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## Improved composable key rates for CV-QKD

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Modern security proofs of quantum key distribution (QKD) must take finite-size effects and composable aspects into consideration. This is also the case for continuous-variable (CV) protocols, which are based on the transmission and detection of bosonic coherent states. In this paper, we refine and advance the previous theory in this area providing a more rigorous formulation for the composable key rate of a generic CV-QKD protocol. Thanks to these theoretical refinements, our general formulas allow us to prove more optimistic key rates with respect to previous literature.

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## I. INTRODUCTION

Quantum key distribution (QKD) is arguably one of the most advanced areas in quantum information, both theoretically and experimentally [1–3], with very well-known limits, such as the fundamental PLOB bound for repeaterless quantum communication [4,5] and its extension to repeaters and networks with arbitrary topologies and routing mechanisms [6]. In particular, the continuous-variable (CV) version of QKD is a preferred option that has been gradually improved in various aspects, such as the rigor of the security proofs, the speed of data processing techniques, and the distance of experimental implementations [1, Secs. 7 and 8]. In terms of CV-QKD theory, the first asymptotic analyses were extended to finite-size effects and, later, to composable security proofs [7–27] (see also Ref. [1, Sec. 9]).

Here we build on previous composable security analyses of CV-QKD [20–22] to provide a more refined and advanced formulation. Our revised formulas enable us to achieve more optimistic key rates for CV-QKD than previous literature. The results apply to a variety of protocols, including schemes with discrete alphabet or continuous (Gaussian [28,29]) modulation of coherent states, with homodyne or heterodyne detection, CV measurement device-independent (MDI) QKD [30,31], and also the postselection versions of these protocols.

The paper is structured as follows. In Sec. II we derive our general formula for the secret key rate of a generic CV-QKD protocol; this is done by refining previous theory and adopting a number of improvements, including a different approach to tensor-product reduction after error correction (proven in Appendix A). In Sec. III, we apply the results to relevant examples of CV-QKD protocols, showing the improvements in terms of key rate with respect to previous literature. Section IV is for conclusions.

## II. COMPOSABLE KEY RATE

In this section we derive an improved formula for the secret key rate of a generic CV-QKD protocol in the finite-size and composable framework. The main derivation is performed under the assumption of collective attacks, but the result will be easily extended to coherent attacks in the case of one of the Gaussian-modulated protocols. We present the various ingredients and aspects of the proof in a number of subsections.

## A. Output state of a CV-QKD protocol

Consider a CV-QKD protocol where  $N$  single-mode systems are transmitted from Alice  $A$  to Bob  $B$ . A portion  $n$  of these systems will be used for key generation, while a portion  $m = N - n$  will be used for parameter estimation. Let us assume that the bosonic communication channel depends on a number  $n_{\text{pm}}$  of parameters  $\mathbf{p} = (p_1, p_2, \dots)$  (e.g., transmissivity and thermal noise of the channel). These parameters are estimated by the parties and we will account for their partial knowledge at the end of the derivations. For now, let us assume that Alice and Bob has perfect knowledge of  $\mathbf{p}$ .

Under the action of a collective attack, the output classical-quantum (CQ) state of Alice ( $A$ ), Bob ( $B$ ), and Eve ( $E$ ) has the tensor-structure form  $\rho^{\otimes n}$ , where

$$\rho = \sum_{k,l \in \{0, \dots, 2^d - 1\}} p(k, l) |k\rangle_A \langle k| \otimes |l\rangle_B \langle l| \otimes \rho_E^{k,l}. \quad (1)$$

Here Alice's variable  $k$  and Bob's variable  $l \in \mathcal{L} = \{0, \dots, 2^d - 1\}$  are both multiary symbols ( $2^d$ -ary, equivalent to  $d$ -bit strings) and  $p(k, l)$  is their joint probability distribution (depending on the interaction used by Eve).

In the case of a protocol based on the Gaussian modulation of coherent states, the multiary symbols are the output of analog-to-digital conversion (ADC) from Alice's and Bob's quadratures,  $x$  and  $y$ , i.e., we have  $x \xrightarrow{\text{ADC}} k$  and  $y \xrightarrow{\text{ADC}} l$ . If the protocol is based on the homodyne detection, we have that  $y$  is randomly created by a random switching between the  $q$  and  $p$  quadrature (with Alice choosing the corresponding quadrature for each instance, upon Bob's classical communication). If

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the protocol is based on the heterodyne detection, both  $q$  and  $p$  quadratures are used, so we have  $y = (q_B, p_B) \xrightarrow{\text{ADC}} (l_q, l_p)$  followed by the concatenation  $l = l_q l_p$  so that the dimension is  $d = d_q + d_p$ , where  $d_q$  ( $d_p$ ) refers to the dimension of  $l_q$  ( $l_p$ ). Finally, in the case of CV-QKD protocols based on discrete-alphabet coherent states, no ADC is necessary and the discretized variables are directly expressed by the encoding variables.

Whatever protocol is used, after  $n$  uses, there will be two sequences of multiary symbols,  $\mathbf{k} = (k_1, k_2, \dots)$  and  $\mathbf{l} = (l_1, l_2, \dots)$ , each with length  $n$  (so their equivalent binary length would be  $nd$ ). These are generated with joint probability  $p(\mathbf{k}, \mathbf{l}) = \prod_{i=1}^n p(k_i, l_i)$ , and the total  $n$ -use state of Alice, Bob and Eve reads

$$\rho^{\otimes n} = \sum_{\mathbf{k}, \mathbf{l} \in \{0, \dots, 2^d - 1\}^n} p(\mathbf{k}, \mathbf{l}) |\mathbf{k}\rangle_{A^n} \langle \mathbf{k}| \otimes |\mathbf{l}\rangle_{B^n} \langle \mathbf{l}| \otimes \rho_{E^n}^{\mathbf{k}, \mathbf{l}}, \quad (2)$$

where  $|\mathbf{k}\rangle = \otimes_{i=1}^n |k_i\rangle$ ,  $|\mathbf{l}\rangle = \otimes_{i=1}^n |l_i\rangle$  and

$$\rho_{E^n}^{\mathbf{k}, \mathbf{l}} = \otimes_{i=1}^n \rho_E^{k_i, l_i}. \quad (3)$$

### B. Error correction and epsilon correctness

Alice and Bob will then perform procedures of error correction (EC) and privacy amplification (PA) over the state  $\rho^{\otimes n}$  with the final goal to approximate the  $s_n$ -bit ideal CQ state, which is of the type

$$\rho_{\text{ideal}}^n = \omega_{AB}^n \otimes \rho_{E^n}, \quad (4)$$

$$\omega_{AB}^n := 2^{-s_n} \sum_{\mathbf{s}} |\mathbf{s}\rangle_{A^n} \langle \mathbf{s}| \otimes |\mathbf{s}\rangle_{B^n} \langle \mathbf{s}|, \quad (5)$$

where Alice's and Bob's classical systems contain the same random binary sequence  $\mathbf{s}$  of length  $s_n$ , from which Eve is completely decoupled (note that the final output is a binary sequence even if we start from multiary symbols  $k$  and  $l$  for Alice and Bob).

In reverse reconciliation, Alice attempts to reconstruct Bob's sequence  $\mathbf{l}$ . During EC, Bob publicly reveals  $\text{leak}_{\text{ec}}$  bits of information to help Alice to compute her guess  $\hat{\mathbf{l}}$  of  $\mathbf{l}$  starting from her local data  $\mathbf{k}$ . In practical schemes of EC (based on linear codes, such as LDPC codes), these  $\text{leak}_{\text{ec}}$  bits of information corresponds to a syndrome  $\text{synd}(\mathbf{l})$  that Bob computes over his sequence  $\mathbf{l}$ , interpreted as a noisy codeword of a linear code agreed with Alice.

Then, as a verification, Alice and Bob publicly compare hashes computed over  $\mathbf{l}$  and  $\hat{\mathbf{l}}$ . If these hashes coincide, the two parties go ahead with the probability  $p_{\text{ec}}$ , otherwise, they abort the protocol. We denote by  $T_{\text{ec}}$  the case of a successful verification (no abort), so that  $\rho_{T_{\text{ec}}}$  represents a conditional post-EC state. More specifically, the hash comparison requires Bob to send  $\lceil -\log_2 \varepsilon_{\text{cor}} \rceil$  bits to Alice for some suitable  $\varepsilon_{\text{cor}}$  (the number of these bits is typically small in comparison to  $\text{leak}_{\text{ec}}$ ). Parameter  $\varepsilon_{\text{cor}}$  is called the  $\varepsilon$ -correctness [32, Sec. 4.3] and it bounds the probability that Alice's and Bob's corrected sequences are different even if their hashes coincide. The probability of such an error is bounded by [33]

$$p_{\text{ec}} \text{Prob}(\hat{\mathbf{l}} \neq \mathbf{l} | T_{\text{ec}}) \leq p_{\text{ec}} 2^{-\lceil -\log_2 \varepsilon_{\text{cor}} \rceil} \leq \varepsilon_{\text{cor}}. \quad (6)$$

### C. Equivalence to a projection process

As discussed above, EC consists of two steps. In the first (correction) step, Bob sends the syndrome information  $\text{synd}(\mathbf{l})$  to Alice. Conditionally on  $\text{synd}(\mathbf{l})$ , she transforms her variable via a function

$$\mathbf{k} \mapsto f_{\text{guess}}(\mathbf{k}, \text{synd}(\mathbf{l})) = \hat{\mathbf{l}} \in \{0, 2^d - 1\}^n. \quad (7)$$

The second (verification) step is the verification of the hashes. If successful, this is equivalent to having a corrected sequence  $\hat{\mathbf{l}}$  that is indistinguishable from  $\mathbf{l}$  with a probability larger than  $1 - \varepsilon_{\text{cor}}$ .

Overall, successful EC is equivalent to filtering the entire set of initial sequences  $(\mathbf{k}, \mathbf{l}) \in \{0, \dots, 2^d - 1\}^n \otimes \{0, \dots, 2^d - 1\}^n$  into a subset of "good" sequences

$$\Gamma = \{(\mathbf{k}, \mathbf{l}) : \text{Prob}(\hat{\mathbf{l}} \neq \mathbf{l} | T_{\text{ec}}) \leq \varepsilon_{\text{cor}}\}, \quad (8)$$

with associated probability  $p_{\text{ec}} = \sum_{(\mathbf{k}, \mathbf{l}) \in \Gamma} p(\mathbf{k}, \mathbf{l})$ . This can equivalently be represented by a projection

$$\rho^{\otimes n} \rightarrow \Pi_{\Gamma} \rho^{\otimes n} \Pi_{\Gamma}, \quad \Pi_{\Gamma} = \sum_{(\mathbf{k}, \mathbf{l}) \in \Gamma} |\mathbf{k}, \mathbf{l}\rangle \langle \mathbf{k}, \mathbf{l}|, \quad (9)$$

restricting the classical states to the labels  $(\mathbf{k}, \mathbf{l}) \in \Gamma$  followed by the application of the quantum operation

$$\mathcal{E}_{\text{guess}}(|\mathbf{k}, \mathbf{l}\rangle \langle \mathbf{k}, \mathbf{l}|) = |\hat{\mathbf{l}}, \mathbf{l}\rangle \langle \hat{\mathbf{l}}, \mathbf{l}|, \quad (10)$$

according to the transformation in Eq. (7). In particular, note that this operation is a completely positive trace-preserving (CPTP) map, i.e., a quantum channel.

Thus, the (normalized) post-EC state is given by

$$\tilde{\rho}_{ABE|T_{\text{ec}}}^n = \sum_{(\mathbf{k}, \mathbf{l}) \in \Gamma} \frac{p(\mathbf{k}, \mathbf{l})}{p_{\text{ec}}} |\hat{\mathbf{l}}, \mathbf{l}\rangle_{A^n B^n} \langle \hat{\mathbf{l}}, \mathbf{l}| \otimes \rho_{E^n}^{\mathbf{k}, \mathbf{l}}. \quad (11)$$

It is clear that the state above, expressed in terms of  $n$ -long sequences of  $2^d$ -ary symbols, can equivalently be rewritten in terms of  $nd$ -long binary strings. It is also important to note that, due to the projection, the state after EC no longer has a tensor product structure.

### D. Privacy amplification and epsilon secrecy

The next step is PA, which realizes the randomness extraction while decoupling Eve. The parties agree to use a function  $f$  randomly chosen from a family  $F$  of two universal hash functions with probability  $p(f)$  among a total of  $|F|$  possible choices (note that it is necessary to randomize over the hash functions as discussed in Ref. [34]). Then, they transform their multiary  $n$ -long sequences into  $nd$ -long binary strings [so the state in Eq. (11) is suitably expressed in terms of these binary strings]. Such strings are individually compressed into a key pair  $\{\hat{\mathbf{s}}, \mathbf{s}\}$  of  $s_n < nd$  random bits.

The process of PA can be described by a CPTP map  $\rho_F \otimes \tilde{\rho}_{ABE|T_{\text{ec}}}^n \rightarrow \tilde{\rho}_{ABEF|T_{\text{ec}}}^n$ , where

$$\tilde{\rho}_{ABEF|T_{\text{ec}}}^n = p_{\text{ec}}^{-1} \sum_{f, \hat{\mathbf{s}}, \mathbf{s}} p(f) p(\hat{\mathbf{s}}, \mathbf{s}) \times |\hat{\mathbf{s}}\rangle_{A^n} \langle \hat{\mathbf{s}}| \otimes |\mathbf{s}\rangle_{B^n} \langle \mathbf{s}| \otimes \rho_{E^n}^{f, \hat{\mathbf{s}}, \mathbf{s}} \otimes |f\rangle_F \langle f|, \quad (12)$$

which is a generalization of Ref. [35, Eq. (5.5)]. This also means that Alice's and Bob's sequences undergo local data processing, which cannot increase their distinguishability, i.e., we have

$$\text{Prob}(\hat{\mathbf{s}} \neq \mathbf{s} | T_{\text{ec}}) \leq \text{Prob}(\hat{\mathbf{I}} \neq \mathbf{I} | T_{\text{ec}}), \quad (13)$$

due to the pigeonhole principle. By tracing out Alice, we can write the reduced state of Bob (containing the key) and Eve

$$\bar{\rho}_{BEF|T_{\text{ec}}}^n = p_{\text{ec}}^{-1} \sum_{f, \hat{\mathbf{s}}, \mathbf{s}} p(f) p(\hat{\mathbf{s}}, \mathbf{s}) |\mathbf{s}\rangle_{B^n} \langle \mathbf{s}| \otimes \rho_{E^n}^{f, \hat{\mathbf{s}}, \mathbf{s}} \otimes |f\rangle_F \langle f|. \quad (14)$$

On the latter state, we impose the condition of  $\varepsilon$ -secrecy for Bob. First note that we may write the ideal state as  $\omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n$ , where

$$\omega_B^n := 2^{-s_n} \sum_{\mathbf{s}} |\mathbf{s}\rangle_{B^n} \langle \mathbf{s}|, \quad \bar{\rho}_{EF|T_{\text{ec}}}^n := \text{Tr}_B(\bar{\rho}_{BEF|T_{\text{ec}}}^n). \quad (15)$$

Then, we impose that the distance from this ideal state must be less than  $\varepsilon_{\text{sec}}$ , i.e., we impose

$$p_{\text{ec}} D(\bar{\rho}_{BEF|T_{\text{ec}}}^n, \omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \leq \varepsilon_{\text{sec}}. \quad (16)$$

### E. Combining correctness and secrecy into epsilon security

Following Ref. [32, Th. 4.1], we can combine the features of correctness and secrecy into a single epsilon parameter. In fact, if Eqs. (6) and (16) hold, then we may write the condition for  $\varepsilon$ -security for Alice and Bob

$$p_{\text{ec}} D(\bar{\rho}_{ABEF|T_{\text{ec}}}^n, \omega_{AB}^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \leq \varepsilon := \varepsilon_{\text{cor}} + \varepsilon_{\text{sec}}. \quad (17)$$

It is instructive to repeat the proof of this result from Ref. [32, Sec. 4.3].

*Proof.* Let us define the following state, similar to  $\bar{\rho}_{ABEF|T_{\text{ec}}}^n$  but where Alice's system is copied from Bob's so they have exactly the same key string

$$\begin{aligned} \bar{\gamma}_{ABEF|T_{\text{ec}}}^n &= p_{\text{ec}}^{-1} \sum_{f, \hat{\mathbf{s}}, \mathbf{s}} p(f) p(\hat{\mathbf{s}}, \mathbf{s}) \\ &\times |\mathbf{s}\rangle_{A^n} \langle \mathbf{s}| \otimes |\mathbf{s}\rangle_{B^n} \langle \mathbf{s}| \otimes \rho_{E^n}^{f, \hat{\mathbf{s}}, \mathbf{s}} \otimes |f\rangle_F \langle f|. \end{aligned} \quad (18)$$

Then, we can use the triangle inequality to write

$$\begin{aligned} D(\bar{\rho}_{ABEF|T_{\text{ec}}}^n, \omega_{AB}^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) &\leq D(\bar{\rho}_{ABEF|T_{\text{ec}}}^n, \bar{\gamma}_{ABEF|T_{\text{ec}}}^n) \\ &+ D(\bar{\gamma}_{ABEF|T_{\text{ec}}}^n, \omega_{AB}^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n). \end{aligned} \quad (19)$$

The first term accounts for the correctness and can be bounded as follows:

$$\begin{aligned} &D(\bar{\rho}_{ABEF|T_{\text{ec}}}^n, \bar{\gamma}_{ABEF|T_{\text{ec}}}^n) \\ &\leq p_{\text{ec}}^{-1} \sum_{f, \hat{\mathbf{s}}, \mathbf{s}} p(f) p(\hat{\mathbf{s}}, \mathbf{s}) D(|\hat{\mathbf{s}}\rangle_{A^n} \langle \hat{\mathbf{s}}|, |\mathbf{s}\rangle_{A^n} \langle \mathbf{s}|) \\ &= \sum_{\hat{\mathbf{s}} \neq \mathbf{s}} \frac{p(\hat{\mathbf{s}}, \mathbf{s})}{p_{\text{ec}}} \\ &= \text{Prob}(\hat{\mathbf{s}} \neq \mathbf{s} | T_{\text{ec}}) \\ &\leq \text{Prob}(\hat{\mathbf{I}} \neq \mathbf{I} | T_{\text{ec}}). \end{aligned} \quad (20)$$

The second term in Eq. (19) accounts for secrecy and can be manipulated as follows:

$$\begin{aligned} &D(\bar{\gamma}_{ABEF|T_{\text{ec}}}^n, \omega_{AB}^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \\ &= D(\bar{\gamma}_{BEF|T_{\text{ec}}}^n, \omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \\ &= D(\bar{\rho}_{BEF|T_{\text{ec}}}^n, \omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n), \end{aligned} \quad (21)$$

where we use the fact that the trace distance does not change if we trace Alice's cloned system in  $\bar{\gamma}^n$ .

Thus we have

$$\begin{aligned} &p_{\text{ec}} D(\bar{\rho}_{ABEF|T_{\text{ec}}}^n, \omega_{AB}^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \\ &\leq p_{\text{ec}} \text{Prob}(\hat{\mathbf{I}} \neq \mathbf{I} | T_{\text{ec}}) + p_{\text{ec}} D(\bar{\rho}_{BEF|T_{\text{ec}}}^n, \omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n). \end{aligned} \quad (22)$$

Using Eqs. (6) and (16) in the right-hand side of Eq. (22) we get Eq. (17). ■

### F. Leftover hash bound

We may now bound the distance of the privacy amplified state  $\bar{\rho}_{BEF|T_{\text{ec}}}^n$  from the ideal state  $\omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n$  containing  $s_n$  random and decoupled bits. For this, we employ the converse leftover hash bound. Following Ref. [36, Th. 6], we may write

$$p_{\text{ec}} D(\bar{\rho}_{BEF|T_{\text{ec}}}^n, \omega_B^n \otimes \bar{\rho}_{EF|T_{\text{ec}}}^n) \leq \varepsilon_s + \frac{1}{2} \sqrt{2^{s_n - H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n}}}, \quad (23)$$

where  $\sigma^n$  is Bob and Eve's subnormalized state before PA and after EC, given by

$$\begin{aligned} \sigma^n &:= \sigma_{BE|T_{\text{ec}}}^n = p_{\text{ec}} \bar{\rho}_{BE|T_{\text{ec}}}^n \\ &= \text{Tr}_A[\mathcal{E}_{\text{guess}}(\Pi_{\Gamma} \rho_{ABE}^{\otimes n} \Pi_{\Gamma})] \\ &= \sum_{(\mathbf{k}, \mathbf{l}) \in \Gamma} p(\mathbf{k}, \mathbf{l}) |\mathbf{l}\rangle_{B^n} \langle \mathbf{l}| \otimes \rho_{E^n}^{\mathbf{k}, \mathbf{l}}. \end{aligned} \quad (24)$$

By imposing the condition

$$\varepsilon_s + \frac{1}{2} \sqrt{2^{s_n - H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n}}} \leq \varepsilon_{\text{sec}}, \quad (25)$$

we certainly realize the secrecy bound in Eq. (16). If we also impose the condition of correctness in Eq. (6), we reach the condition of epsilon security for Alice and Bob expressed by Eq. (17). Setting  $\varepsilon_h := \varepsilon_{\text{sec}} - \varepsilon_s$  and re-arranging Eq. (25), we derive the following upper bound for the binary length of the key (converse leftover hash bound)

$$s_n \leq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} + 2 \log_2(2\varepsilon_h). \quad (26)$$

Thus, for the protocol to be epsilon-secure with  $\varepsilon := \varepsilon_{\text{cor}} + \varepsilon_{\text{sec}} = \varepsilon_{\text{cor}} + \varepsilon_s + \varepsilon_h$ , the binary length of the key cannot exceed the right-hand side of Eq. (26).

### G. Including the leakage due to EC

Let us better describe Eve's system  $E^n$  as  $E^n R$ , where  $E^n$  are the systems used by Eve during the quantum communication while  $R$  is an extra register of dimension  $\dim_R = 2^{\lfloor \text{leak}_{\text{ec}} + \lceil -\log_2 \varepsilon_{\text{cor}} \rceil \rfloor}$ . The latter is used by Eve to store the bits that are leaked during EC. This means that Eq. (26) is more precisely given by

$$s_n \leq H_{\min}^{\varepsilon_s}(B^n|E^n R)_{\sigma^n} + 2 \log_2(2\varepsilon_h). \quad (27)$$

We can then use Ref. [37, Prop. 5.10] for the smooth min-entropy computed over generally subnormalized states, which leads to

$$\begin{aligned} H_{\min}^{\varepsilon_s}(B^n|E^n R)_{\sigma^n} &\geq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} - \log_2 \dim_R \\ &= H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} - \text{leak}_{\text{ec}} - \lceil -\log_2 \varepsilon_{\text{cor}} \rceil \\ &\geq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} - \text{leak}_{\text{ec}} - \log_2(2/\varepsilon_{\text{cor}}). \end{aligned} \tag{28}$$

We then replace the above expression in Eq. (27), which leads to a stricter upper bound for the key length

$$\begin{aligned} s_n &\leq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} + 2\log_2(2\varepsilon_h) \\ &\quad - \text{leak}_{\text{ec}} - \log_2(2/\varepsilon_{\text{cor}}) \\ &= H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} - \text{leak}_{\text{ec}} + \theta, \end{aligned} \tag{29}$$

where we have set

$$\theta := \log_2(2\varepsilon_h^2\varepsilon_{\text{cor}}). \tag{30}$$

Note that we include the more precise term  $\theta$  instead of just  $\log_2(2\varepsilon_h^2)$  as in past derivations [20–22].

### H. Tensor-product reduction and asymptotic equipartition property

We may replace the smooth-min entropy of the subnormalized state  $\sigma^n$  after EC with that of the normalized state  $\rho^{\otimes n}$  before EC. As we show in Appendix A, we may write the following tensor-product reduction

$$H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} \geq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\rho^{\otimes n}}. \tag{31}$$

This is a major improvement with respect to Ref. [20].

Because the state before the EC projection has a tensor product form (under collective attacks), we can now write a simpler (but larger) lower bound that is based on the von Neumann entropy of the single-copy state  $\rho$  in Eq. (1). In fact, we may apply the asymptotic equipartition property (AEP) [37, Cor. 6.5] and write

$$H_{\min}^{\varepsilon_s}(B^n|E^n)_{\rho^{\otimes n}} \geq nH(B|E)_{\rho} - \sqrt{n}\Delta_{\text{aep}}, \tag{32}$$

where

$$\begin{aligned} \Delta_{\text{aep}} &:= 4\log_2(\sqrt{|\mathcal{L}|} + 2)\sqrt{-\log_2(1 - \sqrt{1 - \varepsilon_s^2})} \\ &\simeq 4\log_2(\sqrt{|\mathcal{L}|} + 2)\sqrt{\log_2(2/\varepsilon_s^2)}, \end{aligned} \tag{33}$$

and  $|\mathcal{L}| = 2^d$  is the cardinality of the discretized variable  $l$  (see Ref. [37, Th. 6.4] and Ref. [21, Sec. 2.F.1]).

### I. Upper bound for the secret-key rate

Using Eqs. (31) and (32) in Eq. (29), we may write the following stricter upper bound

$$s_n \leq nH(B|E)_{\rho} - \text{leak}_{\text{ec}} - \sqrt{n}\Delta_{\text{aep}} + \theta, \tag{34}$$

where  $\rho$  is the single-copy state in Eq. (1). We finally expand the conditional entropy as

$$H(B|E)_{\rho} = H(l|E)_{\rho} = H(l) - \chi(l : E)_{\rho}, \tag{35}$$

where  $H(l)$  is the Shannon entropy of  $l$ , and  $\chi(l : E)_{\rho}$  is Eve’s Holevo bound with respect to  $l$ . Therefore, we get

$$s_n \leq n[H(l) - \chi(l : E)_{\rho}] - \text{leak}_{\text{ec}} - \sqrt{n}\Delta_{\text{aep}} + \theta. \tag{36}$$

Alternatively, this can be written as

$$s_n \leq nR_{\infty} - \sqrt{n}\Delta_{\text{aep}} + \theta, \tag{37}$$

where we have introduced the asymptotic key rate

$$R_{\infty} = H(l) - \chi(l : E)_{\rho} - n^{-1}\text{leak}_{\text{ec}}. \tag{38}$$

The result in Eq. (37) is an upper bound to the number of secret random bits that Alice and Bob can extract with epsilon security  $\varepsilon = \varepsilon_{\text{cor}} + \varepsilon_s + \varepsilon_h$ . Note that the secret key rate will need to account for the fact that this amount of bits is generated with probability  $p_{\text{ec}}$  and that only a fraction  $n/N$  of the total systems are used for key generation. Thus, the composable secret key rate (bits per use) of a generic CV-QKD protocol under collective attacks is given by

$$R = \frac{p_{\text{ec}}s_n}{N}. \tag{39}$$

More explicitly, we have the upper bound

$$R \leq R_{\text{UB}} = \frac{p_{\text{ec}}[nR_{\infty} - \sqrt{n}\Delta_{\text{aep}} + \theta]}{N}. \tag{40}$$

### J. Achievable key rate for optimal PA

The result in Eq. (40) means that Alice and Bob cannot exceed  $R_{\text{UB}}$  bits per use if they want to have  $\varepsilon$ -security assured. Assuming they can implement optimal PA, they can reach a rate  $R^{\text{opt}}$ , which is still bounded by  $R_{\text{UB}}$  from above, but we can also guarantee that at least  $R_{\text{LB}}$  bits per use are generated. Basically, for a protocol with optimal extraction of randomness [37, Sec. 8.2], we may have a guaranteed  $\varepsilon$ -security and a rate satisfying  $R_{\text{LB}} \leq R^{\text{opt}} \leq R_{\text{UB}}$ , where  $R_{\text{UB}}$  is given in Eq. (40) and

$$R_{\text{LB}} = \frac{p_{\text{ec}}[nR_{\infty} - \sqrt{n}\Delta_{\text{aep}} + \theta - 1]}{N}. \tag{41}$$

The lower bound in Eq. (41) is proven by repeating the proof and using the direct part of the leftover hash bound [36] {see also Ref. [37, Eq. (8.7)]} for the number of bits  $s_n^{\text{opt}}$  that are achievable by a protocol with optimal data processing. For this number, we may in fact write

$$s_n^{\text{opt}} \geq H_{\min}^{\varepsilon_s}(B^n|E^n)_{\sigma^n} + 2\log_2(\sqrt{2}\varepsilon_h). \tag{42}$$

We can see that the  $-1$  difference between Eqs. (26) and (42) become an extra  $-p_{\text{ec}}/N$  in Eq. (41). Because  $N$  is typically large, we also see that  $R_{\text{LB}} \simeq R_{\text{UB}}$ .

Note that the direct leftover hash bound was used in the derivations of Refs. [20–22], which therefore provided formulas for the rate achievable by protocols with optimal PA. However, these previous works are more pessimistic than our current result due to a different tensor-product reduction with respect to Eq. (31). In particular, the key-rate lower bound from Ref. [21] takes the form

$$R_{\text{LB}}^{\text{old}} = \frac{p_{\text{ec}}[nR_{\infty} - \sqrt{n}\Delta'_{\text{aep}} + \theta' - 1]}{N}, \tag{43}$$

where  $\theta' = \theta + \log_2[p_{\text{ec}}(1 - \varepsilon_s^2/3)]$ , and

$$\Delta'_{\text{aep}} = [\Delta_{\text{aep}}]_{\varepsilon_s \rightarrow p_{\text{ec}}\varepsilon_s^2/3}. \quad (44)$$

(To be precise the formula above is already a refinement since we have also included more precise leakage contribution, as explained in Sec. II G).

### K. Specification to various protocols

#### 1. Formula for discrete-alphabet coherent state protocols

More specific formulas for a discrete-alphabet protocol are immediately derived. Let us define the reconciliation parameter  $\beta \in [0, 1]$  by setting

$$H(l) - n^{-1}\text{leak}_{\text{ec}} = \beta I(k : l), \quad (45)$$

where  $I(k : l)$  is Alice and Bob's mutual information. Then, the asymptotic key rate takes the form

$$R_\infty = \beta I(k : l) - \chi(l : E)_\rho. \quad (46)$$

This is to be replaced in Eq. (40) for the upper bound, and Eq. (41) for the lower bound with optimal PA.

#### 2. Formula for Gaussian-modulated coherent state protocols

In the case of a Gaussian-modulated protocol, we need to express the formulas in terms of quadratures. First, we redefine the reconciliation parameter  $\beta \in [0, 1]$  as

$$H(l) - n^{-1}\text{leak}_{\text{ec}} = \beta I(x : y), \quad (47)$$

where  $I(x : y) \geq I(k : l)$  is Alice and Bob's mutual information computed over their continuous variables. Second, we exploit the data processing inequality for Eve's Holevo bound, so  $\chi(l : E)_\rho \leq \chi(y : E)_\rho$  under digitalization  $y \xrightarrow{\text{ADC}} l$ . Thus, we can use the asymptotic rate

$$R_\infty = \beta I(x : y) - \chi(y : E)_\rho, \quad (48)$$

to be replaced in the previous general formulas.

#### 3. Other protocols

Other protocols can be considered. For example, the composable key rate of CV-MDI-QKD can be expressed using our general formulation once we replace the corresponding asymptotic expression  $R_\infty$ . The same can be stated for postselection protocols, which also involves the introduction of an extra (postselection) probability  $p_{\text{ps}}$ , appearing as a further prefactor in Eqs. (40) and (41), i.e.,  $p_{\text{ec}}[\dots]/N \rightarrow p_{\text{ps}}p_{\text{ec}}[\dots]/N$ . In general, the postselection process can be seen as a global filter that distills the number of runs and is applied before the standard processing of data via EC and PA.

### L. Parameter estimation

The asymptotic key rate  $R_\infty$  depends on a number  $n_{\text{pm}}$  of parameters  $\mathbf{p}$ . By sacrificing  $m$  systems, Alice and Bob can compute maximum likelihood estimators  $\hat{\mathbf{p}}$  and worst-case values  $\mathbf{p}_{\text{wc}}$ , which are  $w$  standard deviations away from the mean values of the estimators. The worst-case value bounds the true value of a parameter up to an error probability  $\varepsilon_{\text{pe}} = \varepsilon_{\text{pe}}(w)$ . This means that, overall,  $n_{\text{pm}}$  worst-case values  $\mathbf{p}_{\text{wc}}$  will bound the

parameters  $\mathbf{p}$  up to a total error probability  $\simeq n_{\text{pm}}\varepsilon_{\text{pe}}$ . Because PE occurs before EC, this probability needs to be multiplied by  $p_{\text{ec}}$ , so we have a total modified epsilon security

$$\varepsilon = \varepsilon_{\text{cor}} + \varepsilon_s + \varepsilon_h + p_{\text{ec}}n_{\text{pm}}\varepsilon_{\text{pe}}. \quad (49)$$

In the composable formulas of Eqs. (40) and (41), the asymptotic term  $R_\infty = R_\infty(\mathbf{p})$  will be computed on the estimators and worst-case values, i.e., replaced by

$$R_\infty^{\text{pe}} := R_\infty(\hat{\mathbf{p}}, \mathbf{p}_{\text{wc}}). \quad (50)$$

In particular, the expressions in Eqs. (46) and (48) will be replaced by

$$R_\infty^{\text{pe}} = \beta [I]_{\hat{\mathbf{p}}} - [\chi_\rho]_{\mathbf{p}_{\text{wc}}}. \quad (51)$$

### M. From one block to a session of blocks

In a typical fiber-based scenario, a QKD session is stable, i.e., the main channel parameters are constant for a substantial period of time. This means that we can consider a session of  $n_{\text{bks}}$  blocks, each block with size  $N$ . In this scenario, the success probability  $p_{\text{ec}}$  becomes the fraction of blocks that survive EC (the value  $1 - p_{\text{ec}}$  is also known as frame error rate). Assuming such a stable QKD session, PE can be performed on a large number of points, namely  $n_{\text{bks}}m$ . This approach leads to better estimators and worst-case values to be used in Eq. (50). Using these improved statistics, Alice and Bob will then implement EC block-by-block. Each block surviving EC will undergo PA, where it is subject to a hash function randomly chosen from a two-universal family. Each block compressed by PA is then concatenated into the final key.

### N. Extension to coherent attacks for heterodyne

One can extend the security of the Gaussian-modulated protocol with heterodyne detection to coherent attacks, following the Gaussian de Finetti reduction of Ref. [19]. The parties need to verify that the Hilbert space of the signal states is suitably constrained. In other words, the energy of Alice's and Bob's states should be less than some threshold values,  $d_A$  and  $d_B$ , respectively. The parties execute a random energy test over  $k$  states to estimate the energy of the other  $n$  signal states that participate in the standard steps of the protocol. Given that the test is successful with probability  $p_{\text{en}}$  and that the protocol is  $\varepsilon$ -secure against collective Gaussian attacks, the new key length is decreased by the following amount of secret bits [19]  $s'_n \leq s_n - \Phi$ , where

$$\Phi := 2 \left[ \log_2 \binom{K+4}{4} \right], \quad (52)$$

and

$$K = \max \left\{ 1, n(d_A + d_B) \frac{1 + 2\sqrt{\frac{\ln(8/\varepsilon)}{2n} + \frac{\ln(8/\varepsilon)}{n}}}{1 - 2\sqrt{\frac{\ln(8/\varepsilon)}{2k}}} \right\}. \quad (53)$$

The number of channel uses per block is extended to  $N' = N + k$ , the epsilon-security is rescaled to

$$\varepsilon' = \frac{K^4}{50} \varepsilon, \quad (54)$$

and the probability of not aborting  $p_{ec}$  is replaced by  $p_{ec} \rightarrow p_{en}p_{ec}$ . One may set Alice's energy threshold to be larger than the mean photon number  $\bar{n}_A = V/2$  of the average thermal state created by her classical modulation  $V$ . More specifically, taking into account statistical calculations due to the use of  $k$  signal states, one may set  $d_A \geq \bar{n}_A + \mathcal{O}(k^{-1/2})$ . Then, under the assumption of a lossy channel with reasonable excess noise, the mean number of photons received by Bob is smaller than  $\bar{n}_A$ , so if we set  $d_B = d_A$ , we certainly have  $d_B \geq \bar{n}_A + \mathcal{O}(k^{-1/2})$ . These conditions lead to an almost successful energy test  $p_{en} \simeq 1$ . Consequently, the secret key rate of the heterodyne protocol under coherent attacks will be given by

$$R' = \frac{p_{ec}s'_n}{N'}, \quad (55)$$

constrained by the upper bound [similar to Eq. (40)]

$$R' \leq R'_{UB} = \frac{p_{ec}[nR_\infty - \sqrt{n}\Delta_{aep} + \theta - \Phi]}{N'}, \quad (56)$$

and the lower bound [similar to Eq. (41)]

$$R' \geq R'_{LB} = R'_{UB} - \frac{p_{ec}}{N'}. \quad (57)$$

### O. Practical considerations

In an experimental implementation of a CV-QKD protocol, the parties have to numerically estimate two crucial parameters: the EC probability  $p_{ec}$  and the reconciliation efficiency  $\beta$ . The EC probability can be computed as the ratio  $p_{ec} = \frac{n_{ec}}{n_{bks}}$  between the  $n_{ec}$  successfully corrected blocks and the total number of blocks of a session  $n_{bks}$ , assuming that the channel is stable (see also [24,25,31]). The reconciliation efficiency can be computed from the leakage of the EC scheme employed. Typically, the EC scheme exploits nonbinary LDPC codes, described by a  $c \times n$  parity check matrix with code rate  $R_{code} = c/n$ , where  $c$  is the number of parity checks. In this case, the leakage can be bounded by

$$n^{-1}\text{leak}_{ec} \leq d_{\text{least}} - R_{code}d_{\text{syn}}, \quad (58)$$

where  $d_{\text{least}}$  is the number of the least significant bits sent on the clear, while  $d_{\text{syn}}$  is the number of syndrome bits (see Refs. [24,25] for details and precise definitions).

Once the leakage is bounded, one may use Eqs. (45) or (47) to compute the reconciliation parameter  $\beta$ . However, in a practical setting, the value of the entropy  $H(l)$  is also not exactly known and must be estimated. During PE, the parties calculate the frequency  $f_l = n_l/n$  of the value  $l$ , starting from its  $n_l$  occurrences in the sequence of length  $n$ . In this way, they construct the estimator

$$\hat{H}(l) = - \sum_{l=0}^{2^d-1} f_l \log_2 f_l. \quad (59)$$

The value of this estimator is then used in Eqs. (45) or (47) to derive an estimate for  $\beta$  [38].

The uncertainty on the value of Bob's entropy has also an effect at the level of the composable key rate, introducing a further epsilon parameter. For the entropy estimator, we have

$$H(l) \geq \mathbb{E}(\hat{H}(l)), \quad (60)$$

and we can write

$$\text{Prob}[|\hat{H}(l) - \mathbb{E}(\hat{H}(l))| \geq \delta_{\text{ent}}] \leq \varepsilon_{\text{ent}}, \quad (61)$$

for

$$\delta_{\text{ent}} = \log_2(n) \sqrt{\frac{2 \ln(2/\varepsilon_{\text{ent}})}{n}}. \quad (62)$$

This means that we have the condition

$$- \delta_{\text{ent}} \leq \hat{H}(l) - \mathbb{E}(\hat{H}(l)) \leq \delta_{\text{ent}} \quad (63)$$

with probability larger than  $1 - \varepsilon_{\text{ent}}$ .

Combining the inequality above with Eq. (60), we get

$$H(l) \geq \hat{H}(l) - \delta_{\text{ent}} \quad (64)$$

up to an error probability  $\varepsilon_{\text{ent}}$ . In other words, we can replace Bob's entropy in the asymptotic rate of Eq. (38) with the lower bound in Eq. (64) computed from the estimator in Eq. (59). This leads to a stricter upper bound for the composable secret key rate. More precisely, Eq. (37) becomes

$$s_n \leq n\hat{R}_\infty - n\delta_{\text{ent}} - \sqrt{n}\Delta_{aep} + \theta, \quad (65)$$

where the asymptotic key rate becomes

$$\hat{R}_\infty = \hat{H}(l) - \chi(l : E)_\rho - n^{-1}\text{leak}_{ec}. \quad (66)$$

Thus, the corresponding composable secret key rate

$$R = \frac{p_{ec}s_n}{N} \quad (67)$$

is upper bounded by

$$R \leq \hat{R}_{UB} = \frac{p_{ec}[n\hat{R}_\infty - n\delta_{\text{ent}} - \sqrt{n}\Delta_{aep} + \theta]}{N}, \quad (68)$$

with overall  $\varepsilon$ -security  $\varepsilon = \varepsilon_{\text{cor}} + \varepsilon_s + \varepsilon_h + p_{ec}\varepsilon_{\text{ent}}$ . Note that  $\varepsilon_{\text{ent}}$  is rescaled by  $p_{ec}$  because Bob's entropy is evaluated during PE and, therefore, before EC. Similarly, according to the discussion in Sec. II J, we may write the lower bound

$$R^{\text{opt}} \geq \hat{R}_{UB} - \frac{p_{ec}}{N} \quad (69)$$

for a protocol with optimal PA.

Including the estimation of the channel parameter  $\mathbf{p}$  via  $\hat{\mathbf{p}}$  and  $\mathbf{p}_{wc}$ , the asymptotic rate in Eq. (66) becomes  $\hat{R}_\infty^{\text{pe}} = \hat{R}_\infty(\hat{\mathbf{p}}, \mathbf{p}_{wc})$ . In particular for the protocols in Sec. II K, the rates in Eqs. (46) and (48) become

$$\hat{R}_\infty^{\text{pe}} = \hat{\beta}[I]_{\hat{\mathbf{p}}} - [\chi_\rho]_{\mathbf{p}_{wc}}. \quad (70)$$

By replacing  $\hat{R}_\infty^{\text{pe}} \rightarrow \hat{R}_\infty$  in Eq. (68), we therefore bound the composable secret key rate, which accounts for the entire PE process, with overall  $\varepsilon$ -security

$$\varepsilon = \varepsilon_{\text{cor}} + \varepsilon_s + \varepsilon_h + p_{ec}\varepsilon_{\text{ent}} + p_{ec}n_{\text{pm}}\varepsilon_{\text{pe}}. \quad (71)$$

Finally, for the heterodyne protocol, we may extend the security to coherent attacks repeating the modifications that lead to Eqs. (56) and (57) of Sec. II N.

### III. EXAMPLES WITH THE MAIN GAUSSIAN-MODULATED PROTOCOLS

In order to use the composable formula, we need to specify the asymptotic key rate and the PE procedure, so that we can

compute the rate  $R_\infty^{\text{pe}}$  in Eq. (50) to be replaced in Eq. (40). Here, we report the known formulas for the asymptotic key rates of the Gaussian-modulated coherent-state protocols (with homodyne and heterodyne detection). These asymptotic formulas can be found in a number of papers (e.g., see Ref. [1] and references therein). Then we consider the modifications due to PE.

### A. Gaussian modulation of coherent states with homodyne detection

We model the link connecting the parties as a thermal-loss channel with transmissivity  $T = 10^{-D/10}$  (where  $D$  is here the loss in dB) and excess noise  $\xi$ . The dilation of the channel is represented by a beam splitter with transmissivity  $T$  that Eve uses to inject one mode of a two-mode squeezed vacuum (TMSV) state with variance

$$\omega = \frac{T\xi}{1-T} + 1. \quad (72)$$

Eve's injected mode is therefore coupled with Alice's incoming mode via the beam splitter and the output is received by Bob, who detects it using a homodyne detector with efficiency  $\eta$  and electronic noise  $u_{\text{el}}$  (both local parameters that can be considered to be trusted in a well-calibrated scenario). The other, environmental, output of the beam splitter is stored by Eve in a quantum memory, together with the kept mode of the TMSV state. In this way, many modes are collected in Eve's quantum memory, which is finally subject to an optimal joint measurement (collective entangling-cloner attack).

Alice and Bob's mutual information is given by

$$I(x : y) = \frac{1}{2} \log_2 \left[ 1 + \frac{V}{\xi + (1 + u_{\text{el}})/(T\eta)} \right], \quad (73)$$

where  $V$  is Alice's modulation. The CM of Eve's output state (her partially transmitted TMSV state) is given by

$$\mathbf{V}_E = \begin{pmatrix} \omega \mathbf{I} & \psi \mathbf{Z} \\ \psi \mathbf{Z} & \phi \mathbf{I} \end{pmatrix}, \quad (74)$$

where  $\mathbf{I} = \text{diag}(1, 1)$ ,  $\mathbf{Z} = \text{diag}(1, -1)$  and

$$\psi = \sqrt{T(\omega^2 - 1)}, \quad (75)$$

$$\phi = T\omega + (1 - T)(V + 1). \quad (76)$$

Then, Eve's conditional CM (conditioned on Bob's outcome) is given by

$$\mathbf{V}_{E|y} = \mathbf{V}_E - b^{-1} \begin{pmatrix} \gamma^2 \mathbf{\Pi} & \gamma \theta \mathbf{\Pi} \\ \gamma \theta \mathbf{\Pi} & \theta^2 \mathbf{\Pi} \end{pmatrix}, \quad (77)$$

where  $\mathbf{\Pi} = \text{diag}\{1, 0\}$  and

$$b = T\eta(V + \xi) + 1 + u_{\text{el}}, \quad (78)$$

$$\gamma = \sqrt{\eta(1 - T)(\omega^2 - 1)}, \quad (79)$$

$$\theta = \sqrt{\eta T(1 - T)(\omega - V - 1)}. \quad (80)$$

By calculating the symplectic eigenvalues of the total CM,  $v_+$  and  $v_-$ , and those of the conditional CM,  $\tilde{v}_+$  and  $\tilde{v}_-$ , we obtain Eve's Holevo information on Bob's outcome

$$\chi(E : y) = h(v_+) + h(v_-) - h(\tilde{v}_+) - h(\tilde{v}_-), \quad (81)$$

where we use the usual CV-based entropy function

$$h(v) := \frac{v+1}{2} \log_2 \frac{v+1}{2} - \frac{v-1}{2} \log_2 \frac{v-1}{2}. \quad (82)$$

Then the asymptotic secret key rate is given by the difference between the mutual information (multiplied by the reconciliation efficiency  $\beta$ ) and Eve's Holevo information as in Eq. (48).

### B. Gaussian modulation of coherent states with heterodyne detection

For the protocol with heterodyne detection, the mutual information is a simple modification of the previous one in Eq. (73) and given by

$$I(x : y) = \frac{V_0}{2} \log_2 \left[ 1 + \frac{V}{\xi + (V_0 + u_{\text{el}})/(\eta T)} \right], \quad (83)$$

where  $V_0 = 2$  (note that for  $V_0 = 1$  we get the expression valid for homodyne detection). Eve's CM is the same as in Eq. (74), but the conditional CM is instead given by

$$\mathbf{V}_{E|y} = \mathbf{V}_E - (b+1)^{-1} \begin{pmatrix} \gamma^2 \mathbf{I} & \gamma \theta \mathbf{Z} \\ \gamma \theta \mathbf{Z} & \theta^2 \mathbf{I} \end{pmatrix}. \quad (84)$$

### C. Parameter estimation and final performance

Let us now include PE, assuming that  $m$  signals are sacrificed for building the estimators of the channel parameters (to be used in the mutual information) and the associated worst-case values (to be used in Eve's Holevo bound). One therefore computes estimators  $\hat{T} \simeq T$ ,  $\hat{\xi} \simeq \xi$  and the following worst-case values:

$$T_{\text{wc}} \simeq T - w\sigma_T, \quad (85)$$

$$\xi_{\text{wc}} \simeq \frac{T}{T_{\text{wc}}} \xi + w\sigma_\xi, \quad (86)$$

where

$$\sigma_T = \frac{2T}{\sqrt{V_0 m}} \sqrt{c_{\text{pe}} + \frac{\xi + \frac{V_0 + u_{\text{el}}}{\eta T}}{V}}, \quad (87)$$

$$\sigma_\xi = \sqrt{\frac{2}{V_0 m} \frac{\eta T \xi + V_0 + u_{\text{el}}}{\eta T_{\text{wc}}}}. \quad (88)$$

In the equations above,  $V_0 = 1$  is for homodyne detection and  $V_0 = 2$  is for heterodyne detection. Then, in Eq. (87), the term  $c_{\text{pe}}$  can be set to zero [14] (in fact, another choice would be  $c_{\text{pe}} = 2$  [15] based on a weaker assumption [39]). The parameter  $w$  that connects the worst-case values with  $\varepsilon_{\text{pe}}$  is simply given by an inverse error function when we assume a Gaussian approximation for the parameters, i.e.,

$$w = \sqrt{2} \text{erf}^{-1}(1 - \varepsilon_{\text{pe}}). \quad (89)$$

However, when stricter conditions are required, e.g., in the case of coherent attacks [see Eq. (54)], we use chi-squared distribution tail bounds where  $w$  is given by

$$w = \sqrt{2 \ln \varepsilon_{\text{pe}}^{-1}}. \quad (90)$$

(See Appendix B for more details.)

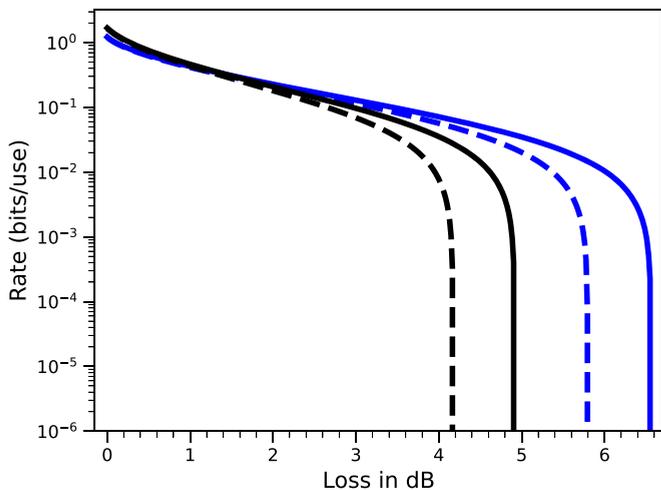


FIG. 1. Improved composable secret key rate [upper bound of Eq. (40)] for the Gaussian modulated coherent-state protocol with homodyne detection (blue solid line) and heterodyne detection (black solid line) with respect to channel loss in dB. These lines overlap with those associated with the lower bound of Eq. (41). The corresponding dashed lines are computed using Eq. (43), based on previous literature. We have set  $\beta = 0.98$  and  $p_{ec} = 0.95$ . Excess noise is  $\xi = 0.01$ , detection efficiency is  $\eta = 0.6$ , and electronic noise is  $u_{el} = 0.1$ . Security epsilons have all been set to  $2^{-32}$ . The cardinality of the alphabet is  $|\mathcal{L}| = 2^7$  for homodyne and  $|\mathcal{L}| = 2^{14}$  for heterodyne. Block size is  $N = 10^7$  and PE is based on  $m = N/10$  sacrificed signals. We have optimized the results over the variance  $V$  of Alice's Gaussian modulation.

Thus, by using  $\hat{\mathbf{p}} = (\hat{T}, \hat{\xi})$  and  $\mathbf{p}_{wc} = (T_{wc}, \xi_{wc})$ , we compute the PE rate as in Eq. (51) to be replaced in Eq. (40) for both the homodyne and heterodyne protocols. Let us assume *ad hoc* values for  $p_{ec}$  and  $\beta$  (the exact numerical values of these parameters are known after a realistic implementation or simulation of EC, as discussed in Sec. II O). Then, we show the performances of the two protocols in Figs. 1 and 2. More specifically, in Fig. 1, we depict the secret key rate versus channel loss, while, in Fig. 2, we show its behavior with respect to block size. For the sake of comparison, we have also included the results based on previous literature [20,21] (refined in Sec. II J). From the figures, we can see a significant improvement in the key rate performance both in terms of robustness to loss and smaller block size.

#### IV. CONCLUSIONS

In this paper, we have introduced an improved formulation for the composable and finite-size secret key rate of a generic CV-QKD protocol. By resorting to previous theory and proving various other tools, such as a refined tensor-product reduction for the state after error correction, we have derived simpler and more optimistic formulas, able to show an improvement in the general performance of CV-QKD. As shown in the examples, this improvement can be appreciated both in terms of increased robustness to loss and/or reduced requirements for the size of the usually larger QKD blocks. In general, this paper contributes to making a step forward in the rigorous deployment of CV-QKD protocols in practical scenarios.

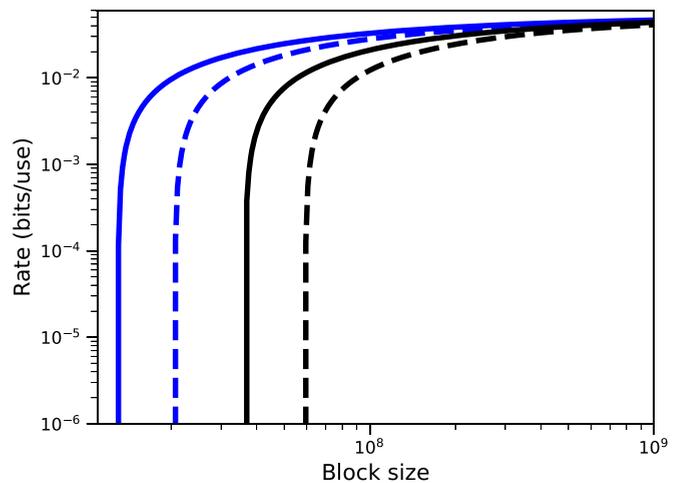


FIG. 2. Improved composable secret key rate [upper bound of Eq. (40)] for the Gaussian modulated coherent-state protocol with homodyne detection (blue solid line) and heterodyne detection (red solid line) with respect to the block size  $N$ . These lines coincide with those computed from the lower bound of Eq. (41). The corresponding dashed lines are computed using Eq. (43), based on previous literature. Loss is set to 7 dB, while all the other parameters are chosen as in Fig. 1.

#### ACKNOWLEDGMENTS

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#### APPENDIX A: PROOF OF THE TENSOR-PRODUCT REDUCTION IN EQ. (31)

Consider an arbitrary Hilbert space  $\mathcal{H}$  and two generally subnormalized states  $\rho, \tau \in S_{\leq}(\mathcal{H})$  with  $\text{Tr}\rho, \text{Tr}\tau \leq 1$ . We may consider the purified distance [40]  $P(\rho, \tau) = \sqrt{1 - F_G(\rho, \tau)^2}$ , where  $F_G$  is the generalized quantum fidelity [37, Def. 3.2, Lemma 3.1].

For any (generally sub-normalized) state  $\rho$  of two quantum systems  $A$  and  $B$ , we may write [37, Def. 5.2]

$$H_{\min}^{\varepsilon}(A|B)_{\rho} = \max_{\tau \in \mathcal{B}^{\varepsilon}(\rho)} H_{\min}(A|B)_{\tau}, \quad (\text{A1})$$

where

$$\mathcal{B}^{\varepsilon}(\rho) := \{\rho' : \text{Tr}\rho' \leq 1, P(\rho', \rho) \leq \varepsilon < 1\} \quad (\text{A2})$$

is a ball of generally subnormalized states around  $\rho$ . In particular, for any generally subnormalized CQ state

$$\rho_{CQ} = \sum_x p_x |x\rangle_C \langle x| \otimes \rho_Q^x, \quad (\text{A3})$$

for  $x$  in the alphabet  $\mathcal{X}$ , we can find another generally subnormalized CQ state  $\tau_{CQ} \in \mathcal{B}^{\varepsilon}(\rho_{CQ})$  such that [37, Prop. 5.8]

$$H_{\min}^{\varepsilon}(C|Q)_{\rho} = H_{\min}(C|Q)_{\tau}. \quad (\text{A4})$$

In particular, we may write

$$\tau_{CQ} = \sum_x q_x |x\rangle_C \langle x| \otimes \tau_Q^x. \quad (\text{A5})$$

Let us now consider a CCQ extension of  $\rho_{CQ}$  denoted by  $\rho_{C'CQ}$  such as  $\rho_{CQ} = \text{Tr}_{C'}(\rho_{C'CQ})$ . More specifically, we may write

$$\rho_{C'CQ} = \sum_{x',x} p_{x',x} |x',x\rangle_{C'C} \langle x',x| \otimes \rho_Q^{x',x}, \quad (\text{A6})$$

$$\rho_{CQ} = \sum_{x',x} p_{x',x} |x\rangle_C \langle x| \otimes \rho_Q^{x',x}, \quad (\text{A7})$$

where the summation takes place over all the elements  $x' \in \mathcal{X}'$  and  $x \in \mathcal{X}$  of the basis  $\{|x',x\rangle_{C'C} \langle x',x|\}$ .

Then there is an extension for  $\tau_{CQ}$  [41, Col. 9], denoted by  $\bar{\tau}_{C'CQ}$ , such that

$$P(\bar{\tau}_{C'CQ}, \rho_{C'CQ}) = P(\tau_{CQ}, \rho_{CQ}) \leq \varepsilon. \quad (\text{A8})$$

Note that, due to the monotonicity of the purified distance [37, Theorem 3.4] under CPTP maps  $\mathcal{E}$ , we have

$$P(\mathcal{E}(\bar{\tau}_{C'CQ}), \mathcal{E}(\rho_{C'CQ})) \leq P(\bar{\tau}_{C'CQ}, \rho_{C'CQ}). \quad (\text{A9})$$

In particular, consider a ‘‘pinching’’ channel [42, Def. 4.4] in the basis  $\{|x',x\rangle_{C'C} \langle x',x|\}$ , i.e.,

$$\mathcal{E}_{\text{pch}}(\rho) = \sum_{x',x} |x',x\rangle_{C'C} \langle x',x| \rho |x',x\rangle_{C'C} \langle x',x|. \quad (\text{A10})$$

This channel transforms an arbitrary input state into an output CCQ state, which is classical in the systems  $C'C$ , i.e., with respect to the basis  $\{|x',x\rangle_{C'C} \langle x',x|\}$ . At the same time, it is clear that this channel does not change  $\rho_{C'CQ}$ . According to Sec. A 1, we may write

$$\begin{aligned} \tau_{C'CQ} &= \mathcal{E}_{\text{pch}}(\bar{\tau}_{C'CQ}) \\ &= \sum_{x',x} q_{x',x} |x',x\rangle_{C'C} \langle x',x| \otimes \tau_Q^{x',x}, \end{aligned} \quad (\text{A11})$$

$$\tau_{CQ} = \sum_{x',x} q_{x',x} |x\rangle_C \langle x| \otimes \tau_Q^{x',x}, \quad (\text{A12})$$

and we have

$$P(\tau_{C'CQ}, \rho_{C'CQ}) = P(\mathcal{E}_{\text{pch}}(\bar{\tau}_{C'CQ}), \mathcal{E}_{\text{pch}}(\rho_{C'CQ})) \leq \varepsilon, \quad (\text{A13})$$

as a consequence of Eqs. (A8) and (A9) specified to the pinching channel.

Consider the joint projection

$$\Pi := \sum_{(x',x) \in \Gamma} |x',x\rangle_{C'C} \langle x',x|, \quad (\text{A14})$$

defined over a reduced alphabet  $\Gamma \subseteq \mathcal{X}' \otimes \mathcal{X}$  for the classical system  $C'C$  and a subsequent guess channel  $\mathcal{E}_{\text{guess}}$  applied to  $C'C$  [cf. Eqs. (9) and (10) in the main text]. Then due to the monotonicity of the purified distance under completely positive trace nonincreasing maps, i.e., projections, CPTP maps, and partial trace operations [37, Theorem 3.4], we have

$$P(\bar{\tau}_{CQ}, \tilde{\rho}_{CQ}) \leq P(\tau_{C'CQ}, \rho_{C'CQ}) \leq \varepsilon, \quad (\text{A15})$$

where

$$\begin{aligned} \tilde{\rho}_{CQ} &= \text{Tr}_{C'}[\mathcal{E}_{\text{guess}}(\Pi \rho_{C'CQ} \Pi)] \\ &= \sum_{(x',x) \in \Gamma} p_{x',x} |x\rangle_C \langle x| \otimes \rho_Q^{x',x}, \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} \bar{\tau}_{CQ} &= \text{Tr}_{C'}[\mathcal{E}_{\text{guess}}(\Pi \tau_{C'CQ} \Pi)] \\ &= \sum_{(x',x) \in \Gamma} q_{x',x} |x\rangle_C \langle x| \otimes \tau_Q^{x',x}. \end{aligned} \quad (\text{A17})$$

This means that  $\bar{\tau}_{CQ} \in \mathcal{B}^\varepsilon(\tilde{\rho}_{CQ})$  and as a consequence of the definition of the smooth min-entropy

$$H_{\min}^\varepsilon(C|Q)_{\tilde{\rho}} \geq H_{\min}(C|Q)_{\bar{\tau}}. \quad (\text{A18})$$

Then we exploit the following formula [37, Sec. 4.2.1] for the min-entropy

$$H_{\min}(A|B)_\tau = -\log_2 \max_{\mathcal{E}} \text{Tr}[\mathcal{E}_{B \rightarrow B'}(\tau_{AB}) \gamma_{AB'}], \quad (\text{A19})$$

where  $\tau_{AB} \in \mathcal{S}_{\leq}(\mathcal{H}_{AB})$  is a subnormalized state for systems  $A$  and  $B$ ,  $\gamma_{AB'} = |\gamma_{AB'}\rangle \langle \gamma_{AB'}|$  is a subnormalized maximally-entangled state for systems  $A$  and  $B'$ , i.e.,

$$|\gamma_{AB'}\rangle = \sum_x |x\rangle \otimes |x\rangle, \quad (\text{A20})$$

and  $\mathcal{E}_{B \rightarrow B'}$  is a CPTP map from  $B$  to  $B'$ , where  $\mathcal{H}_{B'} \cong \mathcal{H}_A$ . In particular, when we assume CQ states as in Eq. (A12), we may write

$$\begin{aligned} &\text{Tr}[\mathcal{E}_{Q \rightarrow Q'}(\tau_{CQ}) \gamma_{CQ'}] \\ &= \sum_{x',x} q_{x',x} \text{Tr}[|x\rangle_C \langle x| \otimes \mathcal{E}_{Q \rightarrow Q'}(\tau_Q^{x',x}) \gamma_{CQ'}] \\ &= \sum_{x',x} q_{x',x} \langle x| \mathcal{E}_{Q \rightarrow Q'}(\tau_Q^{x',x}) |x\rangle \\ &\geq \sum_{(x',x) \in \Gamma} q_{x',x} \langle x| \mathcal{E}_{Q \rightarrow Q'}(\tau_Q^{x',x}) |x\rangle \\ &= \text{Tr}[\mathcal{E}_{Q \rightarrow Q'}(\bar{\tau}_{CQ}) \gamma_{CQ'}], \end{aligned} \quad (\text{A21})$$

where the inequality stems from the fact that we have a summation of a smaller amount of positive terms due to the reduced alphabet  $(x',x) \in \Gamma$  of the projection. By taking the maximum and the minus logarithm of the previous relation, we may write the following relation for the min-entropies of the states  $\tau_{CQ}$  and  $\bar{\tau}_{CQ}$ ,

$$H_{\min}(C|Q)_{\bar{\tau}} \geq H_{\min}(C|Q)_{\tau}. \quad (\text{A22})$$

By replacing Eq. (A4) and (A18) in the previous inequality, we obtain the corresponding inequality for the smooth min-entropies of  $\rho$  and  $\tilde{\rho}$ , i.e.,

$$H_{\min}^\varepsilon(C|Q)_{\tilde{\rho}} \geq H_{\min}^\varepsilon(C|Q)_{\rho}. \quad (\text{A23})$$

Finally, we note that we get Eq. (31), by replacing  $\rho \rightarrow \rho^{\otimes n}$ ,  $\tilde{\rho} \rightarrow \sigma^n$ ,  $C \rightarrow B^n$ , and  $Q \rightarrow E^n$ .

### 1. Form of the CCQ extension in Eq. (A11)

Let us assume a general CCQ state

$$\theta_{C'CQ} = \sum_{x',x} \tilde{q}_{x'} \tilde{q}_{x'|x} |x',x\rangle_{C'C} \langle x',x| \otimes \tilde{\tau}_Q^{x',x} \quad (\text{A24})$$

and we set

$$\tilde{\tau}_Q^x := \sum_x \tilde{q}_{x'|x} \tilde{\tau}_Q^{x',x}. \quad (\text{A25})$$

We impose that the reduced state, after tracing out  $A$ , is equal to Eq. (A5). From the block diagonal form of the states, we obtain

$$q_x \tau_Q^x = \tilde{q}_x \tilde{\tau}_Q^x. \quad (\text{A26})$$

Similarly, by further tracing out  $E$ , we have that

$$q_x \text{tr}\{\tau_Q^x\} = \tilde{q}_x \text{tr}\{\tilde{\tau}_Q^x\}. \quad (\text{A27})$$

By combining Eqs. (A26) and (A27), we obtain

$$\begin{aligned} \tau_Q^x &= \frac{\text{tr}\{\tau_Q^x\}}{\text{tr}\{\tilde{\tau}_Q^x\}} \tilde{\tau}_Q^x \\ &= \sum_{x'} \tilde{q}_{x'|x} \frac{\text{tr}\{\tau_Q^x\}}{\text{tr}\{\tilde{\tau}_Q^x\}} \tilde{\tau}_Q^{x',x}. \end{aligned} \quad (\text{A28})$$

We can freely set

$$q_{x'|x} := \frac{\tilde{q}_{x'|x}}{\text{tr}\{\tilde{\tau}_Q^x\}}, \quad (\text{A29})$$

$$\tau_Q^{x',x} := \text{tr}\{\tau_Q^x\} \tilde{\tau}_Q^{x',x}, \quad (\text{A30})$$

so Eq. (A28) simply becomes

$$\tau_Q^x = \sum_{x'} q_{x'|x} \tau_Q^{x',x}. \quad (\text{A31})$$

Then, by using Eq. (A27) in Eq. (A24), we obtain

$$\begin{aligned} \theta_{CCQ} &= \sum_{x',x} q_x \text{tr}\{\tau_Q^x\}^{-1} \tilde{q}_{x'|x|x',x} c_{CCQ}(x',x) \otimes \text{tr}\{\tau_Q^x\} \tilde{\tau}_Q^{x',x} \\ &= \sum_{x',x} q_x q_{x'|x|x',x} c_{CCQ}(x',x) \otimes \tau_Q^{x',x}, \end{aligned} \quad (\text{A32})$$

where, in the last equation, we have used Eqs. (A29) and (A30). Note that the fact that the state must be CCQ and that its reduced form must be equal to Eq. (A5) completely characterizes the state. Therefore, we can derive the form in Eq. (A11).

## APPENDIX B: DETAILS ON PARAMETER ESTIMATION FOR GAUSSIAN-MODULATED PROTOCOLS

We assume that  $V_0 m$  data points are used for PE, with  $V_0 = 1$  ( $V_0 = 2$ ) for the homodyne (heterodyne) protocol. For simplicity, we assume that the two quadratures have been modulated with the same variance and that the channel transforms them in the same way (phase-insensitive channel, as typical of the standard thermal-loss channel). Then we denote with  $x$  and  $y$  the Gaussian input and output of the channel, respectively, with Gaussian noise variable  $z$  and transmissivity  $T$ , where

$$y = \sqrt{\eta T} x + z. \quad (\text{B1})$$

### 1. Estimating the transmissivity

We write the covariance  $C_{xy} = \text{Cov}(x, y) = \sqrt{\eta T} \sigma_x^2$ , where  $\sigma_x^2$  is the variance of  $x$ . Its estimator is given by

$$\hat{C}_{xy} := \frac{1}{V_0 m} \sum_{i=1}^{V_0 m} [x]_i [y]_i \quad (\text{B2})$$

$$\begin{aligned} &= \frac{1}{V_0 m} \sum_{i=1}^{V_0 m} \sqrt{T} [x]_i^2 + [x]_i [z]_i \\ &\simeq \sqrt{\eta T} \sigma_x^2 + \frac{1}{V_0 m} \sum_{i=1}^{V_0 m} [x]_i [z]_i, \end{aligned} \quad (\text{B3})$$

where, in Eq. (B3), we replaced Alice's known variance.

We calculate  $V_{\text{Cov}} := \text{Var}(\hat{C}_{xy})$  directly from Eq. (B2) and obtain

$$\begin{aligned} V_{\text{Cov}} &= \frac{1}{V_0 m} [\eta T \text{Var}(x^2) + \sigma_x^2 \sigma_z^2] \\ &= \frac{1}{V_0 m} [\eta T 2(\sigma_x^2)^2 + \sigma_x^2 \sigma_z^2] \\ &= \frac{1}{V_0 m} \eta T (\sigma_x^2)^2 \left[ 2 + \frac{\sigma_z^2}{\eta T \sigma_x^2} \right]. \end{aligned} \quad (\text{B4})$$

Otherwise, we can start from Eq. (B3) and obtain

$$V_{\text{Cov}} = \frac{1}{V_0 m} \sigma_x^2 \sigma_z^2. \quad (\text{B5})$$

Both of them can be summarized into

$$V_{\text{Cov}} = \frac{C_{xy}^2}{V_0 m} \left[ c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2} \right], \quad (\text{B6})$$

where, for the first one, we set  $c_{\text{pe}} = 2$  and, for the second one,  $c_{\text{pe}} = 0$ . Then we may write the estimator

$$\hat{T} = \frac{1}{\eta (\sigma_x^2)^2} \hat{C}_{xy}^2 = \frac{V_{\text{Cov}}}{\eta (\sigma_x^2)^2} \left( \frac{\hat{C}_{xy}}{\sqrt{V_{\text{Cov}}}} \right)^2. \quad (\text{B7})$$

Since  $\frac{\hat{C}_{xy}}{\sqrt{V_{\text{Cov}}}}$  follows a standard normal distribution with mean  $\frac{C_{xy}}{\sqrt{V_{\text{Cov}}}}$ , then  $\left(\frac{\hat{C}_{xy}}{\sqrt{V_{\text{Cov}}}}\right)^2$  follows a noncentral chi-squared distribution with degrees of freedom  $d_f = 1$  and noncentrality parameter  $\kappa_{cn} = C_{xy}^2 / V_{\text{Cov}}$ . Consequently  $\hat{T}$  follows the same distribution but rescaled by the factor  $\frac{V_{\text{Cov}}}{\eta (\sigma_x^2)^2}$ . Via the chi-squared distribution parameters, we can calculate its variance

$$\text{Var}(\hat{T}) = \frac{2V_{\text{Cov}}^2}{\eta^2 (\sigma_x^2)^4} \left( 1 + 2 \frac{C_{xy}^2}{V_{\text{Cov}}} \right), \quad (\text{B8})$$

and, by omitting the terms  $\mathcal{O}(1/m^2)$ , we obtain

$$\begin{aligned} \text{Var}(\hat{T}) &= \frac{4V_{\text{Cov}} C_{x,y}^2}{\eta^2 (\sigma_x^2)^4} = \frac{4C_{x,y}^4}{\eta^2 (\sigma_x^2)^4} \frac{[c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2}]}{V_0 m} \\ &= \frac{4\eta^2 T^2 (\sigma_x^2)^4 [c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2}]}{\eta^2 (\sigma_x^2)^4 V_0 m} \\ &= \frac{4T^2}{V_0 m} \left[ c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2} \right] := \sigma_T^2. \end{aligned} \quad (\text{B9})$$

Given that

$$\sigma_z^2 = \eta T \xi + u_{\text{el}} + V_0, \quad (\text{B11})$$

we may write

$$\sigma_T = \frac{2T}{\sqrt{V_0 m}} \sqrt{\left[ c_{\text{pe}} + \frac{\xi + \frac{V_0 + \mu_{\text{el}}}{\eta T}}{\sigma_x^2} \right]}, \quad (\text{B12})$$

as in Eq. (87) of the main text up to replacing  $V = \sigma_z^2$ . For  $m \gg 1$ , we may assume that the distribution of  $\hat{T}$  is Gaussian with variance given by Eq. (B9). Therefore, we may write that

$$T_{\text{wc}} \simeq T - w\sigma_T \quad (\text{B13})$$

with

$$w = \sqrt{2} \text{erf}^{-1}(1 - 2\varepsilon_{\text{pe}}). \quad (\text{B14})$$

To better understand the result above, let us assume a generic estimator  $\hat{p}$  that follows a normal distribution with mean  $p$  and variance  $\sigma_p^2$ . We impose the probability that  $\hat{p} \geq p_{\text{wc}} := p + w\sigma_p$  is less than  $\varepsilon_{\text{pe}}$ . In other words,

$$\text{Prob}[\hat{p} \geq p + w\sigma_p] \leq \varepsilon_{\text{pe}}. \quad (\text{B15})$$

We can rewrite Eq. (B15) as follows:

$$\begin{aligned} \text{Prob}[\hat{p} - p \geq w\sigma_p] &\leq \varepsilon_{\text{pe}}, \\ \text{Prob}\left[\frac{\hat{p} - p}{\sigma_p} \geq w\right] &\leq \varepsilon_{\text{pe}}. \end{aligned} \quad (\text{B16})$$

We can recognize the cumulative distribution

$$\Phi(w) = \frac{1}{2}[1 + \text{erf}(w/\sqrt{2})] \quad (\text{B17})$$

of the normal variable  $\frac{\hat{p} - p}{\sigma_p}$ . We use its connection to the error function  $\text{erf}(\cdot)$  to write

$$1 - \Phi(w) \leq \varepsilon_{\text{pe}}, \quad (\text{B18})$$

$$\frac{1}{2}[1 + \text{erf}(w/\sqrt{2})] \geq 1 - \varepsilon_{\text{pe}}, \quad (\text{B19})$$

$$\text{erf}(w/\sqrt{2}) \geq 1 - 2\varepsilon_{\text{pe}}, \quad (\text{B20})$$

$$w \geq \sqrt{2} \text{erf}^{-1}(1 - 2\varepsilon_{\text{pe}}), \quad (\text{B21})$$

and we use the bound above in Eq. (B14).

Alternatively, we may use tail bounds for the chi-squared distribution. In particular, for the stochastic variable  $X$  following the latter distribution, we have that

$$\text{Prob}[X \leq (d_f + \kappa_{\text{nc}}) - 2\sqrt{(d_f + 2\kappa_{\text{nc}}) \ln \varepsilon_{\text{pe}}^{-1}}] \leq \varepsilon_{\text{pe}}, \quad (\text{B22})$$

$$\text{Prob}[X \geq (d_f + \kappa_{\text{nc}}) + 2\sqrt{(d_f + 2\kappa_{\text{nc}}) \ln \varepsilon_{\text{pe}}^{-1}} + 2 \ln \varepsilon_{\text{pe}}^{-1}] \leq \varepsilon_{\text{pe}}. \quad (\text{B23})$$

Applying this to  $\hat{T}$ , we obtain

$$\begin{aligned} T_{\text{wc}} &= \frac{V_{\text{Cov}} + C_{x,y}^2}{\eta(\sigma_x^2)^2} \\ &\quad - \frac{2}{\eta(\sigma_x^2)^2} \sqrt{(V_{\text{Cov}}^2 + 2C_{x,y}^2 V_{\text{Cov}}) \ln \varepsilon_{\text{pe}}^{-1}}. \end{aligned} \quad (\text{B24})$$

Then, we expand the square root above and omit  $\mathcal{O}(\frac{1}{m})$  terms

$$\begin{aligned} &\sqrt{(V_{\text{Cov}}^2 + 2C_{x,y}^2 V_{\text{Cov}})} \\ &= \sqrt{2C_{x,y}^2 V_{\text{Cov}}} \sqrt{1 + \frac{V_{\text{Cov}}}{2C_{x,y}^2}} \\ &= \sqrt{2C_{x,y}^2 V_{\text{Cov}}} \left[ \left(1 + \frac{V_{\text{Cov}}}{4C_{x,y}^2}\right) + \mathcal{O}\left(\frac{1}{m^2}\right) \right] \\ &= \sqrt{2C_{x,y}^2 V_{\text{Cov}}} + \mathcal{O}\left(\frac{1}{m}\right) \\ &\simeq \sqrt{2 \frac{\eta^2 T^2 (\sigma_x^2)^4}{V_0 m}} \left[ c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2} \right]. \end{aligned} \quad (\text{B25})$$

Finally, we obtain

$$T_{\text{wc}} \simeq T - \sqrt{2 \ln \varepsilon_{\text{pe}}^{-1}} \frac{2T}{\sqrt{V_0 m}} \sqrt{\left[ c_{\text{pe}} + \frac{\sigma_z^2}{\eta T \sigma_x^2} \right]}, \quad (\text{B26})$$

which can be written in the form of Eq. (B13) but with

$$w = \sqrt{2 \ln \varepsilon_{\text{pe}}^{-1}}. \quad (\text{B27})$$

## 2. Estimating the noise

In the same manner, we calculate the estimator for  $\sigma_z^2$ , the variance of the noise variable  $z$ . We have that

$$\hat{\sigma}_z^2 = \frac{1}{V_0 m} \sum_{i=1}^{V_0 m} (y - \sqrt{\eta \hat{T} x})^2 \quad (\text{B28})$$

$$\simeq \frac{1}{V_0 m} \sigma_z^2 \sum_{i=1}^{V_0 m} \left( \frac{y - \sqrt{\eta T x}}{\sigma_z} \right)^2. \quad (\text{B29})$$

The sum above follows a central chi-squared distribution with  $d_f = V_0 m$  and, therefore, with mean  $V_0 m$  and variance  $2V_0 m$ . Then  $\hat{\sigma}_z^2$  follows the same distribution but rescaled by  $\frac{\sigma_z^2}{V_0 m}$ .

Thus, its mean value is  $\sigma_z^2$  while its variance  $V_z = \frac{2(\sigma_z^2)^2}{V_0 m}$ . From this, we may write

$$[\sigma_z^2]_{\text{wc}} \simeq \sigma_z^2 + w\sqrt{V_z} \quad (\text{B30})$$

with  $w$  given by Eq. (B14).

Otherwise, we may use the tail bounds in Eq. (B22) to obtain

$$\begin{aligned} [\sigma_z^2]_{\text{wc}} &= \frac{\sigma_z^2}{V_0 m} (V_0 m + 2\sqrt{V_0 m \ln \varepsilon_{\text{pe}}^{-1}} + 2 \ln \varepsilon_{\text{pe}}^{-1}) \\ &= \sigma_z^2 + \sigma_z^2 \frac{\sqrt{2}}{\sqrt{V_0 m}} \sqrt{2 \ln \varepsilon_{\text{pe}}^{-1}} + \mathcal{O}\left(\frac{1}{m}\right) \\ &\simeq \sigma_z^2 + \sigma_z^2 \frac{\sqrt{2}}{\sqrt{V_0 m}} \sqrt{2 \ln \varepsilon_{\text{pe}}^{-1}}, \end{aligned} \quad (\text{B31})$$

which can be written as in Eq. (B30) but with  $w$  given in Eq. (B27).

Finally, from Eq. (B11), we derive

$$\begin{aligned}\xi_{\text{wc}} &= \frac{[\sigma_z^2]_{\text{wc}}}{\eta T_{\text{wc}}} - \frac{u_{\text{el}} + V_0}{\eta T_{\text{wc}}} \\ &\simeq \frac{\eta T \xi + w \sqrt{V_z} + u_{\text{el}} + V_0}{\eta T_{\text{wc}}} - \frac{u_{\text{el}} + V_0}{\eta T_{\text{wc}}} \\ &= \frac{T}{T_{\text{wc}}} \xi + \frac{w \sqrt{V_z}}{\eta T_{\text{wc}}}.\end{aligned}\quad (\text{B32})$$

This expression can equivalently be written as

$$\xi_{\text{wc}} \simeq \frac{T}{T_{\text{wc}}} \xi + w \sigma_\xi, \quad (\text{B33})$$

where

$$\sigma_\xi = \frac{\sqrt{V_z}}{\eta T_{\text{wc}}} = \sqrt{\frac{2}{V_0 m} \frac{\eta T \xi + V_0 + u_{\text{el}}}{\eta T_{\text{wc}}}}, \quad (\text{B34})$$

as in Eq. (88) of the main text.

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- [38] Note that, for the heterodyne protocol, the estimation of Bob's entropy can also be done on the variables before concatenation as in Refs. [25,31].
- [39] In both Refs. [14,15], the estimator of the transmissivity is based on the estimation of the covariance  $C = \langle xy \rangle$  between Alice's variable  $x$  and Bob's variable  $y = \sqrt{\tau}x + z$ , where  $z$  we denote the Gaussian noise variable. In Ref. [14], the covariance is split into two terms  $C = \sqrt{\tau}\langle x^2 \rangle + \langle xz \rangle$  where  $\langle x^2 \rangle$  is considered to be exactly known, which is a realistic assumption (this is basically Alice's modulation parameter). Ref. [15] does not make this assumption.
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