Proceedings Paper:
Bellas Acosta, I. and Stell, J.G. orcid.org/0000-0001-9644-1908 (2024) Monotone Ω-Sup-Fuzzy Relations: Converse and Complementation. In: Fahrenberg, U., Fussner, W. and Glück, R., (eds.) Relational and Algebraic Methods in Computer Science. 21st International Conference, RAMiCS 2024, 19-22 Aug 2024, Prague, Czechia. Lecture Notes in Computer Science, 14787 . Springer Nature , pp. 65-82. ISBN 978-3-031-68279-7

https://doi.org/10.1007/978-3-031-68279-7_5

Monotone Ω-Sup-Fuzzy Relations: Converse and Complementation

Ignacio Bellas Acosta and John G. Stell[0000−0001−9644−1908]

University of Leeds, Leeds LS2 9JT, U.K.
{sciba,j.g.stell}@leeds.ac.uk

Abstract. L-fuzzy relations on a set X are functions from $X \times X$ to the lattice L and act on the L-fuzzy subsets of X. When L is the lattice of sup-preserving endomaps on a complete lattice Ω, the relations act also on the Ω-fuzzy subsets of X. We call these relations equipped with this action, Ω-sup-fuzzy relations. When X is a preorder, monotone relations of this form act on the lattice of monotone functions from X to Ω. The motivation comes from mathematical morphology in image processing. Grey-scale images are modelled as functions on sets of pixels with Ω as the set of grey levels. More generally, graphs and hypergraphs labelled by grey levels can be handled. Enriching the lattice of Ω-sup-fuzzy relations with a multiplication operation provides a unital quantale that acts on the lattice of grey-scale images via the morphological operations of dilation and erosion. We study the quantale of Ω-sup-fuzzy relations, with particular attention to the concepts of converse and complementation for these relations.

Keywords: Fuzzy relations · Quantales · Sup-preserving endomorphisms.

1 Context and Background

1.1 Fuzzy Relations

The notion of a fuzzy relation on a set X as a function from $X \times X$ to a suitable lattice L, goes back to Goguen [9, p161]. Such relations are there just called 'L-relations' although the terminology 'L-fuzzy relations' is widely used now, for example in [25]. The category-theoretic properties of L-fuzzy relations, where L is a complete Heyting algebra are studied by Winter [25] in the context of Goguen categories.

The converse, which we will denote by $\breve{R}$, of an L-fuzzy relation R is defined [25, p43] by $\breve{R}(x, y) = R(y, x)$. This is a widely used definition, but we contend that converses of L-fuzzy relations are not as straightforward as this might suggest. We argue below, using motivation from mathematical morphology, that fuzzy relations acting on L-fuzzy subsets for a complete lattice L, 'should' correspond to the sup-preserving mappings on the lattice of all L-fuzzy subsets. This means we need to deal with relations R where $R(x, y)$ is no longer an element of L, but a sup-preserving mapping from L to itself. We also argue below, again with motivation from mathematical morphology, that there is a need to account not only for relations which act on L-fuzzy subsets, but for relations which act on monotone functions from a preorder to L. The L-fuzzy subsets are the special case of this where the preorder is discrete, i.e. $x \leq y$ iff $x = y$.

Recent work by Santocanale [19] has investigated involutive quantaloids, and in particular the quantale $\mathcal{Q}(L)$ of sup-preserving (also called join-continuous) endomorphisms of L. This work shows that when L is completely distributive, there is essentially a unique involution (of some special form) on $\mathcal{Q}(L)$. The connection between the involution identified in [19] and the work presented here is, to use an informal analogy rather than to claim a completely general correspondence, that the involution corresponds to a converse-complement operation on relations. That is, an operation consisting of taking converse and then taking the complement of that converse relation. This is certainly the case for Boolean-valued relations on a preorder $\mathbb{S}$, that is monotone functions from $\mathbb{S}^{\mathrm{op}} \times \mathbb{S}$ to the 2-element Boolean algebra. We return to the connection between converse-complement and the involution defined by [19] in Section 1.4 after establishing some background context on the converse operation.

1.2 The need for the converse

The operation of converse on discrete Boolean relations, that is, on subsets of sets of the form $X \times X$ is clearly important as evidenced by its presence as one of the operations in a relation algebra. The converse is a key example in several more abstract settings including dagger categories and allegories [8].

The relational converse appears in the semantics of modal logic. Let $R \subseteq X \times Y$ be a relation between sets and use $\_ \oplus R$ to denote the direct image operation defined on 2^X by $A \mapsto \{y \in Y \mid \exists x \,;\, R(x, y)\}$. If R is the accessibility relation in a Kripke structure and we write $[\![\varphi]\!]$ for the set of states where φ holds in a model, then $[\![\Diamond\varphi]\!] = [\![\varphi]\!] \oplus \breve{R}$. The box is given by $[\![\Box\varphi]\!] = R \ominus [\![\varphi]\!]$ where $R \ominus \_$ is the right adjoint to $\_ \oplus R$. In a tense logic, the two further modalities will need $\breve{R}$ for the box and R itself for the diamond.

In mathematical morphology in image processing the operations $\_ \oplus R$ and $R \ominus \_$ are known respectively as the dilation and the erosion. They are used to transform black and white images by modelling all the pixels of one colour as a subset of all the pixels. The underlying reason for this relationship also surfaces in the logical equivalence in classical modal logic between $\Diamond\varphi$ and $\neg\Box\neg\varphi$, and also more generally in the quantale-theoretic setting where Mulvey and Pelletier define a converse operation (they say 'inverse') on the lattice of sup-preserving endomorphisms of an orthocomplemented lattice [15].

1.3 Converse for relations in an extended sense

The converse for discrete Boolean relations, where we have discrete sets X and Y and relations mapping $X \times Y$ to 2, is very well understood. The converse interacts with the actions of such relation on subsets $X \to 2$ and $Y \to 2$ as seen above. We consider next a generalization of this in two ways at once. One way is the replacement of discrete sets X and Y by preorders. The other way is the replacement of the lattice of two truth values, 2, by an arbitrary complete lattice.

One of these ways on its own, that is Boolean-valued relations on preorders, has already been studied [20]. Such relations on a preorder, $\mathbb{S}$ will be monotone functions from $\mathbb{S}^{op} \times \mathbb{S}$ to 2. The basic converse operation on such a relation R will provide $\breve{R} : \mathbb{S} \times \mathbb{S}^{op} \to 2$ where $\breve{R}(s,t) = R(t,s)$, so clearly R and $\breve{R}$ have different types. One motivation for these monotone Boolean relations is that they allow mathematical morphology on graphs [5] to be treated in a generalization of the algebraic setting for morphology on discrete sets of pixels. This arises by taking the preorder to be the incidence relation between edges and nodes. This automatically provides a generalization to hypergraphs. Monotone Boolean relations are already well known in the semantics of intuitionistic modal logic [17], but the introduction in [20] of a left converse operation, where $\backsim R : \mathbb{S}^{op} \times \mathbb{S} \to 2$ as opposed to $\breve{R} : \mathbb{S} \times \mathbb{S}^{op} \to 2$, allowed the development of a novel bi-intuitionistic modal logic [24] where the left converse plays an essential role in the semantics.

Generalizing from discrete sets to preorders is one way to generalize relations. The other is to generalize the lattice of truth values. For a framework that describes mathematical morphology on fuzzy sets, or more generally fuzzy graphs, the relations need to correspond to sup-preserving mappings on the lattice of fuzzy graphs. When Ω is the lattice of truth-values, the relations which correspond in this way will be monotone relations $\mathbb{S}^{op} \times \mathbb{S} \to [\Omega, \Omega]_\vee$, where $[\Omega, \Omega]_\vee$ is the lattice of sup-preserving endomorphisms of Ω. Another motivation for such a generalization is to ask whether a multi-valued modal logic can be obtained in the style of [24].

1.4 Involutive Quantaloids

For a Boolean-valued relation, R, on a set X, the converse $\breve{R}$ and the complement $\overline{R}$ satisfy $\overline{\breve{R}} = \breve{\overline{R}}$. This combination of the two operations is called the converse-complement. In the case of Boolean-valued relations on a preorder $\mathbb{S}$, that is monotone functions from $\mathbb{S}^{\mathrm{op}} \times \mathbb{S}$ to 2, the converse-complement of the identity monotone relation is a cyclic dualizing element providing a Girard quantale structure. This follows from [23, Prop 21, p449]. In [19] an order-reversing involution is constructed generalizing this converse-complement for $\mathcal{Q}(L)$ the quantale of sup-preserving endomorphisms of any completely distributive lattice L. This fits the above setting by taking L to be the lattice of down sets of $\mathbb{S}$, or in a more specific case the lattice of subgraphs of a graph.

In the paper below we study relations of the form $R : \mathbb{S}^{op} \times \mathbb{T} \to [\Omega, \Psi]_\vee$. To work with the motivating situations of mathematical morphology and modal logic we ultimately need a converse of R that has type that is $\mathbb{T}^{op} \times \mathbb{S} \to [\Psi, \Omega]_\vee$. In other words, between the same structures but in the opposite direction. By considering the example of monotone Boolean relations, it is clear that such a converse is not the converse part of a converse-complement. This would have type $\mathbb{T} \times \mathbb{S}^{op} \to [\Psi^{op}, \Omega^{op}]_{\vee}$, in other words: between the opposite structures and in the opposite direction. A complete analysis of the relationship between our converse (left converse) in Definition 8 and the involutive structure in [19] remains to be carried out.

1.5 Structure of the paper

In Section 2 we introduce the basic definitions and notation that will be used throughout the paper. In Section 3 we introduce Ω-sup-fuzzy relations. We observe that these relations carry a quantale structure, acting on the collection of Ω-fuzzy suborders in a Goguen style generating two isomorphisms. The main observation of this section is that the collection of sup-preserving and inf-preserving endomorphisms on the lattice of Ω-fuzzy suborders is isomorphic to the collection of Ω-sup-fuzzy relations. We conclude this section showing the significance of these observations in the case of mathematical morphology.

In Section 4 we consider the general setting where sup-fuzzy relations are defined over different preorders and lattices. Defining sup-fuzzy relations in terms of quantaloids provides a more holistic perspective while still preserving the properties obtained in Section 3. Based on the techniques defined by Mulvey et al. [15] we propose a formalisation of the complement operation and the converse relation in the context of sup-fuzzy relations.

Finally, we provide a summary in Section 5 with the main results obtained throughout this paper as well as some open questions that arise from the contributions obtained.

2 Preliminaries

This section provides some of the basic notation and terminology used throughout the paper.

We use $\mathbb{S}$ to denote a set with a preorder $\leq_{\mathbb{S}}$. The opposite preorder is denoted by $\mathbb{S}^{op}$. The collection of all monotone functions from $\mathbb{S}$ to $\mathbb{T}$ will be denoted $[\mathbb{S}, \mathbb{T}]$.

For any two monotone functions $f : \mathbb{S} \to \mathbb{T}$ and $g : \mathbb{T} \to \mathbb{S}$ such that $f(s) \leq t$ if and only if $s \leq g(t)$ for every $s \in \mathbb{S}$ and $t \in \mathbb{T}$, we say that f is a left adjoint to g. We use the notation $f \dashv g$ meaning that f is the left adjoint of g.

A lattice $(\Omega, \vee, \wedge)$ is said to be complete if the join and meet of arbitrary subsets of Ω exist. Note that complete lattices are bounded, the upper bound of the complete lattice Ω, usually denoted $\top_{\Omega}$ is defined by $\bigvee^{\Omega} \Omega$, while the bottom element $\bot_{\Omega}$ is defined by $\bigvee^{\Omega} \varnothing$. To simplify notation, the superscripts and subscripts that appear in the join and meet operations, as well as in the top and bottom elements will be dropped when it does not lead to confusion.

The opposite of a lattice Ω, also denoted Ω^{op} is the lattice where the join operation $\vee^{op}$ is the meet operation $\wedge$ in Ω. Similarly, the meet operation over Ω^{op}, namely $\wedge^{op}$, is defined to be the join operation $\vee$ over Ω.

Given two lattices Ω and Ψ, the collection of sup-lattice homomorphisms from Ω to Ψ will be denoted $[\Omega, \Psi]_\vee$. Dually, the collection of inf-lattice homomorphisms from Ω to Ψ will be denoted $[\Omega, \Psi]_\wedge$. We now recall some basic properties of lattices:

Property 1. For any preorder $\mathbb{S}$ and any complete lattices Ω and Ψ the following properties hold:

1. $[\mathbb{S}, \Psi], [\Omega, \Psi]_\vee$ and $[\Omega, \Psi]_\wedge$ are complete lattices,
2. $([\mathbb{S}, \Omega])^{op} = [\mathbb{S}^{op}, \Omega^{op}]$,
3. $([\Omega, \Psi]_\vee)^{op} = [\Omega^{op}, \Psi^{op}]_\wedge$.

Any sup-lattice morphism $f : \Omega \to \Psi$ is a left adjoint to the inf-lattice morphism $f_\vdash : \Psi \to \Omega$ defined by $f_\vdash(r) = \bigvee\{p \in \Omega \mid f(p) \leq r\}$. Dually, every inf-lattice morphism $g : \Psi \to \Omega$ is a right adjoint to the sup-lattice homomorphism $g_\dashv : \Omega \to \Psi$ defined by: $g_\dashv(p) = \bigwedge\{r \in \Psi \mid g(r) \leq p\}$.

Therefore, the function $(-)_\vdash : [\Omega, \Psi]_\vee \to [\Psi, \Omega]_\wedge$ that sends every sup-lattice homomorphism to its right adjoint forms a bijection with respect to the function $(-)_\dashv : [\Psi, \Omega]_\wedge \to [\Omega, \Psi]_\vee$ that maps every inf-lattice homomorphism $g : \Psi \to \Omega$ to its left adjoint $g_\dashv : \Omega \to \Psi$.

A quantale is a tuple $(Q, \otimes)$ where Q is a complete lattice and $\otimes$ is a binary operation over Q, called multiplication, that satisfies the following conditions for any $Q' \subseteq Q$.

$$q \otimes \bigvee Q' = \bigvee_{q' \in Q'} (q \otimes q') \quad \text{and} \quad \bigvee Q' \otimes q = \bigvee_{q' \in Q'} (q' \otimes q).$$

A quantale $(Q, \otimes)$ is unital if there exists an element $e \in Q$ such that $p \otimes e = p = e \otimes p$ for every $p \in Q$.

A quantale homomorphism $f : (Q, \otimes_Q) \to (P, \otimes_P)$ is a sup-lattice homomorphism $f : Q \to P$ that preserves the multiplication. A quantale homomorphism that preserves the unit will be called a unital quantale homomorphism. A quantale homomorphism that has an inverse is a quantale isomorphism.

Given a unital quantale $(Q, \otimes, e)$, a sup lattice M is a right module if there exists a binary operation $* : M \times Q \to M$ such that the following three axioms hold:

1. $m * (q \otimes q') = (m * q) * q'$,
2. $m * \overset{Q}{\bigvee} Q' = \overset{M}{\underset{q \in Q'}{\bigvee}} (m * q)$ and $\overset{M}{\bigvee} A * q = \overset{M}{\underset{m \in A}{\bigvee}} (m * p)$,
3. $m * e = m$

for every $q, q' \in Q, Q' \subseteq Q, m \in M$ and $A \subseteq M$. One can dually define left modules in the usual way.

Since the scalar multiplication $*$ preserves joins in both arguments, there exists two residual operations $\backslash$ and $/$ such that $m * q \leq n \Leftrightarrow q \leq m\backslash n \Leftrightarrow m \leq n/q$ for every $m, n \in M$ and $q \in Q$. The operation $-/q$ is usually called the right residual of $*$ and the operation $m\backslash-$ is called the left residual of $*$ and are defined by $n/q = \bigvee\{m \in M \mid m * q \leq n\}$ and $m\backslash n = \bigvee\{q \in Q \mid m * q \leq n\}$.

3 Ω-sup-fuzzy-relations

L-fuzzy sets have been widely studied in the context of mathematical morphology as a formalisation of colour based images [13,10,2]. A principle that has been discussed in the mathematical morphology community asserts that digital objects need to be formalised, not as collection of pixels, but as structures that, in addition, carry skeletal information regarding the different pixels [3,4]. This can be achieved by modelling digital images as graphs. While positive results have been obtained in the binary case [6,5,1,12], no solution has been provided for more general contexts such as grey-scale or coloured digital images.

In this section we propose a formalisation of fuzzy mathematical morphology over graphs using the theory of quantales. We start by introducing the concepts of Ω-fuzzy suborder and Ω-sup-fuzzy relations, a generalisation of the concepts of H-subgraphs and H-relations discussed in [22]. In Lemma 1 we observe that Ω-sup-fuzzy relations act over the lattice of Ω-fuzzy suborders in terms of a quantale action. Furthermore, we show that the quantale action described induces an isomorphism between Ω-sup-fuzzy relations and the quantale of sup-preserving endofunctions on the Ω-fuzzy suborders.

In Section 3.2 we apply the results obtained in Section 3.1 to mathematical morphology. We observe that fuzzy images over graphs (as in the case of grey-scale or colour based graphical images) can be represented in terms of Ω-fuzzy suborders. Moreover, we observe that the quantale action described in Proposition 1 corresponds to the morphological operation of dilation, while the adjoint quantale action described in formula 2 corresponds to the morphological operation of erosion. From Lemma 1 and Corollary 2 we conclude that dilations/erosions over suborders can be fully characterised in terms of Ω-sup-fuzzy relations.

3.1 Sup-fuzzy relations

Ω-fuzzy suborders represent a generalisation of Ω-fuzzy subsets. Given a Ω-fuzzy suborder $(X, f : X \to \Omega)$, the crisp set X is equipped with a crisp preorder relation $\leq_X$ and the condition of monotonicity is imposed over the fuzzy function $f \in [(X, \leq_X), \Omega]$.

Definition 1. *Given a preorder $\mathbb{S}$ and a complete lattice Ω, an Ω-fuzzy suborder*[1] *is an element $f \in [\mathbb{S}, \Omega]$.*

One might think that sup-preserving endofunctions over Ω-fuzzy suborders would correspond to monotone relations of the form $\mathbb{S}^{op} \times \mathbb{S} \to \Omega$, based on the case when Ω is 2. However, this approach fails even in the case when Ω is the three element chain. Therefore, we introduce Ω-sup-fuzzy relations, a generalisation

[1] The naming Ω-fuzzy-suborder has been chosen following the mathematical morphology tradition where binary images are subsets of a set and binary graphic images are subgraphs of a graph. Thus the name suborder is a shortening of the longer word sub-preorder.

of fuzzy relations that, as we will see throughout this section fully characterises the collection of sup-lattice homomorphisms on Ω-fuzzy suborders.

Definition 2. *Given a preorder $\mathbb{S}$ and a complete lattice Ω, we define the unital quantale of Ω-sup-fuzzy relations to be the triple $([\mathbb{S}^{op} \times \mathbb{S}, [\Omega, \Omega]_\vee], ;, e)$ where $[\mathbb{S}^{op} \times \mathbb{S}, [\Omega, \Omega]_\vee]$ is the sup-lattice of monotone functions from the preorder $\mathbb{S}^{op} \times \mathbb{S}$ to the complete lattice $[\Omega, \Omega]_\vee$.*

The multiplication operation ; is the diagrammatic relation composition defined by $(R \; ; \; S)(s, s') := \bigvee_{z \in \mathbb{S}} S(z, s') \circ R(s, z)$. The identity element is the Ω-sup-fuzzy relation e defined pointwise by:

$$e(s, s') := \begin{cases} id_\Omega & s \leq_{\mathbb{S}} s' \\ \bot_{[\Omega,\Omega]} & otherwise, \end{cases}$$

where $id_\Omega : \Omega \to \Omega$ is the identity function and $\bot_{[\Omega,\Omega]} : \Omega \to \Omega$ is the function that sends every element in Ω to the bottom element.

The collection of Ω-sup-fuzzy relations act on the complete lattice of Ω-fuzzy suborders defining a right quantale module:

Proposition 1. *Given a preorder $\mathbb{S}$ and a complete lattice Ω, the lattice of Ω-fuzzy suborders is a right module of the unital quantale of Ω-sup-fuzzy relations. The scalar multiplication $\oplus$ is defined by:*

$$(f \oplus R)(s) := \bigvee_{s' \in \mathbb{S}} R(s', s)(f(s')) \tag{1}$$

where f is an Ω-fuzzy suborder, R is an Ω-sup-fuzzy relation and $s \in \mathbb{S}$.

Proof. Consider any Ω-fuzzy suborder f and any Ω-sup-fuzzy relation R. That $f \oplus R$ is monotone follows immediately as $R(s', -)(p)$ is monotone for every $s' \in \mathbb{S}$ and $p \in \Omega$. Hence $- \oplus R$ is a well defined endofunction on the lattice of Ω-fuzzy suborders. That $\oplus$ is preserves joins in the left argument follows from the fact that for every $s, s' \in \mathbb{S}$, the map $R(s', s)(-)$ is sup-preserving. That the operation $\oplus$ preserves joins on the right follows from the following set of equalities:

$$\begin{aligned}(f \oplus \bigvee \mathcal{R})(s) &= \bigvee_{s' \in S} (\bigvee \mathcal{R})(s', s)(fs') = \bigvee_{s' \in S} (\bigvee_{R \in \mathcal{R}} R(s', s)(fs')) \\ &= \bigvee_{R \in \mathcal{R}} \bigvee_{s' \in S} R(s', s)(fs') = \bigvee_{R \in \mathcal{R}} (f \oplus R)(s).\end{aligned}$$

That $f \oplus e = f$ can be easily verified by unfolding the definition of the operation $\oplus$, recalling the monotonicity of the function f and the definition of the identity e. We conclude this proof by showing that $f \oplus (R \; ; S) = (f \oplus R) \oplus S$. Consider any $s \in \mathbb{S}$, then:

$$\begin{aligned}(f \oplus (R\,;S))(s) &= \bigvee_{s' \in \mathbb{S}} (R\,;S)(s', s)(fs') = \bigvee_{s' \in \mathbb{S}} \bigvee_{t \in \mathbb{S}} S(t, s)(R(s', t)(fs')) \\ &= \bigvee_{t \in \mathbb{S}} S(t, s)(\bigvee_{s' \in \mathbb{S}} R(s', t)(fs')) = \bigvee_{t \in \mathbb{S}} S(t, s)((f \oplus R)(t)) \\ &= ((f \oplus R) \oplus S)(s).\end{aligned}$$

□

In [7, Prop 3.1.5], the authors observe that the collection of right actions of a quantale Q over a sup-lattice M is bijective with the quantale homomorphisms from the transpose quantale Q^t, i.e. the quantale Q where the multiplication $\otimes$ is reversed, to the quantale of sup-preserving endomorphisms on M. The following result extends this observation in the context of Ω-sup-fuzzy relations and shows that the quantale morphism induced by $\oplus$ is an isomorphism.

Lemma 1. *Given a preorder $\mathbb{S}$ and a complete lattice Ω, the quantale of Ω-sup-fuzzy relations is isomorphic to the quantale of sup-preserving endofunctions over $[\mathbb{S}, \Omega]$ with respect to the transpose composition operation.*

Proof. By [7, Prop 3.1.5] the quantale action $\oplus$ induces a quantale morphism from Ω-sup-fuzzy relations to the quantale of sup-preserving endofunctions on $[\mathbb{S}, \Omega]$. The inverse morphism $\oplus^{-1}$ is defined pointwise as $\oplus^{-1}(F)(s, t)(p) = F(s_p)(t)$ where s_p is the Ω-fuzzy suborder sending elements above s to p and everything else to $\bot$. It can be easily verified that $\oplus^{-1}$ is a well defined quantale homomorphism that preserves the unit. Moreover, it can be easily verified that $\oplus^{-1}$ is the inverse of $\oplus$. □

Furthermore, in [7, Prop 3.1.1.a] it is shown that any right quantale module $* : M \times Q \to M$ induces a left quantale module over the opposite lattice $Q \times M^{op} \to M^{op}$. Such operation corresponds to the action of the right residual operation $/$. In the particular context of Ω-sup-fuzzy relations, the right residual of $\oplus$, that we will denote $\ominus$ is defined by:

$$(R \ominus f)(s) := \bigwedge_{s' \in \mathbb{S}} R(s', s)_{\vdash}(fs') \tag{2}$$

where $f \in [\mathbb{S}, \Omega]$, $s \in \mathbb{S}$ and $R(s', s) \dashv R(s', s)_{\vdash}$

Corollary 1. *Given a preorder $\mathbb{S}$ and a complete lattice Ω, the complete lattice $([\mathbb{S}, \Omega])^{op}$ is a left module of the quantale of Ω-sup-fuzzy relations. The scalar multiplication corresponds to the operation $\ominus$.*

Proof. Follows immediately from [7, Prop 3.1.1.a]. □

We conclude this section, observing that the quantale of Ω-sup-fuzzy relations is isomorphic to the quantale of sup-preserving endofunctions on $[\mathbb{S}, \Omega]^{op}$. Note that by [7, Prop 3.1.3] the left action $\ominus$ induces a quantale homomorphism from Ω-sup-fuzzy relations to the quantale of sup-preserving endofunctions of $([\mathbb{S}, \Omega])^{op}$. Combining the fact that the quantale of sup-preserving morphisms on $[\mathbb{S}, \Omega]$ equipped with the transpose composition is isomorphic to the quantale of sup-preserving endofunctions on $([\mathbb{S}, \Omega])^{op}$ and Lemma 1 we obtain the following result.

Corollary 2. *Given any preorder $\mathbb{S}$ and any complete lattice Ω, the quantale of Ω-sup-fuzzy relations is isomorphic to the quantale of sup-preserving endofunctions on $([\mathbb{S}, \Omega])^{op}$ equipped with the usual function composition.*

Proof. Note that by Lemma 1 the quantale of Ω-sup-fuzzy relations is isomorphic to the quantale of sup-preserving endofunctions on $[\mathbb{S}, \Omega]$ equipped with the transpose function composition operation. Moreover, the quantale of sup-preserving morphisms on $[\mathbb{S}, \Omega]$ equipped with the transposed composition is isomorphic to the quantale of sup-preserving endofunctions on $[\mathbb{S}, \Omega]^{op}$ equipped with the usual function composition operation. This can be easily verified as the operation of taking adjoints lifts to be a quantale morphisms. Combining both results we conclude that the quantale of Ω-sup-fuzzy relations is isomorphic to the quantale of sup-preserving endofunctions on $([\mathbb{S}, \Omega])^{op}$. □

3.2 Sup-fuzzy relations in mathematical morphology

In this subsection we provide a guide that summarises our analysis of graphical fuzzy mathematical morphology in terms of Ω-fuzzy graphs [14]. Given a complete lattice Ω, an Ω-fuzzy graph is a triple (X, μ, ϱ) where X is a set, $\mu : X \to \Omega$ is a function and $\varrho : X \times X \to \Omega$ is a bi-function satisfying the property $\varrho(x, y) \leq \mu(x) \wedge \mu(y)$ for every $x, y \in X$.

In the context of graphical fuzzy mathematical morphology, the complete lattice Ω describes a colour palette. For example, the 8-bit grey-scale can be formalised as the chain of 256 elements while the 24-bit colour palette is represented as the 3-fold cartesian product of the chain of 256 elements.

Fixing a set of pixels X, any graphical image valued in Ω can be represented as an Ω-fuzzy graph, where the function $\mu : X \to \Omega$ represents the colouring function of the image and the bi-function $\varrho : X \times X \to \Omega$ displays the gluing information between the different pixels as mentioned at the beginning of the current section.

An equivalent definition of graphical fuzzy mathematical morphology images can be achieved in terms of Ω-fuzzy suborders following a similar construction as the one displayed in [21,22]. For any Ω-fuzzy graph (X, μ, ϱ), that, as we have seen, represents a graphical image valued in Ω, we define the Ω-fuzzy suborder $f : \mathbb{S} \to \Omega$ where the preorder $\mathbb{S} = (S, \leq_{\mathbb{S}})$ is the set $X \cup \{\{x, y\} \in X \times X \mid x \neq y\}$ equipped with the relation $\leq_{\mathbb{S}}$ defined by $s \leq_{\mathbb{S}} s'$ if and only if $s' \in s$ or $s = s'$. The function f maps those elements x that act as vertices to their colouring $(f(x) = \mu(x))$, while it will map those elements that act as edges $\{x, y\}$ to their gluing information $(f(\{x, y\}) = \varrho(x, y))$.

Therefore, given a set of pixels X and a colour palette Ω, the collection of all graphical images is represented by the sup-lattice of Ω-fuzzy suborders $[\mathbb{S}, \Omega]$. It is well-known [16, p61] that erosion and dilation on sets of pixels can be parameterized by binary relations on the set of pixels, and that the structuring elements widely used in mathematical morphology are essentially a means of presenting such relations. This result has been extended to the scenario in which binary images are formalised as crisp graphs via the introduction of H-stable relations [22].

Ω-sup-fuzzy relations generalise H-stable relations for arbitrary fuzzy scenarios. The scalar multiplication $\oplus$ introduced in Proposition 1 acts on the collection of graphical images over a set X defining a join preserving endomorphism. In the mathematical morphology context, this operation corresponds to the dilation operation [11, p259]. Furthermore by Lemma 1 we observe that the collection of dilations over the collection of graphical images are fully described by the lattice of Ω-sup-fuzzy relations, extending the results in [22] for arbitrary complete lattices.

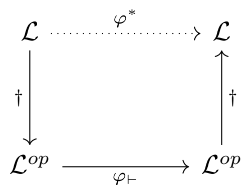
The residual operation $\ominus$ induces an inf-lattice homomorphism on the collection of graphical images. Again, in the context of mathematical morphology, this action corresponds to the erosion operation [11, p259]. As we observed in Corollary 2, the action $\ominus$ draws an isomorphism between the collection of Ω-sup-fuzzy relations and the collection of erosions acting on the collection of graphical images. Thus, concluding that the collection of erosions acting on grey-scale images are fully characterised by the collection of Ω-sup-fuzzy relations. Again, this extends the results obtained in [22] for arbitrary complete lattices.

4 Converse relations

In this section we discuss the notion of converse for Ω-sup-fuzzy relations. Recall from the introduction, that in the case of monotone relations $R : \mathbb{S}^{op} \times \mathbb{S} \to 2$, the operation of reversing the arguments $\breve{R}(s, s') = R(s', s)$ yields a monotone relation in the opposite order, i.e. $\breve{R} : \mathbb{S} \times \mathbb{S}^{op} \to 2$. However, in [22, Def 6, p334] the author introduces two weaker notions of converse, namely the right $\curvearrowleft R$ and left $\curvearrowright R$ converse that are monotone in the original order $\mathbb{S}^{op} \times \mathbb{S} \to 2$.

It is natural to ask what happens to these converse operations when we generalize from 2-valued $\mathbb{S}$-relations to relations in $[\mathbb{S}^{op} \times \mathbb{T}, [\Omega, \Psi]_\vee]$. Given the way that the left and right converse arise from the ordinary converse in the 2-valued case, we first consider a generalization of the ordinary converse that will produce a relation in $[\mathbb{T} \times \mathbb{S}^{op}, [\Psi, \Omega]_\vee]$.

In [15] Mulvey and Pelletier discuss the quantale of sup-lattice endomorphisms of an orthocomplemented lattice $\mathcal{L}$. One contribution of that paper is that the orthocomplement operation induces an involution on the collection of sup-lattice endomorphisms. The overall construction discussed is that given a lattice $\mathcal{L}$ equipped with an orthocomplementation operation $\dagger : \mathcal{L} \to \mathcal{L}^{op}$ and a sup-lattice endomorphism $\varphi : \mathcal{L} \to \mathcal{L}$, we can define an operation $(-)^* : [\mathcal{L}, \mathcal{L}]_\vee \to [\mathcal{L}, \mathcal{L}]_\vee$ defined by the following diagram:



where $\varphi_\vdash$ is the right adjoint of φ, i.e. $\varphi \dashv \varphi_\vdash$. As the authors show, this construction induces an involution over the collection of sup-lattice endomorphisms on the orthocomplemented lattice $\mathcal{L}$, defining a Gelfand quantale [15, p3]. In the context of Gelfand quantales the involution operation acts as a converse operation over the right sided elements.

In this section we introduce a series of constructions based on the work by Mulvey et al [15] to define the right converse operation of an Ω-sup-fuzzy relation.

4.1 Relations and Dilations

Recall that a category $\mathbf{C}$ enriched in the category of sup-lattices and sup-preserving morphisms has objects and arrows as in an ordinary category, but instead of hom-sets each $\mathbf{C}(X, Y)$ is a sup-lattice and composition preserves sups in both arguments. Such sup-enriched categories are also known as quantaloids [18].

Definition 3. *We denote by* Sf-Rel *the quantaloid where:*

Objects are pairs $(\mathbb{S}, \Omega)$ *where* $\mathbb{S}$ *is a preorder and* Ω *a complete lattice,*

A morphism $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$ *is a sup-fuzzy relation* $R \in [\mathbb{S}^{op} \times \mathbb{T}, [\Omega, \Psi]_\vee]$,

Each Sf-Rel(R, S) *carries the usual sup-lattice structure in* $[\mathbb{S}^{op} \times \mathbb{T}, [\Omega, \Psi]_\vee]$.

The composition of $(\mathbb{S}, \Omega) \xrightarrow{R} (\mathbb{T}, \Psi)$ *and* $(\mathbb{T}, \Psi) \xrightarrow{S} (\mathbb{U}, \Delta)$*, denoted* $R\,;S : (\mathbb{S}, \Omega) \to (\mathbb{U}, \Delta)$*, extends the composition operation defined in Definition 2 by:*

$$(R\,;S)(s, u) := \bigvee_{t \in \mathbb{T}} S(t, u) \circ R(s, t).$$

In Sf-Rel a relation $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$ provides for each $(s, t) \in \mathbb{S}^{op} \times \mathbb{T}$ a sup-lattice morphism $R(s, t) : \Omega \to \Psi$. The right adjoint of each $R(s, t)$ is a sup-lattice morphism from Ψ^{op} to Ω^{op}. This construction yields a quantaloid homomorphism (in other words a sup-lattice enriched functor) from Sf-Rel to Sf-Rel^{op} where the *op* indicates reversal of the direction of the morphisms but keeping the sup-lattice order that provides the 2-cells.

Proposition 2. *The map* $\mathcal{F}$: Sf-Rel $\to$ Sf-Rel^{op} *defined by:*

$\mathcal{F}(\mathbb{S}, \Omega) = (\mathbb{S}^{op}, \Omega^{op})$ for every 0-cell $(\mathbb{S}, \Omega)$,

$\mathcal{F}(R) : (\mathbb{T}^{op}, \Psi^{op}) \to (\mathbb{S}^{op}, \Omega^{op})$ where

$$\mathcal{F}(R)(t, s) = (R(s, t))_{\vdash}$$

for every 1-cell $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$ and $(s, t) \in \mathbb{S} \times \mathbb{T}^{op}$,

is a quantaloid homomorphism.

Proof. It suffices to show that for any SF-REL hom object $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$, the function $\mathcal{F}(R) : (\mathbb{T}^{op}, \Psi^{op}) \to (\mathbb{S}^{op}, \Omega^{op})$ preserves joins. However, this follows immediately as $\mathcal{F}(R)(t, s) = (R(s, t))_{\vdash}$ is an inf-preserving function from Ψ to Ω or, equivalently a sup-preserving function from Ψ^{op} to Ω^{op}. □

The following definition generalises quantales of sup-preserving endofunctions over Ω-fuzzy suborders to he quantaloid setting. Furthermore we provide a generalisation of Lemma 1 in this new setting:

Definition 4. *We denote by* F-DIL *the quantaloid where:*

Objects are lattices $[\mathbb{S}, \Omega]$ where $\mathbb{S}$ is a preorder and Ω a complete lattice,

A morphism $F : [\mathbb{S}, \Omega] \to [\mathbb{T}, \Psi]$ is a sup-lattice homomorphism $F \in [[\mathbb{S}, \Omega], [\mathbb{T}, \Psi]]_{\vee}$,

Each F-DIL(F, G) *carries the usual sup-lattice structure $[[\mathbb{S}, \Omega], [\mathbb{T}, \Psi]]_{\vee}$.*

The composition of $F : [\mathbb{S}, \Omega] \to [\mathbb{T}, \Psi]$ and $G : [\mathbb{T}, \Psi] \to [\mathbb{U}, \Delta]$ denoted $G \circ F$ is the usual composition of functions.

Proposition 3. *The map $\oplus$:* SF-REL $\to$ F-DIL *where:*

$\oplus\, (\mathbb{S}, \Omega) = [\mathbb{S}, \Omega]$ for every 0-cell $(\mathbb{S}, \Omega)$,

$\oplus\, R : [\mathbb{S}, \Omega] \to [\mathbb{T}, \Psi]$ where:

$$(f \oplus R)(t) = \bigvee_{s \in \mathbb{S}} R(s, t)(f(s))$$

for every 1-cell $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$, $f \in [\mathbb{S}, \Omega]$ and $t \in \mathbb{T}$.

is a quantaloid isomorphism

Proof. This result is a direct generalisation of Lemma 1 and the same result can be achieved following an identical strategy.

The previous result shows that for any SF-REL relation $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$, the action of the morphism $\oplus$ on R, defines a sup-lattice function $\oplus\, R : [\mathbb{S}, \Omega] \to [\mathbb{T}, \Psi]$. Therefore, the right adjoint of this function, namely $(\oplus\, R)_{\vdash}$ is a sup-lattice homomorphism from $([\mathbb{T}, \Psi])^{op}$ to $([\mathbb{S}, \Omega])^{op}$. The following result characterises the operation $(\oplus\, -)_{\vdash}$ and therefore extends Corollary 1 to the quantaloid scenario. Note that in the following proposition the superscript *op* indicates, as before, the reversal of the direction of the 1-cells but keeping the sup-lattice structure.

Proposition 4. *The function* $\ominus : \text{SF-REL} \to \text{F-DIL}^{op}$ *where:*

$\ominus\,(\mathbb{S},\Omega) = [\mathbb{S}^{op},\Omega^{op}]$ *for every* 0*-cell* $(\mathbb{S},\Omega)$*,*

$R\,\ominus : [\mathbb{T}^{op},\Psi^{op}] \to [\mathbb{S}^{op},\Omega^{op}]$ *where:*

$$(R \ominus g)(s) = \bigwedge_{t\in\mathbb{T}} R(s,t)_{\vdash}(g(t))$$

for every 1*-cell* $R : (\mathbb{S},\Omega) \to (\mathbb{T},\Psi)$*,* $g \in [\mathbb{T},\Psi]$ *and* $s \in \mathbb{S}$*.*

is a quantaloid isomorphism. Moreover, $(\oplus R) \vdash (R\ominus)$ *for every* 1*-cell* $R : (\mathbb{S},\Omega) \to (\mathbb{T},\Psi)$*.*

We now remark some observations regarding the previous results obtained.

Proposition 5. *For every* 1*-cell* $R : (\mathbb{S},\Omega) \to (\mathbb{T},\Psi)$ *in* SF-REL*:*

1. $(\mathcal{F}(R) \ominus^{op} f) = (f \oplus R)$ *for every* $f \in [\mathbb{S},\Omega]$*,*
2. $(g \oplus^{op} \mathcal{F}(R)) = (R \ominus g)$ *for every* $g \in [\mathbb{T}^{op},\Psi^{op}]$*.*

where $\ominus^{op}$ *and* $\oplus^{op}$ *are the opposite functor of* $\ominus$ *and* $\oplus$ *respectively.*

Proof. We only prove the first statement. The second follows immediately since $\oplus\, R \dashv R\,\ominus$. Consider any $t \in \mathbb{T}$, then:

$$\begin{aligned}(\mathcal{F}(R) \ominus^{op} f)(t) &= \bigwedge_{s\in\mathbb{S}}^{op} \mathcal{F}(R)(t,s)_{\dashv}(fs) = \bigvee_{s\in\mathbb{S}} (R(s,t)_{\vdash})_{\dashv}(fs) \\ &= \bigvee_{s\in\mathbb{S}} R(s,t)(fs) = (f \oplus R)(t)\end{aligned}$$

□

From the last observation and considering that the morphisms $\oplus$ and $\ominus$ are isomorphisms, one can easily verify that $\mathcal{F}^{op}$ is the opposite morphism of $\mathcal{F}$.

Corollary 3. *The inverse of the quantaloid morphism* $\mathcal{F} : \text{SF-REL} \to \text{SF-REL}^{op}$ *is its opposite* $\mathcal{F}^{op} : \text{SF-REL}^{op} \to \text{SF-REL}$*.*

4.2 Complement of sup-fuzzy relations

In the context of binary graph mathematical morphology [22], the complement operation $- : 2 \to 2$ induces the pseudocomplement and dual pseudocomplement operations in the bi-Heyting algebra $[\mathbb{S}, 2]$ [22, p333]. Furthermore, these operations allowed Stell to define the left converse relation $\smile\!\! R$ [22, p339] in terms of the dilation and erosion operation. In this section we consider complete lattices equipped with an anti-isomorphism and show, that these conditions are sufficient to define two pseudocomplement and dual pseudocomplement type of operations that will allow us to generalise the concept of left converse in the context of sup-fuzzy relations.

Definition 5. *Let Ω be a complete lattice, an anti-isomorphism is a complete lattice isomorphism $\varphi : \Omega \to \Omega^{op}$.*

Composing an anti-isomorphism $\varphi : \Omega \to \Omega^{op}$ with an Ω-fuzzy suborder $f : \mathbb{S} \to \Omega$ produces a Ω^{op}-fuzzy suborder $\varphi f : \mathbb{S} \to \Omega^{op}$. Therefore, the anti-isomorphism φ lifts to define a complete lattice morphism $\overline{\varphi} : [\mathbb{S}, \Omega] \to [\mathbb{S}, \Omega^{op}]$ that reverses the order of the complete lattice Ω but fixes the order of the preorder $\mathbb{S}$.

Proposition 6. *Let Ω be a complete lattice equipped with an anti-isomorphism, $\varphi : \Omega \to \Omega^{op}$ and let $\mathbb{S}$ be a preorder. The function $\overline{\varphi} : [\mathbb{S}, \Omega] \to [\mathbb{S}, \Omega^{op}]$ defined by $(\overline{\varphi} f)(s) = \varphi(fs)$ is a complete lattice isomorphism and $\overline{\varphi}^{-1} = \overline{\varphi^{-1}}$.*

Proof. Clearly the functions $\overline{\varphi}$ and $\overline{\varphi^{-1}}$ are complete lattice morphisms as joins and meets in $[\mathbb{S}, \Omega]$ and $[\mathbb{S}, \Omega^{op}]$ are computed pointwise. Moreover, since φ^{-1} is the inverse of φ, one can easily see that $\overline{\varphi^{-1}}$ is the inverse of $\overline{\varphi}$. □

The following result allows us to define a pair of complementation type operations that resemble the pseudocomplement and dual-pseudocomplement operations in bi-Heyting algebras.

Proposition 7. *For any preorder $\mathbb{S}$ and any complete lattice Ω, let $[S, \Omega]$ be the lattice of functions from the underlying set S in $\mathbb{S}$ to the complete lattice Ω, then the inclusion map $\iota : [\mathbb{S}, \Omega] \hookrightarrow [S, \Omega]$ has a left adjoint $\mathsf{int} : [S, \Omega] \to [\mathbb{S}, \Omega]$ and a right adjoint $\mathsf{ext} : [S, \Omega] \to [\mathbb{S}, \Omega]$ defined by:*

$$\mathsf{int}(h) := \bigvee \{ f \in [\mathbb{S}, \Omega] \mid f \leq_{[S,\Omega]} h \} \tag{3}$$

$$\mathsf{ext}(h) := \bigwedge \{ f \in [\mathbb{S}, \Omega] \mid h \leq_{[S,\Omega]} f \} \tag{4}$$

Proof. Consider any $f \in [\mathbb{S}, \Omega]$ and any $h \in [S, \Omega]$. Then $\iota(f) \leq_{[S,\Omega]} h$ if and only if $f \leq_{[S,\Omega]} h$ if and only if $f \leq_{[\mathbb{S},\Omega]} \mathsf{int}(h)$. Similarly $h \leq_{[S,\Omega]} \iota(f)$ if and only if $h \leq_{[S,\Omega]} f$ if and only if $\mathsf{ext}(h) \leq_{[\mathbb{S},\Omega]} f$ □

Since the maps $\overline{\varphi}$ and ι preserve joins and meets, the two routes described in the following diagram define a sup and inf preserving morphism $[\mathbb{S}, \Omega] \to [\mathbb{S}^{op}, \Omega^{op}]$:

$$[\mathbb{S}, \Omega] \xrightarrow{\ \overline{\varphi}\ } [\mathbb{S}, \Omega^{op}] \xhookrightarrow{\ \iota\ } [S, \Omega^{op}] \underset{\mathsf{int}}{\overset{\mathsf{ext}}{\rightrightarrows}} [\mathbb{S}^{op}, \Omega^{op}]$$

Definition 6. *Given a preorder $\mathbb{S}$ and a complete lattice Ω equipped with an anti-isomorphism $\varphi : \Omega \to \Omega^{op}$ we let $\neg^{\varphi} : [\mathbb{S}, \Omega] \to [\mathbb{S}^{op}, \Omega^{op}]$ and $\lrcorner^{\varphi} : [\mathbb{S}^{op}, \Omega^{op}] \to [\mathbb{S}, \Omega]$ be the two sup-preserving functions defined by:*

$$\neg^{\varphi} f := (\mathsf{ext})(\iota)(\overline{\varphi})(f) = \bigvee^{[\mathbb{S},\Omega]} \{ f' \in [\mathbb{S}, \Omega] \mid f' \leq_{[S,\Omega]} \overline{\varphi} f \}$$

$$\lrcorner^{\varphi} f := (\mathsf{int})(\iota)(\overline{\varphi})(f) = \bigwedge^{[\mathbb{S},\Omega]} \{ f' \in [\mathbb{S}, \Omega] \mid \overline{\varphi} f \leq_{[S,\Omega]} f' \}$$

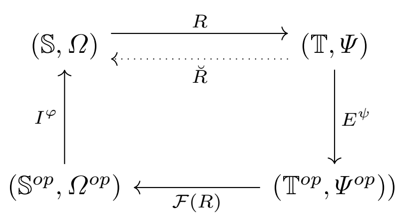
Proposition 8. *Given a preorder $\mathbb{S}$ and a complete lattice Ω equipped with an anti-isomorphism $\varphi : \Omega \to \Omega^{op}$. The sup-lattice morphisms $\neg^\varphi : [\mathbb{S}, \Omega] \to [\mathbb{S}^{op}, \Omega^{op}]$ and $\lrcorner^\varphi : [\mathbb{S}^{op}, \Omega^{op}] \to [\mathbb{S}, \Omega]$ are self adjoint, i.e. $\neg^\varphi \dashv (\neg^\varphi)^{op}$ and $\lrcorner^\varphi \dashv (\lrcorner^\varphi)^{op}$*

Proof. We only show that $\neg^\varphi \dashv (\neg^\varphi)^{op}$. The other case can be proved using a similar argument. Note that by Proposition 7 $\neg^\varphi = (\mathsf{ext})(\iota)(\overline{\varphi}) \dashv (\overline{\varphi^{-1}})(\mathsf{int})(\iota)$. By unfolding the definition of int and since $\overline{\varphi^{-1}}$ is an isomorphism one can easily show that $(\overline{\varphi^{-1}})(\mathsf{int})(\iota)(f) = \neg^\varphi f$. ◻

Observe that the newly defined functions $\neg^\varphi$ and $\lrcorner^\varphi$ preserve joins. Therefore, by Proposition 3 we can characterise these two sup-lattice morphisms in terms of SF-Rel relations.

Definition 7. *For any complete lattice Ω equipped with an anti-isomorphism $\varphi : \Omega \to \Omega^{op}$ and any preorder $\mathbb{S}$ we let $E^\varphi : (\mathbb{S}, \Omega) \to (\mathbb{S}^{op}, \Omega^{op})$ be the unique* SF-Rel *relation such that $\oplus\, E^\varphi = \neg^\varphi$. Similarly, we let $I^\varphi : (\mathbb{S}^{op}, \Omega^{op}) \to (\mathbb{S}, \Omega)$ be the unique* SF-Rel *relation such that $\oplus\, I^\varphi = \lrcorner^\varphi$.*

With the mechanisms defined in Proposition 2 and Definition 7 we have all the ingredients to define the converse operation for SF-Rel relations.



Definition 8. *For any complete lattices Ω and Ψ that are equipped with anti-isomorphisms $\varphi : \Omega \to \Omega^{op}$ and $\psi : \Psi \to \Psi^{op}$ and for any fuzzy relation $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$ we define the left converse* SF-Rel *relation $\curvearrowleft R : (\mathbb{T}, \Psi) \to (\mathbb{S}, \Omega)$ as $\curvearrowleft R = E^\psi \,;\, \mathcal{F}(R) \,;\, I^\varphi$.*

We conclude this section by showing some of the properties of the newly defined SF-Rel converse relation.

Lemma 2. *For any complete lattices Ω and Ψ that are equipped with anti-isomorphisms $\varphi : \Omega \to \Omega^{op}$ and $\psi : \Psi \to \Psi^{op}$ and for any* SF-Rel *relation $R : (\mathbb{S}, \Omega) \to (\mathbb{T}, \Psi)$ the following two equalities hold:*

1. *$(g \oplus \curvearrowleft R) = \lrcorner^\varphi(R \ominus \neg^\psi g)$ for every $g \in [\mathbb{T}, \Psi]$,*
2. *$(\curvearrowleft R \ominus f) = (\neg^\psi)^{op}((\lrcorner^\varphi)^{op} f \oplus R)$ for every $f \in [\mathbb{S}, \Omega]$.*

Proof. Point 1 follows immediately from 5 and the fact that $\oplus\,(R \,;\, S) = (\oplus\, R) \oplus S$ for any SF-Rel relations R and S. Point 2 follows from Definition 8, Proposition 5 and the fact $(R \,;\, S)_\dashv = S_\dashv \,;\, R_\dashv$ for any SF-Rel relations R and S.

5 Conclusion and Future Work

In this paper we have introduced Ω-sup-fuzzy relations and Ω-fuzzy suborders, a generalisation of the concept of stable relations and hypergraphs discussed in [22]. We have provided a connection between these two constructions in terms of a quantale action. Such action has been shown to lift to an isomorphism between Ω-sup-fuzzy relations and the quantale of sup-preserving functions over Ω-fuzzy suborders.

In Section 3.2 we analyse Ω-sup-fuzzy relations and Ω-suborders in the context of mathematical morphology. Firstly, we observe that graphical images over complete lattices (such as 8-bit grey-scale or 24-bit colour) are particular cases of Ω-suborders for a particular complete lattice Ω. Moreover, based on [11] we identify the morphological operations of dilation in this setting as sup and inf-preserving morphisms, respectively. Based on the results obtained in the previous section we are able to characterise the collection of dilations and erosions as Ω-sup-fuzzy relations.

The main contribution of this paper is presented in Section 4, where we re-define sup-fuzzy relations in the context of quantaloids. Abstracting from quantales to quantaloids allows us to generalise the concept of left converse $\backsim R$ that appears in the literature [22,24]. This is achieved by considering complete lattices equipped with an anti-isomorphism. As we observe in Definition 6, anti-isomorphism lift to define two operations that, up to certain degree, act as the pseudocomplement and dual-pseudocomplement operations in bi-Heyting algebras. With the help of these two operations we propose a definition of the left converse that by Lemma 2 coincides with the correct generalisation of left converse proposed in [22, p339].

From the results exposed throughout this paper, we contend that sup-fuzzy relations are an appropriate generalisation of the relational approach to graph based mathematical morphology developed in [21,22]. Further work to characterise some morphological operations should be done.

To define the appropriate left converse of a relation in SF-Rel, we require the existence of an anti-isomorphism on a lattice. While this allows us to define the appropriate left converse operation for some symmetric non-distributive lattices (as the diamond lattice M_3), we believe this requirement could be weakened to account for other complete lattices.

References

1. Bloch, I., Bretto, A.: Mathematical Morphology on Hypergraphs: Preliminary Definitions and Results. In: Debled-Rennesson, I., Domenjoud, E., Kerautret, B., Even, P. (eds.) Discrete Geometry for Computer Imagery. pp. 429–440. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg (2011), DOI: 10.1007/978-3-642-19867-0_36
2. Bloch, I., Maitre, H.: Fuzzy mathematical morphologies: A comparative study. Pattern Recognition **28**(9), 1341–1387 (Sep 1995), DOI: 10.1016/0031-3203(94)00312-A, `https://www.sciencedirect.com/science/article/pii/003132039400312A`
3. Couprie, M., Bertrand, G.: New Characterizations of Simple Points in 2D, 3D, and 4D Discrete Spaces. IEEE Transactions on Pattern Analysis and Machine Intelligence **31**(4), 637–648 (Apr 2009), DOI: 10.1109/TPAMI.2008.117, `http://ieeexplore.ieee.org/document/4522557/`
4. Cousty, J., Bertrand, G., Najman, L., Couprie, M.: Watershed Cuts: Minimum Spanning Forests and the Drop of Water Principle. IEEE Transactions on Pattern Analysis and Machine Intelligence **31**(8), 1362–1374 (Aug 2009), DOI: 10.1109/TPAMI.2008.173, `https://ieeexplore.ieee.org/document/4564470`, conference Name: IEEE Transactions on Pattern Analysis and Machine Intelligence
5. Cousty, J., Najman, L., Dias, F., Serra, J.: Morphological filtering on graphs. Computer Vision and Image Understanding **117**(4), 370–385 (Apr 2013), DOI: 10.1016/j.cviu.2012.08.016, `https://www.sciencedirect.com/science/article/pii/S107731421200183X`
6. Cousty, J., Najman, L., Serra, J.: Some Morphological Operators in Graph Spaces. In: Wilkinson, M.H.F., Roerdink, J.B.T.M. (eds.) Mathematical Morphology and Its Application to Signal and Image Processing. pp. 149–160. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg (2009), DOI: 10.1007/978-3-642-03613-2_14
7. Eklund, P., Gutiérrez García, J., Höhle, U., Kortelainen, J.: Semigroups in Complete Lattices, Developments in Mathematics, vol. 54. Springer International Publishing, Cham (2018), DOI: 10.1007/978-3-319-78948-4, `http://link.springer.com/10.1007/978-3-319-78948-4`
8. Freyd, P., Scedrov, A.: Categories, Allegories, North-Holland Mathematical Library, vol. 39 (1990)
9. Goguen, J.A.: L-fuzzy sets. Journal of Mathematical Analysis and Applications **18**(1), 145–174 (Apr 1967), DOI: 10.1016/0022-247X(67)90189-8, `https://www.sciencedirect.com/science/article/pii/0022247X67901898`
10. Heijmans, H.J.: First principles. In: Morphological Image Operators, Advances in Imaging and Electron Physics, vol. 216, pp. 1–520. Elsevier (2020), DOI: 10.1016/bs.aiep.2020.07.001, `https://linkinghub.elsevier.com/retrieve/pii/S1076567020300550`
11. Heijmans, H.J., Ronse, C.: The algebraic basis of mathematical morphology I. Dilations and erosions. Computer Vision, Graphics, and Image Processing **50**(3), 245–295 (Jun 1990), DOI: 10.1016/0734-189X(90)90148-O, `https://www.sciencedirect.com/science/article/pii/0734189X9090148O`
12. Heumans, H.J.A.M., Nacken, P., Toet, A., Vincent, L.: Graph morphology. Journal of Visual Communication and Image Representation **3**(1), 24–38 (Mar 1992), DOI: 10.1016/1047-3203(92)90028-R, `https://www.sciencedirect.com/science/article/pii/104732039290028R`

13. Maccarone, M.C.: Fuzzy mathematical morphology: Concepts and applications. Vistas in Astronomy **40**(4), 469–477 (Jan 1996), DOI: 10.1016/S0083-6656(96)00031-1, https://www.sciencedirect.com/science/article/pii/S0083665696000311
14. Mordeson, J.N., Nair, P.S.: Fuzzy Graphs. In: Mordeson, J.N., Nair, P.S. (eds.) Fuzzy Graphs and Fuzzy Hypergraphs, pp. 19–81. Physica-Verlag HD, Heidelberg (2000), DOI: 10.1007/978-3-7908-1854-3_2, https://doi.org/10.1007/978-3-7908-1854-3_2
15. Mulvey, C.J., Pelletier, J.W.: A Quantisation of the Calculus of Relations. In: Category Theory 1991. vol. 13, pp. 345–360. American Mathematical Society, Montreal, Canada (1992)
16. Najman, L., Talbot, H.: Mathematical Morphology. From theory to applications. Wiley (2010)
17. Plotkin, G., Stirling, C.: A Framework for Intuitionistic Modal Logics. Journal of Symbolic Logic **53**(2), 669–669 (1988), publisher: Association for Symbolic Logic
18. Rosenthal, K.I.: The theory of quantaloids, Pitman Research Notes in Mathematics, vol. 348. Longman (1996)
19. Santocanale, L.: The Involutive Quantaloid of Completely Distributive Lattices. In: Fahrenberg, U., Jipsen, P., Winter, M. (eds.) Relational and Algebraic Methods in Computer Science. pp. 286–301. Springer International Publishing, Cham (2020), DOI: 10.1007/978-3-030-43520-2_18
20. Stell, J.G.: Symmetric Heyting relation algebras with applications to hypergraphs. Journal of Logical and Algebraic Methods in Programming **84**, 440–455 (2015)
21. Stell, J.G.: Relations in Mathematical Morphology with Applications to Graphs and Rough Sets. In: Winter, S., Duckham, M., Kulik, L., Kuipers, B. (eds.) Spatial Information Theory. pp. 438–454. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg (2007), DOI: 10.1007/978-3-540-74788-8_27
22. Stell, J.G.: Relations on Hypergraphs. In: Kahl, W., Griffin, T.G. (eds.) Relational and Algebraic Methods in Computer Science. pp. 326–341. Lecture Notes in Computer Science, Springer, Berlin, Heidelberg (2012), DOI: 10.1007/978-3-642-33314-9_22
23. Stell, J.G.: Symmetric Heyting relation algebras with applications to hypergraphs. Journal of Logical and Algebraic Methods in Programming **84**(3), 440–455 (May 2015), DOI: 10.1016/j.jlamp.2014.12.001, https://www.sciencedirect.com/science/article/pii/S2352220814000972
24. Stell, J.G., Schmidt, R.A., Rydeheard, D.: A bi-intuitionistic modal logic: Foundations and automation. Journal of Logical and Algebraic Methods in Programming **85**(4), 500–519 (Jun 2016), DOI: 10.1016/j.jlamp.2015.11.003, https://www.sciencedirect.com/science/article/pii/S2352220815001406
25. Winter, M.: Goguen Categories, Trends in Logic, vol. 25. Springer Netherlands, Dordrecht, 1 edn. (2007), DOI: 10.1007/978-1-4020-6164-6, http://link.springer.com/10.1007/978-1-4020-6164-6