson algebra to show that the two corresponding deformations are not equivalent, as an application of Corollary 2.25. See Remark 4.7 below for a third deformation of $\mathcal{A}$. In order to compare the two Poisson algebras, it will be useful to use the same notation for the generators of $Z(\mathcal{A})$, so we will use $U_1, \ldots, U_6$, as in (4.29), which means that

$$U_1 = X_1 = (x_1^2, \overline{0}) \ , \ U_2 = X_2 = (x_2^2, \overline{0}) \ , \ \ U_3 = X_4 = (x_4^2, \overline{0}) \ ,$$
$$U_4 = X_3 = (x_3^2, \overline{0}) \ , \ U_5 = Y = (x_1 x_3, \overline{0}) \ , \ U_6 = Z = (x_2 x_4, \overline{0}) \ .$$

We will also again write $U_i = (u_i, \overline{0})$ for $i = 1, \ldots, 6$, so that $u_1, \ldots, u_N$ generates $Z(\mathcal{A})$. As we have seen in Sect. 4.3, Y and Z are Casimirs of $\Pi(\mathcal{A})$ and the nonzero brackets between the X_i are given by $\{X_k, X_{k\pm1}\} = \pm 2 X_k X_{k\pm1}$. In terms of $u_1, \ldots, u_4$ it leads to the Poisson brackets, given in Table 11; we did not add the brackets with u_5 and u_6 to the table because they are Casimirs.

$\{\cdot\,,\cdot\}' = \{\cdot\,,\cdot\}'_1$	u_1	u_2	u_3	u_4
u_1	0	$4u_1u_2$	$-4u_1u_3$	0
u_2	$-4u_1u_2$	0	0	$4u_2u_4$
u_3	$4u_1u_3$	0	0	$-4u_3u_4$
u_4	0	$-4u_2u_4$	$4u_3u_4$	0

Table 11 Reduced brackets corresponding to the case of the periodic Volterra chain ($N = 4$) and $q \to -1$

We denote this Poisson structure on $Z(\mathcal{A})$ by $\{\cdot\,,\cdot\}'_1$. In order to compare the Poisson structures $\{\cdot\,,\cdot\}_1$ and $\{\cdot\,,\cdot\}'_1$ on $Z(\mathcal{A})$, we look at their singular locus, which is by definition the locus where the rank of the Poisson structure drops. In our examples, the rank is two so the singular locus consists of the points where the rank is zero, which amounts to considering in both cases the ideal generated by the entries of the table. We are therefore led to consider the following two Poisson ideals of $Z(\mathcal{A})$:

$$\mathcal{J} := \langle u_1u_2, u_1u_3, u_2u_4, u_3u_4, u_1u_6, u_4u_6, u_5u_6 \rangle \ ,$$
$$\mathcal{J}' := \langle u_1u_2, u_1u_3, u_2u_4, u_3u_4 \rangle \ .$$

It is clear that $\mathcal{J}'$ is strictly contained in $\mathcal{J}$. Moreover, both ideals have the same radical, since

$$(u_1u_6)^2 = u_1^2u_2u_3 = (u_1u_2)(u_1u_3) \in \mathcal{J}' \ ,$$
$$(u_4u_6)^2 = u_4^2u_2u_3 = (u_2u_4)(u_3u_4) \in \mathcal{J}' \ ,$$
$$(u_5u_6)^2 = u_1u_2u_3u_4 \in \mathcal{J}' \ ,$$

where we have used several times that in $\mathcal{A}$ the following relations hold: $u_2u_3 = u_6^2$ and $u_1u_4 = u_5^2$. It follows that the two reduced Poisson algebras are not isomorphic. In view of Corollary 2.25, the two deformations $(\mathcal{A}[[\nu]], \star)$ and $(\mathcal{A}[[\nu]], \star')$ of $\mathcal{A}$ are nonequivalent.

Remark 4.7 Taking $n = 4$ in Sect. 4.4 we get again the same algebra $\mathcal{A}$, since we find the same relations as (4.27) under the correspondence $a \leftrightarrow x_1$, $d \leftrightarrow x_2$, $c \leftrightarrow x_3$ and $b \leftrightarrow x_4$. It can be shown using the same methods that the deformation of $\mathcal{A}$ is in this case again nonequivalent to the two other deformations of $\mathcal{A}$ of which we have shown in this section that they are nonequivalent.

5 Hamiltonian derivations

Most of the quantum algebras that we have considered in the previous section appear in the literature as quantum algebras for some nontrivial derivation, defining a nonabelian system on a free algebra. These derivations can in general be written as Heisenberg derivations having nontrivial limits for certain values of the deformation parameter q. We will show in this section that the corresponding limiting derivations are Hamiltonian derivations with respect to the commutative Poisson algebra $(\Pi(\mathcal{A}), \{\cdot\,,\cdot\})$ that we have introduced in Sect. 2, and that they can be easily computed using the formulas for the actions of $\Pi(\mathcal{A})$ on the Poisson module $\mathcal{A}$ that we have computed for these examples in Sect. 4. As in the previous section, we take $R = \mathbb{C}$ as our base ring.

5.1 The limiting procedure

In the paragraphs which follow we will apply the above results to several nonabelian chains. We outline the procedure in separate steps.

Step 1 Start from a nonabelian system (derivation) $\partial : \mathfrak{A} \to \mathfrak{A}$ on a free associative algebra $\mathfrak{A} = \mathbb{C}\langle x_1,\ x_2,\ \dots\rangle$. Many interesting such systems are known [4]. Often, they are *evolutionary*, which means that the derivation ∂ is invariant under the shift $x_\ell \mapsto x_{\ell+k}$, for a fixed k. In our case there will be a finite number of variables $x_1, \dots, x_N$ since the index ℓ of x_ℓ is taken modulo N, so it still makes sense for ∂ to be evolutionary.

Step 2 Choose a quantization ideal $\mathcal{I}_q$ of $(\mathfrak{A}(q), \partial)$, depending on a single parameter q, where we recall that $\mathfrak{A}(q) = \mathbb{C}(q)\langle x_1,\ x_2,\ \dots\rangle$. Many such ideals are known [4]. Denote by $\mathcal{B}$ a basis of normally ordered monomials of the quantum $\mathbb{C}(q)$-algebra $\mathfrak{A}(q)/\mathcal{I}_q$. On this algebra, ∂ induces a derivation which may be evolutionary or not, depending on whether or not $\mathcal{I}_q$ is invariant under the shift $x_\ell \mapsto x_{\ell+k}$ for fixed k.

Step 3 Write the equation for ∂ on $\mathcal{A}_q = \mathfrak{A}(q)/\mathcal{I}_q$ in the Heisenberg form

$$\partial a = \frac{1}{\lambda(q)}\,[\mathfrak{H}(q), a]\ , \tag{5.1}$$

where $a \in \mathcal{A}_{\mathbf{q}}$. This is a nontrivial task but again many examples have been written in this form. In this formula, $\mathfrak{H}(q) \in \mathcal{A}_q$; it may be assumed that $\mathfrak{H}(q)$ and $\lambda(q)$ are polynomials in q and have no common nonconstant factor.

Step 4 We can specialize q to any regular value q_0, but as we will see the most interesting choice for q_0 is to choose a simple root of $\lambda(q)$. As before, we write

$q(\nu) = q_0 + \nu$. We recall that $\mathcal{I}_\nu$ stands for the closed ideal of $\mathfrak{A}[[\nu]]$ that corresponds with $\mathcal{I}_{q(\nu)}$. As we have shown in Proposition 3.4, $\mathcal{A}_\nu := \mathfrak{A}[[\nu]]/\mathcal{I}_\nu \simeq \mathcal{A}[[\nu]]$, where $\mathcal{A}$ is the evaluation of $\mathcal{A}_q$ at q_0. Since q_0 is a simple root of $\lambda(q)$, $\frac{\mathfrak{H}(q_0+\nu)}{\lambda(q_0+\nu)} \in \frac{1}{\nu}\mathcal{A}[[\nu]]$ and (5.1) takes in terms of ν the Heisenberg form

$$\delta_H a = \frac{1}{\nu}\,[H, a]_\star = \frac{1}{\nu}\left[H_0 + \nu H_1 + \nu^2 H_2 + \cdots, a\right]_\star \,, \tag{5.2}$$

where $a, H \in \mathcal{A}[[\nu]]$, with $H = H_0 + \nu H_1 + \nu^2 H_2 + \cdots$, i.e., all H_i are elements of $\mathcal{A}$ which belong to the $\mathbb{C}$-span of $\mathcal{B}$. If ∂ is evolutionary (on $\mathcal{A}_{\mathbf{q}}$), then so is δ_H (on $\mathcal{A}[[\nu]]$). Since the left-hand side of (5.2) is a formal power series in ν, it follows that H_0 commutes with any element a of $\mathcal{A} \subset \mathcal{A}[[\nu]]$, hence $H_0 \in Z(\mathcal{A})$ and $\mathbf{H} := \left(H_0, \overline{H_1}\right) \in \Pi(\mathcal{A})$. Notice that, in order to determine $\mathbf{H}$ it suffices to compute in (5.2) the leading terms H_0 and H_1, the latter up to the center $Z(\mathcal{A})$ of $\mathcal{A}$. We will therefore compute and write

$$\frac{\mathfrak{H}(q(\nu))}{\lambda(q(\nu))} = \frac{1}{\nu}(H_0 + \nu H_1) \pmod{\mathcal{H}_\nu} \,, \tag{5.3}$$

which suffices to determine $\mathbf{H}$.

Step 5 As we have shown in Sect. 2.4, the limit $\nu \to 0$ of the Heisenberg derivation (5.2) is the Hamiltonian derivation $\partial_{\mathbf{H}}$ on $\mathcal{A}$, which can be computed directly from $\partial_{\mathbf{H}} a = \{\mathbf{H}\,; a\}$, where we recall that $\{\cdot\,;\cdot\}$ denotes the Lie action of $\Pi(\mathcal{A})$ on $\mathcal{A}$. Let $U_1, \ldots, U_M$ denote a system of algebra generators of $\Pi(\mathcal{A})$ (recall that, as an algebra, $\Pi(\mathcal{A})$ is commutative). We write $\mathbf{H}$ in terms of the generators of $\Pi(\mathcal{A})$, $\mathbf{H} = \mathbf{H}(U_1, \ldots, U_M)$. Since $\{\cdot\,;\cdot\}$ is a derivation in its first argument and since the left and right actions of $\Pi(\mathcal{A})$ on $\mathcal{A}$ coincide (see (2.16)), $\partial_{\mathbf{H}} x_\ell$ can be computed from

$$\partial_{\mathbf{H}} x_\ell = \{\mathbf{H}\,; x_\ell\} = \sum_{i=1}^{M} \frac{\partial \mathbf{H}}{\partial U_i} \cdot \{U_i\,; x_\ell\} \,, \tag{5.4}$$

where we recall that $\cdot$ denotes the left (= right) action of $\Pi(\mathcal{A})$ on $\mathcal{A}$; notice that (2.16)) says in particular that the action $\cdot$ of $\{0\} \times \mathcal{A}/Z(\mathcal{A}) \subset \Pi(\mathcal{A})$ on $\mathcal{A}$ is trivial which permits to largely simplify the use of (5.4) in explicit computations. The brackets $\{U_i\,; x_\ell\}$ between the generators of $\Pi(\mathcal{A})$ and of $\mathcal{A}$ have been computed for several examples in Sect. 4. Notice that the computations are done for the variables x_ℓ (and their projections on the different quotient algebras), rather than for arbitrary elements a of $\mathfrak{A} = \mathbb{C}\langle x_1,\ x_2,\ \ldots\rangle$ or of $\mathfrak{A}(q) = \mathbb{C}(q)\langle x_1,\ x_2,\ \ldots\rangle$. On the one hand, it leads to simpler explicit formulas that are easier to compute and to present, while on the other hand these formulas completely determine the derivation $\partial_{\mathbf{H}}$ of all of $\mathcal{A}$ (hence on $\mathcal{A}[[\nu]]$), because $\partial_{\mathbf{H}}$ is a derivation of $\mathcal{A}$ (Proposition 2.19). Moreover, when $\delta_{\mathbf{H}}$ is evolutionary, say invariant for the shift $x_\ell \mapsto x_{\ell+k}$ then so is $\partial_{\mathbf{H}}$ and it suffices to compute $\delta_{\mathbf{H}} x_\ell$ for $k-1$ successive values of ℓ to know it for all ℓ; as we will see this also simplifies some of the computations.

Remark 5.1 Suppose there exists an element $\hat{\mathfrak{H}}(q) \in \mathcal{A}_q$ that commutes with $\mathfrak{H}(q)$ and is not in $Z(\mathcal{A}(q))$. The corresponding derivation $\hat{\partial} a = \frac{1}{\lambda(q)}\left[\hat{\mathfrak{H}}(q), a\right]$ commutes with ∂ and represents a symmetry for the quantum system (5.1). Furthermore, the leading term in the expansion $\frac{\hat{\mathfrak{H}}(q(\nu))}{\lambda(q(\nu))} = \frac{1}{\nu}(\hat{H}_0 + \nu \hat{H}_1)$ becomes a first integral of the Hamiltonian equation (5.4). If $\hat{H}_0 \neq 0$, then $\partial_{\mathbf{H}}(\hat{H}_0) = 0$ (see Proposition 2.20). In the case where $\hat{H}_0 = 0$, it follows that $\partial_{\mathbf{H}}(\hat{H}_1) = 0$. If $\hat{\mathfrak{H}}(q) \in Z(\mathcal{A}_q)$, then $\hat{\mathbf{H}} := \left(\hat{H}_0, \overline{\hat{H}_1}\right)$ is a Casimir of the Poisson structure (see Proposition 4.2), and therefore, $\hat{H}_0$ is a first integral of (5.4).

5.2 *N*-periodic nonabelian Volterra hierarchy

The *nonabelian Volterra chain* [4] is the derivation of $\mathfrak{A} = \langle x_\ell;\, \ell \in \mathbb{Z}\rangle$, given by

$$\partial_1 x_\ell = x_\ell x_{\ell+1} - x_{\ell-1} x_\ell \ . \tag{5.5}$$

It has an infinite family of commuting derivations $\partial_2, \partial_3, \dots$, forming the so-called *nonabelian Volterra hierarchy*. We consider here the N-periodic case, that is $N \geqslant 3$ and $x_{N+\ell} = x_\ell$ for all ℓ. It was shown in [4] that all members of this hierarchy admit a common quantization ideal, namely the ideal $\mathcal{I}_q$ of $\mathfrak{A}(q)$ generated by all

$$x_{i+1}x_i - q x_i x_{i+1}\ ,\ x_i x_j - x_j x_i\ ,\quad (d_N(i,j) \neq 1)\ .$$

It is the quantization ideal which we studied in Sect. 4.3, see in particular (4.13). Notice that $\mathcal{I}_q$ is invariant under the shift $x_\ell \mapsto x_{\ell+1}$. We determine some nontrivial limits of the (evolutionary) derivations ∂_n of the quantum algebra $\mathcal{A}_q = \mathfrak{A}(q)/\mathcal{I}_q$. Recall that we use the monomials $x_1^{i_1} \dots x_N^{i_N}$ with $i_1, \dots, i_N \in \mathbb{N}$ as a monomial basis for $\mathcal{A}_q$.

To do this we use the results from [5], where the full hierarchy on $\mathcal{A}_q$ is written in the Heisenberg form

$$\partial_n x_\ell = \frac{1}{q^n - 1}\left[\mathfrak{H}^{(n)}, x_\ell\right]\ ,\qquad \ell = 1, \dots, N\ , \tag{5.6}$$

and where the Hamiltonians $\mathfrak{H}^{(n)}$ are given explicitly for any n. Here we will use the first three Hamiltonians

$$\mathfrak{H}^{(1)} = \sum_{k=1}^{N} x_k\ ,$$

$$\mathfrak{H}^{(2)} = \sum_{k=1}^{N} x_k^2 + (1+q)\sum_{k=1}^{N} x_k x_{k+1}\ ,$$

$$\mathfrak{H}^{(3)} = \sum_{k=1}^{N} x_k^3 + (1+q+q^2)\sum_{k=1}^{N}\left(x_k x_{k+1} x_{k+2} + x_k x_{k+1}^2 + x_k^2 x_{k+1}\right)\ .$$

As we already pointed out in Remark 4.4, when $q \to 1$, the algebra $\mathcal{A}$ is commutative and the Poisson brackets are given by $\left\{x_i, x_j\right\} = (\delta_{i,j+1} - \delta_{i,j-1})x_i x_j$, for $1 \leqslant i, j \leqslant N$. We are then in the classical case, see Remark 2.12.

We therefore start with the case of $n = 2$ and $q = -1$, so we put $q(\nu) = \xi + \nu = -1 + \nu$. Then

$$\frac{\mathfrak{H}^{(2)}}{q(\nu)^2 - 1} = -\frac{1}{2\nu}\left(\sum_{k=1}^{N} x_k^2 + \nu \sum_{k=1}^{N} x_k x_{k+1}\right) \quad (\text{mod } \mathcal{H}_\nu) \ .$$

Setting $\mathbf{H}^{(2)} = \left(H_0^{(2)}, \overline{H_1^{(2)}}\right)$, where $H_0^{(2)} = -\frac{1}{2}\sum_{k=1}^{N} x_k^2$ and $H_1^{(2)} = -\frac{1}{2}\sum_{k=1}^{N} x_k x_{k+1}$, we need to compute $\left\{\mathbf{H}^{(2)} \, ; x_\ell\right\}$ for $\ell = 1, \ldots, N$, which we will do for $N > 4$. Since ∂_2 is evolutionary (under $x_\ell \mapsto x_{\ell+1}$), it suffices to do the computation for a particular ℓ, so we may assume that $2 < \ell < N-1$. To do this, we first write $\mathbf{H}^{(2)}$ in terms of the generators X_k and W_I for $\Pi(\mathcal{A})$ that we have constructed in Sect. 4.3. It will be convenient to write W_k as a shorthand for $W_{0,\ldots,0,1,1,0,\ldots,0}$ where the two 1's are at positions k and $k+1$ (with the understanding that when $k = N$ then they are at positions N and 1). Then $\mathbf{H}^{(2)} = -\frac{1}{2}\sum_{k=1}^{N}(X_k + W_k) + W_N$. Since $n = 2$ and $\xi = -1$, the Lie action of $\Pi(\mathcal{A})$ on $\mathcal{A}$, given in (4.16) and (4.17), specializes for $1 < \ell < N$ to

$$\begin{aligned}
\{X_k \, ; x_\ell\} &= 2(\delta_{k,\ell-1} - \delta_{k,\ell+1})x_k^2 x_\ell \ , \\
\{W_k \, ; x_\ell\} &= ((-1)^{i_{\ell+1}} - (-1)^{i_{\ell-1}})x_1^{i_1} \ldots x_\ell^{i_\ell+1} \ldots x_N^{i_N} \ ,
\end{aligned}$$

where $i_k = i_{k+1} = 1$ and all other i_s are zero. Notice that these brackets are zero when k and ℓ are far enough apart, and also that $\{X_\ell \, ; x_\ell\} = 0$. It follows that, if $2 < \ell < N - 1$ then

$$\begin{aligned}
\partial_{\mathbf{H}^{(2)}} x_\ell &= -(\{X_{\ell-1} + X_{\ell+1} \, ; x_\ell\} - \{W_{\ell-2} + W_{\ell-1} + W_\ell + W_{\ell+1} \, ; x_\ell\})/2 \\
&= (x_{\ell+1}^2 - x_{\ell-1}^2)x_\ell - (x_{\ell-2}x_{\ell-1}x_\ell + x_{\ell-1}x_\ell^2 - x_\ell^2 x_{\ell+1} - x_\ell x_{\ell+1} x_{\ell+2}) \\
&= x_\ell x_{\ell+1}(x_{\ell+2} + x_{\ell+1} - x_\ell) - (x_{\ell-2} + x_{\ell-1} - x_\ell)x_{\ell-1}x_\ell \ , \qquad (5.7)
\end{aligned}$$

where we have used in the last step the commutation relations $x_{\ell+1}x_\ell = -x_\ell x_{\ell+1}$ to write the result in a symmetric form. Since ∂_2 is evolutionary, this formula is valid for all ℓ.

When $n = 1$, the denominator of (5.6) does not vanish at $q = -1$, and setting $q = \nu - 1$ leads to the limiting derivation $\partial_{\mathbf{H}^{(1)}}$ on $\mathcal{A}$, where $\mathbf{H}^{(1)} = \left(0, \overline{H_1^{(1)}}\right)$, with $H_1^{(1)} = \sum_{k=1}^{N} x_k$. It is given by $\partial_{\mathbf{H}^{(1)}} x_\ell = x_\ell x_{\ell+1} - x_{\ell-1}x_\ell$ for $\ell = 1, \ldots, N$. Since ∂_1 and ∂_2 commute, so do $\partial_{\mathbf{H}^{(1)}}$ and $\partial_{\mathbf{H}^{(2)}}$; this follows also from the fact that $\left\{\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\right\} = 0$, see Proposition 2.20. The same remark applies to all odd derivations ∂_{2m+1} and their limiting derivations $\partial_{\mathbf{H}^{(2m+1)}}$ on $\mathcal{A}$, where $m \in \mathbb{N}$. The Hamiltonian system (5.7) has first integrals $H_1^{(2k-1)}$, $H_0^{(2k)}$, $k = 1, 2, \ldots$ (see Remark 5.1).

We now consider $n = 3$, with ξ a primitive cubic root of unity. As above we will only do this for $N > 6$. We still use the results of Sect. 4.3 and put $q(\nu) = \xi + \nu$. Then

$$\frac{1}{q(\nu)^3 - 1} = \frac{\xi}{3\nu} + \mathcal{O}(1)\ , \quad \text{and} \quad \frac{1}{q(\nu) - 1} = \frac{1}{\xi - 1} + \mathcal{O}(\nu)\ ,$$

and

$$\frac{\mathfrak{H}^{(3)}}{q(\nu)^3 - 1} = \frac{\xi}{3\nu}\sum_{k=1}^{N} x_k^3 + \frac{1}{\xi - 1}\sum_{k=1}^{N}(x_k x_{k+1} x_{k+2} + x_k x_{k+1}^2 + x_k^2 x_{k+1}) \quad (\text{mod } \mathcal{H}_\nu),$$

so that $\mathbf{H}^{(3)} = \left(H_0^{(3)}, H_1^{(3)}\right)$, with

$$H_0^{(3)} = \frac{\xi}{3}\sum_{k=1}^{N} x_k^3\ , \quad \text{and} \quad H_1^{(3)} = \frac{1}{\xi - 1}\sum_{k=1}^{N}(x_k x_{k+1} x_{k+2} + x_k x_{k+1}^2 + x_k^2 x_{k+1})\ . \tag{5.8}$$

It suffices again to compute $\left\{\mathbf{H}^{(3)}\,;\,x_\ell\right\}$ for a particular value of ℓ since ∂_3 is evolutionary (under $x_\ell \mapsto x_{\ell+1}$). In order to take care of the terms in $H_1^{(3)}$, let us write W_k as a shorthand for $W_{0,\dots,0,1,1,1,0,\dots,0}$, where the three 1's are at positions $k-1$, k, $k+1$, W_k' as a shorthand for $W_{0,\dots,0,1,2,0,\dots,0}$ where the 1 is at position k and W_k'' for $W_{0,\dots,0,2,1,0,\dots,0}$ where the 2 is at position k. In terms of this notation, $\mathbf{H}^{(3)}$ can be written as

$$\begin{aligned}\mathbf{H}^{(3)} = \frac{\xi}{3}\sum_{k=1}^{N} X_k + \frac{1}{\xi - 1}\Bigg(&\sum_{k=1}^{N-2}(W_{k+1} + W_k' + W_k'') + \xi^{-1}(W_1 + W_N) + W_{N-1}' \\ &+ W_{N-1}'' + \xi(W_N' + W_N'')\Bigg).\end{aligned}$$

By our choice of ℓ we will manage that the 6 boundary terms which appear above play no role in the computation. In order to compute $\partial_{\mathbf{H}^{(3)}} x_\ell = \left\{\mathbf{H}^{(3)}\,;\,x_\ell\right\}$, we need the following brackets, which are a specialization of (4.16) and (4.17) for $1 < \ell < N$,

$$\{X_k\,;\,x_\ell\} = 3(\delta_{k,\ell+1} - \delta_{k,\ell-1})\xi^{-1} x_k^3 x_\ell\ , \tag{5.9}$$

$$\{W_k\,;\,x_\ell\} = (\xi^{i_{\ell+1}} - \xi^{i_{\ell-1}}) x_1^{i_1} \dots x_\ell^{i_\ell+1} \dots x_N^{i_N}\ , \tag{5.10}$$

where the latter formula is also valid for W_k' and W_k'', each time upon using the proper values for the indices i_ℓ; for example, in the case of W_k, all indices are zero except $i_{k-1} = i_k = i_{k+1} = 1$. In view of (5.9) and (5.10), $\{X_\ell\,;\,x_\ell\} = \{W_\ell\,;\,x_\ell\} = 0$. Also, the only nonzero brackets $\left\{W_k'\,;\,x_\ell\right\}$ and $\left\{W_k''\,;\,x_\ell\right\}$ are for $k = \ell-2, \dots, \ell+1$, while the only nonzero brackets $\{W_k\,;\,x_\ell\}$ are for $k = \ell \pm 1$ and $k = \ell \pm 2$. Let $3 < \ell < N-2$, where we recall that $N > 6$. Then

$$\partial_{\mathbf{H}^{(3)}} x_\ell = \frac{\xi}{3}\left\{X_{\ell-1} + X_{\ell+1}\,;\,x_\ell\right\} + \frac{1}{\xi - 1}\left\{W_{\ell-2} + W_{\ell-1} + W_{\ell+1} + W_{\ell+2}\,;\,x_\ell\right\}$$

$$+\frac{1}{\xi-1}\left\{W'_{\ell-2}+W'_{\ell-1}+W'_{\ell}+W'_{\ell+1}+W''_{\ell-2}+W''_{\ell-1}+W''_{\ell}+W''_{\ell+1}\,;x_\ell\right\}$$
$$=\left(x_{\ell+1}^3-x_{\ell-1}^3\right)x_\ell+(1+\xi)(x_\ell x_{\ell+1}^2x_{\ell+2}-x_{\ell-2}x_{\ell-1}^2x_\ell+x_\ell^2x_{\ell+1}^2-x_{\ell-1}^2x_\ell^2)$$
$$+x_\ell x_{\ell+1}x_{\ell+2}x_{\ell+3}-x_{\ell-3}x_{\ell-2}x_{\ell-1}x_\ell+x_\ell^2x_{\ell+1}x_{\ell+2}-x_{\ell-2}x_{\ell-1}x_\ell^2$$
$$+x_\ell x_{\ell+1}x_{\ell+2}^2-x_{\ell-2}^2x_{\ell-1}x_\ell+x_\ell^3x_{\ell+1}-x_{\ell-1}x_\ell^3\,.$$

The formulas that we have computed for the limiting derivations are also valid for the infinite (nonperiodic) case are the same, with the same proof (see Remark 4.5).

5.3 2*M*-periodic even nonabelian Volterra hierarchy: another quantization

We now consider another quantization ideal of the even elements $\delta_n=\delta_{2m}$ of the N-periodic nonabelian Volterra hierarchy, in case $N>2$ is even, $N=2M$ (see [4]). The ideal $\mathcal{I}_q$ of $\mathfrak{A}(q)$ is now generated by all

$$x_{i+1}x_i-(-1)^i q x_i x_{i+1}\ ,\ x_ix_j+x_jx_i\ ,\quad (d_N(i,j)\neq 1)\ . \tag{5.11}$$

We have already considered this quantization ideal, the corresponding quantum algebras $\mathcal{A}_q=\mathfrak{A}(q)/\mathcal{I}_q$ and its evaluation at $q=1$ in Sect. 4.4; we will use here the Poisson brackets from that section to obtain the limiting derivation of ∂_2 (see (5.6)) when $q\to 1$. The derivation ∂_2 is given on $\mathcal{A}_q$ in Heisenberg form by

$$\partial_2 x_\ell=\frac{1}{q^2-1}\,[\mathfrak{H},x_\ell]=\frac{1}{q^2-1}\left[\sum_{k=1}^{N}\left(x_k^2+(1+(-1)^kq)x_kx_{k+1}\right),x_\ell\right]\,.$$

We put, as in Sect. 4.4, $q(\nu)=1+\nu$. Then

$$\frac{1}{q(\nu)^2-1}=\frac{1}{2\nu}+\mathcal{O}(\nu^0)\ ,\ \text{and}\ \ 1+(-1)^kq(\nu)=\begin{cases}2+\nu & k\text{ even,}\\ -\nu & k\text{ odd.}\end{cases}$$

and

$$\frac{\mathfrak{H}}{q(\nu)^2-1}=\frac{1}{2\nu}\left(\sum_{k=1}^{N}x_k^2+\sum_{j=1}^{M}x_{2j}x_{2j+1}-\frac{\nu}{2}\sum_{j=1}^{M}x_{2j-1}x_{2j}\right)\quad(\text{mod }\mathcal{H}_\nu)\ ,$$

so that

$$H_0=\frac{1}{2}\sum_{k=1}^{N}x_k^2+\sum_{j=1}^{M}x_{2j}x_{2j+1}\ \text{ and }H_1=-\frac{1}{2}\sum_{j=1}^{M}x_{2j-1}x_{2j}\ .$$

Let us write W_k as a shorthand for $W_{0,\ldots,0,1,1,0,\ldots,0}$ where the two 1's are at positions $2k-1$ and $2k$. Then

$$\mathbf{H} = \frac{1}{2}\sum_{k=1}^{N} X_k + \sum_{j=1}^{M-1} Y_j - Y_M - \frac{1}{2}\sum_{j=1}^{M} W_j \ ,$$

where $X_1, \ldots, X_N, Y_1, \ldots, Y_M$ are the generators of $\Pi(\mathcal{A})$, constructed in Sect. 4.4. We need to compute $\partial_{\mathbf{H}} x_\ell$, which we do for even ℓ only, the computation for odd ℓ being very similar. Then the only nonzero brackets $\{X_k\,;x_\ell\}$, $\left\{Y_j\,;x_\ell\right\}$ and $\{W_k\,;x_\ell\}$ are according to (4.21)–(4.23) given by

$$\begin{aligned}
\{X_{\ell-1}\,;x_\ell\} &= -2x_{\ell-1}^2 x_\ell \ , & \{X_{\ell+1}\,;x_\ell\} &= 2x_\ell x_{\ell+1}^2 \ ,\\
\left\{Y_{\ell/2-1}\,;x_\ell\right\} &= -x_{\ell-2}x_{\ell-1}x_\ell \ , & \left\{Y_{\ell/2}\,;x_\ell\right\} &= x_\ell^2 x_{\ell+1} \ ,\\
\left\{W_{\ell/2-1}\,;x_\ell\right\} &= 2x_{\ell-1}x_\ell^2 \ , & \left\{W_{\ell/2+1}\,;x_\ell\right\} &= -2x_\ell x_{\ell+1}x_{\ell+2} \ .
\end{aligned}$$

Suppose that $1 < \ell < N$ (recall that ℓ and N are even). Then

$$\begin{aligned}
\partial_{\mathbf{H}} x_\ell &= \frac{1}{2}\{X_{\ell-1}+X_{\ell+1}\,;x_\ell\} + \left\{Y_{\ell/2-1}+Y_{\ell/2}\,;x_\ell\right\} - \frac{1}{2}\left\{W_{\ell/2-1}+W_{\ell/2}\,;x_\ell\right\}\\
&= x_\ell x_{\ell+1}^2 - x_{\ell-1}^2 x_\ell + x_\ell^2 x_{\ell+1} - x_{\ell-2}x_{\ell-1}x_\ell + x_\ell x_{\ell+1}x_{\ell+2} - x_{\ell-1}x_\ell^2 \ .
\end{aligned}$$

Since the Volterra hierarchy and the ideal are invariant under the shift $x_i \mapsto x_{i+2}$, the above formula is valid also for $\ell = N$. It is in fact valid for all ℓ, even in the infinite $(N = \infty)$ case.

5.4 A system on the Grassmann algebra

We now consider some simple dynamics on the Grassmann algebra, which we already considered, together with its deformation in Sect. 4.5. On $\mathcal{A}_\nu \simeq \mathcal{A}[[\nu]]$, consider the derivation defined for $a \in \mathcal{A}$ by

$$\partial a = \frac{1}{\nu}[\mathfrak{H}, a] \ , \quad \text{where} \quad \mathfrak{H} = \frac{1}{2}(p^2+x^2) + x\psi\phi \ .$$

It is already written in the Heisenberg form and the corresponding Hamiltonian $\mathbf{H} \in \Pi(\mathcal{A})$ is given by $\mathbf{H} = \frac{1}{2}(P^2+X^2) + XW$. The limiting derivation, for $\nu \to 0$, is given by

$$\partial_{\mathbf{H}}\psi = x\psi \ , \quad \partial_{\mathbf{H}}\phi = -x\phi \ , \quad \partial_{\mathbf{H}} p = x + \psi\phi \ , \quad \partial_{\mathbf{H}} x = -p \ .$$

This is an easy consequence of (5.4), upon using Table 8. For example,

$$\partial_{\mathbf{H}}\psi = \left\{\frac{1}{2}(P^2+X^2) + XW\,;\psi\right\} = P\cdot\{P\,;\psi\} + (X+W)\cdot\{X\,;\psi\} + X\cdot\{W\,;\psi\}$$

$$
\begin{aligned}
&= X \cdot \psi = x\psi \ , \\
\partial_{\mathbf{H}} p &= \left\{ \frac{1}{2}(P^2 + X^2) + XW \, ; \, p \right\} = P \cdot \{P \, ; \, p\} + (X + W) \cdot \{X \, ; \, p\} + X \cdot \{W \, ; \, p\} \\
&= (X + W) \cdot 1 = x + \psi\phi \ .
\end{aligned}
$$

5.5 A hierarchy on the quantum plane

We now consider an example related to the quantum plane which we considered in Sect. 4.2. On the free algebra $\mathbb{C}\langle x, y\rangle$, there is a hierarchy of commuting derivations ∂_n, $n > 0$, defined by

$$\partial_n x = xy(y-x)^{n-1}, \quad \partial_n y = yx(y-x)^{n-1} \ ; \tag{5.12}$$

see [15]. The ideal $\mathcal{I}_q := \langle yx - qxy\rangle$ of $\mathbb{C}\langle x, y\rangle$ is a quantization ideal for each one of these derivations, since $\partial_n(yx - qxy) \in \langle yx - qxy\rangle$. The derivations ∂_n therefore descend to commuting derivations of the quantum plane $\frac{\mathbb{C}(q)\langle x,y\rangle}{\langle yx-qxy\rangle}$, and they can be written in the Heisenberg form

$$\partial_n x = \frac{1}{q^n - 1}[\mathfrak{H}_n, x] \ , \quad \partial_n y = \frac{1}{q^n - 1}[\mathfrak{H}_n, y] \ , \tag{5.13}$$

where $\mathfrak{H}^{(n)} = (y - qx)^n$. Notice that, using the q-binomial formula [10, Prop. IV.2.2],

$$\mathfrak{H}^{(n)} = (y - qx)^n = \sum_{k=0}^{n} \binom{n}{k}_q (-qx)^k y^{n-k} = y^n + (-q)^n x^n + \sum_{k=1}^{n-1} \binom{n}{k}_q (-qx)^k y^{n-k},$$

where the q-binomial coefficients are given by

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-k+1} - 1)}{(q-1)(q^2-1)\dots(q^k - 1)} \ .$$

It is easily checked that they satisfy the following well-known recursion relation

$$\binom{n}{k}_q = \frac{q^n - 1}{q^{n-k} - 1}\binom{n-1}{k}_q \ , \tag{5.14}$$

which we will use. We consider the limiting derivation of (5.13) for $q \to \xi$, where ξ is a primitive n-th root of unity. We therefore set, as in Sect. 4.2, $q(\nu) = \xi + \nu$. Since $q(\nu)^n - 1 = \xi^{-1}n\nu + \mathcal{O}(\nu^2)$, while $\binom{n-1}{k}_q = \binom{n-1}{k}_\xi + \mathcal{O}(\nu)$, where the constant term is nonzero, we get, using (5.14), for $0 < k < n$,

$$\binom{n}{k}_{q(\nu)} = \frac{\xi^{-1}n\nu}{\xi^{n-k} - 1}\binom{n-1}{k}_\xi + \mathcal{O}(\nu^2)$$

and

$$\frac{\mathfrak{H}^{(n)}}{q(\nu)^n-1}=\frac{\xi}{n\nu}(y^n+(-1)^nx^n)+\sum_{k=1}^{n-1}\frac{(-\xi)^k\binom{n-1}{k}_\xi}{\xi^{n-k}-1}x^ky^{n-k}\pmod{\mathcal{H}_\nu}, \tag{5.15}$$

so that

$$H_0^{(n)}=\frac{\xi}{n}(y^n+(-1)^nx^n)\ ,\ \text{and}\ H_1^{(n)}=\sum_{k=1}^{n-1}\frac{(-\xi)^k}{\xi^{n-k}-1}\binom{n-1}{k}_\xi x^ky^{n-k}\ .$$

Recall that the center of $\mathcal{A}$ is generated by $X=(x^n,\overline{0})$, $Y=(y^n,\overline{0})$ and $W_{i,j}=\left(0,\overline{x^iy^j}\right)$, with $0\leqslant i,j\leqslant n,\ i+j\neq 0$ (see Sect. 4.2). Setting $\mathbf{H}^{(n)}=(H_0^{(n)},H_1^{(n)})$ we can write $\mathbf{H}^{(n)}$ in terms of these generators as

$$\mathbf{H}^{(n)}=\frac{\xi}{n}(Y+(-1)^nX)+\sum_{k=1}^{n-1}\frac{(-\xi)^k}{\xi^{n-k}-1}\binom{n-1}{k}_\xi W_{k,n-k}\ .$$

Using Table 2 the limiting derivation $\partial_{\mathbf{H}^{(n)}}=\left\{\mathbf{H}^{(n)}\,;\cdot\right\}$ is given by

$$\begin{aligned}\partial_{\mathbf{H}^{(n)}}x&=\left\{\mathbf{H}^{(n)}\,;x\right\}=xy^n+\sum_{k=1}^{n-1}(-1)^k\xi^k\binom{n-1}{k}_\xi x^{k+1}y^{n-k}\ ,\\ \partial_{\mathbf{H}^{(n)}}y&=\left\{\mathbf{H}^{(n)}\,;y\right\}=(-1)^{n-1}x^ny+\sum_{k=1}^{n-1}(-1)^k\xi^{2k}\binom{n-1}{k}_\xi x^ky^{n-k+1}\ .\end{aligned}$$

5.6 A hierarchy on the quantum torus

We now consider an example related to the quantum torus (see Sect. 4.2, especially Remark 4.3). According to [18], Kontsevich considered on the algebra $\mathfrak{A}:=\mathbb{C}\langle x,y,x^{-1},y^{-1}\rangle$ the derivation, defined by

$$\partial_1x=-y^{-1}+xy-xy^{-1}\ ,\quad \partial_1y=x^{-1}-yx+yx^{-1}\ , \tag{5.16}$$

together with a discrete symmetry, which we will not consider here, and conjectured the integrability of (5.16) (and of the discrete symmetry). The integrability of (5.16) was proven in [18], where a Lax representation for (5.16), as well as a symmetry for it were found; the latter is given by

$$\begin{aligned}\partial_2x&=\ \ xyx+xy^2-xy-(yx)^{-1}-y^{-2}-x^2y^{-1}+xyx^{-1}-xy^{-2}-(yxy)^{-1}-x(yxy)^{-1}\ ,\\ \partial_2y&=-yxy-yx^2+yx+(xy)^{-1}+x^{-2}+y^2x^{-1}-yxy^{-1}+yx^{-2}+(xyx)^{-1}+y(xyx)^{-1}\ .\end{aligned} \tag{5.17}$$

Since $\partial_i(yx - qxy) \subset \langle yx - qxy \rangle \subset \mathfrak{A}(q)$, for $i = 1, 2$, both (5.16) and (5.17) define a derivation of the quantum torus $\mathcal{A}_q = \mathfrak{A}(q)/\langle yx - qxy \rangle$. On $\mathcal{A}_q$, (5.16) can be written in the Heisenberg form

$$\partial_1 x = \frac{1}{q-1}[\mathfrak{H}^{(1)}, x] , \quad \partial_1 y = \frac{1}{q-1}[\mathfrak{H}^{(1)}, y] , \tag{5.18}$$

with $\mathfrak{H}^{(1)} = qx^{-1}y^{-1} + qy^{-1} + y + qx + x^{-1}$. Taking $\mathfrak{H}^{(2)} := \left(\mathfrak{H}^{(1)}\right)^2 - (1+q)\mathfrak{H}^{(1)} - 4q$, we can recast (5.17) also in the Heisenberg form

$$\partial_2 x = \frac{1}{q^2-1}[\mathfrak{H}^{(2)}, x] , \quad \partial_2 y = \frac{1}{q^2-1}[\mathfrak{H}^{(2)}, y] . \tag{5.19}$$

This gives an alternative proof that (5.16) and (5.17) are derivations of $\mathcal{A}_q$. Expanded, and using the commutation relation $yx = qxy$ (which implies, for example, that $y^{-1}x = q^{-1}xy^{-1}$), $\mathfrak{H}^{(2)}$ can be written as

$$\begin{aligned} \mathfrak{H}^{(2)} &= y^2 + q^2x^2 + q^2y^{-2} + x^{-2} + q^3x^{-2}y^{-2} \\ &\quad + (q+1)(-y - qx + qxy + qxy^{-1} + q^{-1}x^{-1}y + q^2x^{-1}y^{-2} + qx^{-2}y^{-1}) . \end{aligned} \tag{5.20}$$

We first consider the limits of (5.18) and (5.19) when $q \to 1$. In this case $\mathcal{A} = \mathbb{C}[x, x^{-1}, y, y^{-1}]$ is the algebra of Laurent polynomials in two variables, in particular it is commutative and we are in the case of a classical limit. Setting $q(\nu) = 1 + \nu$, the limit $\partial_{\mathbf{H}^{(1)}}$ is given by

$$\begin{aligned} \partial_{\mathbf{H}^{(1)}} x &= \left\{x^{-1}y^{-1} + y^{-1} + y + x + x^{-1}, x\right\} = x^{-1}\left\{y^{-1}, x\right\} + \left\{y^{-1} + y, x\right\} \\ &= xy - (x+1)y^{-1}, \end{aligned}$$

and similarly $\partial_{\mathbf{H}^{(1)}} y = xy + (y+1)x^{-1}$. We can use this result for computing $\partial_{\mathbf{H}^{(2)}}$ since $\mathfrak{H}^{(2)}$ is a polynomial in $\mathfrak{H}^{(1)}$. The result is that

$$\partial_{\mathbf{H}^{(2)}} x = (H-1)(xy - (x+1)y^{-1}) , \qquad \partial_{\mathbf{H}^{(2)}} y = (H-1)((y+1)x^{-1} - xy) ,$$

where H is $\mathfrak{H}^{(1)}$ evaluated at $q = 1$, that is $H = x^{-1}y^{-1} + y^{-1} + y + x + x^{-1}$.

To finish, we consider the limiting derivation of ∂_2 when $q \to -1$, setting $q = -1 + \nu$. We easily get from (5.20)

$$\frac{\mathfrak{H}^{(2)}}{q(\nu)^2 - 1} = \frac{1}{\nu}\left(H_0^{(2)} + \nu H_1^{(2)}\right) \pmod{\mathcal{H}_\nu} ,$$

where

$$H_0^{(2)} = -\frac{1}{2}\left(x^2 + y^2 + x^{-2} + y^{-2} - x^{-2}y^{-2}\right) ,$$

$$H_1^{(2)} = -\frac{1}{2}\left(x - y - xy - xy^{-1} - x^{-1}y + x^{-1}y^{-2} - x^{-2}y^{-1}\right) .$$

Let $\mathbf{H}^{(2)} := (H_0^{(2)}, H_1^{(2)})$. Then, in terms of the generators of $\Pi(\mathcal{A})$,

$$\mathbf{H}^{(2)} = -\frac{1}{2}\Big(X + Y + X^{-1} + Y^{-1} - X^{-1}Y^{-1} + U - V - W - Y^{-1}W \\ +X^{-1}Y^{-1}U - X^{-1}Y^{-1}V\Big).$$

Using (5.4) and Table 4,

$$\partial_{\mathbf{H}^{(2)}} x = \left\{\mathbf{H}^{(2)} ; x\right\} = \frac{\partial \mathbf{H}^{(2)}}{\partial Y} \cdot (-2xy^2) + \frac{\partial \mathbf{H}^{(2)}}{\partial V} \cdot (-2xy^2) + \frac{\partial \mathbf{H}^{(2)}}{\partial W} \cdot (-2x^2y) .$$

As we already pointed out, U, V and W act trivially on $\mathcal{A}$ and we get

$$\begin{aligned}\partial_{\mathbf{H}^{(2)}} x &= (1 - Y^{-2} + X^{-1}Y^{-2}) \cdot xy^2 - (1 + X^{-1}Y^{-1}) \cdot xy - (1 + Y^{-1} + X^{-1}) \cdot x^2y \\ &= -y - xy + xy^2 - xy^{-2} + x^{-1}y^{-2} - x^{-1}y^{-1} - x^2y - x^2y^{-1} ,\end{aligned}$$

and similarly

$$\partial_{\mathbf{H}^{(2)}} y = x - xy - x^{-1}y^{-1} + xy^2 - x^2y + x^{-1}y^2 + x^{-2}y - x^{-2}y^{-1} .$$

The derivations ∂_1 and ∂_2 admit higher symmetries ∂_n of the form

$$\partial_n x = \frac{1}{1 - q^n}[\mathfrak{H}^{(n)}, x] \quad \partial_n y = \frac{1}{1 - q^n}[\mathfrak{H}^{(n)}, y] ,$$

for which the limiting derivation as $q \to \xi$, with ξ an n-th root of unity, can be obtained in the same way.

Data availability All data generated or analyzed during this study are contained in this document.

Declarations:

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