Closed form solution to zero coupon bond using a linear stochastic delay differential equation

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ABSTRACT
We present a short rate model that satisfies a stochastic delay differential equation. The model can be considered a delayed version of the Merton model (Merton 1970, 1973) or the Vasiček model (Vasiček 1977). Using the same technique as the one used by Flore and Nappo (2019), we show that the bond price is an affine function of the short rate, whose coefficients satisfy a system of delay differential equations. We give an analytical solution to this system of delay differential equations, obtaining a closed formula for the zero coupon bond price. Under this model, we can show that the distribution of the short rate is a normal distribution whose mean depends on past values of the short rate. Based on the results of Küchler and Mensch (1992), we prove the existence of stationary and limiting distributions.

KEYWORDS
Stochastic delay differential equations, Zero coupon bond, Closed formula, Vasiček model.

1. Introduction

Models based on stochastic delay differential equations have garnered increased attention recently. For instance, Gómez-Valle and Martínez-Rodríguez (2023) introduced a delayed variation of the geometric Brownian motion to price commodity futures. For the valuation of equity options, Arriojas et al. (2007) employed a delayed version of the renowned Black-Scholes-Merton model. Similarly, Kazmerchuk, Swishchuk, and Wu (2007) incorporated a delay parameter into a local volatility model and used it to price European options. Turning to short rate models, Flore and Nappo (2019) offered a delayed version of the Cox-Ingersoll-Ross model, while Coffie (2023) presented a delayed Ait-Sahalia short rate model with jumps. In the context of this paper, we introduce a delayed version of the well-established short rate models, Merton model (Merton 1970, 1973) and Vasiček model (Vasiček 1977). We can show that our model can have an analytical formula for the zero coupon bond price.

The Vasiček model is based on the Ornstein-Uhlenbeck process with Gaussian noise. This paper introduces a novel short rate model, which is a delayed version of the Ornstein-Uhlenbeck process. This model has been explored in existing literature. For

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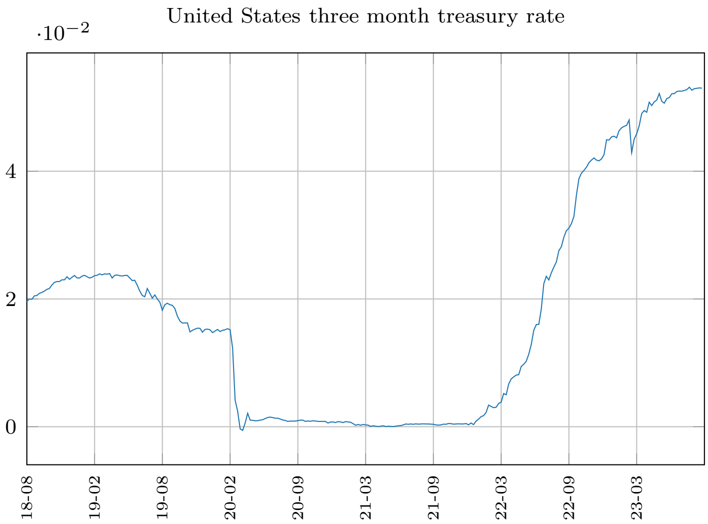

Figure 1. United States three month treasury rate, 1 August 2018—1 October 2023.

example, Mackey and Nechaeva (1995) studied the moment stability of linear stochastic delay differential equations. Basse-O'Connor et al. (2020) proved the existence of strong solutions for the delayed Ornstein-Uhlenbeck process with Lévy noise and compared it with ARMA time series models. Ott (2006) conducted a study on the stability of the paths generated by a delayed Ornstein-Uhlenbeck process. Additionally, Küchler and Mensch (1992) delved into the study of the delayed Ornstein-Uhlenbeck process, proving the existence of a stationary solution.

In the real world, interest rates can vary according to the decisions made by central banks. This generates periods in which the interest rates are low and periods in which the interest rates are high. This behavior can be observed in interest rates given by the United States three month treasury bill, see Figure 1 (Yahoo Finance, `https://finance.yahoo.com/quote/%5EIRX?p=%5EIRX`).

Interest rates tend to change in response to inflation or economic growth. For example, Benhabib (2004) discussed a model in which interest rates vary in response to past inflation. Lilley and Rogoff (2019) argued that lowering interest rates generates economic growth. Inflation and economic growth change could provoke a response in central banks that vary interest rates to increase economic growth and decrease inflation. Hence, interest rates change in response to past events. Because of that, we propose a short rate model that depends on its past values, meaning that the model is not Markovian. Evidence of memory in interest rates can be found in the work of Duan and Jacobs (2001) and Meade and Maier (2003). In Figure 2, we plot the sample auto-correlation function of the daily rate of the United States three month treasury bill. We observe that short rates depend on past values. One characteristic of this model is that it can generate negative values for short rates. However, in recent years, there were periods with negative interest rates (Inhoffen, Pekanov, and Url 2021).

This paper is organized as follows. We initially introduce the model and give an analytical formula for the solution of the stochastic delay differential equation; this is done in Section 2. We compute the joint Laplace transform of the integrated short rate and the short rate in Section 3. This result will allow us to compute the price of the zero coupon bond and the distribution of the short rate. In Section 4, we show

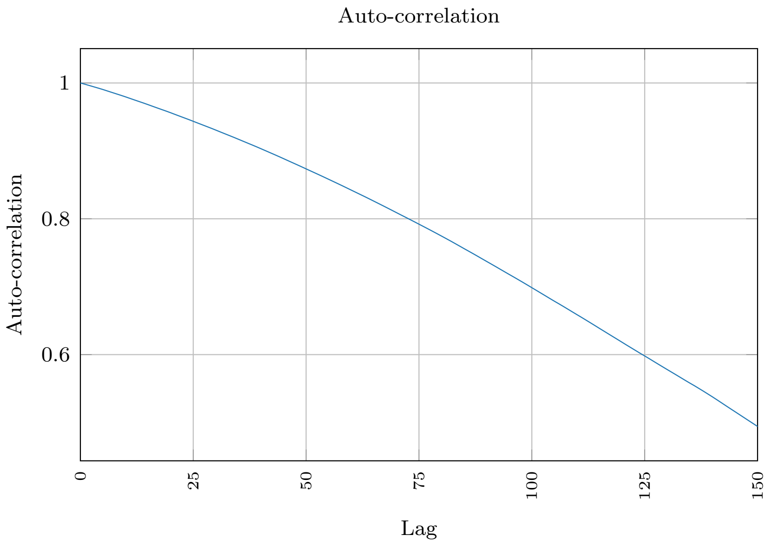

Figure 2. Sample auto-correlation function of the United States three month treasury rate, 1 August 2018—1 October 2023.

that for this short rate model, the price of a zero coupon bond has a closed formula. In Section 5, we study the distribution of the short rate, we adapt the results given by Küchler and Mensch (1992) and give the conditions for the existence of stationary and limiting distributions. We implement the results and perform numerical experiments in Section 6. Finally, in Appendix A we perform a structure-preserving change of measure.

2. Model with delay

Let $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space. Let us assume that the short rate r follows the stochastic delay differential equation

$$dr_t = (a + br_t + cr_{t-\tau})\, dt + \sigma dW_t \tag{1}$$

for all $t \geq 0$, where $a, b, c \in \mathbb{R}$, $\sigma, \tau > 0$ and $W = (W_t)_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$. The initial condition is

$$r_t = \phi(t) \text{ for } t \in [-\tau, 0], \tag{2}$$

where $\phi : [-\tau, 0] \to \mathbb{R}$ is a deterministic integrable function.

We will take $\mathbb{Q}$ to be a risk-neutral probability. It is possible to make a structure-preserving change of measure from the real-world probability $\mathbb{P}$; see Appendix A for details.

This model generalizes the Merton model (which is the special case $b, c = 0$) as well as the Vasiček model (the special case with $b < 0$ and $c = 0$) in a number of different ways, depending on the values of the parameters b and c. For example, taking $b = 0$ and $c < 0$ leads to

$$dr_t = (a + cr_{t-\tau})\, dt + \sigma dW_t,$$

which is a Vasiček model with delay, with mean reversion speed of $\frac{1}{|c|}$ and long-term mean $\frac{a}{|c|}$. In its most general form (when $b < 0$ and $c \neq 0$), it can be interpreted as a Vasiček model with speed of mean reversion $\frac{1}{|b|}$, and short term memory, in the sense that the usual Vasiček long-term mean term $\frac{a}{|b|}$ is replaced by $\frac{a}{|b|} + \frac{c}{|b|} r_{t-\tau}$.

The stochastic delay differential equation (1) has a unique strong solution (von Renesse and Scheutzow 2010, Theorem 2.3). We conclude this section by deriving it in explicit form, thus generalising Proposition 2.2 of Küchler and Mensch (1992). The result makes use of the function $R : [-\tau, \infty) \to \mathbb{R}$ defined as

$$R(s) = \begin{cases} \displaystyle\sum_{n=0}^{\lfloor \frac{s}{\tau} \rfloor} \frac{c^n}{n!} e^{b(s-n\tau)} (s-n\tau)^n & \text{for all } s \geq 0, \\ 0 & \text{for all } s \in [-\tau, 0). \end{cases} \tag{3}$$

We will often make use of the alternative representation

$$R(s) = \sum_{n=0}^{\lfloor x \rfloor} \frac{c^n}{n!} e^{b(s-n\tau)} (s-n\tau)^n H(s-n\tau) \tag{4}$$

where $H = \mathbb{1}_{[0,\infty)}$ is the Heaviside function and x is any real number satisfying $x \geq \max\{\frac{s}{\tau}, 0\}$.

Proposition 2.1. *The strong solution* $(r_t)_{t \geq 0}$ *of the stochastic differential equation* (1) *with initial condition* (2) *satisfies*

$$r_t = R(t) r_0 + a \int_0^t R(t-s) ds + c \int_{-\tau}^0 R(t-s-\tau) \phi(s) ds + \sigma \int_0^t R(t-s) dW_s \tag{5}$$

for all $t \geq 0$, *where* R *is defined in* (3).

Proof. Applying the Itô formula to $e^{-bt} r_t$ in (5) and rearranging gives

$$r_t = e^{bt} r_0 + a \int_0^t e^{b(t-s)} ds + c \int_{-\tau}^{t-\tau} e^{b(t-s-\tau)} r_s ds + \sigma \int_0^t e^{b(t-s)} dW_s \tag{6}$$

for all $t \geq 0$. If $c = 0$, then $R(s) = e^{bs} H(s)$ for all $s \geq 0$ and therefore (6) immediately gives (5).

If $c \neq 0$, then we proceed by induction on intervals of length τ. Consider first $t \in [0, \tau)$. Observe from (3) that $R(u) = e^{bu} H(u)$ for all $u \in [0, \tau)$. Taken together with (2), it follows from (6) that

$$r_t = R(t) r_0 + a \int_0^t R(t-s) ds + c \int_{-\tau}^0 R(t-s-\tau) \phi(s) ds + \sigma \int_0^t R(t-s) dW_s$$

for all $t \in [0, \tau)$. This gives (5) on the interval $[0, \tau)$.

Suppose now that, for some $m \in \mathbb{N}$, the formulation (5) holds for all $t \in [0, m\tau)$, and consider $t \in [m\tau, (m+1)\tau)$. Note that $m = \lfloor \frac{t}{\tau} \rfloor$ and $m - 1 = \lfloor \frac{t-\tau}{\tau} \rfloor$. We use (6), the key idea being to calculate the integral involving past values of r by means of the inductive assumption. First of all,

$$\int_{-\tau}^{t-\tau} e^{b(t-s-\tau)} r_s ds = \int_{-\tau}^{0} e^{b(t-s-\tau)} \phi(s) ds + \int_{0}^{t-\tau} e^{b(t-s-\tau)} r_s ds, \tag{7}$$

where

$$\begin{aligned}\int_{0}^{t-\tau} e^{b(t-s-\tau)} r_s ds = r_0 &\int_{0}^{t-\tau} e^{b(t-s-\tau)} R(s) ds \\ &+ a \int_{0}^{t-\tau} \int_{0}^{s} e^{b(t-s-\tau)} R(s-u) du\, ds \\ &+ c \int_{0}^{t-\tau} \int_{-\tau}^{0} e^{b(t-s-\tau)} R(s-u-\tau) \phi(u) du\, ds \\ &+ \sigma \int_{0}^{t-\tau} \int_{0}^{s} e^{b(t-s-\tau)} R(s-u) dW_u\, ds. \end{aligned} \tag{8}$$

We consider each of the four integrals in (8) in turn. The representation (4) will be used throughout.

(1) It holds for all $0 \leq u \leq s \leq t-\tau$ that

$$e^{b(t-s-\tau)} R(s-u) = \sum_{n=0}^{m-1} \frac{c^n}{n!} e^{b(t-u-(n+1)\tau)} (s-u-n\tau)^n H(s-u-n\tau),$$

and it follows by direct calculation that

$$\int_{u}^{t-\tau} e^{b(t-s-\tau)} R(s-u) ds = \frac{1}{c} \left[R(t-u) - e^{b(t-u)} \right]. \tag{9}$$

In particular,

$$\int_{0}^{t-\tau} e^{b(t-s-\tau)} R(s) ds = \frac{1}{c} \left[R(t) - e^{bt} \right] \tag{10}$$

when $u = 0$.

(2) Applying the Fubini theorem together with (9) gives

$$\begin{aligned}\int_{0}^{t-\tau} \int_{0}^{s} e^{b(t-s-\tau)} R(s-u) du\, ds &= \frac{1}{c} \int_{0}^{t-\tau} \left[R(t-u) - e^{b(t-u)} \right] du \\ &= \frac{1}{c} \int_{0}^{t} \left[R(t-u) - e^{b(t-u)} \right] du, \end{aligned} \tag{11}$$

where the upper limit of the integral can be changed because $R(v) = e^{bv}$ for all $v \in [0, \tau)$.

(3) Note that $R(s-u-\tau)=0$ if $-\tau \le s-u-\tau \le 0$, so that

$$\begin{aligned}\int_0^{t-\tau} e^{b(t-s-\tau)} R(s-u-\tau)ds &= \int_{\tau+u}^{t-\tau} e^{b(t-s-\tau)} R(s-u-\tau)ds \\ &= \frac{1}{c}\left[R(t-u-\tau) - e^{b(t-u-\tau)}\right]\end{aligned}$$

by (9). The Fubini theorem then gives

$$\begin{aligned}&\int_0^{t-\tau}\int_{-\tau}^{0} e^{b(t-s-\tau)} R(s-u-\tau)\phi(u)du\, ds \\ &\qquad = \frac{1}{c}\int_{-\tau}^{0}\left[R(t-u-\tau) - e^{b(t-u-\tau)}\right]\phi(u)du. \quad (12)\end{aligned}$$

(4) Combining the stochastic Fubini theorem with (9) gives

$$\int_0^{t-\tau}\int_0^{s} e^{b(t-s-\tau)} R(s-u)dW_u\, ds = \frac{1}{c}\int_0^{t-\tau}\left[R(t-u) - e^{b(t-u)}\right]dW_u. \quad (13)$$

Substituting (10)-(13) into (8), and then (7), gives

$$\begin{aligned}\int_{-\tau}^{t-\tau} e^{b(t-s-\tau)} r_s ds = \frac{r_0}{c}\left[R(t) - e^{bt}\right] &+ \frac{a}{c}\int_0^t \left[R(t-u) - e^{b(t-u)}\right]du \\ &+ \int_{-\tau}^{0} R(t-u-\tau)\phi(u)du \\ &+ \frac{\sigma}{c}\int_0^{t-\tau}\left[R(t-u) - e^{b(t-u)}\right]dW_u.\end{aligned}$$

Finally, substitution into (6) gives (5) for all $t \in [m\tau,(m+1)\tau)$, which concludes the inductive step. □

3. Laplace transform

In this section we derive an explicit formula for the exponential-affine transform below. Introducing an auxiliary process γ, following an idea of Flore and Nappo (2019), allows us to obtain an affine formula for the conditional expectation of the transform. Explicit formulae for bond prices and the characteristic function of the short rate will be derived in due course as special cases of this formula.

Theorem 3.1. *For any $T > t \ge 0$, $z \in \mathbb{C}$ and $d_0, d_1 \in \mathbb{R}$, we have*

$$E_{\mathbb{Q}}\left(e^{zr_T + \int_t^T (d_0 + d_1 r_s)ds}\middle|\mathcal{F}_t\right) = e^{A(T-t)+D(T-t)r_t + c\int_{t-\tau}^{t} D(T-u-\tau)r_u du} \quad (14)$$

where $A:[0,\infty)\to\mathbb{R}$ and $D:[-\tau,\infty)\to\mathbb{R}$ satisfy the system of delay differential *equations*

$$D'(\ell) = bD(\ell) + cD(\ell - \tau) + d_1, \tag{15}$$
$$A'(\ell) = aD(\ell) + \tfrac{1}{2}\sigma^2 D^2(\ell) + d_0 \tag{16}$$

for all $\ell > 0$, *with initial values* $A(0) = 0$ *and* $D(\ell) = zH(\ell)$ *for all* $\ell \in [-\tau, 0]$, *where* H *is the Heaviside function.*

Proof. Taking any $T > 0$, and taking as given a deterministic (and integrable) function $\Gamma : [0,T] \to \mathbb{R}$ that will be chosen below, define an auxiliary process $\gamma = (\gamma_t)_{t\in[0,T]}$ as

$$\gamma_t = \int_{t-\tau}^{t} \Gamma(s) r_s \mathbb{1}_{[-\tau,T-\tau]}(s) ds \text{ for all } t \in [0,T].$$

Note that $\gamma_T = 0$ and, for all $t \in [0,T]$,

$$\begin{aligned}\gamma_t &= \int_{-\tau}^{t} \Gamma(s) r_s \mathbb{1}_{[-\tau,T-\tau]}(s) ds - \int_0^t \Gamma(s-\tau) r_{s-\tau} ds \\ &= \gamma_0 + \int_0^t \left(\Gamma(s) r_s \mathbb{1}_{[0,T-\tau]}(s) - \Gamma(s-\tau) r_{s-\tau}\right) ds.\end{aligned}$$

Define now

$$\psi_t = E_{\mathbb{Q}}\left(e^{z r_T + \int_t^T (d_0 + d_1 r_s) ds} \middle| \mathcal{F}_t\right) \text{ for all } t \in [0,T].$$

We claim that

$$\psi_t = e^{A(t) + D(t) r_t + \gamma_t} \text{ for all } t \in [0,T], \tag{17}$$

where $A, D : [0,T] \to \mathbb{R}$ are two deterministic functions with $A(T) = 0$ and $D(T) = z$. Application of the Itô formula in (17) gives

$$\begin{aligned}\frac{d\psi_t}{\psi_t} &= \left(A'(t) + aD(t) + \tfrac{1}{2}\sigma^2 D^2(t)\right) dt \\ &\quad + \left(D'(t) + bD(t) + \Gamma(t)\mathbb{1}_{[0,T-\tau]}(t)\right) r_t dt \\ &\quad + \left(cD(t) - \Gamma(t-\tau)\right) r_{t-\tau} dt + \sigma D(t) dW_t.\end{aligned} \tag{18}$$

The function Γ is now chosen so as to allow $r_{t-\tau}$ to be eliminated from (18). Noting that the value of γ does not depend on $\Gamma(t)$ for any $t > T - \tau]$, we simply choose Γ so as to be continuous and constant for such t. In summary, we obtain

$$\Gamma(t) = \begin{cases} cD(t+\tau) & \text{if } t \in [-\tau, T-\tau], \\ z & \text{if } t \in (T-\tau, T]. \end{cases} \tag{19}$$

Define now another auxiliary process $y = (y_t)_{t\in[0,T]}$ as

$$y_t = e^{\int_0^t (d_0 + d_1 r_s) ds} \text{ for all } t \geq 0.$$

It follows that

$$y_t\psi_t = E_{\mathbb{Q}}\left(y_T\psi_T|\mathcal{F}_t\right) \text{ for all } t \in [0,T],$$

in other words, $(y_t\psi_t)_{t\in[0,T]}$ is a martingale. The Itô product rule together with (18) and (19) gives that

$$\begin{aligned}\frac{d(y_t\psi_t)}{y_t\psi_t} &= \left(A'(t) + aD(t) + \tfrac{1}{2}\sigma^2 D^2(t) + d_0\right) dt \\ &\quad + \left(D'(t) + bD(t) + cD(t+\tau)\mathbb{1}_{[0,T-\tau]}(t) + d_1\right) r_t dt + \sigma D(t) dW_t.\end{aligned}$$

The martingale property of $(y_t\psi_t)_{t\in[0,T]}$ then implies that A and D satisfies the system of differential equations

$$\begin{aligned}D'(t) &= -bD(t) - cD(t+\tau)\mathbb{1}_{[0,T-\tau]}(t) - d_1,\\ A'(t) &= -aD(t) - \tfrac{1}{2}\sigma^2 D^2(t) - d_0.\end{aligned}$$

By abuse of notation and the transformation $\ell = T - t$ we obtain (15)–(16) and finally (14). ☐

The key task now is to solve the delay differential equation (15); the following result provides an analytical solution. Once (15) has been solved, the function A can be obtained by integrating (16).

Theorem 3.2. *The solution to the delay differential equation* (15) *with initial condition $D(\ell) = zH(\ell)$ for all $\ell \in [-\tau, 0]$ satisfies*

$$D(\ell) = z + d_1 \sum_{n=0}^{\lfloor \frac{\ell}{\tau} \rfloor} \frac{c^n}{(n+1)!}(\ell - n\tau)^{n+1} + z \sum_{n=1}^{\lfloor \frac{\ell}{\tau} \rfloor} \frac{c^n}{n!}(\ell - n\tau)^n \tag{20}$$

for all $\ell \geq 0$ when $b = 0$ and

$$\begin{aligned}D(\ell) = z + (d_1 + zb) &\sum_{n=0}^{\lfloor \frac{\ell}{\tau} \rfloor} \frac{c^n(-1)^n}{b^{n+1}} \left(e^{b(\ell-n\tau)} \sum_{r=0}^{n} \frac{(-1)^{-r}}{r!} b^r (\ell - n\tau)^r - 1 \right) \\ &+ z \sum_{n=1}^{\lfloor \frac{\ell}{\tau} \rfloor} \frac{c^n(-1)^{n-1}}{b^n} \left(e^{b(\ell-n\tau)} \sum_{r=0}^{n-1} \frac{(-1)^{-r}}{r!} b^r (\ell - n\tau)^r - 1 \right)\end{aligned} \tag{21}$$

for all $\ell \geq 0$ when $b \neq 0$.

Proof. The proof uses the Laplace transform, which, for suitable $f : [0,\infty) \to \mathbb{C}$ and $s \in \mathbb{C}$, is defined as

$$L[f](s) = \int_0^\infty f(u) e^{-su} du.$$

We use a number of elementary properties of the Laplace transform (see, for example Dyke 2014, p. 7, Example 1.2, Theorems 2.1, 2.4, Appendix B), and take as given a value of s for which all expressions below are well defined, i.e. we require $s \notin \{0, b\}$ and $\left|\frac{ce^{-\tau s}}{s-b}\right| < 1$. Observing that (15) is equivalent to

$$D'(\ell) = bD(\ell) + cD(\ell - \tau)H(\ell - \tau) + d_1 \text{ for all } \ell > 0,$$

it follows that

$$sL[D](s) - z = L[D'](s) = bL[D](s) + ce^{-\tau s}L[D](s) + \frac{d_1}{s}.$$

After rearrangement, this becomes

$$L[D'](s) = \frac{d_1 + zb + zce^{-\tau s}}{s - b - ce^{-\tau s}} = \frac{1}{s-b}\left(d_1 + zb + zce^{-\tau s}\right)\frac{1}{1 - \frac{ce^{-\tau s}}{s-b}}.$$

Using the formula for the sum of a geometric series, we obtain

$$\begin{aligned} L[D'](s) &= \frac{1}{s-b}\left(d_1 + zb + zce^{-\tau s}\right)\sum_{n=0}^{\infty}\left(\frac{ce^{-\tau s}}{s-b}\right)^n \\ &= (d_1 + zb)\sum_{n=0}^{\infty}\frac{c^n e^{-n\tau s}}{(s-b)^{n+1}} + z\sum_{n=0}^{\infty}\frac{c^{n+1}e^{-(n+1)\tau s}}{(s-b)^{n+1}}. \end{aligned} \tag{22}$$

Noting that

$$L[(\ell - h)^n e^{b(\ell - h)}H(\ell - h)](s) = \frac{n!e^{-hs}}{(s-b)^{n+1}}$$

for all $n \in \mathbb{N}_0$ and $h \geq 0$, inverting the Laplace transform in (22) gives

$$\begin{aligned} D'(\ell) = (d_1 + zb)&\sum_{n=0}^{\infty}\frac{c^n}{n!}(\ell - n\tau)^n e^{b(\ell - n\tau)}H(\ell - n\tau) \\ &+ z\sum_{n=1}^{\infty}\frac{c^n}{(n-1)!}(\ell - n\tau)^{n-1}e^{b(\ell - n\tau)}H(\ell - n\tau) \text{ for all } \ell \geq 0. \end{aligned}$$

This can be integrated directly, and

$$\begin{aligned} D(\ell) = z + (d_1 + zb)&\sum_{n=0}^{\lfloor \frac{\ell}{\tau} \rfloor}\frac{c^n}{n!}\int_{n\tau}^{\ell}(\ell - n\tau)^n e^{b(\ell - n\tau)}d\ell \\ &+ z\sum_{n=1}^{\lfloor \frac{\ell}{\tau} \rfloor}\frac{c^n}{(n-1)!}\int_{n\tau}^{\ell}(\ell - n\tau)^{n-1}e^{b(\ell - n\tau)}d\ell \text{ for all } \ell \geq 0. \end{aligned} \tag{23}$$

We distinguish between two cases in (23), depending on the value of b. Take first $b \neq 0$. For each $n \in \mathbb{N}$ such that $\ell \geq n\tau$, and $m \in \{n-1, n\} \cap \mathbb{N}$, a change of variable and integration by parts gives that

$$\int_{n\tau}^{\ell}(u-n\tau)^m e^{b(u-n\tau)}du = \frac{1}{b^{m+1}}\int_0^{b(\ell-n\tau)} v^m e^v dv$$
$$= \frac{m!(-1)^m}{b^{m+1}}\left(e^{b(\ell-n\tau)}\sum_{r=0}^{m}\frac{(-1)^{-r}}{r!}b^r(\ell-n\tau)^r-1\right).$$

This formula also applies trivially when $n=0$ or $m=0$, and therefore (21) holds true.

If $b=0$, then the integrals in (23) simplify significantly. For each $n\in\mathbb{N}_0$ such that $\ell\geq n\tau$, and $m\in\{n-1,n\}\cap\mathbb{N}_0$, we obtain

$$\int_{n\tau}^{\ell}(u-n\tau)^m du = \frac{1}{m+1}(\ell-n\tau)^{m+1},$$

which gives (20) as claimed. □

4. Zero coupon bonds

Let us consider zero coupon bonds in this short rate model. The arbitrage-free price of a zero-coupon bond with maturity date $T>0$ is

$$B(t,T)=E_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\,\middle|\,\mathcal{F}_t\right)\ \text{ for all } t\in[0,T] \tag{24}$$

(Musiela and Rutkowski 2004, (9.22)). We have the following result.

Proposition 4.1. *The arbitrage-free price of a zero-coupon bond with maturity date $T>0$ is given for all $t\in[0,T]$ by*

$$B(t,T)=e^{A(T-t)+D(T-t)r_t+c\int_{t-\tau}^{t}D(T-u-\tau)r_u du}, \tag{25}$$

where the functions $A:[0,\infty)\to\mathbb{R}$ and $D:[-\tau,\infty)\to\mathbb{R}$ are defined as $D(\ell)=0$ for all $\ell\in[-\tau,0)$ and

$$D(\ell)=\begin{cases}-\displaystyle\sum_{n=0}^{\lfloor\frac{\ell}{\tau}\rfloor}\frac{c^n}{(n+1)!}(\ell-n\tau)^{n+1} & \text{if } b=0,\\ -\displaystyle\sum_{n=0}^{\lfloor\frac{\ell}{\tau}\rfloor}\frac{c^n(-1)^n}{b^{n+1}}\left(e^{b(\ell-n\tau)}\sum_{r=0}^{n}\frac{(-1)^{-r}}{r!}b^r(\ell-n\tau)^r-1\right) & \text{if } b\neq 0,\end{cases}$$
$$A(\ell)=a\int_0^{\ell}D(u)du+\tfrac{1}{2}\sigma^2\int_0^{\ell}D^2(u)du$$

for all $\ell\geq 0$.

Proof. This is a special case of Theorems 3.1 and 3.2 with $z=d_0=0$ and $d_1=-1$. □

The bond price formula (25) resembles the general exponential-affine structure of bond prices found in affine short rate models (see Musiela and Rutkowski (2004, Section 10.2.2)).

Applying the Itô formula in (25), it is straightforward to establish that

$$dB(t,T) = r_t B(t,T)dt + \sigma D(T-t)B(t,T)dW_t \tag{26}$$

for all $t \in [0,T]$. This is a stochastic differential equation with deterministic volatility, and therefore it is possible to obtain closed formulae for options on bonds (see, for example, Musiela and Rutkowski 2004, Proposition 11.3.1) in terms of D and the parameters of the model, and by extension also for caps, floors and other derivative securities. This includes stock pricing models where the short rate follows (1) under the risk neutral probability $\mathbb{Q}$ (Musiela and Rutkowski 2004, Proposition 11.3.2).

5. Distribution of the short rate

The conditional characteristic function of the short rate, defined for all $T > t \geq 0$ as

$$\phi^r_{T|t}(u) = E_{\mathbb{Q}}\left(e^{iur_T}\middle|\,\mathcal{F}_t\right) \text{ for all } u \in \mathbb{R}. \tag{27}$$

It be obtained as a special case of Theorem 3.1, as follows.

Proposition 5.1. *For any $T > t \geq 0$ we have*

$$\phi^r_{T|t}(u) = e^{iu\left(a\int_0^{T-t} R(u)du + R(T-t)r_t + c\int_{t-\tau}^t R(T-u-\tau)r_u du\right) - \frac{1}{2}u^2\sigma^2\int_0^{T-t} R^2(u)du} \tag{28}$$

for all $u \in \mathbb{R}$, where R is the function defined in (3).

Proof. Fix any $u \in \mathbb{R}$. This corresponds to the special case of Theorems 3.1 and 3.2 with $d_0 = d_1 = 0$ and $z = iu$.

Direct calculation from (20) and (21) give

$$D(\ell) = iuR(\ell) \text{ for all } \ell \geq -\tau.$$

It follows that

$$A(\ell) = iau\int_0^\ell R(u)du - \tfrac{1}{2}u^2\sigma^2\int_0^\ell R^2(u)du \text{ for all } \ell \geq 0$$

by (16) and the result follows immediately. □

It can be inferred from Proposition 2.1 that the (unconditional) distribution of r_T is normal for all $T > 0$. The form of (28) shows that its conditional distribution is also normal, with the parameters being provided by the following corollary.

Corollary 5.2. *For any $0 \leq t < T$, the conditional distribution of r_T given $\mathcal{F}_t$ is normal with mean*

$$\mu_{t,T} = a\int_0^{T-t} R(u)du + R(T-t)r_t + c\int_{t-\tau}^t R(T-u-\tau)r_u du \tag{29}$$

and variance

$$\sigma_{t,T} = \sigma^2 \int_0^{T-t} R^2(u)du. \tag{30}$$

The short rate being normally distributed is useful for both simulating its values and estimating its parameters from historical data (using the maximum likelihood method, for example). Analytical (but complicated) formulae exist for all but one of the integrals in (29)–(30); experience from numerical experiments suggests that numerical quadrature approximate the integrals (including the second integral in (29)) very well.

Remark 1. By the tower property and equation (28), the characteristic function in (27) can be expressed as

$$\begin{aligned}\phi^r_{T|t}(u) &= E_{\mathbb{Q}}\left(e^{iur_T}\,\middle|\,\mathcal{F}_t\right)\\ &= E_{\mathbb{Q}}\left(e^{iur_T}\,\middle|\,\sigma\left(\{r_{t-s}\}_{s=0}^{\tau}\right)\right)\end{aligned}$$

for $T > t \geq 0$ and all $u \in \mathbb{R}$, where $\sigma\left(\{r_{t-s}\}_{s=0}^{\tau}\right)$ is the sigma field generated by the random variables r_{t-s} for $0 \leq s \leq \tau$. This means that the conditional distribution of r_T given $\mathcal{F}_t$, depends only on the values of the short rate from time $t-\tau$ up to time t.

We conclude this section by summarising a number of distributional properties of the short rate. Define first

$$K(t) = \int_0^{\infty} R(|t|+s)R(s)ds \text{ for all } t \in \mathbb{R}. \tag{31}$$

We have the following result due to Küchler and Mensch (1992, Lemma 2.12, Proposition 2.13 with typo's corrected in (2.28)).

Proposition 5.3. *The function K is even and continuously differentiable on $(0,\infty)$. It satisfies*

$$K'(t) = bK(t) + cK(t-\tau) \text{ for all } t \geq 0. \tag{32}$$

In particular,

$$K(0) = \begin{cases} \frac{1}{2d}\frac{c\sinh(d\tau)-d}{b+c\cosh(d\tau)} & \text{if } |c| < -b,\\ \frac{1}{4c}(c\tau-1) & \text{if } c = b,\\ \frac{1}{2d}\frac{c\sin(d\tau)-d}{b+c\cos(d\tau)} & \text{if } c < -|b| \end{cases} \tag{33}$$

where $d = \sqrt{|b^2-c^2|}$, and

$$K(t) = \begin{cases} K(0)\cosh(dt) - \frac{1}{2d}\sinh(dt) & \text{if } |c| < -b,\\ K(0) - \frac{t}{2} & \text{if } c = b,\\ K(0)\cos(dt) - \frac{1}{2d}\sin(dt) & \text{if } c < -|b| \end{cases} \tag{34}$$

for all $t \in [0,\tau]$.

The function K plays an important role in the stationary and limiting distributions of (1), as follows.

Proposition 5.4.

(1) If $b < -c$ *and*

$$b \leq c \text{ and } \tau > 0 \qquad or \qquad b > c \text{ and } 0 < \tau < \tfrac{1}{d}\arccos\left(-\tfrac{b}{c}\right),$$

where $d = \sqrt{|b^2 - c^2|}$*, then the following holds true:*

(a) There exists a unique stationary solution to (1)*, corresponding to* $\phi(t) = -\frac{a}{b+c}$ *for all* $t \in [-\tau, 0]$*. In this case, the short rate is normally distributed with mean and covariance*

$$E(r_t) = -\frac{a}{b+c}, \qquad cov(r_t, r_{t+h}) = \sigma^2 K(h)$$

for all $t > 0$ *and* $h \geq 0$*.*

(b) For all $n \in \mathbb{N}$ *and* $0 \leq h_1 < \cdots < h_n$*, the distribution of* $r_{t+h_1}, \ldots, r_{t+h_n}$ *tends for* $t \to \infty$ *to a normal distribution with mean* $-\frac{a}{b+c}$ *and covariance matrix*

$$\lim_{t\to\infty} cov(r_{t+h_i}, r_{t+h_j}) = \sigma^2 K(|h_j - h_i|) \text{ for all } i, j = 1, \ldots, n.$$

(c) For all $t \geq 0$ *it holds that*

$$\lim_{T\to\infty} \mu_{t,T} = -\frac{a}{b+c}, \qquad \lim_{T\to\infty} \sigma_{t,T} = \sigma^2 K(0).$$

(2) If $b \geq -c$*, then*

$$\lim_{T\to\infty} \sigma_{t,T} = \infty$$

and hence the limiting distribution of the short rate is degenerate.

Proof. The transformation

$$X_t = \tfrac{1}{\sigma}\left(r_t + \tfrac{a}{b+c}\right) \text{ for all } t \geq [-\tau, 0] \tag{35}$$

leads to the stochastic delay differential equation with initial condition

$$dX_t = (bX_t + cX_{t-\tau})\, dt + dW_t \qquad \text{for all } t > 0, \tag{36}$$

$$X_t = \tfrac{1}{\sigma}\left(\phi(t) + \tfrac{a}{b+c}\right) \qquad \text{for all } t \in [-\tau, 0]. \tag{37}$$

Note that (36) is well defined when $b = -c$ even if (35) and (37) are not. Problems of this type were studied by Küchler and Mensch (1992), and the claimed result follows from their Proposition 2.8, Corollary 2.9 and Propositions 2.10 and 2.11. □

Table 1. Values of the parameters which are used in the numerical experiments.

Parameters	Non-degenerate	Degenerate
a	0	0
b	-11	-3
c	4	4
σ	0.01	0.01
τ	2	2

6. Numerical Experiments

In this section, we implement the results of Sections 4–5. To that end, we use two different sets of parameters. One set of parameters generates a model with a limiting distribution, while the other set of parameters generates a model with a degenerate limiting distribution. The values of these parameters are shown in Table 1. In the first column of Table 1, the set of parameters generates a model with a limiting distribution. The second column of Table 1 corresponds to a set of parameters that generates a model with a degenerate limiting distribution; see Proposition 5.4. The initial function ϕ that is used in the numerical experiments appears in equation (38).

$$\phi(t) = \begin{cases} 0.04 & \text{if } t \in [-\tau, -\tau/2] \\ 0.02 & \text{if } t \in (-\tau/2, 0], \end{cases} \tag{38}$$

We simulate the model for 8 years with a time step of 0.0016, using the two sets of parameters shown in Table 1. In Figures 3 and 4, we plot one realization of the short rate model, the theoretical expected value, and the theoretical 95% confidence interval, which have been computed using Corollary 5.2. In Figure 3, we use the set of degenerate parameters and observe that the trajectory does not converge to a limiting distribution. The set of non-degenerate parameters is used in Figure 4; notice that the plotted trajectory converges to a limiting distribution. Apart from that, observe that the long term mean of the trajectory seems to be 0, which coincides with the results shown in Proposition 5.4. Figures 5 and 6 show the term structure of the zero coupon bond price for the two sets of parameters. The bond price is computed using equation (25). Notice that the delay parameter $\tau = 2$ affects the price term structure. The set of degenerate parameters decreases the zero coupon bond price faster with respect to maturity than the set of non-degenerate parameters. Under the set of degenerate parameters, the trajectory of the short rate has a positive periodic trend, making the bond price decrease faster when we increase the maturity date T, see equation (24).

7. Conclusion

In this paper, we obtain an analytical formula for the price of a bond when the short rate satisfies a stochastic delay differential equation. The model presented is a delayed version of the well-known Vasicek model. To our knowledge, this is the first time someone has given an analytical formula to the zero coupon bond under this model. We also give an analytical formula for the strong solution of the stochastic delay differential equation (1). In addition, we showed that the short rate follows a normal distribution and has a stationary distribution under certain conditions. This last result has applications outside of the fixed-income securities context. Since we have a distribution,

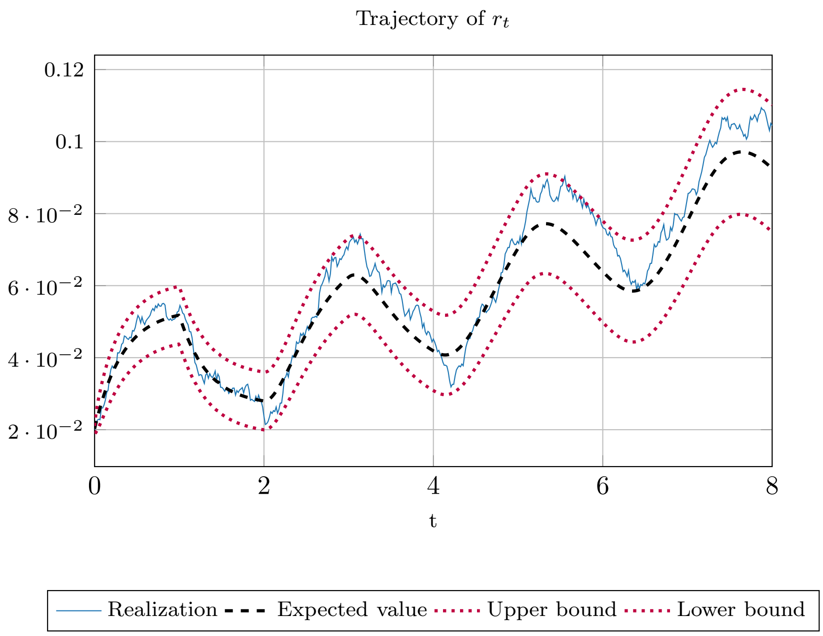

Figure 3. One realization of the short rate model, expected value and the 95% confidence interval of the model using the set of degenerate parameters.

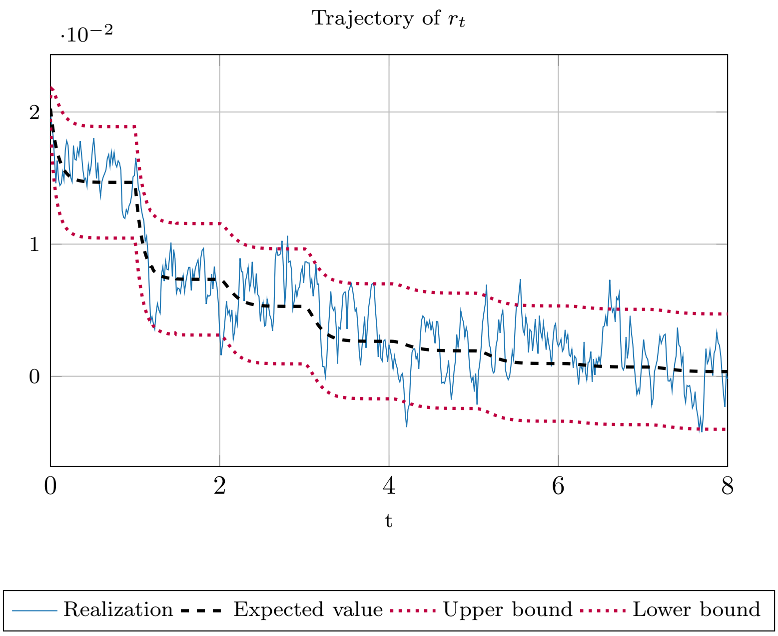

Figure 4. One realization of the short rate model, expected value and the 95% confidence interval of the model using the set of non-degenerate parameters.

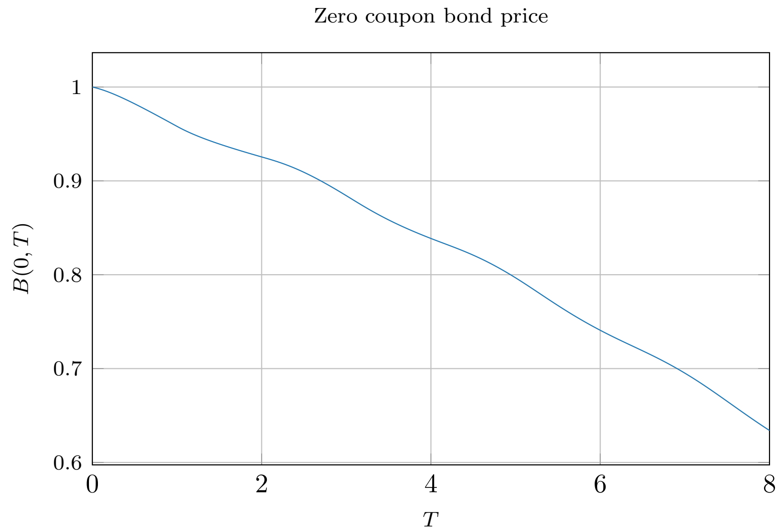

Figure 5. Term structure of the zero coupon bond price using the set of degenerate parameters.

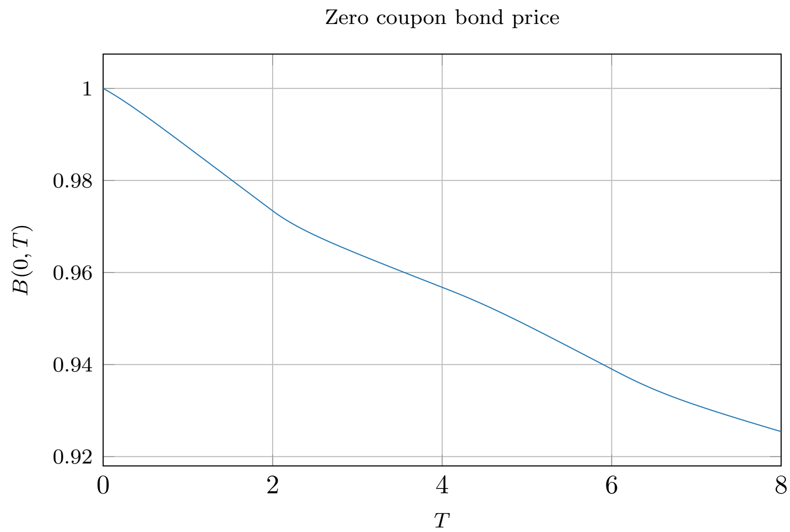

Figure 6. Term structure of the zero coupon bond price using the set of non-degenerate parameters.

we can estimate the parameters of the stochastic delay differential equation (1) from historical data using the maximum likelihood method. For example, the valuation of weather derivatives uses an Ornstein-Uhlenbeck process with a periodic function in the drift to model the temperature (Esunge and Njong 2020). We consider that it would be possible to use equation (1) to model the temperature since the presence of the delay parameter allows the model to capture the past dependencies that appear on the temperature.

Variations of the proposed model can also be applied to other areas of finance. For example, in this model, we assume that the noise of equation (1) is a Brownian motion, but a Lévy process could replace it. In this case, we would have a Lévy-driven Ornstein-Uhlenbeck process with delay. Ornstein-Uhlenbeck processes with Lévy noise are used in several areas of finance like commodities (Li and Linetsky 2014), energy derivatives (Benth, Kallsen, and Meyer-Brandis 2007), or volatility models (Nicolato and Venardos 2003; Barndorff-Nielsen and Shephard 2001b,a). For future work, we would like to study models that include Lévy-driven Ornstein-Uhlenbeck processes with delay.

References

Arriojas, Mercedes, Yaozhong Hu, Salah-Eldin Mohammed, and Gyula Pap. 2007. "A Delayed Black and Scholes Formula." *Stochastic Analysis and Applications* 25 (2): 471–492. https://doi.org/10.1080/07362990601139669.

Barndorff-Nielsen, Ole E., and Neil Shephard. 2001a. *Modelling by Lévy Processess for Financial Econometrics*, 283–318. Boston, MA: Birkhäuser Boston.

Barndorff-Nielsen, Ole E., and Neil Shephard. 2001b. "Non-Gaussian Ornstein-Uhlenbeck-Based Models and Some of Their Uses in Financial Economics." *Journal of the Royal Statistical Society, Series B (Statistical Methodology)* 63 (2): 167–241.

Basse-O'Connor, Andreas, Mikkel Slot Nielsen, Jan Pedersen, and Victor Rohde. 2020. "Stochastic delay differential equations and related autoregressive models." *Stochastics* 92 (3): 454–477. https://doi.org/10.1080/17442508.2019.1635601.

Benhabib, Jess. 2004. "Interest Rate Policy in Continuous Time with Discrete Delays." *Journal of Money, Credit and Banking* 36 (1): 1–15. Accessed 2023-10-20. http://www.jstor.org/stable/3839045.

Benth, Fred Espen, Jan Kallsen, and Thilo Meyer-Brandis. 2007. "A non-Gaussian Ornstein–Uhlenbeck process for electricity spot price modeling and derivatives pricing." *Applied Mathematical Finance* 14 (2): 153–169. https://doi.org/10.1080/13504860600725031.

Coffie, Emmanuel. 2023. "Delay Ait-Sahalia-type interest rate model with jumps and its strong approximation." *Statistics & Risk Modeling* 40 (3-4): 67–89. https://doi.org/10.1515/strm-2022-0013.

Duan, Jin-Chuan, and Kris Jacobs. 2001. *Short and long memory in equilibrium interest rate dynamics.* Technical Report. Centre interuniversitaire de recherche en analyse des organisations (CIRANO). Accessed online 4 October 2023, https://depot.erudit.org/id/000251dd.

Dyke, Phil. 2014. *An introduction to Laplace transforms and Fourier series.* 2nd ed., Springer Undergraduate Mathematics Series. London: Springer.

Esunge, Julius, and James J Njong. 2020. "Weather derivatives and the market price of risk." *Journal of Stochastic Analysis* 1 (3): 7. https://doi.org/10.31390/josa.1.3.07.

Flore, Federico, and Giovanna Nappo. 2019. "A Feynman-Kac type formula for a fixed delay CIR model." *Stochastic Analysis and Applications* 37 (4): 550–573. https://doi.org/10.1080/07362994.2019.1592691.

Gómez-Valle, Lourdes, and Julia Martínez-Rodríguez. 2023. "Estimating and pricing commodity futures with time-delay stochastic processes.” *Mathematical Methods in the Applied Sciences* 1–12. https://doi.org/10.1002/mma.9115.

Inhoffen, Justus, Atanas Pekanov, and Thomas Url. 2021. “Low for long: side effects of negative interest rates.” *Monetary Dialogue Papers* 3–32. Accessed online 4 October 2023, `https://www.europarl.europa.eu/cmsdata/235693/02.DIW_formatted.pdf`.

Kazmerchuk, Yuriy, Anatoliy Swishchuk, and Jianhong Wu. 2007. “The pricing of options for securities markets with delayed response.” *Mathematics and Computers in Simulation* 75 (3): 69–79. https://doi.org/10.1016/j.matcom.2006.09.002.

Klebaner, Fima, and Robert Liptser. 2014. “When a stochastic exponential is a true martingale. Extension of the Beneš method.” *Theory of Probability & Its Applications* 58 (1): 38–62. https://doi.org/10.1137/S0040585X97986382.

Küchler, Uwe, and Beatrice Mensch. 1992. “Langevins stochastic differential equation extended by a time-delayed term.” *Stochastics: An International Journal of Probability and Stochastic Processes* 40 (1-2): 23–42. https://doi.org/10.1080/17442509208833780.

Li, Lingfei, and Vadim Linetsky. 2014. “Time-Changed Ornstein–Uhlenbeck Processes and Their Applications in Commodity Derivative Models.” *Mathematical Finance* 24 (2): 289–330. https://doi.org/10.1111/mafi.12003.

Lilley, Andrew, and Kenneth Rogoff. 2019. “The case for implementing effective negative interest rate policy.” Accessed online 17 October 2023, `https://ssrn.com/abstract=3427388`.

Mackey, Michael C, and Irina G Nechaeva. 1995. “Solution moment stability in stochastic differential delay equations.” *Physical Review E* 52 (4): 3366–3376. https://doi.org/10.1103/PhysRevE.52.3366.

Meade, Nigel, and Margaret R. Maier. 2003. “Evidence of long memory in short-term interest rates.” *Journal of Forecasting* 22 (8): 553–568. https://doi.org/10.1002/for.873.

Merton, Robert C. 1970. *A dynamic general equilibrium model of the asset market and its application to the pricing of the capital structure of the firm.* Working Paper 497. MIT, Cambridge: Sloan School of Management.

Merton, Robert C. 1973. “Theory of rational option pricing.” *The Bell Journal of Economics and Management Science* 4 (1): 141–183.

Musiela, Marek, and Marek Rutkowski. 2004. *Martingale Methods in Financial Modelling.* Springer Science & Business Media.

Nicolato, Elisa, and Emmanouil Venardos. 2003. “Option Pricing in Stochastic Volatility Models of the Ornstein-Uhlenbeck type.” *Mathematical Finance* 13 (4): 445–466.

Ott, Teunis J. 2006. “On the Ornstein-Uhlenbeck process with delayed feedback.” Accessed online 27-Sep-2023, `https://www.teunisott.com/Papers/TCP_Paradigm/{Del_O_U.pdf}`.

Vasiček, Oldrich. 1977. “An equilibrium characterization of the term structure.” *Journal of financial economics* 5 (2): 177–188. https://doi.org/10.1016/0304-405X(77)90016-2.

von Renesse, Max-K., and Michael Scheutzow. 2010. “Existence and uniqueness of solutions of stochastic functional differential equations.” *Random Operators and Stochastic Equations* 18 (3). https://doi.org/10.1515/ ROSE.2010.015.

Appendix A. Change of measure

Instead of starting from a pre-selected risk-neutral measure $\mathbb{Q}$ as we have done in this paper, it is possible to model the short rate under the real-world measure $\mathbb{P}$, and then make a structure-preserving change of measure, the goal being the stochastic differential equation (1). To this end, let us assume that the short rate satisfies the stochastic differential equation

$$dr_t = (\alpha + \beta r_t + \gamma r_{t-\tau})\, dt + \sigma dB_t \tag{A1}$$

under $\mathbb{P}$, where $B = (B_t)$ is a Brownian motion under $\mathbb{P}$ and where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\sigma, \tau > 0$. We take the parameters a, b, c from (1).

For given (possibly large) time horizon $H > 0$, define the processes $\lambda = (\lambda_t)_{t\in[0,H]}$ and $Z = (Z_t)_{t\in[0,H]}$ as

$$\lambda_t = \frac{1}{\sigma}\left[a - \alpha + (b - \beta)r_t + (c - \gamma)r_{t-\tau}\right], \tag{A2}$$

$$Z_t = e^{-\int_0^t \lambda_s dB_s - \frac{1}{2}\int_0^t \lambda_s^2 ds} \tag{A3}$$

for all $t \in [0, H]$. It is sufficient to show that Z is a martingale, because then, by the Girsanov theorem there exists a measure $\mathbb{Q}$ such that $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and a Brownian motion $W = (W_t)_{t=0}^H$ under $\mathbb{Q}$ such that (1) holds true for all $t \in [0, H]$. This achieves the objective.

Proposition A.1. *If the function ϕ is bounded on $[-\tau, 0]$, then Z is a martingale.*

Proof. We use a result by (Klebaner and Liptser 2014, Theorem 5.1). In order to show that Z is a martingale, we need to show that there exists a number $c \geq |r_0|$ such that

$$\max\left\{\lambda_t^2, (a + cr_{t-\tau} + br_t)^2 + \sigma^2(1 + \lambda_t^2)\right\} \leq c\left(1 + \sup_{s\in[0,t)} r_s^2\right) \tag{A4}$$

for all $t \in (0, H]$.

Define

$$c_\phi = \sup_{s\in[-\tau,0]} \phi(s).$$

The identity $(x + y)^2 \leq 2x^2 + 2y^2$ for all $x, y \in \mathbb{R}$ is used repeatedly to obtain

$$\begin{aligned}
\lambda_t^2 &\leq \frac{2}{\sigma^2}\left(2(a-\alpha)^2 + (c-\gamma)^2 c_\phi + 2(b-\beta)^2 r_t^2 + (c-\gamma)^2 r_{t-\tau}^2\right) \\
&\leq \frac{2}{\sigma^2}\left(2(a-\alpha)^2 + (c-\gamma)^2 c_\phi + \left(2(b-\beta)^2 + (c-\gamma)^2\right)\sup_{s\in[0,t)} r_s^2\right) \\
&\leq c_\lambda\left(1 + \sup_{s\in[0,t)} r_s^2\right),
\end{aligned}$$

where

$$c_\lambda = \frac{2}{\sigma^2}\max\left\{2(a-\alpha)^2 + (c-\gamma)^2 c_\phi, 2(b-\beta)^2 + (c-\gamma)^2\right\}.$$

Similarly,

$$\begin{aligned}
(a + br_t + cr_{t-\tau})^2 + \sigma^2 &\leq 2\left(2a^2 + 2b^2 r_t + c^2 r_{t-\tau}\right) + \sigma^2 \\
&\leq 4a^2 + 2c^2 c_\phi + \sigma^2 + \left(4b^2 + 2c^2\right)\sup_{s\in[0,t)} r_s^2,
\end{aligned}$$

and so

$$c = \max\left\{|r_0|, c_\lambda, \max\{4a^2 + 2c^2 c_\phi + \sigma^2, 4b^2 + 2c^2\} + \sigma^2 c_\lambda\right\}$$

has the required properties. $\square$