Article:

Koshuriyan, Zamir (2012) Insight into the two-source structure of the jet noise spectrum using a generalized shell model of turbulence. European Journal of Mechanics - B/Fluids. pp. 129-139. ISSN 0997-7546

https://doi.org/10.1016/j.euromechflu.2011.06.002

Insight into the two-source structure of the jet noise spectrum using a generalized shell model of turbulence

M. Z. Afsar

Ohio Aerospace Institute, 22800 Cedar Point Road, Cleveland, OH 44142, USA.
[mohammed.afsar@cantab.net]

There is a large body of experimental evidence that shows the jet noise spectrum is composed of two sources. Our aim here is to prove, mathematically, that the two-source paradigm can be derived using a minimum number of self-consistent approximations based on our current knowledge of jet turbulence in cold flows. The starting point of the paper is Goldstein's [1] exact re-arrangement of the Navier-Stokes equations, which shows that turbulence enters the acoustic spectrum formula through the Reynolds stress auto-covariance tensor. We extend the shell model of turbulence using a more general symmetry approximation that amounts to assuming the Reynolds stress auto-covariance is, firstly, axi-symmetric, and secondly is equivalent to the same tensor only after it has been averaged (point-wise) over the azimuthal separation.

As a consequence of these two assumptions, the space-time Fourier transform of the Reynolds stress auto-covariance (which we refer to as the spectral tensor) depends on the transverse wave vector only through the square of its magnitude and, moreover, is also an axi-symmetric tensor. This defines the generalized shell model (or GSM) and we apply it to the jet noise problem. The final result shows that the acoustic spectrum can be written as the sum of two groups of terms, one of which corresponds to the peak jet noise in the weakly non-parallel flow limit.

1. Introduction

The physics of sound generation by jet turbulence is a subject of real-world importance. When aircraft take off, the acoustic waves that propagate away from the exhaust jet flow create considerable nuisance if the aircraft is flying from an airport situated in an urban setting.

The Navier Stokes equations provide the most fundamental theoretical starting point for the jet noise problem. They show that momentum transfer by turbulence in the jet produces pressure fluctuations, which propagate away as sound. This result was found by Lighthill [2] who introduced the first systematic theory of jet noise referred to as the acoustic analogy. Ever since Lighthill's work, however, there has been on-going debate about the structure of acoustic "sources" in jet turbulence. The Navier Stokes equations may provide an exact formula for the far field acoustic spectrum, but its complexity (and in particular that of the Reynolds stress auto-covariance tensor) necessitates the introduction of certain rational approximations to make the end formula useable for engineering problems. Hence, the mathematical model defined by these approximations should be based on experiments in jet turbulence—and, ideally, this should be from as wide a data set as possible to maximize the range of applicability of the model

What most experiments have shown is the possibility that the acoustic spectrum may possess a two-source structure. This is in the sense that, at a particular observation angle with respect to the jet axis, the total acoustic spectrum is given by the sum of the individual acoustic spectra from each source. The two source distributions appear to have different spectral properties and different radiation patterns with regard to the position of the observation point relative to the jet axis. The first of these appears to have a relatively narrow spectrum and a highly directional radiation pattern that peaks at small angles to the downstream axis (referred to here as type I behavior) and the second appears to have a more broad band spectrum and a much less directional radiation pattern (referred to here as type II behavior).

Although speculation on the two-source structure began to emerge decades ago, after the experiments on low Mach number flows by Lush [3] and Ahuja [4], this picture has become much more compelling in recent years where the focus has shifted to higher Mach number flows that exhibit a much clearer separation between the type I and type II behaviors. There is now an extensive literature discussing these observations (for example: [5], [6], [7], [8], [9], [10], [11] and [12]). Indeed the recent symposium on sound source mechanisms in turbulent shear flows (ERCOFTAC [13]) was specifically organized to discuss the existence of the two-source structure. Much of the discussion in the ERCOFTAC meeting focused on whether this two-source structure implies that jet flow is composed of two different turbulence scales that radiate sound in distinct ways, or whether it is a natural result of the mean flow interaction effects and differences in the structure of the various components of the Reynolds stress auto-covariance tensor. The aim of this paper is to show that this two-source paradigm of the acoustic spectrum can be derived using a model of jet turbulence based on a minimum number of self-consistent approximations.

There are a number of reasons why a self-consistent theory of jet noise was not derived before. Firstly, the early models were based on Lighthill's theory (Ribner [14] and Goldstein & Rosenbaum [15]), which despite its simplicity, introduces a number of technical complications. For example, it had long been known that Lighthill's equation does not separate out the mean flow interaction effects from turbulence source fluctuations that actually produce the sound and, therefore, is not the best starting point since the type I behavior must in some way depend on the mean flow field (a point argued by Bishop *et al* [16]). Secondly, in the past, there was little data on the space-time properties of the Reynolds stress auto-covariance tensor from which to base models on. Both of these points have, hitherto, been remedied to some extent.

There is now a large data set available on the properties of the Reynolds stress auto-covariance, both from experiments and Large Eddy Simulations (LES); for example: [17], [18], [19] and [20]. Moreover, in the last decade there have been advances in the acoustic analogy approach itself. Goldstein [1] showed that, in the absence of solid surface effects, the far-field pressure auto-covariance can be expressed as the convolution product of a propagator and a two point time-delayed auto-covariance of a fluctuating stress tensor, which reduces to the usual Reynolds stress auto-covariance tensor in the absence of enthalpy fluctuations. The propagator can be calculated once the mean flow is known, since it involves Green's functions that can be determined quite easily for simple flows (e.g. [21]). On the other hand, the Reynolds stress auto-covariance tensor is modeled using its symmetries and the experimentally observed properties of jet flow turbulence. It possesses 36 independent components owing to the pair symmetries in its tensor suffixes. Moreover, for a fixed spatial location and given time delay, it is a function of a vector separating two points in space. Due to this complexity, some logical method is required to simplify the mathematical structure of the tensor, to make it easier to calculate for aero-acoustic purposes in an engineering setting. The challenge for mathematical modeling, however, is to achieve this simplification without compromising on the theoretical consistency of the approximations that are made, which should, therefore, be kept to a minimum.

A number of very important recent discoveries have been made on the problem how best to model the Reynolds stress auto-covariance. Goldstein & Leib [22] (hereafter referred to as G & L) obtained a formula for the pressure auto-covariance based on the formalism developed in Goldstein [1] and certain statistical assumptions about the turbulence. Although their fundamental result was an acoustic spectrum displaying the two-source structure (i.e. two terms that could be added together to obtain the total), it was based on some relatively pragmatic approximations of the Reynolds stress auto-covariance that, simplified the computations, but were more restrictive than necessary. The G & L model was based on the following assumptions: (i) the mean flow is weakly non-parallel; (ii) the four dimensional space-time spectrum of the Reynolds stress auto-covariance (which we call the the spectral tensor) has the same tensorial structure as the corresponding zero wave vector tensor; (iii) the auto-covariance tensor is consistent with the quasi-normality hypothesis and finally, (iv) the turbulence is axi-symmetric. Since we are focusing on the structure of the Reynolds stress auto-covariance tensor, we discuss only assumptions (ii), (iii) & (iv).

Recent turbulence measurements (conducted for jet noise modeling purposes) appear to be limited to components of Reynolds stress auto-covariance tensor and its one-dimensional (frequency) spectrum, i.e. the real space tensor and not its four dimensional space-time spectrum ([17] and [20]). Hence the validity of assumption (ii) cannot be assessed with complete certainty, but the result shown in figure (10) in the paper by Pokora & McGuirk [20] indicates a small variation in the longitudinal component of the Reynolds stress auto-covariance when the transverse separation is increased and the axial separation is held fixed, which would imply assumption (ii) must be re-thought.

Assumption (iii), i.e. the quasi-normal approximation, implies the fluctuating Reynolds stress tensor is a stationary random function whose joint probability distribution is normal (or a multi-dimensional Gaussian random field). G & L only use this assumption to express the fourth rank Reynolds stress auto-covariance tensor as products of second rank correlation tensors ([23] and p. 44 of [24]). Quasi-normality is usually justified by invoking the central limit theorem (see ch. 8 of Batchelor [25]). The arguments appear to be reasonable for single point statistics and the conclusions appear to be verified for grid turbulence by the measurements of [26], [27] and [28] (see p. 241 of [29] for details). The same argument cannot be used to justify normality of two-point statistics, and especially for the shear flow turbulence that is relevant to the jet noise problem (Morris & Zaman [30]). For example, the statistics of the large energy bearing eddies cannot be Gaussian (at least) because the ordered motions prevent application of the central limit theorem.

The normality hypothesis also becomes problematic when dynamical considerations are brought into play (p. 284 of [29], p. 164 of [31] and p. 81 of [32]). This approximation often leads to non-physical behavior that violates physical realizability in this case ([33], [34], [35], [36], [37] and [38]). Even though quasi-normality was used by G & L in the kinematic sense only (and not as a dynamical approximation to the Navier Stokes equations), it still remains desirable to formulate a model of the acoustic spectrum that avoids quasi-normality altogether, and thereby satisfies the requirement of theoretical consistency.

The fourth assumption G & L introduced (of axi-symmetric turbulence) does seem to have a fairly firm basis. Away from jet turbulence, there has been considerable study on axi-symmetric turbulence and justification of its use in areas such as atmospheric science ([39], [40] and [41]) and rotating turbulence ([42] and [43]).

In jet flow turbulence, it has long been known that the strongest correlations occur in the stream wise direction. This would imply that the longitudinal component of the Reynolds stress auto-covariance for example, is significantly smaller for points separated in the transverse direction (this particular finding is clearly evident in figure (10d) in Pokora & McGuirk [20]) than it is for points separated in the stream wise direction. Moreover, recent computational and experimental evidence shows that the amplitudes of the longitudinal and lateral components of the Reynolds stress auto-covariance are un-equal, with the longitudinal component being about three times bigger than the lateral at locations near the end of the potential core of the jet ([18], [19], [20] and [30]). All of which indicates that an assumption of isotropy would not be very accurate.

In this paper, we, therefore, introduce two basic assumptions consistent with the experimental data we have reported. Firstly, we assume the Reynolds stress auto-covariance is an axi-symmetric tensor, and secondly, that it is equivalent to the same tensor only after it has been averaged (point-wise) over the azimuthal separation. The consequences of these assumptions are that the spectral tensor depends on the transverse wave vector only through the square of its magnitude and, finally, that this tensor is also an axi-symmetric tensor. We refer to this model, when applied to the spectral tensor, as the *generalized shell model* (or GSM), since it represents a kinematic generalization of the spherical shell models discussed in §. 8.7 in [24], p. 418 in [44] and ch. 3 in [45]. Hence the presentation of the GSM and its application to the jet noise problem are the aspects of this work that we believe are new. The rest of the paper is organized as follows. The starting equation for the acoustic spectrum is derived very briefly in section (2) using Goldstein's [1] acoustic analogy formalism. The GSM is introduced in section (3) and the acoustic spectrum is analyzed in section (4).

2. Basic equation

The acoustic spectrum,

$$I_\omega(\underline{x}) = \int_{-\infty}^{+\infty} e^{i\omega\tau_0}\overline{p^2}(\underline{x},\tau_0)\,d\tau_0 \tag{2.1}$$

at the observation point $\underline{x}$, i.e. the Fourier transform of the far-field pressure auto-covariance

$$\overline{p^2}(\underline{x},\tau_0) = \frac{1}{2T}\int_{-T}^{+T} p(\underline{x},t)p(\underline{x},t+\tau_0)\,dt \tag{2.2}$$

can be expressed in terms of $I_\omega(\underline{x} \mid \underline{y})$, the acoustic spectrum at $\underline{x}$, due to a unit volume of turbulence at $\underline{y}$, by the equation:

$$I_\omega(\underline{x}) = \int_{V_\infty(\underline{y})} I_\omega(\underline{x} \mid \underline{y})\,d\underline{y} \tag{2.3}$$

where $V_\infty(\underline{y})$ denotes integration over all space with respect to $\underline{y}$. G & L show that this latter quantity is given by

$$I_\omega(\underline{x} \mid \underline{y}) = (2\pi)^2\Gamma_{\nu j}(\underline{x} \mid \underline{y};\omega)\int_{V_\infty(\underline{\eta})} \Gamma^*_{\mu l}(\underline{x} \mid \underline{y}+\underline{\eta};\omega)\mathcal{H}_{\nu j,\mu l}(\underline{y},\underline{\eta},\omega)\,d\underline{\eta} \tag{2.4}$$

where the asterisks denote complex conjugate, Greek suffixes range from 1 to 4, and Latin suffixes from 1 to 3. The second rank tensor defined by,

$$\Gamma_{\nu j}(\underline{x} \mid \underline{y};\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-\tau)}\gamma_{\nu j}(\underline{x} \mid \underline{y}, t-\tau)\,d(t-\tau) \tag{2.5}$$

is the Fourier transform of a "propagator"

$$\gamma_{\nu j}(\underline{x},t \mid \underline{y},\tau) = \frac{\partial g^a_{\nu 4}(\underline{x},t \mid \underline{y},\tau)}{\partial y_j} - (\gamma-1)\delta_{\nu k}\frac{\partial \tilde{v}_k}{\partial y_j}g^a_{44}(\underline{x},t \mid \underline{y},\tau) \tag{2.6}$$

that depends on the adjoint vector Green's function $g^a_{\mu\nu}(\underline{y},\tau \mid \underline{x},t)$, determined by equations (4.8) and (4.11) of G & L. The Green's function and, therefore, $\gamma_{\nu j}(\underline{x},t \mid \underline{y},\tau)$ can be calculated once the mean flow is known. The rank four tensor $\mathcal{H}_{\nu j,\mu l}(\underline{y},\underline{\eta},\omega)$ is related to the spectrum

$$H_{\nu j,\mu l}(\underline{y},\underline{\eta},\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega\tau_0}R_{\nu j,\mu l}(\underline{y},\underline{\eta},\tau_0)\,d\tau_0 \tag{2.7}$$

of the generalized auto-covariance tensor

$$R_{\nu j,\mu l}(\underline{y},\underline{\eta},\tau_0) = \frac{1}{2T}\int\limits_{-T}^{+T}\left[\rho v'_\nu v'_j - \overline{\rho v'_\nu v'_j}\right](\underline{y},\tau)\left[\rho v'_\mu v'_l - \overline{\rho v'_\mu v'_l}\right](\underline{y}+\underline{\eta},\tau+\tau_0)\,d\tau \tag{2.8}$$

by the simple linear transformation,

$$\mathcal{H}_{\nu j,\mu l} = \epsilon_{\nu j,\sigma m} H_{\sigma m,\lambda n}\epsilon_{\mu l,\lambda n} \tag{2.9}$$

where v'_μ denotes a four dimensional velocity fluctuation (with the fourth component defined by equation 2.14 of G & L) and $\tilde{v}_k$ denotes the Favre averaged mean flow velocity, the overbar denotes the time average and,

$$\epsilon_{\nu j,\sigma m} = \left(\delta_{\nu\sigma}\delta_{jm} - \frac{\gamma-1}{2}\delta_{\nu j}\delta_{\sigma m}\right) \tag{2.10}$$

where γ is the ratio of specific heat capacities ($\gamma = 1.4$ for air).

This result was found by G & L, but in a less algebraically compact form, and it also appears in §.(2) in [46]. It shows that there is an exact relation between the far-field acoustic spectrum and the generalized Reynolds stress auto-covariance tensor. The dimensionless ratio, $v'_4/U_{\text{jet}}v'_j$, which determines the relative importance of the fourth component of v'_μ to v'_i, where U_{jet} denotes a characteristic jet velocity must scale as $O(\sqrt{v'^2}/U_{\text{jet}})$ for unheated jets when the Mach number is $O(1)$ (Lilley [47] and Morfey, Szewczyk & Tester [48]) because c'^2 is expected to be $O(\sqrt{v'^2})^2$ in this case. The enthalpy component of v'_μ should, therefore, be small for cold jets and can be set to zero in $R_{\nu j,\mu l}(\underline{y},\underline{\eta},\tau_0)$, which will equal zero whenever $(\mu,\nu) = 4$. The generalized auto-covariance tensor will then reduce to the usual Reynolds stress auto-covariance tensor $R_{ij,kl}(\underline{y},\underline{\eta},\tau_0)$, which is defined by

$$R_{ij,kl}(\underline{y},\underline{\eta},\tau_0) = \frac{1}{2T}\int\limits_{-T}^{+T}\left[\rho v'_i v'_j - \overline{\rho v'_i v'_j}\right](\underline{y},\tau)\left[\rho v'_k v'_l - \overline{\rho v'_k v'_l}\right](\underline{y}+\underline{\eta},\tau+\tau_0)\,d\tau \tag{2.11}$$

The acoustic spectrum then reduces to

$$I_\omega(\underline{x}\mid\underline{y}) = (2\pi)^2\Gamma_{ij}(\underline{x}\mid\underline{y};\omega)\int\limits_{V_\infty(\underline{\eta})}\Gamma^*_{kl}(\underline{x}\mid\underline{y}+\underline{\eta};\omega)\mathcal{H}_{ij,kl}(\underline{y},\underline{\eta},\omega)\,d\underline{\eta} \tag{2.12}$$

2.1. Spectral tensor formalism

The mean flow fields in jets involve two characteristic dimensions: a cross-stream dimension, say D_{jet}, and a much longer stream wise dimension, say L, where ($L \gg D_{\text{jet}}$). In the absence of strong critical layer effects (see G & L), the propagator $\Gamma^*_{kl}(\underline{x}\mid\underline{y}+\underline{\eta};\omega)$ can only depend on the two characteristic scales D_{jet} and L and the acoustic wavelength, $\lambda_{\text{acoustic}} = c_\infty/\omega$ when $|\underline{x}|$ is in the far field. But for a given $(\underline{x} \mid \underline{y})$, the propagator contributes to I_ω through the integral over the correlation volume $V_\infty(\underline{\eta})$, after being weighted by $\mathcal{H}_{ij,kl}(\underline{y},\underline{\eta},\omega)$. Then $\Gamma^*_{kl}(\underline{x} \mid \underline{y}+\underline{\eta};\omega)$ can only change by a significant amount over the correlation volume $V_\infty(\underline{\eta})$ of the turbulence if $D_{\text{jet}}/\lambda_{\text{acoustic}}$ is large when the stream wise and transverse length scales of the turbulence are small compared to the corresponding dimensions L and D_{jet} of the mean flow.

Now, axial correlation lengths are always small compared to the stream wise dimension L of the mean flow and, moreover, figure (10) in Pokora & McGuirk [20] shows that the transverse correlation lengths are small compared to the transverse dimension of the mean flow (a result which was recently corroborated by Morris & Zaman [49]). Hence we can legitimately assume that the stream wise and transverse length scales of the turbulence are small compared to the corresponding dimensions D_{jet} and L of the mean flow. The propagator $\Gamma^*_{kl}(\underline{x} \mid \underline{y}+\underline{\eta};\omega)$ can then be represented by its high frequency, or WKBJ approximation (Khavaran [50]). So for variations on the scale of the correlation volume, $V_\infty(\underline{\eta})$, we have

$$\Gamma^*_{\mu l}(\underline{x} \mid \underline{y}+\underline{\eta};\omega) \approx \Gamma^*_{\mu l}(\underline{x} \mid \underline{y};\omega)\exp\left[i\frac{\omega}{c_\infty}\underline{\eta}.\underline{\nabla}_{\underline{y}}S(\underline{x} \mid \underline{y})\right] \tag{2.13}$$

where

$$\Gamma^*_{\mu l}(\underline{x} \mid \underline{y};\omega) = (\omega/c_\infty)\,A^{(0)}_{\mu l}(\underline{x} \mid \underline{y})\exp\left[i\frac{\omega}{c_\infty}S(\underline{x} \mid \underline{y})\right] \tag{2.14}$$

The term $A^{(0)}_{\mu l}(\underline{x} \mid \underline{y})$ is the first term in the amplitude series and $S(\underline{x} \mid \underline{y})$ is the phase function that satisfies the usual Eikonal equation (see appendix B in [46] for further details).

If we substitute this WKBJ approximation of $\Gamma^*_{kl}(\underline{x} \mid \underline{y}+\underline{\eta};\omega)$ into equation (2.12), it gives a purely algebraic result for the acoustic spectrum:

$$I_\omega(\underline{x} \mid \underline{y}) \approx (2\pi)^2 G_{ij}(\underline{x} \mid \underline{y};\omega)G^*_{kl}(\underline{x} \mid \underline{y};\omega)\Phi^*_{ij,kl}\left(\underline{y},\underline{k},\omega\right) \tag{2.15}$$

where, $G_{ij} = (\Gamma_{ij}+\Gamma_{ji})/2$ is a symmetric rank two tensor. Equation (2.15) depends on the turbulence correlations only through the complex conjugate of the fourth rank spectral tensor $\Phi_{ij,kl}(\underline{y},k_1,\underline{k}_\perp,\omega)$, which is related to the Reynolds stress auto-covariance by the Fourier transform

$$\Phi_{ij,kl}(\underline{y},k_1,\underline{k}_\perp,\omega) = \int_{\underline{\eta}} \mathcal{H}_{ij,kl}(\underline{y},\underline{\eta},\omega)e^{-i\underline{k}\cdot\underline{\eta}}\,d\underline{\eta}. \tag{2.16}$$

The wave vector $\underline{k}$ is defined by its Cartesian components $\underline{k} = (k_1,\underline{k}_\perp)$, where k_1 is the axial component and $\underline{k}_\perp = (k_2,k_3)$ is in the transverse direction. $\mathcal{H}_{ij,kl}(\underline{y},\underline{\eta},\omega)$ is related to the Reynolds stress auto-covariance through equations (2.7)—(2.9), and $\underline{k}$ can be identified as $(\omega/c_\infty)\underline{\nabla}_{\underline{y}}S(\underline{x} \mid \underline{y})$ (see [46]). The spectral tensor has two pair symmetries since $\Phi_{ij,kl} = \Phi_{ji,kl}$ and $\Phi_{ij,kl} = \Phi_{ij,lk}$. Therefore, it possesses 36 independent components. Our aim now is to reduce this number by developing a generalized shell model in axi-symmetric turbulence.

3. Generalized shell model (GSM)

3.1. Two fundamental approximations

Pokora & McGuirk [20] measured turbulence in an incompressible jet flow. Their results revealed three very important properties of the Reynolds stress auto-covariance tensor. Firstly, $R_{11,11}$ remains the biggest component of the tensor, and the amplitudes of $R_{11,11}$, $R_{22,22}$ and $R_{33,33}$ are not equal to one another. This is shown in their figure (15) at a downstream location at the end of the potential core. Secondly, figure (10) in Pokora & McGuirk [20] shows that $R_{11,11}(\underline{y}, \underline{\eta}, \tau_0)$ is correlated over much longer distances in the stream wise direction η_1 than the transverse direction $\underline{\eta}_\perp$. And, finally, there is a small variation of $R_{11,11}(\underline{y}, \underline{\eta}, \tau_0)$ with circumferential separation. These particular trends in the amplitudes of the Reynolds stress auto-covariance tensor were corroborated in the LES computations by McMullen *et al* [19]; moreover, similar space-time behavior was also found in the work of Harper-Bourne [17].

Since the measurements seem to suggest that the turbulence is approximately axi-symmetric, we introduce the following approximation:

Fundamental Assumption 3.1. *We suppose:*

$$R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0) \textit{ is an axi-symmetric tensor,} \tag{3.1}$$

in the sense that the tensor form defined by $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$ remains invariant to proper and improper rotations. Or, in the language of Lie group theory (see [62] and [63]), the tensor form defined by $R_{ij,kl}$ is invariant under the **O(3)** *symmetry group.*

However, the structure of $R_{ij,kl}(\underline{y}, \eta_1, \underline{\eta}_\perp, \tau_0)$ is not known with great certainty when the azimuthal separation $\psi = \tan^{-1}(\eta_2/\eta_3)$ is increased (with η_1 held fixed). We, therefore, introduce a second approximation.

Fundamental Assumption 3.2. *We suppose:*

$$R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0) = R_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \tau_0) \approx \overline{R}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \tau_0), \tag{3.2}$$

where $\eta_\perp = |\underline{\eta}_\perp|$ and

$$\overline{R}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi} R_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \tau_0)\, d\psi. \tag{3.3}$$

When taken together, assumptions (3.1) and (3.2) imply:

$$\overline{R}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \tau_0) \text{ is an axi-symmetric tensor.} \tag{3.4}$$

Now since a time Fourier transform performed on $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$ will not affect assumptions (3.1) and (3.2), we can make equivalent statements for the tensor $H_{ij,kl}(\underline{y}, \underline{\eta}, \omega)$; that is,

$$H_{ij,kl}(\underline{y}, \underline{\eta}, \omega) \text{ is an axi-symmetric tensor,} \tag{3.5}$$

and,

$$H_{ij,kl}(\underline{y}, \underline{\eta}, \omega) = H_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \omega) \approx \overline{H}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \omega) \tag{3.6}$$

where

$$\overline{H}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \omega) = \frac{1}{2\pi} \int_0^{2\pi} H_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \omega)\, d\psi. \tag{3.7}$$

Whence,

$$\overline{H}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \omega) \text{ is an axi-symmetric tensor.} \tag{3.8}$$

3.2. Consequences of assumptions (3.1) and (3.2)

Assumptions (3.1) and (3.2) imply the following two lemmas:

Lemma 3.1. *If $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$ is an axi-symmetric tensor, then the spectral tensor $\Phi_{ij,kl}(\underline{y}, \underline{k}, \omega)$ is also an axi-symmetric tensor, under the same symmetry group as $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$. The proof of this lemma is immediate because it is the tensor form of $R_{ij,kl}$ that must remain invariant as part of assumption (3.1).*

Lemma 3.2. *$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \omega)$ depends upon the transverse wave vector, $\underline{k}_\perp$, only through the square of its magnitude, $k_\perp^2$, when assumption (3.2) is satisfied. That is,*

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \omega) = \Phi_{ij,kl}(\underline{y}, k_1, k_\perp, \Psi, \omega) \approx \overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_\perp^2, \omega), \tag{3.9}$$

where

$$\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_\perp, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_{ij,kl}(\underline{y}, k_1, k_\perp, \Psi, \omega)\, d\Psi \tag{3.10}$$

$\Psi = \tan^{-1}(k_2/k_3)$ *and* $k_\perp = |\underline{k}_\perp|$.

Proof. Equation (2.9) shows that $\mathcal{H}_{ij,kl}$ is linearly related to $H_{ij,kl}$. Therefore, assumption (3.2) implies:

$$\overline{\mathcal{H}}_{ij,kl} = \epsilon_{ij,pq} \overline{H}_{pq,rs} \epsilon_{kl,rs} \tag{3.11}$$

That is:

$$\mathcal{H}_{ij,kl}(\underline{y}, \eta, \omega) = \mathcal{H}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \omega) \approx \overline{\mathcal{H}}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \omega), \tag{3.12}$$

where

$$\overline{\mathcal{H}}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{ij,kl}(\underline{y}, \eta_1, \eta_\perp, \psi, \omega)\, d\psi. \tag{3.13}$$

After using assumption (3.2) in the form of (3.12) we can write the integral definition of the spectral tensor as follows:

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \omega) = \int_{\eta_{\perp}} \int_{\eta_1} \overline{\mathcal{H}}_{ij,kl}(\underline{y}, \eta_1, \eta_{\perp}, \omega) e^{-ik_1\eta_1} \int_0^{2\pi} e^{-i\underline{k}_{\perp}\cdot\underline{\eta}_{\perp}}\, d\psi\, d\eta_1 \eta_{\perp}\, d\eta_{\perp}. \tag{3.14}$$

The ψ-integral can be easily evaluated to give:

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \omega) = 2\pi \int_{\eta_{\perp}} \int_{\eta_1} \overline{\mathcal{H}}_{ij,kl}(\underline{y}, \eta_1, \eta_{\perp}, \omega) e^{-ik_1\eta_1} J_0(k_{\perp}\eta_{\perp})\, d\eta_1 \eta_{\perp}\, d\eta_{\perp}, \tag{3.15}$$

which is just the Hankel transform of

$$2\pi \int_{\eta_1} \overline{\mathcal{H}}_{ij,kl}(\underline{y}, \eta_1, |\underline{\eta}_{\perp}|, \omega) e^{-ik_1\eta_1}\, d\eta_1 \tag{3.16}$$

(p.962 in [51]). Since the expansion of $J_0(k_{\perp}\eta_{\perp})$ involves only even powers of its argument (p.57 in [52]), the spectral tensor depends on $\underline{k}_{\perp}$ only through the square of its magnitude $k_{\perp}^2$. Hence:

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \omega) = \Phi_{ij,kl}(\underline{y}, k_1, k_{\perp}, \Psi, \omega) \approx \overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_{\perp}^2, \omega) \tag{3.17}$$

where

$$\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_{\perp}, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_{ij,kl}(\underline{y}, k_1, k_{\perp}, \Psi, \omega)\, d\Psi. \tag{3.18}$$

□

Finally, we can combine Lemmas (3.1) and (3.2) to arrive at the result that

$$\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_{\perp}^2, \omega) \text{ is an axi-symmetric tensor.} \tag{3.19}$$

Equation (3.19) is a direct consequence of assumptions (3.1) and (3.2) and is an interpretation of axi-symmetric turbulence theory that would seem to be most suitable for jet noise modeling purposes. However, by averaging over the azimuthal separation at the outset, we have, in fact, generalized the spherical shell models that appear as the starting point of various dynamical theories in turbulence (e.g. Obukhov [53]; Gledzer [54]; Yamada & Okhitani [55] & [56]; L'vov *et al* [57] etc., see ch. 3 in Ditlevsen [45]), to cylindrical shells, without altering the kinematic consequences of axi-symmetry appropriate to jet turbulence. We, therefore, refer to equation (3.19) as the *generalized shell model* (or GSM) to emphasize this apparent generality.

It is shown in appendix A that the algebraic representation of the GSM under $\mathsf{O}(3)$ (i.e. the full rotation group – proper and improper rotations) will be the same even if the symmetry group is projected down to the $\mathsf{SO(3)}$ sub-group (i.e. invariance to proper rotations only). Appendix A shows that $\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_\perp^2, \omega)$ is given by:

$$
\begin{aligned}
\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_2^2 + k_3^2, \omega) \quad = \quad & [\delta_{ij}\delta_{kl} - \delta_{i1}\delta_{j1}\delta_{kl} - \delta_{k1}\delta_{l1}\delta_{ij} + \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{22,22}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
+ \quad & \big[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} + 2\delta_{i1}\delta_{j1}\delta_{kl} + 2\delta_{k1}\delta_{l1}\delta_{ij} \\
- \quad & \delta_{i1}\delta_{l1}\delta_{jk} - \delta_{j1}\delta_{l1}\delta_{ik} - \delta_{j1}\delta_{k1}\delta_{il} - \delta_{i1}\delta_{k1}\delta_{jl}\big]\,\overline{\Phi}_{23,23}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
+ \quad & \big[\delta_{i1}\delta_{l1}\delta_{jk} + \delta_{j1}\delta_{l1}\delta_{ik} + \delta_{j1}\delta_{k1}\delta_{il} \\
+ \quad & \delta_{i1}\delta_{k1}\delta_{jl} - 4\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\big]\,\overline{\Phi}_{12,12}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
+ \quad & [\delta_{i1}\delta_{j1}\delta_{kl} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{11,22}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
+ \quad & [\delta_{k1}\delta_{l1}\delta_{ij} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{22,11}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
+ \quad & \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\overline{\Phi}_{11,11}(\underline{y}, k_1, k_\perp^2, \tau_0).
\end{aligned} \tag{3.20}
$$

Equation (3.20) satisfies Millionshchikov's identity in spectral space,

$$
\overline{\Phi}_{22,22}(\underline{y}, k_1, k_\perp^2, \tau_0) = \overline{\Phi}_{22,33}(\underline{y}, k_1, k_\perp^2, \tau_0) + 2\overline{\Phi}_{23,23}(\underline{y}, k_1, k_\perp^2, \tau_0). \tag{3.21}
$$

The real space equivalent (equation 12.131 on p. 68 in Monin & Yaglom [29]) was analyzed numerically in [58]; it was shown there to be in excellent agreement with LES of a round cold jet flow and the experimental data reported in Pokora & McGuirk [20].

4. Acoustic spectrum

Inserting equation (3.20) into the acoustic spectrum formula (2.15) shows that the far field acoustic spectrum is given by:

$$
\begin{aligned}
\frac{I_\omega(\underline{x} \mid \underline{y})}{(2\pi)^2} \quad \approx \quad & [G_{ii}G_{kk}^* - 2\mathrm{Re}(G_{11}G_{kk}^*) + G_{11}G_{11}^*]\,\overline{\Phi}_{22,22}^* \\
+ \quad & \mathrm{Re}\,[G_{11}G_{kk}^* - G_{11}G_{11}^*]\,\big(\overline{\Phi}_{11,22}^* + \overline{\Phi}_{22,11}^*\big) \\
+ \quad & G_{11}G_{11}^*\overline{\Phi}_{11,11}^* \\
+ \quad & 2\,[G_{ik}G_{ik}^* - G_{ii}G_{kk}^* + 2\mathrm{Re}(G_{11}G_{kk}^*) - 2G_{k1}G_{k1}^*]\,\overline{\Phi}_{23,23}^* \\
+ \quad & 4\,[G_{k1}G_{k1}^* - G_{11}G_{11}^*]\,\overline{\Phi}_{12,12}^*
\end{aligned} \tag{4.1}
$$

(Re denotes the real part). The $\overline{\Phi}_{23,23}^*$ spectral tensor component can be re-written using the Millionshchikov identity (equation 3.21) and the result can be rearranged to obtain the following remarkably simple formula for the acoustic spectrum.

$$
\begin{aligned}
\frac{I_\omega(\underline{x} \mid \underline{y})}{(2\pi)^2} &\approx \left[|G_{22}|^2 + |G_{33}|^2 + 2|G_{23}|^2\right]\overline{\Phi}^*_{22,22} \\
&+ \mathrm{Re}\left[G_{11}(G^*_{22} + G^*_{33})\right]\left(\overline{\Phi}^*_{11,22} + \overline{\Phi}^*_{22,11}\right) \\
&+ |G_{11}|^2\overline{\Phi}^*_{11,11} + 2\left[\mathrm{Re}\,(G_{22}G^*_{33}) - |G_{23}|^2\right]\overline{\Phi}^*_{22,33} \\
&+ 4\left[|G_{12}|^2 + |G_{13}|^2\right]\overline{\Phi}^*_{12,12}
\end{aligned} \tag{4.2}
$$

where the arguments of the symmetric propagator, G_{ij}, are $(\underline{x} \mid \underline{y}; \omega)$ and the components of spectral tensor are functions of $(\underline{y}, k_1, k_2^2 + k_3^2, \omega)$. Equation (4.2) applies to jets of any cross section because equation (3.19) is likely to remain accurate in the rectangular jet case as well.

Equation (4.2) depends on the six independent components $\overline{\Phi}^*_{22,22}$, $\overline{\Phi}^*_{11,22}$, $\overline{\Phi}^*_{22,11}$, $\overline{\Phi}^*_{11,11}$, $\overline{\Phi}^*_{22,33}$ and $\overline{\Phi}^*_{12,12}$, of the spectral tensor $\overline{\Phi}^*_{ij,kl}$ which are related to the components of the Reynolds stress auto-covariance tensor by equations (2.7)—(2.10) and (2.16). It is a generalization of equation (6.27) in G & L that does not require the mean flow to be weakly non-parallel as can easily be seen by putting

$$
G_0 \equiv G_{22} + G_{33} \tag{4.3}
$$

and using the identity

$$
|G_{22}|^2 + |G_{33}|^2 \equiv |G_0|^2 - 2\mathrm{Re}\,(G_{22}G^*_{33}) \tag{4.4}
$$

to re-write equation (4.2) in the form,

$$
\begin{aligned}
\frac{I_\omega(\underline{x} \mid \underline{y})}{(2\pi)^2} &\approx |G_0|^2\overline{\Phi}^*_{22,22} + \mathrm{Re}\,(G_{11}G^*_0)\left[\overline{\Phi}^*_{11,22} + \overline{\Phi}^*_{22,11}\right] + |G_{11}|^2\overline{\Phi}^*_{11,11} \\
&+ 2\left[|G_{23}|^2 - \mathrm{Re}\,(G_{22}G^*_{33})\right]\left(\overline{\Phi}^*_{22,22} - \overline{\Phi}^*_{22,33}\right) \\
&+ 4\left[|G_{12}|^2 + |G_{13}|^2\right]\overline{\Phi}^*_{12,12}
\end{aligned} \tag{4.5}
$$

(G & L replaced the $(\Phi^*_{11,22} + \Phi^*_{22,11})/2$ term in (4.2) by $\Phi^*_{11,22}$, but the present analysis shows that $\Phi^*_{11,22}$ & $\Phi^*_{22,11}$ are actually independent components of $\Phi^*_{ij,kl}$).

All of the spectral tensor coefficients would be positive definite in equation (4.5) if $|G_{23}|^2 \geq \mathrm{Re}\,(G_{22}G^*_{33})$, since the remaining coefficients are clearly positive for all values of $(\underline{x} \mid \underline{y}; \omega)$. The numerical simulations in G & L show that this inequality will be satisfied for a weakly non-parallel flow. But all of the spectral tensor coefficients will only be positive definite in equation (4.2) if $|G_{23}|^2 < \mathrm{Re}\,(G_{22}G_{33}^*)$. However, experiments and LES data both indicate $R_{22,33}/R_{11,11}$ is negligibly small and, therefore, that the $\overline{\Phi}_{22,33}^*$ term can be neglected. This also implies that the magnitude of $R_{22,22}$ will be large compared to the magnitude of $R_{22,33}$. Figure (16) in McMullen *et al* [19] and figure (15) of Pokora & McGuirk [20] show that this is true at a particular point near the end of the potential core on the nozzle lip line, while further analysis of McMullen's LES solution shows that this ratio is negligible almost everywhere in a region extending from $y_1/D_{\mathrm{jet}} = 4$ to $y_1/D_{\mathrm{jet}} = 8$ in the axial direction, and $r/D_{\mathrm{jet}} = 0$ to $r/D_{\mathrm{jet}} = 1$ in the radial direction. Hence equation (4.2) can be approximated to obtain the even simpler result:

$$
\begin{aligned}
\frac{I_\omega(\underline{x} \mid \underline{y})}{(2\pi)^2} \approx\; & \left[|G_{22}|^2 + |G_{33}|^2 + 2|G_{23}|^2\right]\overline{\Phi}_{22,22}^* \\
+\; & \mathrm{Re}\,(G_{11}G_0^*)\left[\overline{\Phi}_{11,22}^* + \overline{\Phi}_{22,11}^*\right] + |G_{11}|^2\overline{\Phi}_{11,11}^* \\
+\; & 4\left[|G_{12}|^2 + |G_{13}|^2\right]\overline{\Phi}_{12,12}^*
\end{aligned}
\tag{4.6}
$$

that is a linear combination with positive definite coefficients of only five components of $\overline{\Phi}_{ij,kl}^*$ (i.e. $\overline{\Phi}_{22,22}^*$, $\overline{\Phi}_{11,22}^*$, $\overline{\Phi}_{22,11}^*$, $\overline{\Phi}_{11,11}^*$ and $\overline{\Phi}_{12,12}^*$).

4.1. Thinking of the spectrum as two sources

The acoustic spectrum takes the simple form,

$$I_\omega \sim G_{ii}G_{kk}^*\overline{\Phi}_{11,11}^* + 2\,(G_{ik}G_{ik}^* - G_{ii}G_{kk}^*)\,\overline{\Phi}_{12,12}^* \tag{4.7}$$

when the spectral tensor (equation 3.20) is replaced by the isotropic model

$$\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_2^2 + k_3^2, \omega) = \delta_{ij}\delta_{kl}\alpha(\underline{y}, \omega) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\beta(\underline{y}, \omega) \tag{4.8}$$

discussed in G&L (equations 6.11 & 6.12), since (as noted in that paper, and in terms of the spectral tensor $\overline{\Phi}_{ij,kl}^*$),

$$\overline{\Phi}_{11,11}^* = \overline{\Phi}_{22,22}^* = 2\overline{\Phi}_{12,12}^* + \overline{\Phi}_{11,22}^* \quad \text{and} \quad \overline{\Phi}_{22,33}^* = \overline{\Phi}_{11,22}^* \tag{4.9}$$

holds in the isotropic case. And since $G_{ik}G_{ik}^*$ is a quadratic form which possesses Hermitian symmetry and $G_{ii}G_{kk}^*$ is a quadratic form with diagonal symmetry, the coefficient of each spectral tensor component in equation (4.7) is positive definite. This implies that the sound field will behave as if it were generated by two statistically independent source distributions, $\overline{\Phi}_{12,12}^*$ and $\overline{\Phi}_{11,11}^*$, if $\mathrm{Re}\left[\overline{\Phi}_{12,12}^*(\omega)\right]$ and $\mathrm{Re}\left[\overline{\Phi}_{12,12}^*(\omega)\right]$ are both positive. Using equation (5.1) in Afsar [58] (and the results in §.(5.2) of that paper), the Hermitian quadratic form $G_{ik}G_{ik}^*$, which will be $O(\omega^2)$ as $\omega \to 0$, will be proportional to $\cos^4\theta(1 - M\cos\theta)^{-6}$, where $M = M(\underline{y}_\perp)$ is the Mach number profile and θ is the observation angle with respect to the downstream jet axis, and will, therefore, produce a highly directional radiation pattern that vanishes at 90^0 and peaks at small observation angles to the jet axis, for the parallel flow limit considered in G & L. This source distribution is expected to be highly localized since it will tend to be concentrated in regions where the mean flow velocity and the turbulence intensity are both large (i.e. on the jet centerline just downstream of the potential core). It is therefore expected to produce a relatively narrow spectrum. On the other hand, the diagonal quadratic form $G_{ii}G^*_{kk}$, which, will be $O(\omega^4)$ as $\omega \to 0$ (using the equation (5.2) in Afsar [58]), will depend on a somewhat lower inverse power of the Doppler factor and, therefore, be less directional in the weakly non-parallel limit. These are the type I and type II behaviors discussed in the introduction.

The situation is more complicated in the more general statistically axi-symmetric case, but the conclusion is basically the same. The acoustic spectrum again reduces to an algebraic form that can be interpreted as being the sum of two independent sources (each having groups of terms). In order to show this more clearly equation (4.2) is written as:

$$\frac{I_\omega(\underline{x} \mid \underline{y})}{(2\pi)^2} = \text{type I terms } + \text{ type II terms} \tag{4.10}$$

where

$$\begin{aligned} \text{type I terms} \quad &= \quad |G_{11}|^2\overline{\Phi}^*_{11,11} + \mathrm{Re}\,(G_{11}G_0^*)\left[\overline{\Phi}^*_{11,22} + \overline{\Phi}^*_{22,11}\right] \qquad \text{(4.11a)} \\ &+ \quad 4\Bigg[|G_{12}|^2 + |G_{13}|^2\Bigg]\overline{\Phi}^*_{12,12} \\ \text{type II terms} \quad &= \quad \Bigg[|G_{22}|^2 + |G_{33}|^2 + 2|G_{23}|^2\Bigg]\overline{\Phi}^*_{22,22} \qquad \text{(4.11b)} \end{aligned}$$

and, where, all of the coefficients of the spectral components are positive definite. Equations (4.10) & (4.11) therefore imply that the sound field will behave as if it were generated the source distributions $\overline{\Phi}^*_{11,11}$, $\overline{\Phi}^*_{22,22}$, $\overline{\Phi}^*_{11,22}$, $\overline{\Phi}^*_{22,11}$ and $\overline{\Phi}^*_{12,12}$. But since any group of sources can be thought of as a single source, we can again think of these as two distinct source distributions. And since the coefficients are all positive definite these sources will behave as if they were statistically independent source distributions if the real parts of the spectral tensor components are positive as they are for the Reynolds stress auto-covariance model used in G & L (see appendix B). Notice that this implies that all of the terms will be positive definite in the more exact equation (4.5) since the LES simulations and experimental data ([19] and [20]) imply that $\mathrm{Re}\left(\overline{\Phi}^*_{22,22}\right) - \mathrm{Re}\left(\overline{\Phi}^*_{22,33}\right)$ will be positive in this case. Moreover, [19], [20] and recent work by Morris & Zaman [49] all indicate that $R_{11,11}(\underline{y}, \underline{\eta}, \tau_0)$ has the longest correlation length. But more importantly, for a parallel mean flow, all of the wave propagation terms in type I are dominant at small observation angles from the $\cos\theta$ dependence. For example, $\Bigg[|G_{12}|^2 + |G_{13}|^2\Bigg]$, which will be of $O(\omega^2)$ in the parallel flow, as $\omega \to 0$, is the only term proportional to $\cos^4\theta(1 - M\cos\theta)^{-6}$, and will therefore produce the most highly directional radiation pattern and have a relatively narrow spectrum in the weakly non-parallel limit. These are again the type I and type II behaviors discussed in the introduction. Source coherence effects are also expected to play a role here, but that is beyond the scope of this paper. Equation (4.11) is then the main results of this paper. It shows that the acoustic spectrum will behave as if it were produced by two independent sources if the Reynolds stress auto-covariance is simplified by the introduction of assumptions (3.1) and (3.2). This result puts the two-source paradigm of jet noise on a more rigorous theoretical foundation and can hopefully be used as the new starting point for the next generation of jet noise models.

5. Conclusions

The structure of noise sources in turbulence has been the subject of long standing debate in the aero-acoustics community. Goldstein & Leib [22] (referred to in this paper as G & L) recovered the two-source behavior from equation (2.4) by showing that it can be reduced to an algebraic form consisting of two types of terms that can be interpreted as being independent sources. But their analysis makes a number of relatively restrictive assumptions about the turbulence. The present paper shows that this can be accomplished under much more general symmetry approximations that experiments show are less restrictive.

The paper begins with equation (2.4), which is an exact result that expresses the acoustic spectrum as a convolution product of a propagator and the two-point time delayed auto-covariance $R_{\nu j,\mu l}(\underline{y},\underline{\eta},\tau_0)$ of generalized Reynolds stress. As in G & L, the present paper is restricted to cold jets and neglects the enthalpy terms in $R_{\nu j,\mu l}$ order to reduce to the usual Reynolds stress auto-covariance tensor $R_{ij,kl}$. It also assumes that the stream wise and transverse length scales of the turbulence are small compared to the corresponding length scales of the mean flow in order to reduce equation (2.4) to a purely algebraic result that depends on the turbulence only through the four-dimensional space time Fourier transform $\Phi_{ij,kl}(\underline{y},\underline{k},\omega)$ of $R_{ij,kl}(\underline{y},\underline{\eta},\tau_0)$ (referred to here as the spectral tensor). But G & L used the quasi-normality approximation and assumed $\Phi_{ij,kl}(\underline{y},\underline{k},\omega)$ possessed the same tensorial structure as it would if the transverse component of the wave number vector $\underline{k}$ were zero in order to simplify this tensor. However, the experiments by Pokora & McGuirk [20] cast some doubt on the quasi-normality approximation and, moreover, there are a number of technical complications associated with that approximation that prevent it from being used unequivocally. In the present paper we, therefore, extended the shell model of turbulence under a general symmetry approximation that amounts to assuming the Reynolds stress auto-covariance is, firstly, axi-symmetric, and secondly is equivalent to the same tensor only after it has been averaged (point-wise) over the azimuthal separation. The model is given by equation (3.19) and referred to as the *generalized shell model* (or GSM).

Unlike the G & L result, the final formula for the acoustic spectrum (given by equation 4.11) which generalizes equation (6.27) of G & L is valid for any mean flow. It shows that the acoustic spectrum can be interpreted as being the result of two more-or-less independent source distributions (referred to here as type I and type II source distributions). The relatively localized type I distribution (equation 4.11a) produces a fairly narrow spectrum that tends to dominate the sound field at small observation angles to the downstream jet axis. The type II distribution term (equation 4.11b) implies that it would be greatest at larger angles and higher angular frequencies. The type II distribution tends to produce a much less directional sound field with somewhat broader spectrum.

Acknowledgements
M.Z.A acknowledges the financial support of the David Crighton Fellowship from Department of Applied Mathematics & Theoretical Physics, Cambridge, UK (2008); the Post-doctoral Fellowship from the Center for Turbulence Research at Stanford University, USA (2009) and the NASA Post-doctoral Program Fellowship (2010). He is grateful to Dr. S. J. Leib for pointing out the relations in appendix B and would also like to thank the anonymous referee for making helpful suggestions to improve the paper.

Appendix A. Proof of Equation (3.20)

In this appendix we use the invariant theory of tensors expounded in Robertson [60] and Chandrasekhar [61].

A.1. Axi-symmetry in $\Phi_{ij,kl}(\underline{y}, \underline{k}, \omega)$ *under the* $\mathsf{O(3)}$ *symmetry group*

Let $\underline{a}$, $\underline{b}$, $\underline{c}$ & $\underline{d}$ be arbitrary unit vectors. At a particular field point $\underline{y}$, and for a given time delay τ_0, the symmetric quartilinear tensor form $\mathsf{R}(\underline{\eta}, \underline{\lambda}; \underline{a}, \underline{b}, \underline{c}, \underline{d}) = R_{ij,kl}(\underline{\eta}, \underline{\lambda})a_i b_j c_k d_l$ will be a scalar field depending on these six vectors: $\underline{\eta}$, $\underline{\lambda}$, $\underline{a}$, $\underline{b}$, $\underline{c}$ & $\underline{d}$, where $\underline{\lambda}$ is a unit vector that signifies the direction of preference. So R will be an invariant of the full rotation group with respect to $\underline{\lambda}$ when the Cartesian tensor $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$ is axi-symmetric. This means that the quartilinear tensor form R must remain the same under a proper rotation of the co-ordinate axes about $\underline{e}_1$, when $\underline{\lambda}$ is pointing in the axial direction ($\underline{\lambda} = \underline{e}_1$) and to an improper rotation, such as a reflection of the entire configuration, but now through any co-ordinate plane containing $\underline{\lambda}$ and perpendicular to $\underline{\lambda}$ (see figure A.1). Invariance to proper and improper rotations constitutes the $\mathsf{O(3)}$ symmetry group of R, of which $\mathsf{SO(3)}$ is a sub-group (see figure 2.5 on p. 47 in Gilmore [62]).

Since $\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \tau_0)$ has the same pair symmetries in its tensor suffixes as $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$, $\Phi_{ij,kl}$ must be an axi-symmetric tensor if $R_{ij,kl}$ is axi-symmetric since the tensor form $\Phi(\underline{k}, \underline{\lambda}; \underline{a}, \underline{b}, \underline{c}, \underline{d}) = \Phi_{ij,kl}(\underline{k}, \underline{\lambda})a_i b_j c_k d_l$ is related to R by equations (2.7)—(2.10) and (2.16). So Φ must be invariant to the full rotation group if R is invariant. In other words, Φ and R are homomorphic under the $\mathsf{O(3)}$ symmetry group.

Now since the field point $\underline{y}$ and the time delay τ_0 are treated as fixed, the axi-symmetric representation for $\Phi_{ij,kl}$ is obtained by requiring that the quartilinear symmetric (spectral) form Φ to be invariant to the full rotation group involving $\underline{\lambda}$, means that it can be expressed in terms of its integrity basis by the first main theorem of invariant theory (pp. 36-39 in Weyl [64]). The integrity basis (or the *integral rational basis*, after Weyl [64], p. 30) of Φ is the collection of basic

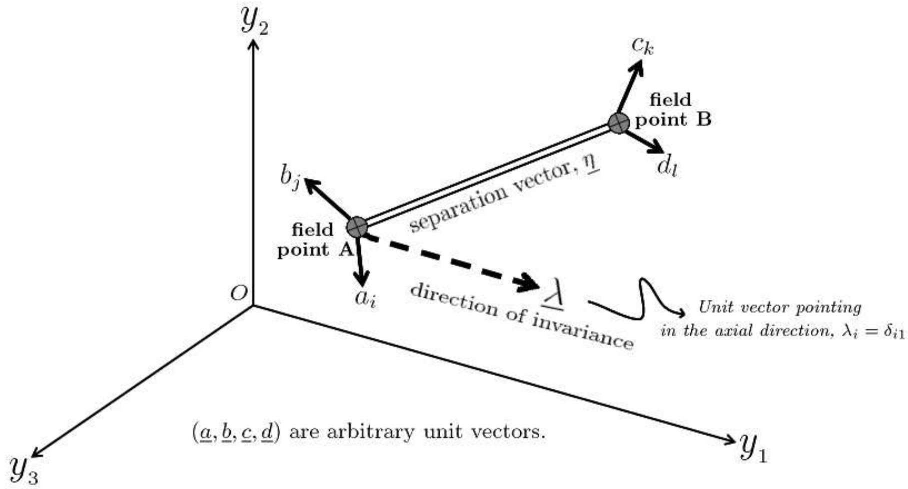

Figure A.1: Vector configuration (c.f. fig. 6 in Monin & Yaglom [29] and fig. 1 in Robertson [60]).

invariants that can be formed from its vector arguments. The first main theorem of invariant theory can be used to prove that Φ can be expressed as a scalar multiple of the basic invariants that can be formed by the various inner tensor products in the list of its vector arguments $(\underline{k}, \underline{\lambda}; \underline{a}, \underline{b}, \underline{c}, \underline{d})$.

Generally, we can write the integrity basis of Φ as,

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \tau_0) = \sum_{perm(i,j,k,l)} B^{(p,m,n)}_{ij,kl}(\underline{k}, \underline{\lambda}; \underline{a}, \underline{b}, \underline{c}, \underline{d}) \alpha_{I(p,m,n)}(\underline{y}, k^2, k_1, \tau_0) \tag{A.1}$$

where the scalar field, $\alpha_{I(p,m,n)}(\underline{y}, k^2, k_1, \tau_0)$, is an arbitrary function of $k^2 = k_i k_i = |\underline{k}|^2$ and $k_i \lambda_i = k_1$ (treating field point $\underline{y}$ and the time delay τ_0 as fixed) for the n-th permutation in (i, j, k, l). The notation $perm(i, j, k, l)$ denotes a permutation over all possible combinations of the tensor suffixes (i, j, k, l). The function $I(p, m, n)$ is a mapping of the 3-dimensional subspace $p = 1, 2, 3, ..., P, m, n = 1, 2, 3$ of positive integers into the space of positive integers where the index p individuates the unique permutation of $perm(i, j, k, l)$ with P being the total number of independent permutations possible. Equation (A.1) is basically the same as equation (8) in Arad *et al* [41] and equation (2.11) in Kurien & Sreenivasan [65], but for a tensor of rank 4. At this point we could use the Clebsh-Gordan method known in Quantum Mechanics to develop the O(3) representation. This is described in detail in Arad *et al* [41]. In this paper, however, we develop this by hand using invariant theory, since the problem is still not too complicated.

The tensor form of $B^{(p,m,n)}_{ij,kl}(\underline{k}, \underline{\lambda})$, with unit vectors $(\underline{a}, \underline{b}, \underline{c}, \underline{d})$, is composed of the following,

$$\left.\begin{array}{l}\delta_{ij}\left[\delta_{kl}, k_k\lambda_l, \lambda_k\lambda_l\right] a_i b_j c_k d_l \\ k_i k_j\left[\delta_{kl}, k_k\lambda_l, k_k k_l\right] a_i b_j c_k d_l \\ \lambda_i\lambda_j\left[k_k k_l, k_k\lambda_l, \lambda_k\lambda_l\right] a_i b_j c_k d_l\end{array}\right\}\quad \text{Basic invariants} \tag{A.2}$$

but in all possible combinations of the tensor suffixes (i, j, k, l). (This can be proved formally by mathematical induction, as Mouron [66] did in his proof of the first main theorem of invariant theory. One could also formulate a proof using Lie group formalism by, for example, extending the work of Rajan [67] on algebraic groups, to the orthogonal groups, and in particular to the $\mathsf{O(3)}$ symmetry group).

So if $\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \tau_0)$ is axi-symmetric it can be expressed as a linear combination of each basic invariant in the integrity basis multiplied by a scalar field. This allows one to write equation (A.1) as:

$$\begin{aligned}\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \tau_0) = & \sum_{perm(i,j,k,l)}\Big[\delta_{ij}\left(\delta_{kl}\alpha_{I(p,1,1)} + k_k\lambda_l\alpha_{I(p,1,2)} + \lambda_k\lambda_l\alpha_{I(p,1,3)}\right) + \\ + & \qquad k_i k_j\left(\delta_{kl}\alpha_{I(p,2,1)} + k_k\lambda_l\alpha_{I(p,2,2)} + k_k k_l\alpha_{I(p,2,3)}\right) + \\ + & \qquad \lambda_i\lambda_j\left(k_k k_l\alpha_{I(p,3,1)} + k_k\lambda_l\alpha_{I(p,3,2)} + \lambda_k\lambda_l\alpha_{I(p,3,3)}\right)\Big]\end{aligned} \tag{A.3}$$

The arguments in Φ reduce to $(\underline{k}; \underline{a}, \underline{b}, \underline{c}, \underline{d})$ when $\underline{\lambda} = \underline{0}$. The integrity basis will, therefore, only involve permutations of the three basic invariants,

$$\left.\begin{array}{l}\delta_{ij}\delta_{kl} a_i b_j c_k d_l \\ k_i k_j\left[\delta_{kl}, k_k k_l\right] a_i b_j c_k d_l\end{array}\right\}\quad \text{Basic invariants when } \underline{\lambda} = \underline{0} \tag{A.4}$$

in all possible combinations of the tensor suffixes (i, j, k, l) when Φ is invariant under the $\mathsf{O(3)}$ symmetry group. This corresponds to the isotropic result given by equation (3.3.7) on p. 42 in Batchelor [25].

Since the spectral tensor has two pair symmetries:

$$\Phi_{ij,kl} = \Phi_{ji,kl} \quad \text{and} \quad \Phi_{ij,kl} = \Phi_{ij,lk}, \tag{A.5}$$

equation (A.3) can be simplified to show the axi-symmetric tensor $\Phi_{ij,kl}(\underline{y}, \underline{k}, \tau_0)$ depends upon 22 independent scalar fields and can be written as:

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \tau_0) =$$

$$\begin{aligned}
&\delta_{ij}\delta_{kl}\alpha_1 + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\alpha_2 + \delta_{ij}(k_k\lambda_l + k_l\lambda_k)\alpha_4 + \delta_{kl}(k_i\lambda_j + k_j\lambda_i)\alpha_6 + \\
&[\delta_{ik}(k_j\lambda_l + k_l\lambda_j) + \delta_{jl}(k_i\lambda_k + k_k\lambda_i) + \delta_{il}(k_j\lambda_k + k_k\lambda_j) + \delta_{jk}(k_i\lambda_l + k_l\lambda_i)]\,\alpha_8 + \\
&\delta_{ij}\lambda_k\lambda_l\alpha_{16} + \delta_{kl}\lambda_i\lambda_j\alpha_{17} + (\delta_{ik}\lambda_j\lambda_l + \delta_{jl}\lambda_i\lambda_k + \delta_{il}\lambda_j\lambda_k + \delta_{jk}\lambda_i\lambda_l)\alpha_{18} + \\
&k_ik_j\delta_{kl}\alpha_{22} + k_kk_l\delta_{ij}\alpha_{23} + (k_ik_k\delta_{jl} + k_jk_l\delta_{ik} + k_ik_l\delta_{jk} + k_jk_k\delta_{il})\alpha_{24} + \\
&k_ik_j(k_k\lambda_l + k_l\lambda_k)\alpha_{28} + k_kk_l(k_i\lambda_j + k_j\lambda_i)\alpha_{30} + \\
&[k_ik_k(k_j\lambda_l + k_l\lambda_j) + k_jk_l(k_i\lambda_k + k_k\lambda_i) + k_ik_l(k_j\lambda_k + k_k\lambda_j) + k_jk_k(k_i\lambda_l + k_l\lambda_i)]\,\alpha_{32} + \\
&k_ik_jk_kk_l\alpha_{40} + \\
&\lambda_i\lambda_jk_kk_l\alpha_{41} + \lambda_k\lambda_lk_ik_j\alpha_{42} + (\lambda_i\lambda_kk_jk_l + \lambda_j\lambda_lk_ik_k + \lambda_i\lambda_lk_jk_k + \lambda_j\lambda_kk_ik_l)\alpha_{43} + \\
&\lambda_i\lambda_j(k_k\lambda_l + k_l\lambda_k)\alpha_{47} + \lambda_k\lambda_l(k_i\lambda_j + k_j\lambda_i)\alpha_{49} + \\
&[\lambda_i\lambda_k(k_j\lambda_l + k_l\lambda_j) + \lambda_j\lambda_l(k_i\lambda_k + k_k\lambda_i) + \lambda_i\lambda_l(k_j\lambda_k + k_k\lambda_j) + \lambda_j\lambda_k(k_i\lambda_l + k_l\lambda_i)]\,\alpha_{51} + \\
&\lambda_i\lambda_j\lambda_k\lambda_l\alpha_{59} +
\end{aligned} \tag{A.6}$$

This result is too complicated to be of engineering use. However, in many physical instances, departures from isotropy are usually weak, such that the dependence of $\Phi_{ij,kl}$ on $\underline{k} = (k_1, \underline{k}_\perp)$ can be approximated to $\underline{k}_\perp = \underline{0}$, i.e.

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \tau_0) \approx \Phi_{ij,kl}(\underline{y}, k_1, \underline{0}, \tau_0) \tag{A.7}$$

Physically, eddies in jet turbulence are usually quite elongated in the stream wise direction, which means that $R_{ij,kl}(\underline{y}, \underline{\eta}, \tau_0)$ will be a rapidly varying function of $\underline{\eta}_\perp$ normalized by the stream wise correlation length. Rapid variation in the real space field variable $\underline{\eta}_\perp$, implies a slow variation in spectral space field variable $\underline{k}_\perp$. The limiting condition, therefore, corresponds to the "infinitely long" stream wise eddy with zero thickness in spectral space. In which case, $|\Phi_{ij,kl}|$, has no variation with $\underline{k}_\perp$, thus leaving the result that, $\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \omega) \approx \Phi_{ij,kl}(\underline{y}, k_1, \underline{0}, \omega)$.

The integrity basis (equation A.3) can then be simplified to

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_\perp, \tau_0) =$$

$$\begin{aligned}
&= \sum_{\text{perm}} \delta_{ij}\delta_{kl}\alpha_{I(p,1,1)} + \sum_{\text{perm}} \delta_{ij}\delta_{k1}\delta_{l1}\left[k\alpha_{I(p,1,2)} + \alpha_{I(p,1,3)}\right] \\
&+ \sum_{\text{perm}} \delta_{i1}\delta_{j1}\delta_{kl}k^2\alpha_{I(p,2,1)} \\
&+ \sum_{\text{perm}} \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\left[k^3\alpha_{I(p,2,2)} + k^4\alpha_{I(p,2,3)} + k^2\alpha_{I(p,3,1)} + k\alpha_{I(p,3,2)} + \alpha_{I(p,3,3)}\right]
\end{aligned} \tag{A.8}$$

If we write this out in full, we can see there are only six basic invariants in the reduced integrity basis.

$$
\begin{aligned}
\Phi_{ij,kl}(\underline{y},k_1,\underline{0},\tau_0) &= \delta_{ij}\delta_{kl}A_1(\underline{y},k^2,k_1,\tau_0)+[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}]\,A_2(\underline{y},k^2,k_1,\tau_0)\\
&+ \delta_{i1}\delta_{j1}\delta_{kl}A_4(\underline{y},k^2,k_1,\tau_0)+\delta_{k1}\delta_{l1}\delta_{ij}A_5(\underline{y},k^2,k_1,\tau_0)\\
&+ [\delta_{i1}\delta_{k1}\delta_{jl}+\delta_{j1}\delta_{l1}\delta_{ik}+\delta_{i1}\delta_{l1}\delta_{jk}+\delta_{j1}\delta_{k1}\delta_{il}]\,A_6(\underline{y},k^2,k_1,\tau_0)\\
&+ \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}A_{10}(\underline{y},k,k_1,\tau_0). \qquad \text{(A.9)}
\end{aligned}
$$

This result is similar to equation (25a) in Gaite [42] and to equation (10) in Gaite [43]. However, both their results were in the homogeneous rotating turbulence context, and moreover, the Gaite [43] result was for a fourth rank tensor that was totally symmetric in its suffixes; whereas, in this paper, the spectral tensor has only pair symmetries and the starting point is inhomogeneous turbulence.

The scalar fields (the $A's$) are related to the $\alpha's$ by:

$$
\begin{aligned}
A_1(\underline{y},k^2,k_1,\tau_0) &= \alpha_1(\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10a)}\\
A_2(\underline{y},k^2,k_1,\tau_0) &= \alpha_2(\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10b)}\\
A_4(\underline{y},k^2,k_1,\tau_0) &= \left[\alpha_{17}+2k\alpha_6+k^2\alpha_{22}\right](\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10c)}\\
A_5(\underline{y},k^2,k_1,\tau_0) &= \left[\alpha_{16}+2k\alpha_4+k^2\alpha_{23}\right](\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10d)}\\
A_6(\underline{y},k^2,k_1,\tau_0) &= \left[\alpha_{18}+2k\alpha_8+k^2\alpha_{24}\right](\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10e)}\\
A_{10}(\underline{y},k^2,k_1,\tau_0) &= \Big[\alpha_{59}+k\left[2(\alpha_{47}+\alpha_{49})+8\alpha_{51}\right]+k^2\left[(\alpha_{41}+\alpha_{42})+4\alpha_{43}\right]\\
&+ k^3\left[2(\alpha_{28}+\alpha_{30})+8\alpha_{32}\right]+k^4\alpha_{40}\Big](\underline{y},k^2,k_1,\tau_0) \qquad \text{(A.10f)}
\end{aligned}
$$

The $A's$ in equation (A.9) are not really important because we can re-write them in terms of components of $\Phi_{ij,kl}$ (substituting $(i,j,k,l)=(1,2,3)$ into equation 4.17). The explicit form of $\Phi_{ij,kl}(\underline{y},k_1,\underline{0},\tau_0)$ then follows, and shows that it depends upon the six components: $\Phi_{11,11}$, $\Phi_{22,22}$, $\Phi_{11,22}$, $\Phi_{22,11}$, $\Phi_{12,12}$ & $\Phi_{23,23}$; i.e.

$$
\begin{aligned}
\Phi_{ij,kl}(\underline{y},k_1,\underline{0},\tau_0) &= [\delta_{ij}\delta_{kl}-\delta_{i1}\delta_{j1}\delta_{kl}-\delta_{k1}\delta_{l1}\delta_{ij}+\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\Phi_{22,22}(\underline{y},k_1,\underline{0},\tau_0)\\
&+ \big[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ij}\delta_{kl}+2\delta_{i1}\delta_{j1}\delta_{kl}+2\delta_{k1}\delta_{l1}\delta_{ij}\\
&- \delta_{i1}\delta_{l1}\delta_{jk}-\delta_{j1}\delta_{l1}\delta_{ik}-\delta_{j1}\delta_{k1}\delta_{il}-\delta_{i1}\delta_{k1}\delta_{jl}\big]\,\Phi_{23,23}(\underline{y},k_1,\underline{0},\tau_0)\\
&+ \big[\delta_{i1}\delta_{l1}\delta_{jk}+\delta_{j1}\delta_{l1}\delta_{ik}+\delta_{j1}\delta_{k1}\delta_{il}\\
&+ \delta_{i1}\delta_{k1}\delta_{jl}-4\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\big]\,\Phi_{12,12}(\underline{y},k_1,\underline{0},\tau_0)\\
&+ [\delta_{i1}\delta_{j1}\delta_{kl}-\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\Phi_{11,22}(\underline{y},k_1,\underline{0},\tau_0)\\
&+ [\delta_{k1}\delta_{l1}\delta_{ij}-\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\Phi_{22,11}(\underline{y},k_1,\underline{0},\tau_0)\\
&+ \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\Phi_{11,11}(\underline{y},k_1,\underline{0},\tau_0) \qquad \text{(A.11)}
\end{aligned}
$$

A.2. Projecting the symmetry group down to the SO(3) *sub-group*

Under SO(3), the tensor form Φ must remain invariant to proper rotations only (i.e. $\det L_{ij} = +1$ where L_{ij} is the transformation matrix). However, invariance will continue to hold for certain types of improper rotations because $\Phi_{ij,kl}(\underline{y}, \underline{k}, \tau_0)$ is a tensor of even parity. For example, the tensor form Φ will continue to remain invariant to a complete inversion ($L_{ij} = -\delta_{ij}$), since the L-matrix occurs an even number of times when $\Phi_{ij,kl}$ is transformed (see p. 143 in Butcher & Cotter [69]).

On the other hand, an improper rotation where $\underline{e}_3$ is inverted, i.e. $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, would not maintain invariance of Φ because an odd grouping of the suffixes $(i, j, k, l) = 3$, for example, would result in a change of sign (a simple consequence of the Cartan-Dieudonne theorem: pp. 10-12 in Cartan [70] and p. 37 in Stillwell [71]).

To allow for the change in sign under the SO(3) sub-group, the algebraic expression of $\Phi_{ij,kl}(\underline{y}, \underline{k}, \tau_0)$ must include permutations of the unit alternating tensor ϵ_{ijk} with vectors $(\underline{k}, \underline{\lambda})$ in all possible combinations of suffixes (i, j, k, l) in the integrity basis (equation A.3) . However, since ϵ_{ijk} is anti-symmetric in any two of its suffixes (with one held fixed or summed), the only extra terms would involve: $\epsilon_{ijp}\epsilon_{klq}\,(k_p k_q + \lambda_p \lambda_q)$ and Grassmann products of ϵ_{ijk} and $(\underline{k}, \underline{\lambda})$ (i.e. skew-symmetric products; see p. 92 in Bishop & Goldberg [72]). Therefore, we would have the new integrity basis:

$$\Phi_{ij,kl}(\underline{y}, k_1, \underline{k}_{\perp}, \tau_0)$$

$$= \text{Equation (A.3)}$$

$$+ \sum_{\text{perm}} \epsilon_{ijp} k_p \epsilon_{klq} k_q \alpha_{I(p,4,1)} + \sum_{\text{perm}} \epsilon_{ijp} \lambda_p \epsilon_{klq} \lambda_q \alpha_{I(p,4,2)}$$

$$+ \sum_{\text{perm}} \text{Grassmann algebra} + ... \qquad \text{(A.12)}$$

But these additions to equation (A.3) are all zero when the weak axi-symmetry condition (equation A.7) is applied. Hence, the decomposition of the spectral tensor given by equation (A.11) remains the same if the symmetry group used to derive it is, O(3), or the SO(3) sub-group.

A.3. Axi-symmetry as required by equation (3.19)

The GSM approximation requires $\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_{\perp}^2, \omega)$ to be an axi-symmetric tensor. But if $\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_2^2 + k_3^2, \tau_0)$ is assumed to be an axi-symmetric tensor, no further basic invariants can be formed by the various inner tensor products in the reduced list of vector arguments $(k\underline{e}_1, \underline{\lambda}; \underline{a}, \underline{b}, \underline{c}, \underline{d})$, where $\underline{k} = k\underline{e}_1$ and $k^2 = k_1^2 + k_2^2 + k_3^2$ and $k_2^2 + k_3^2$ since $k_2^2 + k_3^2$ is a scalar. The integrity basis of $\Phi_{ij,kl}(\underline{y}, k_1, k_2^2 + k_3^2, \tau_0)$

is, therefore, kinematically identical to $\Phi_{ij,kl}(\underline{y}, k_1, \underline{0}, \tau_0)$. Therefore, the algebraic expression for the GSM is also given by equation (A.11), and is written as follows:

$$
\begin{aligned}
\overline{\Phi}_{ij,kl}(\underline{y}, k_1, k_2^2 + k_3^2, \omega) &= [\delta_{ij}\delta_{kl} - \delta_{i1}\delta_{j1}\delta_{kl} - \delta_{k1}\delta_{l1}\delta_{ij} + \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{22,22}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
&+ \big[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} + 2\delta_{i1}\delta_{j1}\delta_{kl} + 2\delta_{k1}\delta_{l1}\delta_{ij} \\
&- \delta_{i1}\delta_{l1}\delta_{jk} - \delta_{j1}\delta_{l1}\delta_{ik} - \delta_{j1}\delta_{k1}\delta_{il} - \delta_{i1}\delta_{k1}\delta_{jl}\big]\overline{\Phi}_{23,23}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
&+ \big[\delta_{i1}\delta_{l1}\delta_{jk} + \delta_{j1}\delta_{l1}\delta_{ik} + \delta_{j1}\delta_{k1}\delta_{il} \\
&+ \delta_{i1}\delta_{k1}\delta_{jl} - 4\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\big]\overline{\Phi}_{12,12}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
&+ [\delta_{i1}\delta_{j1}\delta_{kl} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{11,22}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
&+ [\delta_{k1}\delta_{l1}\delta_{ij} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}]\,\overline{\Phi}_{22,11}(\underline{y}, k_1, k_\perp^2, \tau_0) \\
&+ \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\overline{\Phi}_{11,11}(\underline{y}, k_1, k_\perp^2, \tau_0)
\end{aligned}
\tag{A.13}
$$

Appendix B. Spectral tensor components

Since $\Phi_{ij,kl}$ is related to $\mathcal{H}_{ij,kl}$ through the Fourier transform, equation (2.16), we can write out the components of $\Phi_{ij,kl}$ in the GSM (equation 3.20) in terms of $\mathcal{H}_{ij,kl}$ using equations (2.9) & (2.10). Hence:

$$
\begin{aligned}
\overline{\mathcal{H}}_{11,11} &= \left[1 - (\gamma - 1) + \left(\frac{\gamma - 1}{2}\right)^2\right]\overline{H}_{11,11} \\
&+ \left[2\left(\frac{\gamma - 1}{2}\right)^2 - (\gamma - 1)\right]\left(\overline{H}_{11,22} + \overline{H}_{11,33}\right) + \left(\frac{\gamma - 1}{2}\right)^2\left(\overline{H}_{22,22} + \overline{H}_{33,33}\right) \\
&+ 2\left(\frac{\gamma - 1}{2}\right)^2\overline{H}_{22,33}
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\overline{\mathcal{H}}_{22,22} &= \left(\frac{\gamma - 1}{2}\right)^2\overline{H}_{11,11} + \left[2\left(\frac{\gamma - 1}{2}\right)^2 - (\gamma - 1)\right]\overline{H}_{11,22} \\
&+ 2\left(\frac{\gamma - 1}{2}\right)^2\overline{H}_{11,33} + \left[1 - (\gamma - 1) + \left(\frac{\gamma - 1}{2}\right)^2\right]\overline{H}_{22,22} \\
&+ \left[2\left(\frac{\gamma - 1}{2}\right)^2 - (\gamma - 1)\right]\overline{H}_{22,33} + \left(\frac{\gamma - 1}{2}\right)^2\overline{H}_{33,33}
\end{aligned}
\tag{B.2}
$$

$$\begin{aligned}
\overline{\mathcal{H}}_{11,22} &= \left[\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\overline{H}_{11,11} \\
&+ \left[1-(\gamma-1)+2\left(\frac{\gamma-1}{2}\right)^2\right]\overline{H}_{11,22} \\
&+ \left[2\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\overline{H}_{11,33} + \left[\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\overline{H}_{22,22} \\
&+ \left[2\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\overline{H}_{22,33} + \left(\frac{\gamma-1}{2}\right)^2\overline{H}_{33,33}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\overline{\mathcal{H}}_{22,33} &= \left(\frac{\gamma-1}{2}\right)^2\overline{H}_{11,11} + \left[2\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\left(\overline{H}_{11,22}+\overline{H}_{11,33}\right) \\
&+ \left[2\left(\frac{\gamma-1}{2}\right)^2 - \frac{\gamma-1}{2}\right]\left(\overline{H}_{22,22}+\overline{H}_{33,33}\right) \\
&+ \left[1-(\gamma-1)+2\left(\frac{\gamma-1}{2}\right)^2\right]\overline{H}_{22,33}
\end{aligned} \tag{B.4}$$

$$\overline{\mathcal{H}}_{12,12} = \overline{H}_{12,12} \tag{B.5}$$

$$\overline{\mathcal{H}}_{23,23} = \overline{H}_{23,23} \tag{B.6}$$

G & L showed that components of the spectral tensor components, defined above in terms of $\mathcal{H}_{ij,kl}$, have positive real parts.

References

[1] M. E. Goldstein, A generalized acoustic analogy, J. Fluid Mech. 488 (2003), 315–333.

[2] M. J. Lighthill, On sound generated aerodynamically: I. General theory, Proc. Roy. Soc. A, 222 (1952), 564–587.

[3] P. A. Lush, Measurements of sub-sonic jet noise and comparison with theory, J. Fluid Mech. 46 (1971), 477–500.

[4] K. K. Ahuja, Correlation and prediction of jet noise, J. Sound Vib. 29 (1973), 155–168.

[5] C. K. W. Tam, M. Golebioski and J. M. Seiner, Two components of turbulent mixing noise from supersonic jets, AIAA Paper, 96-1716, (1996).

[6] C. K. W. Tam, Influence of nozzle geometry on noise of high speed jets, AIAA J. 36 (1998), 1396–1400.

[7] C. K. W. Tam and K. M. B. Q. Zaman, Subsonic jet noise from axi-symmetric and tabbed nozzles, AIAA J. 38 (2000), 592–599.

[8] M. D. Dahl and D. Papamoschou, Analytical predictions and measurements of the noise radiated from supersonic coaxial jets, AIAA J. 38 (2000), 584–591.

[9] K. Viswanathan, Analysis of the two similarity components of turbulent mixing noise, AIAA J. 40 (2002), 1735–1744.

[10] K. Viswanathan, Aeroacoustics of hot jets, J. Fluid Mech. 516 (2004), 39–82.

[11] C. K. W. Tam, K. Viswanathan, K. Ahuja and J. Panda, The sources of jet noise: experimental evidence, J. Fluid Mech. 615 (2008), 253–292.

[12] P. J. Morris, A note on noise generation by large scale turbulent structures in subsonic and supersonic jets, Int. J. Aero-ac. 8 (2009), 301–316.

[13] ERCOFTAC, Symposium on sound source mechanisms in turbulent shear flows. University of Poiters, France, See http://ercoftac-ssmtsf.ceat.univ-poitiers.fr/. (2008).

[14] H. S. Ribner, Quadrupole correlations governing the pattern of jet noise, J. Fluid Mech. 38 (1969), 1–8.

[15] M. E. Goldstein and B. Rosenbaum, Effect of anisotropic turbulence on aerodynamic noise, J. Sound Vib. 54 (1973), 630–645.

[16] K. A. Bishop, J. E. Ffowc-Williams and W. Smith, On the sources of the unsuppressed high-speed jet, J. Fluid Mech. 50 (1971), 21–31.

[17] M. Harper-Bourne, Jet Noise Turbulence Measurements, AIAA Paper, 2003-3214, (2003).

[18] S. A. Karabasov, M. Z. Afsar, T. P. Hynes, A. P. Dowling, W. A. McMullen, C. D. Pokora, G. J. Page and J. J. McGuirk, Using large eddy simulation within an acoustic analogy approach for jet noise modeling, AIAA Paper, 2008-2985, (2008).

[19] W. A. McMullen, C. D. Pokora, G. J. Page and J. J. McGuirk, Large eddy simulation of a high Reynolds number subsonic turbulent jet for acoustic source capture, AIAA Paper, 2974-768, (2008).

[20] C. D. Pokora and J. J. McGuirk, Spatio-temporal turbulence correlation using a high speed PIV in an axi-symmetric jet, AIAA Paper, 3028-687, (2008).

[21] M. Z. Afsar, Solution of the parallel shear layer Green's function using conservation equations, Int. J. Aero-ac. 8 (2009), 585–602.

[22] M. E. Goldstein and S. J. Leib, The aero-acoustics of slowly diverging supersonic jets, J. Fluid Mech. 600 (2008), 291–337.

[23] L. Isserlis, On a formula for the product moment coefficient of any order of a normal frequency distribution in any number of variables, Biometrika 12 (1918), 134–139.

[24] U. Frisch, Turbulence: The Legacy of A. N. Kolmogorov, Cambridge University Press (1996).

[25] G. K. Batchelor, The Theory of Homogeneous Turbulence, Cambridge University Press (1953).

[26] F. N. Frenkiel and P. S. Klebanoff, Higher-order correlations in a turbulent field, Physics of Fluids 10 (1967), 507–520.

[27] C. W. Van Atta and W. Y. Chen, Correlation measurements in grid turbulence using digital harmonic analysis, J. Fluid Mech. 34 (1968), 497–515.

[28] C. W. Van Atta and T. T. Yeh, Some measurements of multi-point time correlations in grid turbulence, J. Fluid Mech. 41 (1970), 169–178.

[29] A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence. Vol. II, Dover Publications (2007).

[30] P. J. Morris and K. M. B. Q. Zaman, Velocity measurements in jets with application to noise source modeling, AIAA Paper, 2009-17, (2009).

[31] M. Lesiuer, Turbulence in Fluids, Kluwer Academic Publishers (1990).

[32] W. D. McComb, The Physics of Fluid Turbulence, Oxford University Press (2003).

[33] S. A. Orszag, Analytical theories of turbulence, J. Fluid Mech. 41 (1970), 363–386.

[34] I. Proudman and W. H. Reid, On the decay of a normally distributed and homogeneous turbulent velocity field, Proc. Roy. Soc. A, 247 (1954), 163–189.

[35] R. H. Kraichnan, Relation of the fourth order to the second order moments in stationary isotropic turbulence, Phys. Rev. 107 (1957), 1485–1490.

[36] Y. Ogura, A consequence of the zero-fourth-cummalant approximation in the decay of isotropic turbulence, J. Fluid Mech. 16 (1963), 38–40.

[37] E. E. O' Brien and G. C. Francis, A consequence of the zero fourth cummalant approximation, J. Fluid Mech. 13 (1963), 369–382.

[38] A. P. Mirabel, An example of the application of the Millionshchikov hypothesis to the problem of isotropic turbulence, Izv. Akad. Nauk, SSSR, Ser. Mech. Zhidk. i Gaza, (1969), 171–175 (cited in Monin & Yaglom [29]).

[39] I. Arad, B. Dhruva, S. Kurien, V. S. L'vov, I. Procaccia and K. R. Sreenivasan, Extraction of the anisotropic contribution in turbulent flows, Phys. Rev. Lett. 81 (1998), 5330.

[40] S. Kurien and K. R. Sreenivasan, Anisotropic scaling contributions to high-order structure functions in high Reynolds number turbulence, Phys. Rev. E. 62 (2000), 2206.

[41] I. Arad, V. S. L'vov and I. Procaccia, Correlation functions in isotropic and anisotropic turbulence: The role of the symmetry group, Phys. Rev. E. 59 (1999), 6753.

[42] J. Gaite, D. Hochberg and C. Molina-Paris, Large-scale features of rotating forced turbulence, Phys. Rev. E. 67 (2003), 0206304.

[43] J. Gaite, Anisotropy in homogeneous rotating turbulence, Phys. Rev. E. 68 (2005), 1063.

[44] P. Saguat and C. Cambon, Homogeneous Turbulence Dynamics, Cambridge University Press, UK, (2008).

[45] P. D. Ditlevsen, Turbulence and Shell Models, Cambridge University Press, UK, (2011).

[46] M. Z. Afsar, M. E. Goldstein and A. Fagan, Enthalpy flux – momentum flux coupling in the acoustic spectrum of heated jets. Accepted for publication in AIAAJ. (2011).

[47] G. M. Lilley, The sound radiated by isotropic turbulence with application to the theory of jet noise, J. Sound Vib. 190, (1996), 463–476.

[48] C. L. Morfey, V. M. Szewczyk and B. J. Tester, New Scaling laws for hot and cold jet mixing noise based on a geometric acoustics model, J. Sound Vib. 61, (1978), 255–292.

[49] P. J. Morris and K. M. B. Q. Zaman, Two component velocity correlations in jets and noise source modeling, AIAA Paper 2010-3781, (2010).

[50] A. Khavaran, Private communication,(2009).

[51] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw Hill, USA, (1953).

[52] F. Bowman, Introduction to Bessel Functions, Dover Publications, USA, (1958).

[53] A. M. Oboukhov, On some general characteristics of the equations of the dynamics of the atmosphere, Izv. Akad. Nauk SSSR. Fiz. Atmos. Okeana, 7, (1971), 695–704.

[54] E. B. Gledzer, System of hydrodynamic type admitting two quadratic integrals of motion, Sov. Phys. Dokl. SSSR. 18, (1973), 216–7.

[55] M. Yamada and K. Okhitani, Lyapunov spectrum of a model of two-dimensional turbulence, Phys. Rev. Lett. 60, (1988a), 983–6.

[56] M. Yamada and K. Okhitani, The inertial sub-range and non-positive Lyapunov exponents in fully-developed turbulence, Progr. Theo. Phys. 79, (1988b), 1265–68.

[57] V. S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia and D. Vandembroueq, Improved shell model of turbulence, Phys. Rev. E, 58 (1998), 1811-22.

[58] M. Z. Afsar, J. J. McGuirk and C. D. Pokora, Statistical axi-symmetry of the two-point time-delayed Reynolds stress auto-covariance tensor for jet noise modeling, 19th Polish Fluid Mechanics Conference, Poznan, Poland.(2010).

[59] M. Z. Afsar, Asymptotic properties of the overall sound pressure level of subsonic jet flows using isotropy as a paradigm, J. Fluid Mech. 664 (2010), 510–539.

[60] H. P. Robertson, Theory of isotropic turbulence, Proc. Cam. Phil. Soc. 36 (1939), 209–223.

[61] S. Chandrasekhar, The theory of axisymmetric turbulence, Proc. Roy. Soc. A, 242 (1950), 557–577.

[62] R. Gilmore, Lie Groups, Lie Algebras and some of their Applications, Dover, New York, USA, (2002).

[63] M. Hamermesh, Group Theory and its Application to Physical Problems, Dover, New York, USA, (1962).

[64] H. Weyl, The Classical Groups: their Invariants and Representations, Princeton University Press, New Jersey, USA, (1997).

[65] S. Kurien and K. R. Sreenivasan, Measures of anisotropy and the universal properties of turbulence, Chapter in New trends in turbulence. NATO Advanced Study Institute. (2001), 55–111.

[66] C. G. Mouron, The first fundamental theorem of invariant theory for the unimodular and orthogonal groups, MSc Thesis. Texas Tech University, USA, (1996).

[67] C. S. Rajan, Unique decomposition of tensor products of irreducible representations of simple algebraic groups, Ann. Math. 160(2) (2004), 683–704.

[68] M. J. Lighthill, Waves in Fluids, Cambridge University Press, UK, (1978).

[69] P. N. Butcher and D. Cotter, The Elements of Non-linear Optics, Cambridge University Press, UK, (1990).

[70] E. Cartan, The Theory of Spinors, Dover, New York, USA, (1981).

[71] J. Stillwell, Naive Lie Theory, Springer Publications, New York, USA, (2008).

[72] R. L. Bishop and S. I. Goldberg, Tensor Analysis on Manifolds, Dover publications, USA, (1980).