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Polynomial-exponential equations — Some new cases of solvability

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Abstract

Recently, Brownawell and the second author proved a 'non-degenerate' case of the (unproved) 'Zilber Nullstellensatz' in connexion with 'Strong Exponential Closure'. Here, we treat some significant new cases. In particular, these settle completely the problem of solving polynomial-exponential equations in two complex variables. The methods of proof are also new, as is the consequence, for example, that there are infinitely many complex z with $e^z + e^{1/z} = 1$.

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1 | INTRODUCTION

In a recent paper [5], Brownawell and the second author proved a result in connexion with Zilber's 'Strong Exponential Closure Axiom' for 'pseudo-exponential fields'. Over $\mathbf{C}$, this axiom becomes a conjecture, and it is formulated for irreducible algebraic varieties $\mathcal{V}$ in $\mathbf{C}^n \times \mathbf{C}^{*n}$. Its non-strong form, mentioned by Zilber [27, Corollary 4.5, p. 83] in connexion with $\mathbf{C}_{\exp}$ (see also Bays and Kirby [4, pp. 495 and 538, 539] on 'exponential-algebraic closure'), states that if $\mathcal{V}$ is 'normal' (or ex-normal) and 'free' (see later for discussions of these concepts), then $\mathcal{V}$ has a point of the shape

$$(X_1, \dots, X_n, \hat{X}_1, \dots, \hat{X}_n) = (z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}). \tag{1.1}$$

This shape makes evident the connexion with Schanuel's Conjecture (see [14] and Zilber [26] for much more, as well as the work [16] of the first author and also the more recent papers [7, 8] of D'Aquino, Fornasiero and Terzo).

No data are associated with this article

Proposition 2 of [5, p. 448] proves the existence of (1.1) with these two hypotheses on $\mathcal{V}$ replaced by a single one. Denote by π the projection from $\mathbf{C}^n \times \mathbf{C}^{*n}$ to $\mathbf{C}^n$. Then, if (the Zariski closure) of $\pi(\mathcal{V})$ has dimension n, there is always such a point (1.1). This single hypothesis implies that $\mathcal{V}$ is normal (but not that $\mathcal{V}$ is free). We remark that the proposition as stated in [5] appears to require that $\mathcal{V}$ has dimension n; however, if the dimension is $n' > n$, then life just gets simpler and we can adjoin $n' - n$ suitably chosen equations $\hat{X}_i = 1$. For $n = 1$, the proposition seems to be reasonably well known; see, for example, Marker [17] (or Henson and Rubel [13] earlier). But already for $n = 2$, it was new.

Our main purpose in this paper is to relax the condition that $\dim \pi(\mathcal{V}) = n$. We may note that if this dimension is 0, then $\pi(\mathcal{V})$ is a single point $(z_1, \ldots, z_n)$ and now the existence of (1.1) is obvious, provided that $\mathcal{V}$ itself has dimension at least n.

Our main result concerns the case $\dim \pi(\mathcal{V}) = 1$. Here too it is reasonable to assume that $\mathcal{V}$ has dimension at least n: for example, it is not difficult to prove that if $\mathcal{L}$ is a line in $\mathbf{C}^n$ and $\mathcal{K}$ is a translate of a group subvariety of $\mathbf{G}_{\mathrm{m}}^n = \mathbf{C}^{*n}$ of dimension $n - 2$, both generic in a perfectly explicit sense, then $\mathcal{L} \times \mathcal{K}$ contains no point (1.1).

However, there are more subtle obstructions to solvability. A simple example is $\mathcal{V}$ defined in $\mathbf{C}^2 \times \mathbf{C}^{*2}$ by

$$X_1 + X_2 = 1, \quad \hat{X}_1 \hat{X}_2 = 1, \tag{1.2}$$

because if $z_1 + z_2 = 1$, then $e^{z_1} e^{z_2} = e \neq 1$. There are similar examples with $X_1 + X_2 = 1$ replaced by

$$m_1 X_1 + m_2 X_2 = c \tag{1.3}$$

for any complex c and integers m_1, m_2 not both zero. One can build analogous examples in $\mathbf{C}^n \times \mathbf{C}^{*n}$ where $\pi(\mathcal{V})$ is contained in the hyperplane defined by

$$m_1 X_1 + \cdots + m_n X_n = c. \tag{1.4}$$

It turns out that this is essentially the only obstruction under our new assumption. Thus, we shall prove the following result.

Theorem 1.1. *Suppose that $\mathcal{V}$ is an irreducible algebraic variety in $\mathbf{C}^n \times \mathbf{C}^{*n}$, of dimension at least n, such that the Zariski closure of $\pi(\mathcal{V})$ in $\mathbf{C}^n$ has dimension 1. If* (1.4) *does not hold on $\mathcal{V}$ with any complex c and any integers $m_1, \ldots, m_n$ not all zero, then $\mathcal{V}$ contains a point* (1.1).

For $n = 1$, this reduces again to the result in [17]; but already for $n = 2$, it is again new.

Note that the absence of relations (1.4) is what Zilber calls 'free of additive dependencies (over **C**)' [27, p. 74]. When an obstruction (1.4) does arise, one can recover further information inductively by means of 'back-substitution'.

For example in $\mathbf{C}^2 \times \mathbf{C}^{*2}$ with $\mathcal{V}$ defined by

$$X_1 + X_2 = 1, \quad X_1 X_2 = \hat{X}_1 + \hat{X}_2, \tag{1.5}$$

any point (1.1) on $\mathcal{V}$ must lie on $\hat{X}_1\hat{X}_2 = e$; thus, we can eliminate $X_2, \hat{X}_2$ to get down to

$$\hat{X}_1^2 + e = X_1\hat{X}_1(1 - X_1) \tag{1.6}$$

in $\mathbf{C} \times \mathbf{C}^*$.

In Theorem 1.1, we focus on the existence of a single point (1.1), but our proof gives rather more, as was the case in [5]. From the viewpoint of exponential polynomials, it is natural to consider just the projections $(z_1, \dots, z_n)$, although we will mention another possibility later. In [5], we remarked that the set of all such projections is not only infinite but even Zariski dense in $\mathbf{C}^n$ (which is $\pi(\mathcal{V})$ there). This was proved by a simple trick (see later); unfortunately, that does not work in our situation if $n > 1$. Nevertheless, our method of proof shows that the set of $(z_1, \dots, z_n)$ is indeed infinite and therefore Zariski dense in the curve $\pi(\mathcal{V})$.

In fact, for $n = 1$, the known results already lead to a fairly explicit description of the set $Z = Z_C$ of points (1.1) on an irreducible curve C in $\mathbf{C} \times \mathbf{C}^*$ based only on the geometry of C. Namely, if $\dim \pi(C) = 0$, then $\pi(Z)$ is a single point, while if $\dim \pi(C) = 1$, then $\pi(Z)$ is infinite, hence Zariski dense in $\pi(C)$.

Now for $n = 2$, we can combine Theorem 1.1 with back-substitution and [5] to give a conclusive result for $\pi(Z)$ and surfaces $\mathcal{S}$ in $\mathbf{C}^2 \times \mathbf{C}^{*2}$ (we remind the reader that Schanuel's Conjecture itself remains unknown for $n = 2$). To state this, the following notation will be useful.

If the Zariski closure of $\pi(\mathcal{S})$ is a line $\mathcal{L}$ as in (1.3), which we may call a line with rational slope, we write $\mathcal{K}$ for the set of (e^{z_1}, e^{z_2}) in $\mathbf{C}^{*2}$ with (z_1, z_2) in $\mathcal{L}$. This is an algebraic curve (and even a translate of a group subvariety). Then, we write $\mathcal{G} = \mathcal{L} \times \mathcal{K}$ (still a translate of a group subvariety) and

$$\mathcal{T} = \mathcal{S} \cap \mathcal{G}. \tag{1.7}$$

We will see in Section 5 that if the variety $\mathcal{T}$ is non-empty, then it is infinite.

Our result for $n = 2$ is as follows.

Theorem 1.2. *Suppose that $\mathcal{S}$ is an irreducible surface in $\mathbf{C}^2 \times \mathbf{C}^{*2}$, and let $Z = Z_{\mathcal{S}}$ be the set of points (1.1) in $\mathcal{S}$.*

(a) *If $\dim \pi(\mathcal{S}) = 0$, then $\pi(Z)$ is a single point, so Zariski dense in $\pi(\mathcal{S})$.*
(b) *If $\dim \pi(\mathcal{S}) = 2$, then $\pi(Z)$ is infinite, and even Zariski dense in $\pi(\mathcal{S})$.*
(c) *If $\dim \pi(\mathcal{S}) = 1$ and the Zariski closure of $\pi(\mathcal{S})$ is not a line of rational slope, then $\pi(Z)$ is infinite, so Zariski dense in $\pi(\mathcal{S})$.*
(d) *If $\dim \pi(\mathcal{S}) = 1$ and the Zariski closure of $\pi(\mathcal{S})$ is a line of rational slope $\mathcal{L}$, then the following subcases depending on $\mathcal{T}$ in (1.7) are possible.*
    (d_1) *If $\mathcal{T}$ is empty, then $\pi(Z)$ is empty.*
    (d_2) *If $\dim \mathcal{T} = 2$, then $\pi(Z)$ is infinite, so Zariski dense in $\pi(\mathcal{S})$.*
    (d_3) *If $\dim \mathcal{T} = 1$, then the following subcases are possible:*
        (d_{31}) *If $\dim \pi(\mathcal{T}) = 0$, then $\pi(Z)$ is non-empty and finite, so not Zariski dense in $\pi(\mathcal{S})$.*
        (d_{32}) *If $\dim \pi(\mathcal{T}) = 1$, then $\pi(Z)$ is infinite, so Zariski dense in $\pi(\mathcal{S})$.*

In particular, the only situation where there are no points (1.1) in $\mathcal{S}$ is (d_1) of (d); a typical example is (1.2), where now $\mathcal{L}$ is defined by $X_1 + X_2 = e$ and $\mathcal{K}$ by $\hat{X}_1\hat{X}_2 = e$. We stress that all possible cases and subcases may happen.

1.1 A proof sketch

The key ideas of our proofs build on those of [17] for $n = 1$. We proceed to recall the arguments there for the example $X_1 = \hat{X}_1$.

To solve the resulting $e^z = z$, we look at the function $\Phi(z) = e^z - z$, an entire function of order at most 1. Using Hadamard's Factorisation Theorem as in [17] or better [15, XIII 3.5], we find that if Φ has no zeroes, then it is e^ϕ for ϕ entire. A standard application of Borel–Carathéodory (see below) shows that ϕ is a polynomial of degree at most 1. So, there would be a, b in $\mathbf{C}$ with

$$e^z - z = e^{az+b}. \tag{1.8}$$

This can be disproved in an elementary way by repeated differentiation, or less elementary using algebraic structure theorems of van den Dries [22] and of Henson and Rubel [13]; in our situation for general n, it suffices to apply a well-known result of Ax [1] (which, in fact, is a functional analogue of Schanuel's Conjecture).

Let us examine more closely why our results are new for $n = 2$. Consider the example

$$X_1 X_2 = 1, \quad \hat{X}_1 + \hat{X}_2 = 1.$$

Solving for (1.1) is equivalent to solving the single (non-polynomial-exponential) equation

$$e^z + e^{1/z} = 1 \tag{1.9}$$

in complex numbers $z \neq 0$, which does not seem trivial but one sees no obvious obstruction. The basic argument in [5], finding a fairly obvious approximate solution (like $z_0 = 200\pi i$ to $e^z = z$), then refining it and then using Newton's Method to home in on an actual solution, seems numerically to give convergence (see also Section 7). However, for

$$e^z + e^{z^2} = 1 \tag{1.10}$$

and $z_0 = (200\pi i)^{1/2} = \sqrt{100\pi}(1 + i)$, the matter is less clear, and numerically, there are hints of the 'chaos' which is well known to exist in Newton's Method, with convincing convergence only after 20 iterations. Curiously enough, it works fine for $z_0 = -\sqrt{100\pi}(1 + i)$.

In fact, the method used in [17] works very well for (1.10): we find instead of (1.8)

$$e^z + e^{z^2} - 1 = e^{az^2+bz+c},$$

which can be disproved as before. Our main contribution in this paper is to show that it extends to (1.9) and our general situation.

As it stands, this argument fails for (1.9), because we have an essential singularity at $z = 0$. Thus, we have to restrict to $\mathbf{C} \setminus \{0\}$. But generally Φ analytic on $\mathbf{C} \setminus \{0\}$ with no zeroes on $\mathbf{C} \setminus \{0\}$ need not be e^ϕ for ϕ analytic on $\mathbf{C} \setminus \{0\}$. A simple counterexample is $\Phi(z) = z$.

Littlewood [10, p. 392] said 'it can pay to find out what is the worst enemy of what you want to prove, and then induce him to change sides'. This we do here; the argument to prove $\Phi = e^\phi$ constructs ϕ as

$$\int \frac{\Phi'(z)}{\Phi(z)} \mathrm{d}z,$$

and all we have to do is stay inside $\mathbf{C} \setminus \{0\}$ and ensure that the integral along the homology loops containing $z = 0$ vanishes, which can be arranged by multiplying Φ by a power of the 'enemy' z. The upshot for (1.9) is that

$$e^z + e^{1/z} - 1 = z^m e^{\phi(z)} \tag{1.11}$$

for some integer m and some ϕ analytic on $\mathbf{C} \setminus \{0\}$.

But now we will have to be more careful with Borel–Carathéodory and $z = 0$, and, in fact, it yields only

$$\phi(z) = az^2 + bz + c + \frac{d}{z} + \frac{e}{z^2} \tag{1.12}$$

now with an exponent of z bigger than one might expect and of course of $1/z$ too. Nevertheless, still Ax's Theorem leads to a contradiction.

For examples like

$$e^z + e^{1/(z^3+z+1)} = 1,$$

we have to avoid three points, so three enemies, namely the three factors of $z^3 + z + 1$, and leading to homology of rank 3.

More generally, we have the following.

Example 1.3. Suppose that Φ is analytic on $\mathbf{C} \setminus \{p_1, \dots, p_{s-1}\}$ and never vanishes there. Then there are integers $m_1, \dots, m_{s-1}$ such that

$$\Phi(z) = (z - p_1)^{m_1} \cdots (z - p_{s-1})^{m_{s-1}} e^{\phi(z)}$$

for some ϕ also analytic on $\mathbf{C} \setminus \{p_1, \dots, p_{s-1}\}$.

Consider next the surface

$$X_1^3 + X_1 + 1 = X_2^2, \quad \hat{X}_1 + \hat{X}_2 = 1$$

leading to

$$e^z + e^{\sqrt{z^3+z+1}} = 1.$$

As the projection to $\mathbf{C}^2$ is the affine part of an elliptic curve $\mathcal{E}$, we can no longer work with $\mathbf{C} \setminus S$ for a finite set S, and this particular problem concerns $\mathcal{E} \setminus \{O\}$ for the origin O. In fact, the homology is the same as that of $\mathcal{E}$, with rank 2, and so, we have to find two enemies. These can be written down explicitly on the universal cover $\mathbf{C}$ of $\mathcal{E}$ in terms of Weierstrass $\wp$ and ζ functions, or by integrating suitable differentials of the first and second kind on $\mathcal{E} \setminus \{O\}$. They are, in fact, the very simplest examples of Baker–Akhiezer functions (see [3, Ch. XIV] and the foreword by Krichever), although they were known to Weierstrass. For these, there seem to be no algebraic structure theorems, but again, Ax suffices for a contradiction.

To see this in action, represent $\mathcal{E} \setminus \{O\}$ in the form $y^2 = 4x^3 - g_2x - g_3$ (a simple change of variables suffices). Recall that $\mathcal{E}$ can be seen as the quotient of $\mathbf{C}$ by a two-dimensional lattice Ω, so we may think of functions on $\mathcal{E} \setminus \{O\}$ as doubly periodic functions on $\mathbf{C} \setminus \Omega$. The associated Weierstrass ζ has the property that for every $\omega \in \Omega$, there is a quasi-period η such that $\zeta(z+\omega) = \zeta(z) + \eta$. Explicit enemies are then the functions $e^{\omega\zeta(z)-\eta z}$ for any non-zero ω. The outcome is the following analogue of (1.11).

Example 1.4. Suppose that Φ is doubly periodic with respect to Ω, analytic on $\mathbf{C} \setminus \Omega$ and never vanishes there. Then there is a period ω, with quasi-period η, such that

$$\Phi(z) = e^{\omega\zeta(z)-\eta z} e^{\phi(z)}$$

for some ϕ also doubly periodic with respect to Ω and analytic on $\mathbf{C} \setminus \Omega$.

Going further, to solve

$$e^z + e^{1/\sqrt{z^3+z+1}} = 1,$$

we have to avoid an additional three points (the zeroes of $z^3 + z + 1$), leading to homology of rank 5 and differentials of the third kind or the Weierstrass sigma function (also Baker–Akhiezer). Here too we get an analogue of (1.11). See (6.5), (6.13) and (6.16) in Section 6 for more examples.

Finally, consider

$$X_1^9 + X_2^9 = 1, \quad \hat{X}_1 + \hat{X}_2 = 1 \tag{1.13}$$

leading to a curve C of genus 28 and homology rank 56. To write down the enemies as complex functions (on the covering space) is not so easy without the aid of theta functions (in 28 variables), but on the curve, the 56 enemies correspond again to a suitable choice of differentials of the first and second kind (and generally, we need the third kind too).

After working out our proofs in terms of these explicitly constructed enemies, we realised that there is a more abstract proof based on the canonical isomorphism between algebraic and analytic de Rham cohomology of complex affine varieties, dating back to Grothendieck [12]. That paper actually uses Hironaka's resolution of singularities, but we found that this was not needed in our situation (see Section 3 for more details). So, in the end, we were able to avoid any appeal to [12] by using instead suitable differentials on the underlying curve. This is the proof that we present here; nevertheless, we do give an account of the original more explicit constructions, also because these seem to be helpful in obtaining effective versions of our results in which, for example, the zeroes can be localised.

1.2 | Further remarks

In [5], with the points (1.1) on $\mathcal{V}$ projecting to a Zariski dense subset of $\mathbf{C}^n$, we noted that this holds even in a strong sense of being 'relatively near' to any one of 'sufficiently many' points on $(2\pi i\mathbf{Z})^n$. It would be interesting to obtain similar strengthenings in our present setup.

It is also natural to consider the distribution of the unprojected points (1.1). In the situation of [5], the trick extends at once to show that they are Zariski dense in $\mathcal{V}$ itself. For if G in

$\mathbf{C}[X_1, \dots, X_n, \hat{X}_1, \dots, \hat{X}_n]$ does not vanish on $\mathcal{V}$, we may apply [5] to the variety in $\mathbf{C}^{n+1} \times \mathbf{C}^{*n+1}$ defined by the equations of $\mathcal{V}$ together with $G = \hat{X}_{n+1}$.

In our situation, the analogous statement is unclear, even for $n = 2$. This is illustrated by case (c) in Theorem 1.2. Just for the example (1.13) the density in $\mathcal{S}$ would amount to the fact that there is no $G \neq 0$ in $\mathbf{C}[X_1, \hat{X}_1]$ such that $G(z, e^z) = 0$ for all z with $e^z + e^{\sqrt[9]{1-z^9}} = 1$, which does not seem obvious.

In fact, this case (c) is the only problem, as will be established during the proof.

It would also be interesting to extend the investigations to $\pi(\mathcal{V})$ of other dimensions. The simplest case of dimension 2 in $\mathbf{C}^3 \times \mathbf{C}^{*3}$ leads to systems of equations such as

$$e^z + e^{z^2 - w^2} = z, \quad e^w + e^{z^2 - w^2} = -w.$$

But it may be more difficult to find corresponding extensions of Theorem 1.2. This is because of the obstructions coming from the concept of 'normal' (see [27, p. 75]). For $\mathbf{C}^n \times \mathbf{C}^{*n}$, it amounts to the following. For $k = 1, \dots, n$ and a matrix of k independent rows and n columns with integer entries m_{ij}, we define a map μ from $\mathbf{C}^n \times \mathbf{C}^{*n}$ to $\mathbf{C}^k \times \mathbf{C}^{*k}$ by

$$\mu(X_1, \dots, X_n, \hat{X}_1, \dots, \hat{X}_n) = \left(\sum_{j=1}^n m_{1j} X_j, \dots, \sum_{j=1}^n m_{kj} X_j, \prod_{j=1}^n \hat{X}_j^{m_{1j}}, \dots, \prod_{j=1}^n \hat{X}_j^{m_{kj}} \right).$$

Then one imposes the condition that $\dim \mu(\mathcal{V}) \geqslant k$ for all k and μ.

If this condition fails, then some $\mu(\mathcal{V})$ may be too small to contain the analogues of points (1.1).

Note that normality is not a necessary condition; when $n = 2$, it fails for the example

$$X_1 + X_2 = 1, \qquad \hat{X}_1 \hat{X}_2 = e$$

with $k = 1$ and $m_{11} = m_{12} = 1$ whereas here Z is clearly infinite. We note that this surface is not 'free' (see [27, pp. 74–75]); in fact, it turns out that for $n = 2$, 'freeness' implies 'normality'.

We briefly mention some aspects of decidability and effectivity in our Theorem 1.1.

Already in Theorem 1.2 for $n = 2$, it may not be possible to decide for a given $\mathcal{S}$ which of the various possibilities actually arises. Say the defining equations are just $X_1 = 1, \hat{X}_1 = \theta$ for some θ in $\mathbf{C}$, so we are in (d); we find that we are in (d_1) or (d_2) according to whether $e \neq \theta$ or $e = \theta$. Now if θ is an explicitly given element of $\overline{\mathbf{Q}(\pi)}$, this may not be so easy, as, for example, with

$$\sqrt[6]{\pi^5 + \pi^4} = 2.7182818086 \dots;$$

and, in general, it involves Schanuel's Conjecture of course (see [14, p. 31]).

Furthermore, for three-folds in $\mathbf{C}^3 \times \mathbf{C}^{*3}$, another obstacle arises. Take any absolutely irreducible polynomial P in two variables over $\mathbf{Q}$, and $\mathcal{V}$ defined by $P(X_1/(2\pi i), X_2/(2\pi i)) = 0$ together with $\hat{X}_1 = 1, \hat{X}_2 = 1$. Then, the points (1.1) correspond exactly to the integral solutions of $P(x_1, x_2) = 0$. If the genus here is at least 2, then we know no algorithm for finding these, as, for example, with

$$x_1^4 - 2x_2^4 + x_1 x_2 + x_1 - n = 0,$$

and, in general, it involves Hilbert's Tenth Problem (see, e.g. [9]).

We note also that just to decide if a variety is normal involves Zilber-Pink matters (specifically the conjectures of intersection with tori or cosets as considered in [26, 27] for example).

Nevertheless, we shall give some simple effectivity arguments for special cases of Theorem 1.1 such as (1.9).

1.3 | Structure of the paper

The rest of our paper is arranged as follows.

In Section 2, we record some preliminary observations towards the proof of Theorem 1.1, including the 'punctured' version of Borel–Carathéodory which avoids $z = 0$ and also the version of Ax's Theorem that we need.

Then, in Section 3, we construct suitable differentials on the underlying curve $\pi(\mathcal{V})$ and deduce our special form of the Grothendieck result.

The proof of Theorem 1.1 follows in Section 4, and that of Theorem 1.2 in Section 5.

Then, in Section 6, we present the explicit versions of (1.11) for small genus, and finally, in Section 7, we briefly explain about effectivity and Theorem 1.1.

We are grateful to David Grant for his help in connexion with the genus 2 constructions in Section 6.

2 | PRELIMINARIES

From now on, $\mathcal{V}$ will be as in Theorem 1.1. When $\dim \mathcal{V} = n$, it will be important to know that $\mathcal{V}$ is defined, apart from the equations defining the curve $\pi(\mathcal{V})$, by a single additional equation.

Lemma 2.1. *Suppose that $\mathcal{V}$ is an irreducible algebraic variety in $\mathbf{C}^n \times \mathbf{C}^{*n}$ of dimension n such that the Zariski closure of $\pi(\mathcal{V})$ in $\mathbf{C}^n$ is a curve C_0. Let $\mathfrak{P}_0$ be the prime ideal of C_0 in $\mathfrak{R}_0 = \mathbf{C}[X_1, \ldots, X_n]$. Similarly, write $\mathfrak{P}$ for the prime ideal of $\mathcal{V}$ in $\mathfrak{R} = \mathbf{C}[X_1, \ldots, X_n, \hat{X}_1, \ldots, \hat{X}_n]$.*

Then, there are F in $\mathfrak{P}$, not in $M\mathfrak{R}_0 + \mathfrak{P}_0\mathfrak{R}$ for any monomial M in $\hat{X}_1, \ldots, \hat{X}_n$, and G_0 in $\mathfrak{R}_0$, not in $\mathfrak{P}_0$, such that $\mathfrak{P}$ is contained in $G_0^{-1}(F\mathfrak{R} + \mathfrak{P}_0\mathfrak{R})$. Furthermore, if C_0 is a line, then we can take $G_0 = 1$; in particular, if $n = 2$ and C_0 is defined by the vanishing of a polynomial F_0 of degree 1, then $\mathfrak{P} = F\mathfrak{R} + F_0\mathfrak{R}$.

Proof. Let $x_1, \ldots, x_n$ denote the coordinate functions on C_0, and call $\mathfrak{r}_0$ its coordinate ring $\mathbf{C}[x_1, \ldots, x_n]$. By considering elements of $\mathfrak{R}$ as polynomials in $\hat{X}_1, \ldots, \hat{X}_n$, we see that the specialisation σ from $\mathfrak{R}$ to $\mathfrak{r}_0[\hat{X}_1, \ldots, \hat{X}_n]$ has kernel $\mathfrak{P}_0\mathfrak{R}$ (which lies in $\mathfrak{P}$). We now pass into $K_0[\hat{X}_1, \ldots, \hat{X}_n]$ with the quotient field $K_0 = \mathbf{C}(x_1, \ldots, x_n)$ of $\mathfrak{r}_0$. We claim that $\sigma(\mathfrak{P})$ is prime in $\mathfrak{r}_0[\hat{X}_1, \ldots, \hat{X}_n]$, and then by clearing denominators, we see that $K_0\sigma(\mathfrak{P})$ is prime in $K_0[\hat{X}_1, \ldots, \hat{X}_n]$ too.

Indeed, suppose that P_1, P_2 are in $\mathfrak{r}_0[\hat{X}_1, \ldots, \hat{X}_n]$ with P_1P_2 in $\sigma(\mathfrak{P})$. Clearly, $P_1 = \sigma(Q_1)$, $P_2 = \sigma(Q_2)$ with Q_1, Q_2 in $\mathfrak{R}$, and $P_1P_2 = \sigma(Q)$ for Q in $\mathfrak{P}$. Thus, $Q_1Q_2 - Q$ is in the kernel $\mathfrak{P}_0\mathfrak{R}$ so in $\mathfrak{P}$. So, also Q_1Q_2 is in $\mathfrak{P}$. If Q_1 is in $\mathfrak{P}$, then P_1 is in $\sigma(\mathfrak{P})$ and similarly for P_2; this gives the above claim.

Now $\mathfrak{R}/\mathfrak{P}$ has transcendence degree n over $\mathbf{C}$, and it has a subfield $\mathfrak{R}_0/\mathfrak{P}_0$ of transcendence degree 1 over $\mathbf{C}$; thus, $\mathfrak{R}/\mathfrak{P}$ has transcendence degree $n-1$ over $\mathfrak{R}_0/\mathfrak{P}_0$. It follows (see, e.g. [25, p. 91]) that $K_0\sigma(\mathfrak{P})$ is minimal in the sense of [24, p. 238]. This latter reference ('Principal Ideal Theorem') shows that $K_0\sigma(\mathfrak{P})$ is principal, as $K_0[\hat{X}_1, \ldots, \hat{X}_n]$ is a unique factorisation domain. We can further assume that the generator is in $\sigma(\mathfrak{P})$, so it is $\sigma(F)$ for some F in $\mathfrak{P}$. If F were in

$M\mathfrak{R}_0 + \mathfrak{P}_0\mathfrak{R}$ for some monomial M as above, then $\sigma(F) = \sigma(M)f$ for some non-zero f in $\mathfrak{r}_0$, so we could have taken the generator as M; however, that would imply that the projection of $\mathcal{V}$ to $\mathbf{C}^{*n}$ is empty, which is certainly not the case.

Finally, for each A in $\mathfrak{P}$, there is h_A in $K_0[\hat{X}_1, \dots, \hat{X}_n]$ with $\sigma(A) = h_A\sigma(F)$ and there is $g_A \neq 0$ in $\mathfrak{r}_0$ with $g_A h_A$ in $\mathfrak{r}_0[\hat{X}_1, \dots, \hat{X}_n]$. In particular, $g_A = \sigma(G_A)$ for some G_A in $\mathfrak{R}_0$, and $g_A h_A = \sigma(B_A)$ for some B_A in $\mathfrak{R}$. Hence, A is in $G_A^{-1}(F\mathfrak{R} + \mathfrak{P}_0\mathfrak{R})$ and the result follows on taking a finite basis for $\mathfrak{P}$.

If C_0 is a line, then $\mathfrak{r}_0$ is isomorphic to some $\mathbf{C}[x]$, thus a unique factorisation domain. Then, $\mathbf{C}[x][\hat{X}_1, \dots, \hat{X}_n]$ is a unique factorisation domain, so $\sigma(\mathfrak{P})$ is principal and we may assume $\sigma(F)$ to be its generator, in which case we find $\mathfrak{P} = F\mathfrak{R} + \mathfrak{P}_0\mathfrak{R}$, or in other words, we may always take $g_A = 1$. In the special case $n = 2$ with $\mathfrak{P}_0$ generated by a polynomial F_0 of degree 1, we find $\mathfrak{P} = F\mathfrak{R} + F_0\mathfrak{R}$.

This completes the proof. □

Thus, to find a point (1.1) on $\mathcal{V}$, it suffices to solve $F(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) = 0$ with $(z_1, \dots, z_n)$ on C_0 but $G_0(z_1, \dots, z_n) \neq 0$.

Next, we recall a standard version of Borel–Carathéodory which estimates the absolute value $|\phi_0|$ in terms of the real part $\Re\phi_0$.

Lemma 2.2. *For* $0 \leqslant r < R$ *and any* ϕ_0 *analytic on the disc* $|w| \leqslant R$*, we have*

$$\sup_{|w|\leqslant r} |\phi_0(w)| \leqslant \frac{2r}{R-r} \sup_{|w|\leqslant R} \Re\phi_0(w) + \frac{R+r}{R-r}|\phi_0(0)|.$$

Proof. See, for example, [15, XII 3.1]. □

This is used in the classical theory to deduce from an inequality $\Re\phi_0(w) \leqslant c|w|^\kappa$ $(\kappa > 0)$, for ϕ_0 entire and all $|w|$ large, a similar inequality $|\phi_0(w)| \leqslant c'|w|^\kappa$. We use it here to obtain the following consequence for functions ϕ analytic only near (but not at) a finite point, which we can take as $z = 0$.

Lemma 2.3. *Let* ϕ *be a function analytic on a punctured neighbourhood of 0 on which*

$$\Re\phi(z) \leqslant \frac{c}{|z|^\kappa}$$

for some real $c, \kappa \geqslant 0$ *independent of* z*. Then, there is a punctured neighbourhood of 0 on which*

$$|\phi(z)| \leqslant \frac{c'}{|z|^{\kappa+1}}$$

for some real c' *independent of* z*.*

Proof. Let $0 < |z| \leqslant 2\delta$ be a punctured neighbourhood as in the assumption. The conclusion will be about the neighbourhood $0 < |z| \leqslant \delta$. Write $c_1 = \sup_{|z|=\delta} |\phi(z)|$, and choose any z_1 with $0 < |z_1| \leqslant \delta$. We are going to apply Lemma 2.2 to carefully chosen w-discs with

$$w = z - \delta\frac{z_1}{|z_1|},$$

so that their centre $w = 0$ lies on $|z| = \delta$. Accordingly, define

$$\phi_0(w) = \phi\left(w + \delta\frac{z_1}{|z_1|}\right)$$

and

$$r = \delta - |z_1|, \quad R = \delta - \frac{|z_1|}{2},$$

so that $0 \leqslant r < R$. Now $|w| \leqslant R$ implies

$$|z| = \left|w + \delta\frac{z_1}{|z_1|}\right| \leqslant R + \delta \leqslant 2\delta,$$

and so the larger w-disc $|w| \leqslant R$ is contained in the larger z-disc $|z| \leqslant 2\delta$.

Then, for

$$w_1 = z_1 - \delta\frac{z_1}{|z_1|} = z_1\left(1 - \frac{\delta}{|z_1|}\right),$$

we have

$$|w_1| = |z_1|\left|1 - \frac{\delta}{|z_1|}\right| = -(|z_1| - \delta) = r,$$

and so w_1 lies on the smaller w-disc $|w| = r$. Thus, Lemma 2.2 implies

$$|\phi(z_1)| = |\phi_0(w_1)| \leqslant \frac{2\delta}{|z_1|/2}\frac{2^\kappa c}{|z_1|^\kappa} + \frac{2\delta}{|z_1|/2}\left|\phi\left(\delta\frac{z_1}{|z_1|}\right)\right| \leqslant \frac{2^{\kappa+2}\delta c}{|z_1|^{\kappa+1}} + \frac{4\delta c_1}{|z_1|},$$

and the result follows. □

Finally, here is the result of Ax that we shall use.

Lemma 2.4. *In characteristic zero, let K be a differential field with derivation D and constant field C. Let $\xi_1, \ldots, \xi_n$ be in K and let $\hat{\xi}_1, \ldots, \hat{\xi}_n$ be in K^* such that:*

(1) $D\xi_i = D\hat{\xi}_i/\hat{\xi}_i$ *($i = 1, \ldots, n$);*
(2) $m_1\xi_1 + \cdots + m_n\xi_n$ *is not in C for any $m_1, \ldots, m_n$ in $\mathbf{Z}$ not all zero.*

Then at least $n + 1$ among $\xi_1, \ldots, \xi_n, \hat{\xi}_1, \ldots, \hat{\xi}_n$ are algebraically independent over C.

Proof. This follows at once from Theorem 3 of [1, p. 253] (with a single derivation); note that the rank there is 1 because, for example, $D\xi_n = 0$ would imply ξ_n in C, contradicting (2). Note also that (2) implies that $\xi_1, \ldots, \xi_n$ are all not in C, and, in particular, are all non-zero, which seems to be an additional assumption of this Theorem 3 (and is superfluous anyway). □

In fact, we will not need the full force of [1], because we apply it with $\xi_1, \ldots, \xi_n$ in the function field K_0 of a curve with $D = \mathrm{d}/\mathrm{d}z$ for some (non-constant) z in K_0. Then, in fact, $\hat{\xi}_1 =$

$e^{\xi_1}, \dots, \hat{\xi}_n = e^{\xi_n}$ are algebraically independent over K_0 (so we are rather with what is known in the trade as 'Ax–Lindemann–Weierstrass'). This could be seen directly by considering a relation $\sum_{m=1}^{M} \beta_m e^{\alpha_m} = 0$ for $\beta_1, \dots, \beta_M$ in K_0 and $\alpha_1, \dots, \alpha_M$ in K_0 different modulo $\mathbf{C}$, dividing by the last term, differentiating, using induction on M and finally comparing poles of $D\beta_M/\beta_M$ and $D\alpha_M$.

It is convenient to record the following obvious consequence for $n = 1$ ('Ax–Hermite–Lindemann').

Corollary 2.5. *In characteristic zero, let K be a differential field with derivation D and constant field C. Let ξ and $\hat{\xi} \neq 0$ be in K with $D\xi = D\hat{\xi}/\hat{\xi}$ and ξ not in C. Then, $\xi, \hat{\xi}$ are algebraically independent over C.*

3 | DUALITY AND PERIODS

Here we record a number of results on functions and differentials, first of all rational and then only meromorphic, on an algebraic curve. Thus, let C be a complete smooth complex algebraic curve of genus $g \geqslant 0$, and let S be a finite subset of C with cardinality $s \geqslant 1$. Denote by C_S the (affine) curve $C \setminus S$. By (meromorphic) *differential* on C_S, we mean a differential 1-form, that is, $\Psi \mathrm{d}\Phi$ for Φ, Ψ meromorphic on C_S; we call such a differential *regular* on C_S if it has no poles, and *rational* if we can take Ψ, Φ rational.

Let Δ_S denote the space of all rational differentials that are regular on C_S, modulo the ones of the form $\mathrm{d}\phi$ for ϕ rational and with no poles on C_S (we call those *exact*). In other words, Δ_S is the first cohomology group of the algebraic de Rham complex of C_S.

Lemma 3.1. *The linear space Δ_S has dimension $2g + s - 1$. Furthermore, given some fixed P_0 in S, each element of Δ_S can be represented by a differential with at most simple poles in $S \setminus \{P_0\}$.*

Proof. This is fairly well known, but we show how to recover it easily from the Riemann–Roch Theorem. For a given $\mathbf{Z}$-divisor D, let $\mathcal{L}(D)$ be the space of all rational functions on C with divisor at least $-D$, and $\ell(D)$ be its dimension. Let K_z be the divisor of some rational differential form on C, say $\mathrm{d}z$ for some non-constant rational function z. Then, Riemann–Roch states that

$$\ell(D) = \ell(K_z - D) + \deg(D) + 1 - g. \tag{3.1}$$

Furthermore, recall that $\ell(K_z - D)$ is also the dimension of the rational differential forms with divisor at least D. See, for example, [21, p. 17].

Fix some P_0 in S. For positive m sufficiently large, consider the space of differentials with pole at P_0 of order at most m, and at most simple poles on the rest of S (and no other poles). This has dimension $\ell(K_z - D)$ for

$$D = -\sum_{P \in S} P - (m-1)P_0$$

and since here $\ell(D) = 0$, we get dimension $m + s + g - 2$. And the subspace of the exact ones (which, of course, have no simple poles) has dimension $\ell((m-1)P_0) - 1 = m - g - 1$. Thus, the quotient, which embeds naturally into Δ_S, has dimension $2g + s - 1$.

We now claim that all other differentials are equivalent to the ones above. Let δ be a rational regular differential on C_S. Suppose that δ has pole of order $k > 1$ at some P in $S \setminus \{P_0\}$. By (3.1), for every m large enough, there must be a function h in $\mathcal{L}((k-1)P + mP_0)$ which is not in $\mathcal{L}((k-2)P + mP_0)$, thus with pole of order exactly $k-1$ at P and no other poles except possibly at P_0. It follows that $\delta + \alpha \mathrm{d}h$, for some α in $\mathbf{C}$, has pole of order $\leqslant k-1$ at P, and the same poles as δ on the rest of $S \setminus \{P_0\}$. By an easy induction on the orders of the poles of δ outside of P_0, one finds f such that $\delta + \mathrm{d}f$ has poles of order at most one on $S \setminus \{P_0\}$, as required. □

For C and S as above, it is well known that the homology $H_1(C_S)$ of $C_S = C \setminus S$ is free of rank $2g + s - 1$ (see, e.g. [21, p. 101]).

Lemma 3.2. *The pairing $H_1(C_S) \times \Delta_S \to \mathbf{C}$ induced by integration is non-degenerate.*

Proof. Since $H_1(C)$ and Δ_S have both dimension $2g + s - 1$, it suffices to observe the following: if a rational regular differential δ on C_S is such that $\oint_\Gamma \delta = 0$ over every closed path Γ on C_S, then δ is exact, that is, $\delta = \mathrm{d}\phi$ for some rational function ϕ.

Let δ be one such form. Then, for instance, $\int_{Q_0}^{Q} \delta$, with Q_0 in C_S fixed, defines a regular function ϕ on C_S, and we have $\delta = \mathrm{d}\phi$. Write $\delta = f\mathrm{d}z$, where f, z are rational and z is a local parameter at some Q in S. Clearly, if f has a pole of order $k \geqslant 1$ at Q, then for any $\kappa > k-1$, there is c such that $|\phi(P)| \leqslant c|z(P)|^{-\kappa}$ for all P in a neighbourhood of Q. It follows that ϕ extends to a meromorphic function on $C = C_S \cup S$; thus, ϕ is rational by the Riemann Existence Theorem or Chow's Theorem. □

Lemma 3.3. *Let Φ, Ψ be functions analytic on C_S. Then, there is a rational differential δ regular on C_S and a function ϕ analytic on C_S such that*

$$\Psi \mathrm{d}\Phi = \delta + \mathrm{d}\phi. \tag{3.2}$$

Proof. Consider the periods given by the integrals $\oint_\Gamma \Psi \mathrm{d}\Phi$. By Lemma 3.2, there is a regular differential δ on C_S with exactly the same periods. In particular, $\oint_\Gamma \Psi \mathrm{d}\Phi = \oint_\Gamma \delta$ for every Γ in $H_1(C_S)$. It follows that

$$\phi(Q) = \int_{Q_0}^{Q} (\Psi \mathrm{d}\Phi - \delta)$$

for some fixed Q_0 in C_S defines an analytic function on C_S, hence $\Psi \mathrm{d}\Phi - \delta = \mathrm{d}\phi$, as desired. □

One can also deduce Lemma 3.3 from Grothendieck's paper [12] (which is actually an extract from a letter to Atiyah). It relates the complex cohomology (denoted by $H^*(\mathcal{X}, \mathbf{C})$ there) to the de Rham cohomology (denoted by $H^*(\mathcal{X}, \Omega_{\mathcal{X}})$ there), even for an algebraic variety $\mathcal{X}$ of any dimension. The proof uses Hironaka's resolution of singularities, which for curves $\mathcal{X}$ is classical.

Further, one can describe the pairing in Lemma 3.2 more explicitly by looking at period matrices. Enumerate $S = \{P_0, P_1, \ldots, P_{s-1}\}$. We can make a $\mathbf{Z}$-basis

$$\mathcal{L}_1, \ldots, \mathcal{L}_{2g}, \mathcal{M}_1, \ldots, \mathcal{M}_{s-1} \tag{3.3}$$

for $H_1(C_S)$ out of basis elements $\mathcal{L}_1, \ldots, \mathcal{L}_{2g}$ of $H_1(C)$ and small loops $\mathcal{M}_1, \ldots, \mathcal{M}_{s-1}$ around the points $P_1, \ldots, P_{s-1}$, respectively, of $S \setminus \{P_0\}$.

By Lemma 3.1 applied to $S_0 = \{P_0\}, S_1 = \{P_0, P_1\}, \ldots$ and linear algebra, we can make a $\mathbf{C}$-basis

$$\rho_1, \ldots, \rho_{2g}, \sigma_1, \ldots, \sigma_{s-1} \tag{3.4}$$

for Δ_S out of differentials of the second kind $\rho_1, \ldots, \rho_{2g}$ (with pole at most in P_0 and residue zero) together with $\sigma_1, \ldots, \sigma_{s-1}$ (with simple poles exactly at $P_1, \ldots, P_{s-1}$, respectively, and only other pole at P_0). Indexing the rows by the differentials and the columns by the loops, we get a period matrix. The block Π of size $2g$ in the top left corner corresponds to the period matrix for the case $S = \{P_0\}$; hence, it is non-singular, and to its right there is a zero block. The block underneath Π we do not know, but to its right we get the diagonal matrix whose diagonal entries are the (non-zero) residues of $\sigma_1, \ldots, \sigma_{s-1}$ at $P_1, \ldots, P_{s-1}$. We discuss in Section 6 some examples in which such differentials can be made explicit via well-known special functions.

4 | PROOF OF THEOREM 1.1

It suffices to prove the result when $\mathcal{V}$ has dimension n. In fact, the hypotheses imply that the dimension is n or $n+1$, and in the latter case, we can simply adjoin $\hat{X}_n = c$.

Let C_0 be the Zariski closure of $\pi(\mathcal{V})$ in $\mathbf{C}^n$, and let F, G_0 be as in Lemma 2.1. Let C be a complete smooth model of C_0, so that the coordinate functions $x_1, \ldots, x_n$ on C_0 can be regarded as rational functions $\xi_1, \ldots, \xi_n$ on C. Choose a non-empty finite subset S of C containing all the poles of $\xi_1, \ldots, \xi_n$ and the zeroes of $G_0(\xi_1, \ldots, \xi_n)$ (which is not identically zero). It will then suffice to find a point of $C_S = C \setminus S$ at which the function

$$\Phi = F(\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}) \tag{4.1}$$

vanishes.

Note that we can take S arbitrarily large, and this will show that the points $(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ of $\mathcal{V}$ project to a set which is Zariski dense in C_0, as mentioned in Section 1.

We assume that there are no such points, and we will reach a contradiction. Thus, Φ does not vanish on C_S. By Lemma 3.3 applied to $\Psi = 1/\Phi$, we have (3.2) for some rational differential δ on C, regular on C_S, and a function ϕ, analytic on C_S.

Claim 4.1. Under the above assumptions, ϕ is rational on C.

Proof. We note that all periods of $\mathrm{d}\Phi/\Phi$ are in $2\pi i\mathbf{Z}$, because any integral is the variation of $\log \Phi$ continuously along the contour ('Principle of the Argument'). As these are also the periods of $\delta = \mathrm{d}\Phi/\Phi - \mathrm{d}\phi$, we can define $\exp(\int \delta)$ as a function on C_S, for example, as

$$\Phi_0(Q) = \exp\left(\int_{Q_0}^{Q} \delta\right) = \exp\left(\int_{Q_0}^{Q} \left(\frac{\mathrm{d}\Phi}{\Phi} - \mathrm{d}\phi\right)\right) \tag{4.2}$$

for any fixed Q_0 in C_S. Thus, $\mathrm{d}\Phi_0/\Phi_0 = \delta$. Comparing this with (3.2), we deduce that

$$\Phi(Q) = c_0 \Phi_0(Q) e^{\phi(Q)}, \qquad (c_0 = \Phi(Q_0)e^{-\phi(Q_0)} \neq 0), \tag{4.3}$$

and from this, we will estimate $\phi(Q)$ as Q approaches some point P of S.

Let z be a local parameter at this P, so that we can regard everything in (4.3) as functions of z. Suppose $\xi_1, \dots, \xi_n$ have poles of orders at most $k \geqslant 0$ at P. Then, $|\Phi(Q)| \leqslant c \exp(c|z(Q)|^{-k})$ by (4.1) for some c independent of Q. Next, if we write $\delta = \psi \mathrm{d}z$, then ψ has a pole of order at most $k_0 \geqslant 0$ at P. In (4.2), it is not difficult to see that we can choose the contour to have length bounded independently of Q (near P) and with $|\psi| \leqslant c|z|^{-k_0}$, and it follows that $|\int_{Q_0}^{Q} \delta| \leqslant c|z|^{-k_0}$ for some c independent of Q (where for short $z = z(Q)$). Thus, $|\Phi_0(Q)|^{-1} = \exp(-\Re \int_{Q_0}^{Q} \delta) \leqslant \exp(c|z|^{-k_0})$. So, we get similar bounds for $|e^{\phi(Q)}| = |c_0^{-1}\Phi(Q)\Phi_0(Q)^{-1}|$, and it follows that

$$\Re\phi(Q) = \log|e^{\phi(Q)}| \leqslant c|z|^{-\kappa}$$

for $\kappa = \max\{k, k_0\}$ and some c independent of Q. By Lemma 2.3, we deduce

$$|\phi(Q)| \leqslant c|z|^{-\kappa-1}.$$

Thus, ϕ is meromorphic at P, and so (as in the proof of Lemma 3.2, say by Riemann Existence) is rational on C as claimed. □

We can now obtain our contradiction using Ax (Lemma 2.4) on (4.3). We can identify the function field $K_0 = \mathbf{C}(x_1, \dots, x_n)$ of the affine part of C with $\mathbf{C}(\xi_1, \dots, \xi_n)$, and we take $K = K_0(\hat{\xi}_1, \dots, \hat{\xi}_n)$ with $\hat{\xi}_1 = e^{\xi_1}, \dots, \hat{\xi}_n = e^{\xi_n}$.

For a non-constant rational function z on C, we have a derivation $D = \mathrm{d}/\mathrm{d}z$ on K with constant field $C = \mathbf{C}$, also acting on K_0. Note that (1) of Lemma 2.4 holds. With $\delta = \psi \mathrm{d}z$ as above, we have $D\Phi_0/\Phi_0 = \psi$ from (4.2) and so

$$D\Phi = \chi\Phi \tag{4.4}$$

for $\chi = \psi + D\phi$ in K_0. Writing $F = \sum_{\mathbf{i}} F_{\mathbf{i}} M_{\mathbf{i}}$ for $\mathbf{i} = (i_1, \dots, i_n)$, where $F_{\mathbf{i}}$ is in $\mathfrak{R}_0 = \mathbf{C}[X_1, \dots, X_n]$ and $M_{\mathbf{i}} = \hat{X}_1^{i_1} \cdots \hat{X}_n^{i_n}$, we get $\Phi = \sum_{\mathbf{i}} \gamma_{\mathbf{i}} \Phi_{\mathbf{i}}$ for $\gamma_{\mathbf{i}}$ in K_0 and

$$\Phi_{\mathbf{i}} = \hat{\xi}_1^{i_1} \cdots \hat{\xi}_n^{i_n}.$$

Also $D\Phi_{\mathbf{i}}/\Phi_{\mathbf{i}} = \xi_{\mathbf{i}}$ for

$$\xi_{\mathbf{i}} = i_1 D\xi_1 + \cdots + i_n D\xi_n.$$

We find from (4.4) the equations $\sum_{\mathbf{i}} \beta_{\mathbf{i}} \Phi_{\mathbf{i}} = 0$ for

$$\beta_{\mathbf{i}} = D\gamma_{\mathbf{i}} + \xi_{\mathbf{i}}\gamma_{\mathbf{i}} - \chi\gamma_{\mathbf{i}}$$

also in K_0.

If some $\beta_{\mathbf{i}} \neq 0$, this shows that the transcendence degree of $K_0(\hat{\xi}_1, \dots, \hat{\xi}_n)$ over K_0 is at most $n-1$, so over $\mathbf{C}$ at most n, contradicting Lemma 2.4 (note that (2) there holds because we are assuming that $m_1X_1 + \cdots + m_nX_n$ is not constant on $\mathcal{V}$). Thus, we may assume that all $\beta_{\mathbf{i}} = 0$.

Next, suppose that there are two different $\mathbf{i}, \mathbf{i}'$ with

$$\gamma = \gamma_{\mathbf{i}} \neq 0 \neq \gamma_{\mathbf{i}'} = \gamma'.$$

Then,

$$0 = \frac{\beta_{\mathbf{i}}}{\gamma} - \frac{\beta_{\mathbf{i}'}}{\gamma'} = \frac{D\gamma}{\gamma} - \frac{D\gamma'}{\gamma'} + (\xi_{\mathbf{i}} - \xi_{\mathbf{i}'}),$$

and it follows that $D\xi = D\hat{\xi}/\hat{\xi}$ for

$$\hat{\xi} = \frac{\gamma'}{\gamma}, \qquad \xi = m_1\xi_1 + \cdots + m_n\xi_n$$

with $(m_1, \ldots, m_n) = \mathbf{i} - \mathbf{i}' \neq 0$. Here, $\xi, \hat{\xi}$ are both in K_0, so algebraically dependent over $\mathbf{C}$. Thus, by Corollary 2.5, ξ lies in $\mathbf{C}$. However, this is also ruled out by our assumption on $m_1X_1 + \cdots + m_nX_n$.

Thus there is at most one non-zero $\gamma_{\mathbf{i}}$, and the $F_{\mathbf{i}'}$ ($\mathbf{i}' \neq \mathbf{i}$) are in the prime ideal $\mathfrak{P}_0$ of C in $\mathfrak{R}_0$. But then $F = F_{\mathbf{i}}M_{\mathbf{i}} + \sum_{\mathbf{i}'\neq\mathbf{i}} F_{\mathbf{i}'}M_{\mathbf{i}'}$ would be in $M_{\mathbf{i}}\mathfrak{R}_0 + \mathfrak{P}_0\mathfrak{R}$, excluded in Lemma 2.1.

This completes the proof of Theorem 1.1.

5 | PROOF OF THEOREM 1.2

We go through it case-by-case-by-subcase-by-subsubcase.

Case (a)

Here $\dim \pi(\mathcal{S}) = 0$, and the conclusion is clear. An example is $X_1 = 0, X_2 = 0$ with Z as the single point (0,0,1,1).

Case (b)

Here $\dim \pi(\mathcal{S}) = 2$, and this follows from [5], even with Z dense in $\mathcal{S}$. An elementary example is $\hat{X}_1 = 1, \hat{X}_2 = 1$ with $Z = (2\pi i\mathbf{Z})^2 \times \{1\}^2$.

Case (c)

For $\dim \pi(\mathcal{S}) = 1$ and $\pi(\mathcal{S})$ not contained in a line of rational slope, the conclusion is essentially our Theorem 1.1 for $n = 2$; we just have to recall the remark at the beginning of Section 4 that the finite set S can be taken arbitrarily large. We have already given some examples but an elementary one is $X_1X_2 = 1, \hat{X}_1\hat{X}_2 = 1$ with Z as the set of all $(z, 1/z, e^z, e^{1/z})$ with $z + 1/z$ in $2\pi i\mathbf{Z}$.

We now consider $\pi(\mathcal{S})$ of dimension 1 and contained in a line of rational slope $\mathcal{L}$. Say that $\mathcal{L}$ is defined by $m_1X_1 + m_2X_2 = c$ with m_1, m_2 integers not both zero and c complex, as in (1.3). Then, $\mathcal{K}$ is given by

$$\hat{X}_1^{m_1}\hat{X}_2^{m_2} = \hat{c} \tag{5.1}$$

with $\hat{c} = e^c$. Recall that $\mathcal{T}$ from (1.7) is $\mathcal{T} = \mathcal{S} \cap \mathcal{G} = \mathcal{S} \cap (\mathcal{L} \times \mathcal{K})$.

Subcase (d_1) of case (d)

Here, the set $\mathcal{T}$ is empty, and the set Z of $(z_1, z_2, e^{z_1}, e^{z_2})$ in $\mathcal{S}$ lies in $\mathcal{G}$ and so in $\mathcal{T}$, hence Z is empty. We already gave the example (1.2).

Subcase (d_2)

Here $\dim \mathcal{T} = 2$, and we can even show that Z is dense in $\mathcal{S}$. Note that in this case, $\mathcal{S} = \mathcal{G} = \mathcal{T}$ (because $\mathcal{S}$ and $\mathcal{G}$ are irreducible surfaces), and thus, Z coincides with the set of $(z_1, z_2, e^{z_1}, e^{z_2})$ with (z_1, z_2) in $\mathcal{L}$. It is now not difficult to see, using the algebraic independence of z and e^z, that Z is Zariski dense in $\mathcal{S}$; for example, if $m_2 \neq 0$, it contains the set of $(z, z', e^z, e^{z'})$ as z varies in $\mathbf{C}$, where $z' = (c - m_1 z)/m_2$ (see also the parametrisations (5.2), (5.3) below). An example is $X_1 + X_2 = 1$, $\hat{X}_1\hat{X}_2 = e$ with Z as the set of $(z, 1-z, e^z, e^{1-z})$ for z in $\mathbf{C}$.

Finally, we settle the remaining cases with $\dim \mathcal{T} = 1$.

Subsubcase (d_{31}) of subcase (d_3)

Here $\dim \pi(\mathcal{T}) = 0$. Since the set $\pi(Z)$ is contained in $\pi(\mathcal{T})$, it must be finite or empty. Thus, Z is finite or empty as well.

In fact, Z cannot be empty. Namely, for each Q in $\pi(\mathcal{T})$, the fibre $\pi^{-1}(Q)$ in $\mathcal{T}$ has $\dim \pi^{-1}(Q) \leqslant 1$, and there must be Q with equality. As $\mathcal{T}$ is in $\mathcal{G} = \mathcal{L} \times \mathcal{K}$, the fibre must be the whole of $Q \times \mathcal{K}$. Writing $Q = (z_1, z_2)$, we see that in particular $(z_1, z_2, e^{z_1}, e^{z_2})$ is in the fibre, so in $\mathcal{T}$ and therefore $\mathcal{S}$.

An example is $X_1 + X_2 = 1$, $X_1 + \hat{X}_1\hat{X}_2 = e$ with Z as the single point $(0, 1, 1, e)$. However, changing the second equation here to $X_1 - X_1^2 + \hat{X}_1\hat{X}_2 = e$ gives an extra point $(1, 0, e, 1)$, and so, Z need not be a single point as in case (a).

Subsubcase (d_{32})

This final possibility, $\dim \pi(\mathcal{T}) = 1$, involves the operation of 'back-substitution' to land in $\mathbf{C} \times \mathbf{C}^*$, so we give the full details. For the moment, we assume only the hypotheses in (d), that is, $\pi(\mathcal{S})$ has dimension 1 and is contained in a line $\mathcal{L}$ of rational slope.

We can suppose that m_1, m_2 are coprime. Fix integers a_1, a_2 with $a_1 m_1 + a_2 m_2 = 1$. We can parametrise $\mathcal{L}$ by

$$X_1 = a_1 c + m_2 Y, \quad X_2 = a_2 c - m_1 Y, \tag{5.2}$$

and correspondingly $\mathcal{K}$ by

$$\hat{X}_1 = \hat{c}^{a_1}\hat{Y}^{m_2}, \quad \hat{X}_2 = \hat{c}^{a_2}\hat{Y}^{-m_1}. \tag{5.3}$$

By Lemma 2.1, our $\mathcal{S}$ is defined by $m_1 X_1 + m_2 X_2 = c$ and $F = 0$. Thus, $\mathcal{T}$ is defined by $m_1 X_1 + m_2 X_2 = c$, (5.1) and $F = 0$. We now check, as mentioned in the introduction, that if $\mathcal{T}$ is non-empty, then it is infinite.

For this, we define a morphism f from $\mathcal{S}$ to $\mathbf{G}_{\mathrm{m}}$ by $f(X_1, X_2, \hat{X}_1, \hat{X}_2) = \hat{X}_1^{m_1}\hat{X}_2^{m_2}$. If f were not dominant, then $\hat{X}_1^{m_1}\hat{X}_2^{m_2}$ would be a constant $\hat{c}'$ on $\mathcal{S}$. Then $\mathcal{S}$ would be contained in $\mathcal{L} \times \mathcal{K}'$ for some translate $\mathcal{K}'$ of $\mathcal{K}$. As m_1, m_2 are coprime, $\mathcal{K}'$ is irreducible; so $\mathcal{S} = \mathcal{L} \times \mathcal{K}'$. Now, in fact, $\mathcal{K}' = \mathcal{K}$, else $\mathcal{S}$ and $\mathcal{G} = \mathcal{L} \times \mathcal{K}$ would not intersect and $\mathcal{T}$ would be empty. Thus, $\mathcal{S} = \mathcal{L} \times \mathcal{K} = \mathcal{G}$ and $\mathcal{T} = \mathcal{S}$ is certainly infinite (and we end up in subcase (d_2)).

Thus, we can suppose that f is dominant. Now the Fibre Dimension Theorem [6, p. 228] says that $f^{-1}(\hat{c}) = \mathcal{T}$ has dimension at least 1; and this finishes the checking.

Define $\tilde{\mathcal{D}}$ in $\mathbf{C} \times \mathbf{C}^*$ as the set of $(Y, \hat{Y})$ with $G = 0$, where

$$G = G(Y, \hat{Y}) = F(a_1 c + m_2 Y, a_2 c - m_1 Y, \hat{c}^{a_1}\hat{Y}^{m_2}, \hat{c}^{a_2}\hat{Y}^{-m_1}).$$

Then, (5.2), (5.3) define a map φ from $\tilde{\mathcal{D}}$ to $\mathcal{T}$. Its inverse is given by, for example,

$$Y = a_2 X_1 - a_1 X_2, \qquad \hat{Y} = \hat{X}_1^{a_2}\hat{X}_2^{-a_1},$$

and so, we have isomorphisms.

Therefore, returning to our subsubcase (d_{32}), we must have $\dim \tilde{\mathcal{D}} = 1$. In particular, G is not identically constant.

However, it may not be irreducible; but any irreducible factor gives an irreducible curve. There is at least one of these curves, say $\mathcal{D}$, on which Y is not constant, else Y would take at most finitely many values on $\tilde{\mathcal{D}}$ and then (X_1, X_2) would take at most finitely many values in $\phi(\tilde{\mathcal{D}}) = \mathcal{T}$ by (5.1), contrary to our assumption $\dim \pi(\mathcal{T}) = 1$. Therefore, by Theorem 1.1, the set W of points (w, e^w) on $\mathcal{D}$ project to a Zariski dense subset of $\mathbf{C}$. It follows that $\pi(\varphi(W))$ is Zariski dense in $\pi(\mathcal{S})$. Finally, $\varphi(W)$ is contained in Z by (5.2) and (5.3).

An example is (1.5); the choice $a_1 = 0$, $a_2 = 1$ leads to $Y = X_1$, $\hat{Y} = \hat{X}_1$ and so (1.6).

6 | EXAMPLES

We can actually go further with (4.3) in the style $e^z + e^{1/z} - 1 = z^m e^{\phi(z)}$ of (1.11). Namely, the period matrix just after (3.4) is some invertible M. Thus, we can act on (3.4) by $2\pi i M^{-1}$ to get $2\pi i I_h$ for the identity matrix of order $h = 2g + s - 1$, and then integrating these and exponentiating as in (4.2) gives $\Phi_1, \ldots, \Phi_h$ analytic on C_S and never vanishing there. Now the period (row) vector of $\mathrm{d}\Phi/\Phi$ is $2\pi i\mathbf{m}$ for some $\mathbf{m} = (m_1, \ldots, m_h)$ in $\mathbf{Z}^h$, and so, we find

$$\Phi = \Phi_1^{m_1} \cdots \Phi_h^{m_h} e^{\phi} \tag{6.1}$$

in (4.3).

Originally, we proved (6.1) for small genus actually by constructing $\Phi_1, \ldots, \Phi_h$ directly in an *ad hoc* fashion. As some amusing formulae turned up, we feel that it may be of some interest to present our constructions here.

Case $g = 0$

Now C may be taken as $\mathbf{P}_1$, which we identify with $\mathbf{C} \cup \{\infty\}$.

If $s = 1$, then $h = 0$ and there is nothing to do. So, we assume $s \geqslant 2$.

If S contains ∞, then $S = \{\infty, p_1, \dots, p_{s-1}\}$, and clearly $z - p_1, \dots, z - p_{s-1}$ are the desired Φ_i's. And indeed, we find $\mathrm{d}(z - p_i)/(z - p_i) = \mathrm{d}z/(z - p_i) = \sigma_i$ satisfying exactly the same conditions as in the original (3.4). The period matrix in Lemma 3.2, with the homology basis (3.3), is $2\pi i I_{s-1}$ (provided that we choose the appropriate orientations). Thus, δ in Lemma 3.3 for general Φ must have a decomposition

$$\delta = m_1\sigma_1 + \cdots + m_{s-1}\sigma_{s-1}$$

now with integer coefficients. And so, $\Phi_0(z) = (z - p_1)^{m_1} \cdots (z - p_{s-1})^{m_{s-1}}$ in the proof of Claim 4.1. Thus, if Φ is any function analytic on $\mathbf{C}_S$ and not vanishing there, it has the form

$$\Phi(z) = (z - p_1)^{m_1} \cdots (z - p_{s-1})^{m_{s-1}} e^{\phi(z)}$$

for some ϕ also analytic on $\mathbf{C}_S$, exactly as in Example 1.3. The special case $s = 2$ and $p_1 = 0$ is (1.11).

If $S = \{p_0, p_1, \dots, p_{s-1}\}$ does not contain ∞, then we can use in a similar way, for example,

$$\frac{z - p_1}{z - p_0}, \dots, \frac{z - p_{s-1}}{z - p_0}.$$

Case $g = 1$

Now C can be taken as an elliptic curve $\mathcal{E}$, with origin O, whose affine part is

$$y^2 = 4x^3 - g_2x - g_3.$$

It is parametrised by the Weierstrass functions $x = \wp(z), y = \wp'(z)$ with corresponding period lattice Ω.

Now examples of Φ are not so easy to write analytically as $z - p$; but we found the following, at first for $S = \{O\}$. Take any period ω in Ω. It has a corresponding quasi-period η defined by $\zeta(z + \omega) = \zeta(z) + \eta$ for the associated Weierstrass zeta function. Then,

$$\Phi^{(\omega)}(z) = e^{\omega\zeta(z) - \eta z} \tag{6.2}$$

is doubly periodic. This is because

$$\omega\zeta(z + \tilde{\omega}) - \eta(z + \tilde{\omega}) = \omega\zeta(z) - \eta z + \omega\tilde{\eta} - \eta\tilde{\omega}$$

for any other period $\tilde{\omega}$ with quasi-period $\tilde{\eta}$; and the Legendre relations show that $\omega\tilde{\eta} - \eta\tilde{\omega}$ is in $2\pi i\mathbf{Z}$. In fact, we have here the very simplest form of a Baker–Akhiezer function, with an essential singularity at $z = 0$; see [3] Chapter XIV and in particular page xxviii of Krichever's foreword for more general versions, although this particular example does occur, even for arbitrary genus, in Weierstrass [23, p. 312]. It clearly does not vanish on $\mathcal{E}_S$ for $S = \{O\}$.

See also Pellarin [20] for analogues of Baker–Akhiezer in positive characteristic.

In fact, we get two for the price of one by taking basis elements of $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ and corresponding η_1, η_2 so Φ_1, Φ_2 (note that these depend on the choice of basis, but the multiplicative group they generate does not). As $\zeta' = -\wp$, we find

$$\frac{\mathrm{d}\Phi_i}{\Phi_i} = -(\omega_i \wp(z) + \eta_i)\mathrm{d}z = -(\omega_i x + \eta_i)\frac{\mathrm{d}x}{y} \qquad (i = 1, 2).$$

There are loops $\mathcal{L}_1, \mathcal{L}_2$ with

$$\int_{\mathcal{L}_i} \frac{\mathrm{d}x}{y} = \omega_i, \quad \int_{\mathcal{L}_i} \frac{x\mathrm{d}x}{y} = -\eta_i \qquad (i = 1, 2)$$

(of course, the differentials here are possible ρ_1, ρ_2 in the modified (3.4) above), and so, the periods we want are

$$\int_{\mathcal{L}_1} \frac{\mathrm{d}\Phi_1}{\Phi_1} = 0, \quad \int_{\mathcal{L}_2} \frac{\mathrm{d}\Phi_1}{\Phi_1} = \pm 2\pi i$$
$$\int_{\mathcal{L}_1} \frac{\mathrm{d}\Phi_2}{\Phi_2} = \pm 2\pi i, \quad \int_{\mathcal{L}_2} \frac{\mathrm{d}\Phi_2}{\Phi_2} = 0$$

by the more precise form of Legendre (depending on orientation). This does not quite give $2\pi i I_h = 2\pi i I_2$ for the case $S = \{O\}$; but at least we get $2\pi i U$ for unimodular U with $\det U = \pm 1$. This is already enough to imply that any function Φ, analytic on $\mathcal{E}_S$ and never vanishing there, has the form $\Phi = \Phi_1^{m_1}\Phi_2^{m_2}e^{\phi}$ for ϕ also analytic on $\mathcal{E}_S$; which is equivalent to Example 1.4.

For more general $S = \{O, P_1, \ldots, P_{s-1}\}$ containing O, we found other examples as follows. Suppose $P \neq O$; then we can write $P = (\wp(u), \wp'(u))$ and define

$$\Psi^{(u)}(z) = \frac{\sigma(z-u)}{\sigma(z)} e^{u\zeta(z)} \tag{6.3}$$

for the Weierstrass sigma function (note that this depends on the choice of u but if we change u by a period ω, then $\Psi^{(u)}$ changes by $\Phi^{(\omega)}$ up to constants). It is analytic on $\mathcal{E}$ with O, P removed and never vanishes there. It too is a Baker–Akhiezer function (and almost certainly known to Weierstrass) with its essential singularity at $z = 0$. That distinguishes it from a similar expression occurring in the exponential map for a multiplicative extension of $\mathcal{E}$, which has $e^{\zeta(u)z}$ in place of $e^{u\zeta(z)}$.

As $\sigma'/\sigma = \zeta$, we obtain

$$\frac{\mathrm{d}\Psi^{(u)}}{\Psi^{(u)}} = (\zeta(z-u) - \zeta(z) - u\wp(z))\mathrm{d}z, \tag{6.4}$$

which by the addition theorem for ζ is

$$\left(-\zeta(u) + \frac{1}{2}\frac{\wp'(z) + \wp'(u)}{\wp(z) - \wp(u)} - u\wp(z)\right)\mathrm{d}z = -\zeta(u)\frac{\mathrm{d}x}{y} - u\frac{x\mathrm{d}x}{y} + \theta_P$$

for the perhaps more classically familiar

$$\theta_P = \frac{1}{2}\frac{y + \wp'(u)}{x - \wp(u)}\frac{\mathrm{d}x}{y}.$$

Both have residue divisor $P - O$.

Thus, with $P = P_1, \dots, P_{s-1}$ (of course, giving rise to $\sigma_1, \dots, \sigma_{s-1}$ in (3.4) above) and $\mathcal{M}_1, \dots, \mathcal{M}_{s-1}$ as in the proof of Lemma 3.2, we obtain in the bottom right block of the period matrix a diagonal matrix with entries $\pm 2\pi i$. So, again something unimodular and corresponding $\Psi_1, \dots, \Psi_{s-1}$.

Therefore, any function Φ, analytic on $\mathcal{E}_S$ and never vanishing there, has the form

$$\Phi = \Phi_1^{m_1}\Phi_2^{m_2}\Psi_1^{n_1} \cdots \Psi_{s-1}^{n_{s-1}} e^{\phi} \tag{6.5}$$

for ϕ also analytic on $\mathcal{E}_S$.

Occasionally, we can find simpler Ψ. For example, if $P = (e, 0)$ is a point of order 2, then $\Psi = x - e = \wp(z) - e$ is analytic on $\mathcal{E}_S$ for $S = \{O, P\}$ and never vanishes there (even without essential singularity). And if say $e = \wp(\omega/2)$, then indeed

$$\wp(z) - e = \frac{(\Psi^{(\omega/2)}(z))^2}{(\sigma(\omega/2))^2\Phi^{(\omega)}(z)}.$$

And if our $S = \{P_0, P_1, \dots, P_{s-1}\}$ does not contain O, then we can simply use the group law to reduce to $\{O, P_1 - P_0, \dots, P_{s-1} - P_0\}$ thus obtaining for example $\Phi^{(\omega)}(z - u_0)$ in place of (6.2).

Case $g = 2$

Here it will suffice to deal with a complex hyperelliptic curve $\mathcal{H}$ whose affine part is defined by

$$y^2 = x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5 \tag{6.6}$$

with the discriminant of the right-hand side non-zero. We are therefore using the notation of Grant [11]. Of course, there is no longer a parametrisation by $\mathbf{C}$. To obtain the analogue of $\wp$ and so on, we must embed $\mathcal{H}$ into its Jacobian, which is parametrised by $\mathbf{C}^2$.

We originally constructed examples of Φ using theta functions. We fix a matrix $T = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix}$ in the Siegel upper half space. We have a standard theta function $\theta(\mathbf{z})$ defined for $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by

$$\theta(\mathbf{z}) = \sum_{\mathbf{p} \in \mathbf{Z}^2} \exp(\pi i(\mathbf{p}^t T\mathbf{p} + 2\mathbf{p}^t\mathbf{z}))$$

with column vectors $\mathbf{p}$. It satisfies

$$\theta(\mathbf{z} + \mathbf{e}_1) = \theta(\mathbf{z}), \quad \theta(\mathbf{z} + \mathbf{e}_2) = \theta(\mathbf{z})$$

for $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, as well as

$$\theta(\mathbf{z} + \mathbf{t}_1) = c_1 \exp(-2\pi i z_1)\theta(\mathbf{z}), \quad \theta(\mathbf{z} + \mathbf{t}_2) = c_2 \exp(-2\pi i z_2)\theta(\mathbf{z})$$

for $\mathbf{t}_1 = \begin{pmatrix} \tau_1 \\ \tau \end{pmatrix}$, $\mathbf{t}_2 = \begin{pmatrix} \tau \\ \tau_2 \end{pmatrix}$ and constants c_1, c_2 (see, e.g. [18, pp. 118–120]).

So, the ‘Baker zeta functions’

$$\zeta_1 = \frac{1}{\theta}\frac{\partial\theta}{\partial z_1}, \quad \zeta_2 = \frac{1}{\theta}\frac{\partial\theta}{\partial z_2}$$

satisfy

$$\zeta_1(\mathbf{z}+\mathbf{e}_1)=\zeta_1(\mathbf{z}),\ \zeta_1(\mathbf{z}+\mathbf{e}_2)=\zeta_1(\mathbf{z}),\ \zeta_1(\mathbf{z}+\mathbf{t}_1)=-2\pi i+\zeta_1(\mathbf{z}),\ \zeta_1(\mathbf{z}+\mathbf{t}_2)=\zeta_1(\mathbf{z}), \tag{6.7}$$

$$\zeta_2(\mathbf{z}+\mathbf{e}_1)=\zeta_2(\mathbf{z}),\ \zeta_2(\mathbf{z}+\mathbf{e}_2)=\zeta_2(\mathbf{z}),\ \zeta_2(\mathbf{z}+\mathbf{t}_1)=\zeta_2(\mathbf{z}),\ \zeta_2(\mathbf{z}+\mathbf{t}_2)=-2\pi i+\zeta_2(\mathbf{z}). \tag{6.8}$$

So, miraculously

$$\Phi_1=e^{\zeta_1},\quad \Phi_2=e^{\zeta_2} \tag{6.9}$$

are 'quadruply periodic' (also known to Weierstrass). Suitably translated, these generalise one of the $\Phi^{(\omega)}=e^{\omega\zeta(z)-\eta z}$; in fact, here $\omega=1$ and $\eta=0$ in the new normalisation.

We need another pair corresponding to $\omega=\tau$. A short calculation shows that

$$\zeta_3=\tau_1\zeta_1+\tau\zeta_2+2\pi i z_1,\quad \zeta_4=\tau\zeta_1+\tau_2\zeta_2+2\pi i z_2$$

satisfy

$$\zeta_3(\mathbf{z}+\mathbf{e}_1)=\zeta_3(\mathbf{z})+2\pi i,\ \zeta_3(\mathbf{z}+\mathbf{e}_2)=\zeta_3(\mathbf{z}),\ \zeta_3(\mathbf{z}+\mathbf{t}_1)=\zeta_3(\mathbf{z}),\ \zeta_3(\mathbf{z}+\mathbf{t}_2)=\zeta_3(\mathbf{z}), \tag{6.10}$$

$$\zeta_4(\mathbf{z}+\mathbf{e}_1)=\zeta_4(\mathbf{z}),\ \zeta_4(\mathbf{z}+\mathbf{e}_2)=\zeta_4(\mathbf{z})+2\pi i,\ \zeta_4(\mathbf{z}+\mathbf{t}_1)=\zeta_4(\mathbf{z}),\ \zeta_4(\mathbf{z}+\mathbf{t}_2)=\zeta_4(\mathbf{z}). \tag{6.11}$$

Thus,

$$\Phi_3=e^{\tau_1\zeta_1+\tau\zeta_2+2\pi i z_1},\quad \Phi_4=e^{\tau\zeta_1+\tau_2\zeta_2+2\pi i z_2} \tag{6.12}$$

will do (known, of course, to Weierstrass).

These are functions on open subsets of $\mathbf{C}^2$, and we get functions on $\mathcal{H}$ by taking restrictions to a one-dimensional analytic set. So, we have to understand their poles, which just come from the zeroes of θ. In the usual notation, we choose basis elements $\mathcal{A}_1,\mathcal{A}_2,\mathcal{B}_1,\mathcal{B}_2$ (aka $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4$ in (3.3) above) for the homology of $\mathcal{H}$ in the standard way and then differentials ρ_1,ρ_2 of the first kind on $\mathcal{H}$ normalised such that the respective integrals of $\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}$ are the columns $\mathbf{e}_1,\mathbf{e}_2,\mathbf{t}_1,\mathbf{t}_2$. Then,

$$\varepsilon(Q)=\begin{pmatrix}\int_\infty^Q\rho_1\\ \int_\infty^Q\rho_2\end{pmatrix}$$

embeds $\mathcal{H}$ into $\mathbf{C}^2/\Lambda$ for the lattice generated by these columns.

The Riemann Vanishing Theorem implies that there is some $\mathbf{u}_0$ such that $\theta(\mathbf{z})=0$ if and only if there is Q in $\mathcal{H}$ with $\mathbf{z}=\mathbf{u}_0-\varepsilon(Q)$ modulo Λ (see, e.g. Corollary 3.6 of [18, p. 160]). In particular, $\theta(\mathbf{u}_0)=0$ (in fact, because our $\mathcal{H}$ is hyperelliptic, we have $\mathbf{u}_0=\mathbf{e}_1+\frac{1}{2}\mathbf{e}_2+\frac{1}{2}\mathbf{t}_1+\frac{1}{2}\mathbf{t}_2$, in $\frac{1}{2}\Lambda$ but not Λ — see, e.g. [19, 3.80, 3.82] — however, if we wanted to progress to curves of genus $g\geqslant 3$ which are not hyperelliptic, then we should forget this explicit value).

Now the trouble with (6.9) and (6.12) is that theta functions tend to have two zeroes when restricted to $\varepsilon(\mathcal{H})$ (or even infinitely many, such as $\theta(\mathbf{u}_0-\varepsilon(Q))$ for example). We can overcome this problem for $S = \{P_0\}$ provided that P_0 is not one of the six Weierstrass points, which are ∞ and the five points with $y = 0$ on (6.6). Namely, consider

$$\lambda^{(P_0)}(Q) = \theta(\mathbf{u}_0 - 2\varepsilon(P_0) + \varepsilon(Q)).$$

This vanishes if and only if there is Q' in $\mathcal{H}$ with $\mathbf{u}_0 - 2\varepsilon(P_0) + \varepsilon(Q) = \mathbf{u}_0 - \varepsilon(Q')$ modulo Λ; that is,

$$\varepsilon(Q) + \varepsilon(Q') = \varepsilon(P_0) + \varepsilon(P_0) \quad \mathrm{mod}\ \Lambda.$$

If at least one of Q, Q' is not P_0, then by Abel–Jacobi, there is a rational function on $\mathcal{H}$ with a double or single pole at P_0 and no other poles. However, by the definition of Weierstrass point, that is impossible. Thus, $Q = P_0$ $(= Q')$ and in particular $\lambda^{(P_0)}$ is not identically zero on $\mathcal{H}$.

Thus, (6.9) and (6.12) restricted to $\mathbf{z} = \mathbf{u}_0 - 2\varepsilon(P_0) + \varepsilon(Q)$ provide functions

$$\Phi_1^{(P_0)},\ \Phi_2^{(P_0)},\ \Phi_3^{(P_0)},\ \Phi_4^{(P_0)}$$

analytic on $\mathcal{H}_S$ (for this singleton S) never vanishing there (also Weierstrass–Baker–Akhiezer).

As for the periods, we have, for example, $\mathrm{d}\Phi_1/\Phi_1 = \mathrm{d}\zeta_1$, and so, the integral of $\Phi_1^{(P_0)}$ around say $\mathcal{A}_1$ corresponds to the change in $\zeta_1(\mathbf{u}_0 - 2\varepsilon(P_0) + \varepsilon(Q))$, which is zero by the first of (6.7). So, we see that the periods of $\mathrm{d}\Phi_1^{(P_0)}/\Phi_1^{(P_0)}$ are

$$0,\ 0,\ \pm 2\pi i,\ 0$$

(depending on orientation) and likewise from (6.8), the periods of $\mathrm{d}\Phi_2^{(P_0)}/\Phi_2^{(P_0)}$ are

$$0,\ 0,\ 0,\ \pm 2\pi i.$$

And from (6.10) and (6.11), we find that the periods of $\mathrm{d}\Phi_3^{(P_0)}/\Phi_3^{(P_0)}$, $\mathrm{d}\Phi_4^{(P_0)}/\Phi_4^{(P_0)}$ are

$$\pm 2\pi i,\ 0,\ 0,\ 0$$

$$0,\ \pm 2\pi i,\ 0,\ 0,$$

respectively. So, once again, we get a matrix $M_0 = 2\pi i U$ for unimodular U. Thus, any function Φ, analytic on $\mathcal{H}_S$ for this $S = \{P_0\}$ and never vanishing there, has the form

$$\Phi = \left(\Phi_1^{(P_0)}\right)^{m_1}\left(\Phi_2^{(P_0)}\right)^{m_2}\left(\Phi_3^{(P_0)}\right)^{m_3}\left(\Phi_4^{(P_0)}\right)^{m_4} e^{\phi} \tag{6.13}$$

for ϕ also analytic on $\mathcal{H}_S$.

But what about $S = \{P_0\}$ for a Weierstrass point P_0? The above fails because $\lambda^{(P_0)}$ is then identically zero on $\mathcal{H}$ thanks to the function x or $1/(x - e)$ for the zeroes e of the right-hand side of (6.6).

If $P_0 = \infty$, for example, then we could try to flip back into the construction in Section 3, because $\rho_1, \rho_2, \rho_3, \rho_4$ can be taken as the well known

$$\frac{\mathrm{d}x}{y},\ \frac{x\mathrm{d}x}{y},\ \frac{x^2\mathrm{d}x}{y},\ \frac{x^3\mathrm{d}x}{y}. \tag{6.14}$$

The right linear combinations appear to be connected to the 'Legendre relations' of [2, p. 14], and then, one would have to integrate and exponentiate. However, the analogue of (6.14) for a Weierstrass point $P_0 \neq \infty$ seems messy.

Alternatively here is a dirty trick that works for any Weierstrass point P_0. We choose any Q_0 not a Weierstrass point and as above

$$\lambda^{(Q_0)}(Q) = \theta(\mathbf{u}_0 - 2\varepsilon(Q_0) + \varepsilon(Q))$$

has a double zero at $Q = Q_0$ and no other zero. Similar arguments show that

$$\mu^{(Q_0)}(Q) = \theta(\mathbf{u}_0 - \varepsilon(Q_0) - \varepsilon(P_0) + \varepsilon(Q))$$

has simple zeroes at $Q = Q_0, P_0$ and no other zero. Thus, $(\mu^{(Q_0)})^2/\lambda^{(Q_0)}$ has a double zero at $Q = P_0$ and no other zeroes or poles. And so,

$$\exp\big(2\zeta_i(\mathbf{u}_0 - \varepsilon(Q_0) - \varepsilon(P_0) + \mathbf{z}) - \zeta_i(\mathbf{u}_0 - 2\varepsilon(Q_0) + \mathbf{z})\big) \qquad (i = 1, 2)$$

are the analogues of (6.9) for example.

And the new period matrix is just $2M_0 - M_0 = M_0$ the old period matrix.

This settles (6.13) for singletons $S = \{P_0\}$. For general $S = \{P_0, P_1, \dots, P_{s-1}\}$ ($s \geqslant 2$), we write down the analogue of (6.3) as

$$\Psi^{(\mathbf{u})}(\mathbf{z}) = \frac{\theta(\mathbf{z} - \mathbf{u})}{\theta(\mathbf{z})} e^{u_1\zeta_1(\mathbf{z}) + u_2\zeta_2(\mathbf{z})}, \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{6.15}$$

also quadruply periodic. If $P \neq \infty$, then

$$\theta(\mathbf{u}_0 - \varepsilon(P) + \varepsilon(Q))$$

vanishes at $Q = P, \infty$ and nowhere else. So, if also $P_0 \neq \infty$, then taking $\mathbf{z} = \mathbf{u}_0 - \varepsilon(P_0) + \varepsilon(Q)$ and $\mathbf{u} = \varepsilon(P) - \varepsilon(P_0)$ in (6.15), we get a function $\Psi^{(P_0,P)}$ analytic on $\mathcal{H}$ with P_0, P removed and never vanishing there (also Baker–Akhiezer).

Then, $\mathrm{d}\Psi^{(P_0,P)}/\Psi^{(P_0,P)}$ is a rational differential on $\mathcal{H}$ whose residue divisor is the divisor $P - P_0$ of $\Psi^{(P_0,P)}$ itself, and so integrating over a small loop around P gives $\pm 2\pi i$. In fact, it is only $\theta(\mathbf{z} - \mathbf{u})$ in (6.15) that causes the pole of the differential at P.

One could go further as in (6.4) by using the 'Baker $\wp$ functions'

$$\wp_{ij} = -\frac{\partial \zeta_i}{\partial z_j} \qquad (i, j = 1, 2)$$

and even $\wp_{ijk}$ (not quite as in [2, p. 38] or [11, p. 99]), and this would lead to the corresponding differentials (3.4) on $\mathcal{H}$.

Anyway, doing the above for $P = P_1, \dots, P_{s-1}$ leads to the usual unimodular matrix, so we conclude that if $S = \{P_0, P_1, \dots, P_{s-1}\}$ does not contain ∞, then any function Φ, analytic on $\mathcal{H}_S$ and never vanishing there, has the form

$$\Phi = \left(\Phi_1^{(P_0)}\right)^{m_1} \left(\Phi_2^{(P_0)}\right)^{m_2} \left(\Phi_3^{(P_0)}\right)^{m_3} \left(\Phi_4^{(P_0)}\right)^{m_4} \left(\Psi^{(P_0,P_1)}\right)^{n_1} \cdots \left(\Psi^{(P_0,P_{s-1})}\right)^{n_{s-1}} e^{\phi} \tag{6.16}$$

for ϕ also analytic on $\mathcal{H}_S$.

Presumably, ∞ can be handled with more dirty tricks.

Probably, these constructions extend to any genus $g \geqslant 3$, at least if $P_0, P_1, \ldots, P_{s-1}$ are in 'general position'.

7 EFFECTIVITY

Here, we sketch some possibilities for effective versons of Theorem 1.1. For simplicity, we restrict ourselves to genus $g = 0$.

The basic idea can be illustrated with $n = 1$ and $\mathcal{V}$ in $\mathbf{C} \times \mathbf{C}^*$ defined by $X_1 = \hat{X}_1$; so, we want a zero of $\Phi(z) = e^z - z$. We will localise in the sense of finding some explicit R such that there is a zero z_0 with $|z_0| \leqslant R$. Of course, there are many direct ways of doing this, but they do not extend to general n.

So, let us suppose to the contrary that for some $R > 0$, there is no z_0 with $\Phi(z_0) = 0$ and $|z_0| \leqslant R$. Now the (classical) argument shows that $\Phi = e^{\phi}$ for some ϕ analytic on the disc $|z| \leqslant R$. So,

$$\Re\phi(z) = \log|e^z - z| \leqslant \log(e^R + R)$$

on this disc.

As $e^{\phi(0)} = \Phi(0) = 1$, we can assume $\phi(0) = 0$. Then, Lemma 2.2 with $r = R/2$ gives

$$\sup_{|z| \leqslant R/2} |\phi(z)| \leqslant 2\log(e^R + R).$$

This says that $|\phi| = O(R)$ on 'large' discs; thus, ϕ ought to be 'almost' a linear polynomial $az + b$. More precisely, if $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, then we could show that the a_k ($k = 2, 3, \ldots$) are 'small', so that $\Phi(z)$ is 'near' e^{az+b}. As one might guess from (1.8) in the discussion of Section 1, this could be disproved with an appropriate 'effective' extension of Ax's Theorem. In fact, such an extension can be supplied; however, here we can take a short cut as follows.

We look at just

$$|a_2| = \left| \frac{1}{2\pi i} \int_{|z|=R/2} \frac{\phi(z)}{z^3} \mathrm{d}z \right| \leqslant 8\frac{\log(e^R + R)}{R^2}. \tag{7.1}$$

On the other hand (recall $a_0 = 0$),

$$1 + \frac{1}{2}z^2 + \cdots \;=\; e^z - z \;=\; e^{\phi(z)} \;=\; 1 + a_1 z + \left(a_2 + \frac{1}{2}a_1^2\right)z^2 + \cdots \tag{7.2}$$

and we deduce $a_1 = 0$ and $a_2 = 1/2$. This contradicts (7.1) for $R = 17$.

For $n = 2$ and say $X_1X_2 = 1$, $\hat{X}_1 + \hat{X}_2 = 1$, we propose to find R such that there is a zero z_0 of $\Phi(z) = e^z + e^{1/z} - 1$ with $1/R \leqslant |z_0| \leqslant R$. Now we have to use Laurent series, and we can show that $\Phi(z) = z^m e^{\phi(z)}$ holds as in (1.11) with $\phi(z) = \sum_{k=-\infty}^{\infty} a_k z^k$, and that a_k for $k \geqslant 3$ and $k \leqslant -3$ are 'small'. Thus, ϕ is near $az^2 + bz + c + d/z + e/z^2$ as in (1.12). This time a short cut by equating coefficients does not seem so easy as in (7.2), but 'effective Ax' can be used, or also repeated differentiation (here five times suffice) to deduce a contradiction for large R; provided that we can also estimate the exponent m.

This latter problem seems not entirely trivial. If we take $R > 1$, then the method shows that

$$m = \pm\frac{1}{2\pi i} \int_{|z|=1} \frac{\mathrm{d}\Phi}{\Phi}. \tag{7.3}$$

We politely asked Maple to compute this but it refused to answer. Finally, we realised that this was due to zeroes z_0 with $|z_0| = 1$. Writing $z = e^{it}$, drawing a graph to guess a rough solution and then refining with Newton gives indeed the pair

$$-0.0828555773300646822 3 \ldots \pm .9965615652358371338 \ldots i.$$

So, we did actually stumble on zeroes!

Incidentally, this leads to a one-sentence proof that there is a zero, because $\Phi(e^{it})$ is continuous from $\mathbf{R}$ to $\mathbf{R}$ with value $2e - 1 > 0$ at $t = 0$ and value $2e^{-1} - 1 < 0$ at $t = \pi$.

If $R > 2$ and for some reason we had taken say $|z| = 2$ in (7.3), then Maple would have obliged with something looking suspiciously like $m = 1$ (and other more theoretical considerations lead to a rigorous bound — we found $|m| \leqslant 62$ for example). Such shady calculations suggest that there is a pair of zeroes with $7 < |z_0| < 8$ (our own value for R was about 10^8).

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