Nucl. Phys. B 1006 (2024) 116648

Contents lists available at ScienceDirect

Nuclear Physics, Section B

journal homepage: www.elsevier.com/locate/nuclphysb

Quantum Field Theory and Statistical Systems

Zhukovsky-Volterra top and quantisation ideals

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ARTICLE INFO

Editor: Hubert Saleur

Keywords:
Quantum top
Quantisation ideal
Quadratic Poisson brackets
Sklyanin algebra

ABSTRACT

In this letter, we revisit the quantisation problem for a fundamental model of classical mechanics—the Zhukovsky-Volterra top. We have discovered a four-parametric pencil of compatible Poisson brackets, comprising two quadratic and two linear Poisson brackets. Using the quantisation ideal method, we have identified two distinct quantisations of the Zhukovsky-Volterra top. The first type corresponds to the universal enveloping algebras of $so(3)$, leading to Lie-Poisson brackets in the classical limit. The second type can be regarded as a quantisation of the four-parametric inhomogeneous quadratic Poisson pencil. We discuss the relationships between the quantisations obtained in our paper, Sklyanin's quantisation of the Euler top, and Levin-Olshanetsky-Zotov's quantisation of the Zhukovsky-Volterra top.

1. Introduction

The classical and quantum tops are fundamental models in physics. The anisotropic Zhukovsky-Volterra [1,2] and Euler tops [3] stand out as the simplest yet non-trivial examples. In the classical case, they describe the motion of a free rigid body in the presence or absence of an external field. In the quantum domain, they characterise the dynamics of an isolated spinning particle, an atom or a nucleus subjected to a constant external field, and contribute to the description of phase transitions in atomic nuclei in the Lipkin-Meshkov-Glick model (see [4] and references therein). Their significance in classical and quantum mechanics motivates us to revisit the problem of quantisation using a novel approach.

The classical Zhukovsky-Volterra system [1,2] is a dynamical system in three-dimensional phase space with coordinates S_α, $\alpha \in \overline{1,3}$ (representing rotational momenta or classical spin), described by the following system of three ordinary differential equations:

$$\frac{dS_\alpha}{dt} = 2(j_\beta - j_\gamma)S_\beta S_\gamma + 2(k_\beta S_\gamma - k_\gamma S_\beta), \tag{1}$$

where α, β, γ represent a cyclic permutation of the indices $1,2,3$, j_α are parameters of anisotropy (reciprocals of the components of the inertia tensor), and k_α are the constant components of the external field.

It is well-known that the Zhukovsky-Volterra (and Euler) top admits a pencil of compatible Lie–Poisson brackets. By extending the phase space with a new coordinate S_0, which is a constant of motion of the dynamical system, one can also construct quadratic Poisson brackets on the resulting four-dimensional space. For the Euler top, the later coincides with the famous Sklyanin algebra [10]. In the case of the Zhukovsky-Volterra top, it represents a known modification of the Sklyanin algebra [6].

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https://doi.org/10.1016/j.nuclphysb.2024.116648
Received 25 May 2024; Accepted 29 July 2024

Available online 5 August 2024

In this paper, we demonstrate that Zhukovsky-Volterra and Euler tops admit a second inhomogeneous quadratic Poisson bracket. Furthermore, we establish that all four Poisson brackets are mutually compatible in the sense of Magri [9], and define a family of Poisson brackets:

$$
\begin{aligned}
\{S_\alpha, S_\beta\}_{a,b,c,d} &= 2\left((c + d\, j_\alpha)S_0 S_\gamma + (a + b\, j_\gamma)S_\gamma + d\, k_\gamma S_0 + b\, k_\gamma\right),\\
\{S_0, S_\alpha\}_{a,b,c,d} &= 2e\,(j_\beta - j_\gamma)S_\beta S_\gamma + 2e\,(k_\beta S_\gamma - k_\gamma S_\beta),
\end{aligned}
\tag{2}
$$

where a, b, c, d and e are arbitrary parameters of the family (if $e \neq 0$, then by a rescaling of the bracket one can set $e = 1$ without a loss of generality). The existence of the second quadratic structure in these models, along with the compatibility of all four Poisson brackets, seems to be a novel result to the best of our knowledge. If $e \neq 0$, the variable S_0 plays the role of the Hamiltonian for the entire family of the brackets.

We address the quantisation problem through a novel approach based on the concept of quantisation ideal, initially introduced in [12] and further developed in [13–16]. This approach is tailored for dynamical systems defined on free associative algebras. As a preliminary step, we lift the Zhukovsky-Volterra system to the free algebra $\mathcal{A} = \mathbb{C}\langle \hat S_0, \hat S_1, \hat S_2, \hat S_3\rangle$ using the same Lax representation as applied to the matrix-valued system [17]. The resulting system defines a derivation $\frac{d}{dt} : \mathcal{A} \to \mathcal{A}$ of the algebra. Subsequently, we seek a quantisation ideal $\mathcal{J} \subset \mathcal{A}$, which is an ideal satisfying two conditions:

(i) the ideal $\mathcal{J}$ is $\frac{d}{dt}$–stable: $\frac{d}{dt}\mathcal{J} \subset \mathcal{J}$;
(ii) the quotient algebra $\mathcal{A}/\mathcal{J}$ admits a basis of normally ordered monomials.

The quotient algebra $\mathcal{A}/\mathcal{J}$ is then said to be a quantum algebra for the system. The first condition implies that $\frac{d}{dt}$ descends to a derivation of the quantum algebra, defining the quantum Zhukovsky-Volterra system with commutation relations determined by the generators of the quantisation ideal.

As a candidate for a quantisation ideal, we consider the ideal $\mathcal{I}$ generated by the set of polynomials:

$$
\begin{aligned}
\mathcal{I} = \langle\, &\hat S_\alpha \hat S_\beta - \hat S_\beta \hat S_\alpha + A_\gamma(\hat S_0 \hat S_\gamma + \hat S_\gamma \hat S_0) + K_\gamma \hat S_0 + M_\gamma \hat S_\gamma + N_\gamma,\\
&\hat S_\gamma \hat S_0 - \hat S_0 \hat S_\gamma + B_\gamma(\hat S_\alpha \hat S_\beta + \hat S_\beta \hat S_\alpha) + (l_\alpha \hat S_\beta - l_\beta \hat S_\alpha) \mid \alpha, \beta, \gamma \in \overline{1,3}\rangle.
\end{aligned}
$$

These polynomials generalise the commutation relations in the Sklyanin algebra [10] and the algebra obtained in [6]. The ideal $\mathcal{I}$ is parametrised by the set of 18 constants $A_\alpha, B_\alpha, K_\alpha, M_\alpha, N_\alpha$ and l_α, $\alpha \in \overline{1,3}$. The fulfillment of conditions (i) and (ii) leads to a system of algebraic equations on these parameters, which we have solved in the paper to derive the most general quantisation ideal of the form $\mathcal{I}$.

In the simplest case $A_\gamma = B_\gamma = 0$ condition (ii) is satisfied due to the Poincare-Birkchoff-Witt Theorem. Ideals satisfying condition (i) almost immediately give rise to a quantum algebra isomorphic to the universal enveloping algebra given by (2) with $c = d = e = 0$, and with the central element $\hat S_0$.

We treat the generic case $A_\gamma \neq 0, B_\gamma \neq 0$ separately. Firstly, we identify and solve equations for the coefficients of the ideal to fulfill condition (ii) for quadratic and cubic monomials (Theorem 3.1). Condition (i) imposes further constraints on the coefficients of the ideal (Propositions 3.2 and 3.3). Finally we propose a reparametrisation of the coefficients that resolve all the constraints and is convenient for a classical limit.

Ultimately, we obtain a general quantisation ideal that leads to commutation relations:

$$
\begin{aligned}
[\hat S_\alpha, \hat S_\beta] &= -\frac{h}{1 + h^2 j_\gamma\left(C + (j_\beta + j_\alpha)D\right)}\Big((C + Dj_\gamma)(\hat S_0 \hat S_\gamma + \hat S_\gamma \hat S_0) + 2(A + Bj_\gamma)\hat S_\gamma + 2Dk_\gamma \hat S_0 + 2Bk_\gamma\Big),\\
[\hat S_\gamma, \hat S_0] &= -h\Big((j_\alpha - j_\beta)(\hat S_\alpha \hat S_\beta + \hat S_\beta \hat S_\alpha) + 2(k_\alpha \hat S_\beta - k_\beta \hat S_\alpha)\Big).
\end{aligned}
\tag{3}
$$

Here $h = -i\hbar$ can be regarded as a quantisation (Planck) constant.

The five parameters h, A, B, C, D define the quantum algebra. The classical limit corresponds to $h = 0$, in which the algebra becomes commutative. We may assume that the parameters $A = A(h)$, $B = B(h)$, $C = C(h)$, $D = D(h)$ are analytic functions of the variable h at $h = 0$. They represent a trajectory to the classical boundary in the space of parameters. Evaluating the standard classical limit of the commutation relations (3) results in the Poisson brackets family (2), where $a = A(0)$, $b = B(0)$, $c = C(0)$, $d = D(0)$, $e = 1$.

The quantum Sklyanin algebra [10] corresponds to the case of the Euler top ($k_\alpha = 0$), with homogeneous quadratic commutation relations having $A = B = 0$, $e = 1$ and certain choices of functions $C(h)$ and $D(h)$ (see Section 3.6.1). In the quantisation of the Zhukovsky-Volterra by Levin-Olshanetsky-Zotov [6], the commutation relations do not have the form (3), and they do not satisfy condition (i) for the lifted equation on $\mathcal{A}$, but satisfy condition (ii). Their quantum system is a deformation of the classical one, i.e. they deform both the commutative algebra and the equation of motion.

The structure of the present paper is as follows: in Section 2, we explore the classical Zhukovsky-Volterra (and Euler) top, its integrals and four compatible Poisson structures. In Section 3, we apply the quantization ideal method and identify a five-dimensional variety of quantizations. Finally, in Section 4, we provide a brief summary and discuss the open problems.

2. Classical Zhukovsky-Volterra and Euler tops

2.1. Equation of motion and its first integrals

Motion of a rigid body in a constant external field, known as the classical Zhukovsky-Volterra top, can be characterised by the vector of angular momenta $(S_1, S_2, S_3) \in \mathbb{R}^3$, whose components satisfy the following system of ordinary differential equations

$$\frac{dS_\alpha}{dt} = 2(j_\beta - j_\gamma)S_\beta S_\gamma + 2(k_\beta S_\gamma - k_\gamma S_\beta). \tag{4}$$

Here (k_1, k_2, k_3) denotes a vector of a constant external field and j_1, j_2, j_3 are reciprocals of inertia momentum of the diagonal inertia tensor. In system (4) and thereafter we assume that indices (α, β, γ) represent a cyclic permutation of the table $(1, 2, 3)$.

The classical Zhukovsky-Volterra top (4) admits two first integrals

$$C = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2, \qquad H = \frac{1}{2}\sum_{\alpha=1}^{3} j_\alpha S_\alpha^2 + \sum_{\alpha=1}^{3} k_\alpha S_\alpha. \tag{5}$$

When the external field vanishes ($k_\alpha = 0$), system (4) reduces to the classical Euler top.

2.2. Linear Poisson pencil

It is well known that system (4) is Hamiltonian

$$\frac{dS_\alpha}{dt} = \{S_\alpha, H\}_1,$$

with respect to the standard linear Poisson structure on $so^*(3)$:

$$\{S_\alpha, S_\beta\}_1 = 2S_\gamma, \tag{6}$$

and the Hamiltonian H. It is also Hamiltonian

$$\frac{dS_\alpha}{dt} = \{C, S_\alpha\}'_1,$$

with respect to the linear inhomogeneous Poisson structure

$$\{S_\alpha, S_\beta\}'_1 = 2(j_\gamma S_\gamma + k_\gamma), \tag{7}$$

and the Hamiltonian C.

The functions C and H are Casimir functions of the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}'_1$, respectively, i.e.,

$$\{C, S_\alpha\}_1 = \{H, S_\alpha\}'_1 = 0, \quad \alpha \in \overline{1,3}.$$

The brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}'_1$ are compatible in the sense of Magri [9], meaning that a linear combination of the brackets

$$\{S_\alpha, S_\beta\}_{a,b} = 2(a + j_\gamma b)S_\gamma + 2k_\gamma b, \tag{8}$$

with arbitrary constant parameters a and b is a Poisson bracket. The Casimir function of this bracket is the following function

$$C_{a,b} = aC + bH.$$

2.3. Extension of the phase space and quadratic Poisson structures

Let us now consider quadratic Poisson structure for the Zhukovsky-Volterra top. For this purpose we need to extend the phase space of the model with a new variable S_0, which is a constant of motion ($\frac{dS_0}{dt} = 0$) and a central element of two linear brackets:

$$\{S_0, S_\alpha\}_1 = 0, \quad \{S_0, S_\alpha\}'_1 = 0, \qquad \alpha \in \overline{1,3}.$$

In the extended phase space we are looking for an inhomogeneous quadratic Poisson structure of the form:

$$\{S_\alpha, S_\beta\} = 2a_\gamma S_0 S_\gamma + 2K_\gamma S_0, \tag{9a}$$

$$\{S_0, S_\alpha\} = e\big(2(j_\beta - j_\gamma)S_\beta S_\gamma + 2(k_\beta S_\gamma - k_\gamma S_\beta)\big), \tag{9b}$$

where a_α, K_γ are some constants.

Proposition 2.1. *(i) The brackets (9) satisfy the Jacobi identity iff*

$$a_\alpha = c + j_\alpha d, \quad K_\alpha = k_\alpha d, \quad \alpha \in \overline{1,3}. \tag{10}$$

(ii) The Poisson brackets given by (9) with the structure constants defined by (10) are compatible with the pencil of linear-constant Poisson brackets $\{\ ,\ \}_{a,b}$ *defined by (8) for any values of* $c,\ d,\ a,\ b,\ e$.

Furthermore, in order for the equation (9b) to coincide exactly with the equation of motion we will further impose the normalization condition $e=1$. This can be always achieved if $e\neq 0$. The case $e=0$ can be effectively reduced to the previously considered linear case. Thus we obtained the Poisson brackets:

$$\{S_\alpha,S_\beta\}_{a,b,c,d}=2\big((c+j_\alpha d)S_0S_\gamma+(a+j_\gamma b)S_\gamma+k_\gamma dS_0+k_\gamma b\big), \tag{11a}$$

$$\{S_0,S_\alpha\}_{a,b,c,d}=2(j_\beta-j_\gamma)S_\beta S_\gamma+2(k_\beta S_\gamma-k_\gamma S_\beta), \tag{11b}$$

which depend on four arbitrary parameters a,b,c,d.

The above Proposition can be proven by solving equations on the coefficients a_γ, K_γ that are obtained from the Jacobi identity. We derive the brackets (11) as a classical limit of the general commutation relations (44) outlined in Proposition 3.5. Consequently, the fulfillment of the Jacobi identity is guaranteed for any choice of a,b,c,d.

Remark 1. Note that (11) represents a linear combination of two quadratic brackets if we set $c+d=1$.

Remark 2. Observe that in the case $da-bc=0,\ c\neq 0,\ d\neq 0$, the linear term in the equation (11a) can be obtained simply by shift of the element S_0: $S_0\to S_0+v$, where $v=ac^{-1}=bd^{-1}$.

Applying the method of indeterminate coefficients, we have identified two Casimir functions of the brackets (11):

$$C_1=\sum_{\alpha=1}^{3}(c+j_\alpha d)S_\alpha^2+2d\sum_{\alpha=1}^{3}k_\alpha S_\alpha-2(da-bc)S_0, \tag{12a}$$

$$C_2=\sum_{\alpha=1}^{3}S_\alpha^2+dS_0^2+2bS_0. \tag{12b}$$

3. Quantum Zhukovsky-Volterra top and quantisation ideals

In order to apply the quantisation ideal approach we lift equations of the classical commutative Zhukovsky–Volterra top to a free associative algebra $\mathcal{A}=\mathbb{C}\langle\hat S_0,\dots,\hat S_3\rangle$ using a Lax representation, which is similar to the commutative case.

In the simplest case of Lie type ideals the existence of a basis of normally ordered monomials $\mathcal{B}=\langle S_0^{n_0}S_1^{n_1}S_2^{n_2}S_3^{n_3}\mid n_k\in\mathbb{N}\rangle$ in the quotient algebra $\mathcal{A}/\mathcal{I}$, or a PBW basis, is following from the Poincaré–Birkhoff–Witt Theorem. The stability condition (i) leads to a quantisation that in the classical limit results in the pencil of compatible linear Poisson brackets (8).

For a quadratic ideal $\mathcal{I}\subset\mathcal{A}$, we examine the conditions emerging from the requirement of existence of a PBW basis in the quotient algebra $\mathcal{A}/\mathcal{I}$ (condition (ii)) in order to find equations on the parameters of the ideal. We focus on the subspaces spanned by monomials $\mathcal{B}_N=\langle\hat S_0^{n_0}\hat S_1^{n_1}\hat S_2^{n_2}\hat S_3^{n_3}\mid n_0+n_1+n_2+n_3\leqslant N,\ n_k\in\mathbb{N}\rangle$ with $N=2$ and $N=3$.

Subsequently, we consider the stability condition (i). The quantisation obtained in this way depends on five parameters, and in the commutative classical limit results in a four parametric family of compatible Poisson brackets (11). Finally we compare the results obtained with Sklyanin's quantisation of the Euler top and Levin–Olshanetsky–Zotov quantisation of the Zhukovsky–Volterra top.

3.1. *Equation of motion on free associative algebra*

In the classical commutative case the Lax representation for the Zhukovsky-Volterra model was discovered in [5] and studied in [6], [7] and [8]. The Lax pair with matrix valued entries S_α was discussed in [17]. Here, we employ the same Lax pair, replacing matrices S_α by elements $\hat S_\alpha$ from the free algebra $\mathcal{A}$:

$$\hat L=i\sum_{\alpha=1}^{3}(u_\alpha\hat S_\alpha+\frac{k_\alpha}{u_\alpha})\sigma_\alpha,\quad \hat M=i\sum_{\alpha=1}^{3}u_\beta u_\gamma\hat S_\alpha\sigma_\alpha, \tag{13}$$

where u_α, $\alpha\in\overline{1,3}$ are coordinates of a point on the elliptic spectral curve

$$u_1^2-u_2^2=j_1-j_2,\qquad u_2^2-u_3^2=j_2-j_3,$$

and σ_α are standard Pauli matrices. The Lax equation

$$\frac{d\hat L}{dt}=[\hat L,\hat M], \tag{14}$$

leads to the dynamical system

$$\frac{d\hat{S}_\alpha}{dt} = (j_\beta - j_\gamma)(\hat{S}_\beta\hat{S}_\gamma + \hat{S}_\gamma\hat{S}_\beta) + 2(k_\beta\hat{S}_\gamma - k_\gamma\hat{S}_\beta), \qquad \alpha \in \overline{1,3}, \tag{15}$$

that together with the equations

$$\frac{d\hat{S}_0}{dt} = 0, \tag{16}$$

represent a lift of the classical commutative Zhukovsky-Volterra top (4) to the free algebra $\mathcal{A}$.

3.2. Quantisation ideals of Lie type

Let us at first consider a Lie type ideal $\mathcal{J}$, generated by polynomials:

$$\mathcal{J} = \langle f_\gamma = \hat{S}_\alpha\hat{S}_\beta - \hat{S}_\beta\hat{S}_\alpha - a_\gamma\hat{S}_\gamma - b_\gamma,\ g_\gamma = \hat{S}_0\hat{S}_\gamma - \hat{S}_\gamma\hat{S}_0 \,|\, \alpha,\beta,\gamma \in \overline{1,3}\rangle, \tag{17}$$

where a_γ, b_γ are six arbitrary constants and $\hat{S}_0$ is a central element of the algebra $\mathcal{A}$. It follows from the Poincare-Birkchoff-Witt theorem that the quotient algebra $\mathcal{A}/\mathcal{J}$ admits a basis of normally ordered monomials $\mathcal{B} = \langle S_0^{n_0} S_1^{n_1} S_2^{n_2} S_3^{n_3} \,|\, n_k \in \mathbb{N}\rangle$, which is referred as a PBW basis.

The ideal (17) is a quantisation ideal for system (15), (16) if $\mathcal{J}$ it is stable with respect to the dynamics.

Proposition 3.1. *The ideal $\mathcal{J}$ is $\frac{d}{dt}$–stable, i.e. $\frac{d\mathcal{J}}{dt} \subset \mathcal{J}$ if and only if*

$$a_\gamma = 2(a + j_\gamma b), \quad b_\gamma = 2k_\gamma b, \tag{18}$$

where a, b are arbitrary constants.

Sketch of the proof. The stability conditions can be obtained from the requirement that the time derivatives of the ideal generators belong to the ideal. The conditions of stability of the generators g_γ do not impose any constraints on the constants a_γ, b_γ. It follows from the conditions $\frac{df_\gamma}{dt} \subset \mathcal{J},\ \gamma \in \overline{1,3}$ that

$$(j_1 - j_2)a_3 + (j_2 - j_3)a_1 + (j_3 - j_1)a_2 = 0, \qquad b_\gamma(j_\alpha - j_\beta) = k_\gamma(a_\alpha - a_\beta), \quad \alpha,\beta,\gamma \in \overline{1,3}.$$

The above implies (18). □

The statement of the Proposition means that the quantisation ideal of the Lie type effectively depends on two parameters

$$\mathcal{J}_{(a,b)} = \langle \hat{S}_\alpha\hat{S}_\beta - \hat{S}_\beta\hat{S}_\alpha - 2(a + j_\gamma b)\hat{S}_\gamma - 2k_\gamma b,\ \hat{S}_0\hat{S}_\gamma - \hat{S}_\gamma\hat{S}_0 \,|\, \alpha,\beta,\gamma \in \overline{1,3}\rangle,$$

that in the classical limit reduces to the Poisson pencil (8). The center of the quantum algebra $\mathcal{A}/\mathcal{J}_{(a,b)}$ is generated by $\hat{S}_0$ and $a\hat{C} + b\hat{H}$, where

$$\hat{C} = \frac{1}{2}\sum_{\alpha=1}^{3}\hat{S}_\alpha^2, \qquad \hat{H} = \frac{1}{2}\sum_{\alpha=1}^{3} j_\alpha\hat{S}_\alpha^2 + \sum_{\alpha=1}^{3} k_\alpha\hat{S}_\alpha. \tag{19}$$

The specification $a = i\hbar$ and $b = 0$ leads to the standard commutation relations

$$[\hat{S}_\alpha, \hat{S}_\beta] = 2i\hbar\hat{S}_\gamma, \qquad [\hat{S}_0, \hat{S}_\alpha] = 0$$

for $so(3)$ quantum systems. On the algebra $\mathcal{A}/\mathcal{J}_{(i\hbar,0)}$ the quantum Zhukovsky-Volterra system (15), (16) can be presented in the Heisenberg from

$$i\hbar\frac{d\hat{S}_\alpha}{dt} = [\hat{S}_\alpha, \hat{H}]. \tag{20}$$

The center of the quantum algebra $\mathcal{A}/\mathcal{J}_{(i\hbar,0)}$ is generated by the elements $\hat{S}_0$ and $\hat{C}$. The classical limit in this case results in the Poisson brackets (6), Hamiltonian H and Casimir element C (5).

The second choice of specification $a = 0$ and $b = i\hbar$ leads to commutation relations

$$[\hat{S}_\alpha, \hat{S}_\beta]' = 2i\hbar(j_\gamma\hat{S}_\gamma + k_\gamma), \qquad [\hat{S}_0, \hat{S}_\alpha]' = 0$$

on the algebra $\mathcal{A}/\mathcal{J}_{(0,i\hbar)}$. Here we use "prime" in $[\cdot,\cdot]'$ to emphasize that the multiplication rules the algebras $\mathcal{A}/\mathcal{J}_{(i\hbar,0)}$ and $\mathcal{A}/\mathcal{J}_{(0,i\hbar)}$ are different.

In the algebra $\mathcal{A}/\mathcal{J}_{(0,i\hbar)}$ the center is generated by the elements $\hat{S}_0, \hat{H}$ (19), and the element $\hat{C}$ becomes the Hamiltonian for the quantum Zhukovsky-Volterra system

$$i\hbar\frac{d\hat{S}_\alpha}{dt} = [\hat{C}, \hat{S}_\alpha]'. \tag{21}$$

The classical limit in the case of algebra $\mathcal{A}/\mathcal{J}_{(0,i\hbar)}$ results in the Poisson brackets (7), Hamiltonian C and Casimir element H (5).

3.3. Quadratic ideals: the PBW condition

We start with consideration of a quite general ideal $\mathcal{I} \subset \mathcal{A}$ generated by quadratic polynomials:

$$\begin{aligned} \mathcal{I} = \langle F_\gamma = \hat{S}_\alpha \hat{S}_\beta - \hat{S}_\beta \hat{S}_\alpha + A_\gamma(\hat{S}_0 \hat{S}_\gamma + \hat{S}_\gamma \hat{S}_0) + K_\gamma \hat{S}_0 + M_\gamma \hat{S}_\gamma + N_\gamma, \\ G_\gamma = \hat{S}_\gamma \hat{S}_0 - \hat{S}_0 \hat{S}_\gamma + B_\gamma(\hat{S}_\alpha \hat{S}_\beta + \hat{S}_\beta \hat{S}_\alpha) + (l_\alpha \hat{S}_\beta - l_\beta \hat{S}_\alpha) \,|\, \alpha,\beta,\gamma \in \overline{1,3}\rangle. \end{aligned} \tag{22}$$

These polynomials generalise the commutation relations in the Sklyanin algebra [10] and the algebra obtained in [6].

The first problem is to find conditions on 18 parameters $A_\alpha, B_\alpha, K_\alpha, M_\alpha, N_\alpha$ and l_α, $\alpha \in \overline{1,3}$ which guarantee the existence of the normally ordered monomial basis in the subspaces of quadratic and cubic polynomials in the quotient algebra $\mathcal{A}/\mathcal{I}$.

Theorem 3.1.

1. *Quadratic polynomials in variables $\hat{S}_\alpha$, $\alpha \in \{0,1,2,3\}$ admit normal ordering, modulo the ideal $\mathcal{I}$, i.e. a unique representation in the monomial basis $\mathcal{B}_2$, iff*

$$A_1 B_1 \neq -1, \quad A_2 B_2 \neq 1, \quad A_3 B_3 \neq -1. \tag{23}$$

2. *Cubic polynomials admit normal ordering, modulo the ideal $\mathcal{I}$, if conditions (23) satisfied and*

$$\sum_{\gamma=1}^{3} A_\gamma B_\gamma + \prod_{\gamma=1}^{3} A_\gamma B_\gamma = 0, \tag{24}$$

$$B_1 + B_2 + B_3 = 0, \tag{25}$$

$$K_\alpha = \frac{(A_\beta - A_\gamma + B_\alpha A_\beta A_\gamma)}{B_\alpha(1 + A_\beta B_\beta A_\gamma B_\gamma)} l_\alpha, \tag{26}$$

$$M_\alpha = 2\nu A_\alpha + \mu\Big(\frac{3 + A_\beta B_\beta - A_\gamma B_\gamma + A_\beta B_\beta A_\gamma B_\gamma}{1 + A_\beta B_\beta A_\gamma B_\gamma}\Big), \tag{27}$$

$$N_\alpha = \nu K_\alpha, \qquad \alpha \in \overline{1,3}, \tag{28}$$

$$B_\alpha \neq 0, \quad A_\alpha^2 B_\alpha^2 \neq 1, \qquad \alpha \in \overline{1,3}, \tag{29}$$

$$\big(1 + A_1 B_1 - A_2 B_2 - A_1 A_2 B_2 B_1\big)^2 + 16 A_1 A_2 B_1 B_2 \neq 0, \tag{30}$$

where μ, ν are arbitrary parameters, and indices α, β, γ are cyclic permutation of the set $1,2,3$.

Sketch of the Proof. (1.) We regard $F_\gamma = 0,\ G_\gamma = 0, \gamma \in \{1,2,3\}$ as a system of six linear equations with respect to the quadratic monomials $\hat{S}_i \hat{S}_j,\ i > j$ which are not normally ordered. This system admits a unique solution if and only if the conditions (23) are satisfied. Its solution enables us to span any polynomial of degree less or equal to two in the basis $\mathcal{B}_2$ of the normally ordered monomials, modulo the ideal $\mathcal{J}$.

(2) The set of possible 64 cubic monomials contains 20 normally ordered monomials. The rest 44 unordered monomials can be expressed in the basis $\mathcal{B}_3$ solving the system of 48 polynomial equations

$$\hat{S}_\beta F_\alpha = 0, \quad F_\alpha \hat{S}_\beta = 0, \quad \hat{S}_\beta G_\alpha = 0, \quad G_\alpha \hat{S}_\beta = 0, \quad \alpha \in \{1,2,3\},\ \beta \in \{0,1,2,3\}.$$

The solution of the above system enables one to represent any cubic polynomial in $\mathcal{A}$ in the basis $\mathcal{B}_3$ uniquely, modulo the ideal $\mathcal{I}$. The resolvability conditions for this overdetermined system of linear equations lead to (24)-(30). □

Remark 3. The quantum ideals and PBW conditions for the quadratic structures of the quantum Euler top are obtained by putting $l_\alpha = 0,\ K_\alpha = 0,\ N_\alpha = 0,\ \alpha \in \overline{1,3}$ in the formulae above.

3.4. Stability of the quadratic ideal

3.4.1. Dynamical stability of the ideal and projective parametrisation

We will denote $\mathcal{J}$ the ideal $\mathcal{I}$ (22), whose parameters satisfy conditions (23)-(30) (Theorem 3.1). The stability of the ideal with respect to the dynamics (15), (16) impose further constraints.

Proposition 3.2. *The ideal $\mathcal{J}$ is $\frac{d}{dt}$–stable iff*

$$B_\alpha = h(j_\beta - j_\gamma), \quad l_\alpha = 2hk_\alpha, \tag{31}$$

where h is an arbitrary constant.

There is a convenient parametrisation of parameters A_α, satisfying (24) with B_α satisfying (31).

Proposition 3.3. *The coefficients A_α satisfying condition (24) with the constants B_α defined by (31) can be parametrized as follows:*

$$A_\alpha = \frac{1}{hJ_\alpha}\frac{J_\beta - J_\gamma}{j_\beta - j_\gamma}, \qquad \alpha \in \overline{1,3}, \tag{32}$$

where J_α satisfy the following inequalities:

$$J_\alpha \neq 0, \quad J_\alpha \neq J_\beta + J_\gamma, \qquad \alpha \in \overline{1,3}, \tag{33}$$

$$\left(J_1 + J_2 - J_3\right)^4 + 16 J_1 J_2 \left(J_1 - J_3\right)\left(J_2 - J_3\right) \neq 0 \tag{34}$$

and are arbitrary otherwise.

The statement of Proposition 3.3 can be checked by a direct substitution of (31), (32) in (24). Conditions (33), (34) represent inequalities (29), (30), where A_α is given by (32).

Remark 4. Observe that there exists another then (32) parametrization of A_α, namely:

$$A_\alpha = -\frac{1}{h\tilde{J}_\alpha}\frac{\tilde{J}_\beta - \tilde{J}_\gamma}{j_\beta - j_\gamma}, \qquad \alpha \in \overline{1,3}. \tag{35}$$

The parametrizations (32), (35) are equivalent. The equivalence is achieved by an invertible map [18]:

$$\tilde{J}_\alpha = J_\alpha(J_\alpha - J_\beta - J_\gamma).$$

The structure constants K_δ, as it follows from the formula (26) and the above form of A_α, B_α, l_α, are:

$$K_\delta = -\frac{2}{h}\frac{k_\delta}{J_\delta}\frac{\sum_{\alpha=1}^{3} j_\alpha(J_\beta - J_\gamma)}{\prod_{\alpha=1}^{3}(j_\beta - j_\gamma)}, \qquad \delta \in \overline{1,3}. \tag{36}$$

The structure constants M_α, as it follows from the formula (27) are the following:

$$M_\alpha = \frac{2\nu}{hJ_\alpha}\frac{J_\beta - J_\gamma}{j_\beta - j_\gamma} + \mu\frac{(J_1 + J_2 + J_3)}{J_\alpha}, \qquad \alpha \in \overline{1,3}. \tag{37}$$

In the result the generators of the ideal $\mathcal{J}$ acquire the following explicit form:

$$\hat{S}_\alpha\hat{S}_\beta - \hat{S}_\beta\hat{S}_\alpha + \frac{1}{hJ_\gamma}\frac{J_\alpha - J_\beta}{j_\alpha - j_\beta}\left((\hat{S}_0 + \nu)\hat{S}_\gamma + \hat{S}_\gamma(\hat{S}_0 + \nu)\right) + K_\gamma(\hat{S}_0 + \nu) + \frac{\mu\sum_{\alpha=1}^{3} J_\alpha}{J_\gamma}\hat{S}_\gamma, \tag{38a}$$

$$\hat{S}_\gamma\hat{S}_0 - \hat{S}_0\hat{S}_\gamma + h\big((j_\alpha - j_\beta)(\hat{S}_\alpha\hat{S}_\beta + \hat{S}_\beta\hat{S}_\alpha) + 2(k_\alpha\hat{S}_\beta - k_\beta\hat{S}_\alpha)\big), \qquad \gamma \in \overline{1,3}, \tag{38b}$$

where the indices α, β, γ constitute the cyclic permutations of the indices $1,2,3$ and the ideal parameters $(J_1 : J_2 : J_3, \nu, \mu, h)$ belong to the space $\mathbb{C}P^2 \times \mathbb{C}^3$.

3.4.2. The Casimir elements

Let us now describe the Casimir elements of the algebra (22) with the structure constants (31)-(36).

Proposition 3.4. *The following elements:*

$$\hat{C}_1 = -\frac{\prod_{\alpha=1}^{3}(J_\beta - J_\gamma)}{\prod_{\alpha=1}^{3}(j_\beta - j_\gamma)}\frac{(\hat{S}_0 + \nu)^2}{h^4\prod_{\delta=1}^{3} J_\delta} + \sum_{\alpha=1}^{3}\frac{J_\beta - J_\gamma}{j_\beta - j_\gamma}\frac{\hat{S}_\alpha^2}{h^2 J_\alpha} + \sum_{\alpha=1}^{3}\frac{K_\alpha}{h}\hat{S}_\alpha + \mu\frac{(\sum_{\alpha=1}^{3} J_\alpha)(\sum_{\alpha=1}^{3} j_\alpha J_\alpha(J_\beta - J_\gamma))}{h^3\prod_{\delta=1}^{3} J_\delta\prod_{\alpha=1}^{3}(j_\beta - j_\gamma)}\hat{S}_0, \tag{39}$$

and

$$\hat{C}_2 = -\sum_{\alpha=1}^{3}\frac{1}{h^2}\frac{J_\beta - J_\gamma}{j_\beta - j_\gamma}(J_\alpha - J_\beta - J_\gamma)\hat{S}_\alpha^2 +$$

$$2\sum_{\alpha=1}^{3}\frac{k_\alpha}{h^2}\frac{j_\alpha(J_\beta-J_\gamma)(J_\alpha-J_\beta-J_\gamma)-j_\beta(J_\gamma-J_\alpha)J_\gamma-j_\gamma(J_\alpha-J_\beta)J_\beta}{(j_1-j_2)(j_3-j_1)(j_2-j_3)}\hat{S}_\alpha+\frac{\mu(\sum_{\alpha=1}^{3}J_\alpha)(\sum_{\alpha=1}^{3}j_\alpha(J_\beta-J_\gamma))}{h^3\prod_{\alpha=1}^{3}(j_\beta-j_\gamma)}\hat{S}_0 \tag{40}$$

are central elements of the algebra of the algebra (22) with the structure constants (31)-(37).

Idea of the Proof. The Casimir elements are found using the method of the indeterminate coefficients in the assumption of the linear-quadratic form of the Casimirs. □

Remark 5. From (38b) it follows that Heisenberg equation of motion with respect to $\hat{S}_0$:

$$i\hbar\frac{d\hat{S}_\gamma}{dt}=[\hat{S}_\gamma,\hat{S}_0],\quad \gamma\in\overline{1,3}$$

coincides — on the quotient algebra — with the dynamical equations (15) if and only if $h=-i\hbar$.

3.5. The variety of quantum algebras and the classical limit

3.5.1. Affine re-parametrisation of the ideal

In this subsection we give another parametrisation of the ideal that yields a quantum analogue of the Poisson pencil structure. It is based on the observation that a simultaneous re-scaling of the parameters $J_\alpha\to\hat{J}_\alpha=QJ_\alpha,\ Q\neq 0$ does not affect the ideal generated by the polynomials (38).

Lemma 3.1. *Let* $\sum_{\alpha=1}^{3}J_\alpha j_\beta j_\gamma(j_\beta-j_\gamma)\neq 0$. *Then up to projective equivalence* $\hat{J}_\alpha=QJ_\alpha$ *the structure constants* $\hat{J}_\alpha$ *can be parametrised as follows:*

$$\hat{J}_\alpha=1+h^2j_\alpha\Big(C+(j_\beta+j_\gamma)D\Big),\qquad \alpha\in\overline{1,3}, \tag{41}$$

where C *and* D *are arbitrary complex parameters.*

Proof. The system of three equations on variables C,D and Q

$$QJ_\alpha=1+h^2j_\alpha\Big(C+(j_\beta+j_\gamma)D\Big),\qquad \alpha\in\overline{1,3}$$

admits a unique solution

$$C=\sum_{\alpha=1}^{3}\frac{J_\alpha j_\alpha(j_\beta-j_\gamma)}{h^2\sum_{\alpha=1}^{3}J_\alpha j_\beta j_\gamma(j_\beta-j_\gamma)},\quad D=-\frac{\sum_{\alpha=1}^{3}J_\alpha(j_\beta-j_\gamma)}{h^2\sum_{\alpha=1}^{3}J_\alpha j_\beta j_\gamma(j_\beta-j_\gamma)}, \tag{42}$$

and

$$Q=\frac{\prod_{\alpha=1}^{3}(j_\beta-j_\gamma)}{\sum_{\alpha=1}^{3}J_\alpha j_\beta j_\gamma(j_\beta-j_\gamma)}.\quad \square$$

Remark 6. In terms of the parameters C,D the inequalities (33), (34) take the form

$$\begin{gathered}\left(h^2\left(C\left(j_1+j_2-j_3\right)+2Dj_1j_2\right)+1\right)^4+\\ 16h^4\left(j_1-j_3\right)\left(j_2-j_3\right)\left(C+Dj_1\right)\left(C+Dj_2\right)\left(h^2j_2\left(C+D\left(j_1+j_3\right)\right)+1\right)\left(h^2j_1\left(C+D\left(j_2+j_3\right)\right)+1\right)\neq 0,\\ 1+h^2\left(C\left(j_\alpha+j_\beta-j_\gamma\right)+2Dj_\alpha j_\beta\right)\neq 0,\quad 1+h^2j_\alpha\left(C+D\left(j_\beta+j_\gamma\right)\right)\neq 0,\quad \alpha\in\overline{1,3}.\end{gathered} \tag{43}$$

The following Proposition is the main result of the present article:

Proposition 3.5. *A quantisation ideal of the form (22) for the Zhukovsky–Volterra top (15), (16) leads to a quadratic quantum algebra with the commutation relations:*

$$[\hat{S}_\alpha,\hat{S}_\beta]=-\frac{2h}{1+h^2j_\gamma\big(C+(j_\beta+j_\alpha)D\big)}\Big((C+Dj_\gamma)\frac{(\hat{S}_0\hat{S}_\gamma+\hat{S}_\gamma\hat{S}_0)}{2}+(A+Bj_\gamma)\hat{S}_\gamma+Dk_\gamma\hat{S}_0+Bk_\gamma\Big), \tag{44a}$$

$$[\hat{S}_\gamma, \hat{S}_0] = -h\Big((j_\alpha - j_\beta)(\hat{S}_\alpha\hat{S}_\beta + \hat{S}_\beta\hat{S}_\alpha) + 2(k_\alpha\hat{S}_\beta - k_\beta\hat{S}_\alpha)\Big), \qquad \gamma \in \overline{1,3}, \tag{44b}$$

where parameters h, A, B, C, D satisfy the inequalities (43) and arbitrary otherwise.

Remark 7. The parameters ν, μ are related with the constants A, B, C, D, h as follows:

$$\nu D = B, \quad 2\nu h C + \mu\Big(3 + h^2\big(2D\sum_{\alpha=1}^{3} j_\beta j_\gamma + C\sum_{\alpha=1}^{3} j_\alpha\big)\Big) = 2hA. \tag{45}$$

3.5.2. The classical limit

Let us now assume that the functions A, B, C, D are analytical functions of h:

$$A = a + \mathcal{O}(h), \quad B = b + \mathcal{O}(h), \quad C = c + \mathcal{O}(h), \quad D = d + \mathcal{O}(h). \tag{46}$$

Under such the assumption the quantum algebra (44) is a quantum deformation of the classical inhomogeneous quadratic Poisson algebra with the Poisson brackets (11) labeled by four parameters a, b, c, d. Indeed, using (44) and the expansions (46), it is easy to see that in the limit $h \to 0$ the right-hand-side of (44) is exactly $\{\ ,\ \}_{a,b,c,d}$ multiplied by $-h$ (i.e. by $i\hbar$). The inequalities (43) are obviously satisfied in the neighbourhood of $h = 0$.

In terms of parameters A, B, C, D, the central elements (47)-(48) of the quantum algebra take the form:

$$\hat{C}_1 = -\frac{h^2\prod_{\alpha=1}^{3}(C + Dj_\alpha)(D\hat{S}_0^2 + 2B\hat{S}_0)}{D\prod_{\alpha=1}^{3}(1 + h^2 j_\alpha(C + (j_\beta + j_\gamma)D))} + \sum_{\alpha=1}^{3}\frac{(C + Dj_\alpha)\hat{S}_\alpha^2 + 2Dk_\alpha\hat{S}_\alpha}{(1 + h^2 j_\alpha(C + (j_\beta + j_\gamma)D))} - 2(AD - CB)\frac{(D + h^2(C^2 + CD\sum_{\alpha=1}^{3} j_\alpha + D^2\sum_{\alpha=1}^{3} j_\beta j_\gamma))}{D\prod_{\alpha=1}^{3}(1 + h^2 j_\alpha(C + (j_\beta + j_\gamma)D))}\hat{S}_0, \tag{47}$$

$$\hat{C}_2 = \sum_{\alpha=1}^{3}(C + Dj_\alpha)(1 + h^2(C(j_\beta + j_\gamma - j_\alpha) + 2Dj_\beta j_\gamma))\hat{S}_\alpha^2 + 2\sum_{\alpha=1}^{3}(D - h^2(C^2 + DCj_\alpha - D^2 j_\beta j_\gamma))k_\alpha\hat{S}_\alpha - 2(AD - CB)\hat{S}_0. \tag{48}$$

The classical limit of the central elements $\hat{C}_1$ and $\hat{C}_2$ yield Casimir elements (12) of the quadratic Poisson bracket (11):

$$C_1 = \lim_{\hbar\to 0}\hat{C}_1 = \lim_{\hbar\to 0}\hat{C}_2 = \sum_{\alpha=1}^{3}(c + dj_\alpha)S_\alpha^2 + 2d\sum_{\alpha=1}^{3}k_\alpha S_\alpha - 2(ad - bc)S_0 \tag{49}$$

$$C_2 = \lim_{\hbar\to 0}\frac{1}{h^2}\frac{D}{\prod_{\alpha=1}^{3}(C + Dj_\alpha)}\Big(\hat{C}_2 - \frac{D\prod_{\alpha=1}^{3}(1 + h^2 j_\alpha(C + (j_\beta + j_\gamma)D))}{(D + h^2(C^2 + CD\sum_{\alpha=1}^{3} j_\alpha + D^2\sum_{\alpha=1}^{3} j_\beta j_\gamma))}\hat{C}_1\Big) = dS_0^2 + \sum_{\alpha=1}^{3}S_\alpha^2 + 2bS_0. \tag{50}$$

3.6. Comparison with the existing algebras

3.6.1. Sklyanin algebra

The quantum Sklyanin algebra [10] corresponds to the case of the purely quadratic structure of quantum anisotropic Euler's top, i.e. to the case $\mu = \nu = 0$ and $k_\alpha = 0$, $\alpha \in \overline{1,3}$. The corresponding ideal generators (38) are simplified to the following form:

$$F_\gamma = \hat{S}_\alpha\hat{S}_\beta - \hat{S}_\beta\hat{S}_\alpha + \frac{1}{hJ_\gamma}\frac{J_\alpha - J_\beta}{j_\alpha - j_\beta}\big(\hat{S}_0\hat{S}_\gamma + \hat{S}_\gamma\hat{S}_0\big), \qquad \gamma \in \overline{1,3}, \tag{51a}$$

$$G_\gamma = \hat{S}_\gamma\hat{S}_0 - \hat{S}_0\hat{S}_\gamma + h(j_\alpha - j_\beta)(\hat{S}_\alpha\hat{S}_\beta + \hat{S}_\beta\hat{S}_\alpha), \tag{51b}$$

The algebra with commutation relations (51) is equivalent to the Sklyanin algebra obtained from the quantum group considerations, where

$$J_\alpha = \frac{1 + 2j_\alpha h^2 + ((j_\beta + j_\gamma)j_\alpha - j_\beta j_\gamma)h^4}{(1 + h^2 j_\alpha)}, \qquad \alpha \in \overline{1,3}. \tag{52}$$

The above expression for J_α coincides with the one obtained by Sklyanin after the re-parametrisation $h^2 = 1/\wp(i\hbar)$, where $\wp$ is a Weierstrass elliptic function with $g_2 = -4(j_1j_2 + j_2j_3 + j_3j_1)$, $g_3 = -4j_1j_2j_3$, assuming $j_1 + j_2 + j_3 = 0$.

It follows from Lemma 3.1 that corresponding functions $C = C(h)$ and $D = D(h)$ are:

$$C = \frac{1 - h^4 \sum_{\alpha=1}^{3} j_\beta j_\gamma - 2h^6 j_1 j_2 j_3}{1 + h^6(\sum_{\alpha=1}^{3} j_\alpha^3 + j_1 j_2 j_3) - h^8 \sum_{\alpha=1}^{3} j_\beta^2 j_\gamma^2},$$

$$D = \frac{h^2(3 + h^4 \sum_{\alpha=1}^{3} j_\beta j_\gamma)}{1 + h^6(\sum_{\alpha=1}^{3} j_\alpha^3 + j_1 j_2 j_3) - h^8 \sum_{\alpha=1}^{3} j_\beta^2 j_\gamma^2}.$$

In this case $c = 1$, $d = 0$, i.e. in the classical limit we obtain the first quadratic (Sklyanin) brackets.

3.6.2. Algebra of Levin-Olshanetsky-Zotov

Levin, Olshanetsky and Zotov proposed a quantisation of the Zhukovsky-Volterra top ($k_\alpha \neq 0$) [6]. Basing on the reflection equation algebra they found a quantum algebra defined by the following ideal:

$$\mathcal{J}_{LOZ} = \langle \hat{S}_\alpha \hat{S}_\beta - \hat{S}_\beta \hat{S}_\alpha - i(\hat{S}_0 \hat{S}_\gamma + \hat{S}_\gamma \hat{S}_0), \quad \hat{S}_\gamma \hat{S}_0 - \hat{S}_0 \hat{S}_\gamma + i\frac{(J_\alpha - J_\beta)}{J_\gamma}(\hat{S}_\alpha \hat{S}_\beta + \hat{S}_\beta \hat{S}_\alpha) + \frac{i}{J_\gamma}(l_\alpha \hat{S}_\beta - l_\beta \hat{S}_\alpha)\rangle. \tag{53}$$

The re-scaling of the variables

$$\hat{S}_0 \to ih\hat{S}_0, \quad \hat{S}_\alpha \to \sqrt{J_\alpha J_\beta}\hat{S}_\alpha, \quad l_\alpha \to h\sqrt{J_\alpha J_\beta} l_\alpha, \quad \alpha \in \overline{1,3}$$

transforms the ideal (53) to the form

$$\mathcal{J} = \langle \hat{S}_\alpha \hat{S}_\beta - \hat{S}_\beta \hat{S}_\alpha + \frac{h}{J_\gamma}(\hat{S}_0 \hat{S}_\gamma + \hat{S}_\gamma \hat{S}_0), \quad \hat{S}_\gamma \hat{S}_0 - \hat{S}_0 \hat{S}_\gamma + \frac{(J_\alpha - J_\beta)}{h}(\hat{S}_\alpha \hat{S}_\beta + \hat{S}_\beta \hat{S}_\alpha) + (l_\alpha \hat{S}_\beta - l_\beta \hat{S}_\alpha)\rangle. \tag{54}$$

The ideal $\mathcal{J}$ is a partial case of the ideal $\mathcal{I}$ (22), corresponding to the following choice of the parameters

$$A_\alpha = \frac{h}{J_\alpha}, \; B_\alpha = \frac{(J_\beta - J_\gamma)}{h}, \; K_\alpha = 0, \; M_\alpha = 0, \; N_\alpha = 0, \qquad \alpha \in \overline{1,3}. \tag{55}$$

The parametrisation (55) satisfies the condition (24), (25) of our Theorem 3.1, and therefore the quantum algebra of Levin, Olshanetsky and Zotov possess PBW property up to monomials of the third order.

The Heisenberg equations with the Hamiltonian $\hat{S}_0$ in [6] are not equivalent to the dynamical system (15) on the free algebra $\mathcal{A} = \mathbb{C}\langle \hat{S}_0, \ldots, \hat{S}_3 \rangle$, in other words, the coefficients B_α do not have the form (31), since the ideal (54), (55) is not invariant with respect to the non-Abelian dynamics (15). The quantisation presented in [6] can be regarded as a simultaneous deformation of both the commutative algebra of functions on the phase space and the constants j_α of the dynamical system (1). This deformation depends on a single parameter $\hbar$

$$J_\alpha = \frac{\sqrt{(1 + h^2 j_\beta)(1 + h^2 j_\gamma)}}{\sqrt{(1 + h^2 j_\alpha)}}, \qquad \alpha \in \overline{1,3}, \tag{56}$$

where $h^2 = 1/\wp(i\hbar)$ with $g_2 = -4(j_1 j_2 + j_2 j_3 + j_3 j_1)$, $g_3 = -4 j_1 j_2 j_3$, and $j_1 + j_2 + j_3 = 0$. In the classical limit $\hbar \to 0$ we get

$$J_\alpha = 1 + \hbar^2 j_\alpha + \mathcal{O}(\hbar^4), \qquad \alpha \in \overline{1,3},$$

the commutation relations yield the quadratic Poisson structure (11) with $c = 1$, $a = b = d = 0$, and the limiting system coincides with (1).

4. Conclusion and discussion

The results of this paper pose several interesting mathematical and physical problems. The quantisation obtained depends on five parameters, one of which can be identified with the Planck constant. This quantisation is a generalisation of the commonly used deformation of the $so(3)$ standard Poisson bracket and Sklyanin's quadratic Poisson structure in the case of the Euler top. In the classical limit, it results in a four-parametric family of Poisson brackets, which lead to the same dynamical system as the Zhukovsky-Volterra (and Euler) top, thereby yielding identical dynamics.

In the quantum case, the problem is more subtle. Although the equations of motion formally coincide, the observables $\hat{S}_\alpha$ satisfy commutation relations that essentially depend on a choice of the quantisation parameters. We have reasons to believe that the resulting spectrum of the Hamiltonian also depends on the choice of the parameters. In order to compare our results with experimental data, it is necessary to develop a representation theory for the obtained five-parametric algebra (see [11] for the Sklyanin algebra subcase) and solve the spectral problem for the corresponding Hamiltonian.

CRediT authorship contribution statement

T. Skrypnyk: Conceptualization, Methodology, Draft preparation. **A. Mikhailov:** Conceptualization, Methodology, Reviewing and Editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Taras Skrypnyk reports financial support was provided by University of Cambridge Isaac Newton Institute for Mathematical Sciences. Taras Skrypnyk reports financial support was provided by London Mathematical Society. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors are grateful to E. Sklyanin, N. Reshetikhin, D. Gurevich, and A. Zotov for their valuable discussions. The work of the second author was made possible by an Isaac Newton Institute and London Mathematical Society Solidarity grant, for which he expresses his gratitude. This article is partially based on work from COST Action CaLISTA CA21109, supported by COST (European Cooperation in Science and Technology), www.cost.eu.

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