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Recursive Constrained Sine Second-Order Error Promoting Adaptive Algorithm

Yong Chen, Yingsong Li, *Senior Member, IEEE*, Yongchun Miao, Yuriy Zakharov, *Senior Member, IEEE*, Zhixiang Huang, *Senior Member, IEEE*

Abstract—This brief proposes a recursive constrained sine second-order error promoting adaptive (RCSOEPA) algorithm. Compared with classical recursive method, the RCSOEPA algorithm can achieve better steady state performance in impulsive-noise. In general, the sine second-order error (SSOE) is constructed to devise a new recursive constrained adaptive-filtering within the least-squares framework for solving linear constrained optimization problems. The mean-square (MS) stability of the RCSOEPA and its theoretical instantaneous MS deviation under Gaussian and non Gaussian noise are analyzed, numerically investigated and discussed in detail. Simulated results are reported to give a confirmation of the theoretical analysis, and show that the RCSOEPA outperforms recent developed constrained adaptive filtering algorithms in the estimation misalignment and when used for system identification under impulsive-noise.

Index Terms—impulsive-noise, impulsive-noise, Recursive constrained adaptive filtering (CAF), Sine second-order error (SSOE).

I. INTRODUCTION

IN the field of signal processing in wireless communication, array signal processing, interference suppression and system identification, the parameter vector estimation is often gotten by using adaptive filters [1], [2] and is subject to linear constraints. To solve the above problem, the constrained-adaptive-filtering (CAF) algorithms have been proposed and well developed in recent years [3], [4]. The greatest advantage of a CAF algorithm lies in its error correction property avoiding error accumulation and thus achieving good performance.

The second-order error (SOE) criterion is widely used in adaptive filtering (AF) since it results in low complexity and simple structure algorithms [5], and it is typically an optimal criterion under the Gaussian assumption. The constrained-least-mean-square (CLMS) algorithm, which is a classical CAF constructed based on the SOE criterion [6], can achieve high robustness in Gaussian noise. However, in practice, there are various non-Gaussian noise scenarios, where the behaviors of the CLMS will decrease, especially when it is used in impulsive-noise with heavy-tail distributions (such as alpha-stable noise [7] and Cauchy noise [8]).

In order to effectively combat the adverse effects of non-Gaussian noises, the maximum-correntropy-criterion (MCC) is reported [9], [10], which is a non-linear measure used to maximize the correlation between outputs of the unknown

system and the AF algorithm to combat the local similarity in non Gaussian noise. In addition, continuous hybrid p -norm algorithm and constrained minimum M -estimation algorithm are presented in [11], [12]. Using the MCC criterion instead of the SOE criterion, the constrained MCC (CMCC) algorithm [13] achieves high robustness under non-Gaussian noise by maximizing the correlation. Then, using the matrix-inversion theorem [14], the recursive-CMCC (RCMCC) is presented to get a faster convergence and a stronger performance in non-Gaussian noise than the basic CMCC.

Here, we propose a recursive constrained sine second-order error promoting adaptive (RCSOEPA) algorithm. The sine second-order error (SSOE) criterion used for developing the algorithm helps in suppressing large errors caused by impulsive-noise. The MS stability of the RCSOEPA and its theoretical instantaneous mean-square-deviation (MSD) under Gaussian and non-Gaussian noises are analyzed. Simulation results illustrated to give a confirmation of the theoretical analysis show that the RCSOEPA outperforms other recent developed constrained adaptive filtering algorithms in the estimation misalignment when used for system-identification under impulsive-noises.

The brief is structured below. Section II presents the RCSOEPA whose convergence and instantaneous MSD are analyzed in Section III. In Section IV, we present simulation results and compare them with that of known CAF algorithms. Section V concludes this brief.

Notation: We use plain font letters as scalars, and we also use capital bold and lowercase bold letters to represent matrices and vectors, respectively. All vectors are column-vectors. The inverse, transpose, expectation, and trace operators are expressed as $(\cdot)^{-1}$, $(\cdot)^T$, $E(\cdot)$, and $tr(\cdot)$, respectively.

II. THE PROPOSED RCSOEPA ALGORITHM

A. Channel Model

An unknown system output at time l is $d(l) = \mathbf{h}_0^T \mathbf{u}(l) + \eta(l)$, where the system vector is $\mathbf{h}_0 = [h_1, h_2, \dots, h_M]^T$, $\mathbf{u}(l) = [u_1, u_2, \dots, u_M]^T$ represents the input-vector at l , $\eta(l)$ is the background noise with variance σ_η^2 , and the estimation error is defined as

$$e(l) = d(l) - \mathbf{h}^T(l-1) \mathbf{u}(l), \quad (1)$$

where $\mathbf{h}(l-1)$ denotes the system vector estimate at $l-1$.

The SSOE cost function is defined as

$$J(e) = \begin{cases} 4\sin^2\left(\frac{e}{2c}\right) & \text{if } |e| \leq \pi c \\ 4 & \text{otherwise} \end{cases}, \quad (2)$$

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where c is a positive constant. The SSOE function is introduced into the CAF scheme to develop a CSSDA algorithm, by solving the following problem:

$$\min_{\mathbf{h}} \mathbb{E} \left[4 \sin^2 \left(\frac{e}{2c} \right) \right] \quad s.t. \quad \mathbf{g} = \mathbf{S}^T \mathbf{h}, \quad (3)$$

where \mathbf{S} represents constraint matrix, having an size of $M \times K$ and $M > K$, and \mathbf{g} represents an $K \times 1$ constraint vector. Utilizing the Lagrange-method, the cost-function for the CSSDA is obtained as

$$J_{\text{CSSDA}}(e) = \mathbb{E} \left[4 \sin^2 \left(\frac{e}{2c} \right) \right] + \gamma^T [\mathbf{g} - \mathbf{S}^T \mathbf{h}], \quad (4)$$

where γ is the $K \times 1$ Lagrange multiplier.

B. Optimal Solution

Considering the constraint $\mathbf{g} = \mathbf{S}^T \mathbf{h}$ and taking $\frac{\partial J_{\text{CSSDA}}}{\partial \mathbf{h}} = \mathbf{0}$ into consideration, the CSSDA algorithm for finding the optimized weight-vector \mathbf{h}_{opt} can be derived as follows

$$\begin{aligned} & \mathbb{E} \left[\frac{2}{c} \sin \left(\frac{e(l)}{c} \right) \mathbf{u}(l) \right] + \mathbf{S} \gamma = \mathbf{0}, \\ \Rightarrow & \mathbb{E} [w(e(l)) e(l) \mathbf{u}(l)] + \mathbf{S} \gamma = \mathbf{0}, \\ \Rightarrow & \mathbb{E} [w(e(l)) (d(l) - \mathbf{h}_{\text{opt}}^T(l) \mathbf{u}(l)) \mathbf{u}(l)] + \mathbf{S} \gamma = \mathbf{0}, \\ \Rightarrow & \mathbb{E} [w(e(l)) \mathbf{u}(l) \mathbf{u}^T(l)] \mathbf{h}_{\text{opt}} = \mathbb{E} [w(e(l)) d(l) \mathbf{u}(l)] + \mathbf{S} \gamma, \\ \Rightarrow & \mathbf{R}_w \mathbf{h}_{\text{opt}} = \mathbf{p}_w + \mathbf{S} \gamma, \\ \Rightarrow & \mathbf{h}_{\text{opt}} = \mathbf{R}_w^{-1} \mathbf{p}_w + \mathbf{R}_w^{-1} \mathbf{S} \gamma, \end{aligned} \quad (5)$$

where $w(e(l)) = \frac{2 \sin(\frac{e(l)}{c})}{ce(l)}$, $\mathbf{R}_w = \mathbb{E} [w(e(l)) \mathbf{u}(l) \mathbf{u}^T(l)]$, $\mathbf{p}_w = \mathbb{E} [w(e(l)) d(l) \mathbf{u}(l)]$. Thereby, the constraint vector $\mathbf{g} = \mathbf{S}^T \mathbf{h}$ is represented as

$$\begin{aligned} & \mathbf{g} = \mathbf{S}^T \mathbf{h}_{\text{opt}}, \\ \Rightarrow & \mathbf{g} = \mathbf{S}^T [\mathbf{R}_w^{-1} \mathbf{p}_w + \mathbf{R}_w^{-1} \mathbf{S} \gamma], \\ \Rightarrow & \gamma = (\mathbf{S}^T \mathbf{R}_w^{-1} \mathbf{S})^{-1} (\mathbf{g} - \mathbf{S}^T \mathbf{R}_w^{-1} \mathbf{p}_w). \end{aligned} \quad (6)$$

Using (6), \mathbf{h}_{opt} in (5) is updated to

$$\mathbf{h}_{\text{opt}} = \mathbf{R}_w^{-1} \mathbf{p}_w + \mathbf{R}_w^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_w^{-1} \mathbf{S})^{-1} (\mathbf{g} - \mathbf{S}^T \mathbf{R}_w^{-1} \mathbf{p}_w). \quad (7)$$

We use the following approximations of \mathbf{R}_w and \mathbf{p}_w , respectively:

$$\hat{\mathbf{R}}_w = \frac{1}{l} \sum_{i=1}^l w(e(i)) \mathbf{u}(i) \mathbf{u}^T(i), \quad (8)$$

$$\hat{\mathbf{p}}_w = \frac{1}{l} \sum_{i=1}^l w(e(i)) d(i) \mathbf{u}(i), \quad (9)$$

which can be recursively computed as

$$\begin{aligned} \hat{\mathbf{R}}_w(l) &= \hat{\mathbf{R}}_w(l-1) + w(e(l)) \mathbf{u}(l) \mathbf{u}^T(l), \\ \hat{\mathbf{p}}_w(l) &= \hat{\mathbf{p}}_w(l-1) + w(e(l)) d(l) \mathbf{u}(l). \end{aligned} \quad (10)$$

Thus, we finally obtain the system vector estimate

$$\begin{aligned} \mathbf{h}(l) &= \left(\frac{1}{l} \hat{\mathbf{R}}_w(l) \right)^{-1} \frac{1}{l} \hat{\mathbf{p}}_w(l) + \left(\frac{1}{l} \hat{\mathbf{R}}_w(l) \right)^{-1} \mathbf{S} \\ &\times \left(\mathbf{S}^T \left(\frac{1}{l} \hat{\mathbf{R}}_w(l) \right)^{-1} \mathbf{S} \right)^{-1} \left(\mathbf{g} - \mathbf{S}^T \left(\frac{1}{l} \hat{\mathbf{R}}_w(l) \right)^{-1} \frac{1}{l} \hat{\mathbf{p}}_w(l) \right) \\ &= \hat{\mathbf{R}}_w^{-1}(l) \hat{\mathbf{p}}_w(l) + \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} \left(\mathbf{S}^T \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} \right)^{-1} \\ &\times \left(\mathbf{g} - \mathbf{S}^T \hat{\mathbf{R}}_w^{-1}(l) \hat{\mathbf{p}}_w(l) \right). \end{aligned} \quad (11)$$

C. RCSOEP Algorithm

Using matrix-inversion lemma [15], from (10), we get

$$\begin{aligned} \hat{\mathbf{R}}_w^{-1}(l) &= \hat{\mathbf{R}}_w^{-1}(l-1) - \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l) (w^{-1}(e(l)) \\ &+ \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l))^{-1} \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1). \end{aligned} \quad (12)$$

Define

$$\begin{aligned} \mathbf{n}(l) &= \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l) (w^{-1}(e(l)) \\ &+ \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l))^{-1}. \end{aligned} \quad (13)$$

Then, we can write

$$\begin{aligned} \mathbf{n}(l) &= w(e(l)) \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l) \\ &- \mathbf{n}(l) w(e(l)) \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{u}(l) \\ &= w(e(l)) \left(\hat{\mathbf{R}}_w^{-1}(l-1) - \mathbf{n}(l) \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1) \right) \mathbf{u}(l) \\ &= w(e(l)) \hat{\mathbf{R}}_w^{-1}(l) \mathbf{u}(l). \end{aligned} \quad (14)$$

By defining the unconstrained optimal solution as

$$\begin{aligned} \mathbf{h}_{\text{unc}}(l-1) &= \hat{\mathbf{R}}_w^{-1}(l-1) \hat{\mathbf{p}}_w(l-1), \\ e_{\text{unc}}(l) &= d(l) - \mathbf{h}_{\text{unc}}^T(l-1) \mathbf{u}(l), \end{aligned} \quad (15)$$

and substituting (10), (12) and (14) into (11), the updated RCSOEP recursion is given by

$$\begin{aligned} \mathbf{h}(l) &= \hat{\mathbf{R}}_w^{-1}(l) [\hat{\mathbf{p}}_w(l-1) + w(e(l)) d(l) \mathbf{u}(l)] \\ &+ \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} \gamma(l) \\ &= \mathbf{h}_{\text{unc}}(l-1) + e_{\text{unc}}(l) \mathbf{n}(l) + \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} \gamma(l), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \gamma(l) &= \left(\mathbf{S}^T \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} \right)^{-1} (\mathbf{g} - \mathbf{S}^T (\mathbf{h}_{\text{unc}}(l-1) \\ &+ e_{\text{unc}}(l) \mathbf{n}(l))). \end{aligned} \quad (17)$$

From the iteration of the proposed RCSOEP algorithm in (16), it has a complexity of $o(M^2)$.

III. THEORETICAL ANALYSIS OF CONVERGENCE AND MSD OF THE RCSOEP ALGORITHM

A. Assumptions

To obtain theoretical analysis convergence and MSD of the proposed RCSOEP, the following assumptions are adopted to simplify the analysis [8], [16].

(1) The noise $\eta(l)$ is an i. i. d. non-Gaussian with zero-mean (ZM), and it is independent of any other signals.

(2) The prior error $e^c(l)$, defining later, is ZM Gaussian.

(3) $\mathbf{u}(l)$ is an i. i. d. ZM Gaussian process with positive definite covariance-matrix $\mathbf{R}_u = \mathbb{E} [\mathbf{u}(l) \mathbf{u}^T(l)]$.

(4) When the system operates in its steady-state, $w(e(l))$ is uncorrelated with $\|\mathbf{u}(l)\|^2$.

B. Mean Square Stability

Based on assumption (4), we obtain an approximation

$$\begin{aligned} \hat{\mathbf{R}}_w^{-1}(l) &= \left[l \left[\frac{1}{l} \sum_{i=1}^l w(e(i)) \mathbf{u}(i) \mathbf{u}^T(i) \right] \right]^{-1} \\ &\approx \frac{1}{l} \left[\mathbb{E} [w(e(l)) \mathbf{u}(l) \mathbf{u}^T(l)] \right]^{-1} \\ &= \frac{b}{l} \mathbf{R}_u^{-1}, \end{aligned} \quad (18)$$

where $b = 1/\mathbb{E} [w(e(l))]$.

Using the above definitions of $e(l)$ and $e_{\text{unc}}(l)$ and substituting (14) and (18) into (16), we obtain

$$\begin{aligned} e_{unc}(l) - e(l) &= \mathbf{u}^T(l) \mathbf{h}(l-1) - \mathbf{u}^T(l) \mathbf{h}_{unc}(l-1) \\ &= \mathbf{u}^T(l) (\mathbf{h}(l-1) - \mathbf{h}_{unc}(l-1)) \\ &= \mathbf{u}^T(l) \hat{\mathbf{R}}_w^{-1}(l-1) \mathbf{S} \gamma(l-1). \end{aligned} \quad (19)$$

Substituting (14) and (19) into (16), we obtain

$$\begin{aligned} \mathbf{h}(l) &= \mathbf{h}(l-1) + w(e(l)) e(l) \hat{\mathbf{R}}_w^{-1}(l) \mathbf{u}(l) \\ &+ \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} (\gamma(l) - \gamma(l-1)) \\ &= \mathbf{h}(l-1) + w(e(l)) e(l) \hat{\mathbf{R}}_w^{-1}(l) \mathbf{u}(l) \\ &+ \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S} (\mathbf{S}^T \hat{\mathbf{R}}_w^{-1}(l) \mathbf{S})^{-1} \\ &\times \left(\mathbf{g} - \mathbf{S}^T (\mathbf{h}(l-1) + w(e(l)) e(l) \hat{\mathbf{R}}_w^{-1}(l) \mathbf{u}(l)) \right). \end{aligned} \quad (20)$$

Combining with (18), we arrive at

$$\begin{aligned} \mathbf{h}(l) &\approx \mathbf{h}(l-1) + \frac{b}{l} w(e(l)) e(l) \mathbf{R}_u^{-1} \mathbf{u}(l) \\ &+ \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} \\ &\times \left(\mathbf{g} - \mathbf{S}^T (\mathbf{h}(l-1) + \frac{b}{l} w(e(l)) e(l) \mathbf{R}_u^{-1} \mathbf{u}(l)) \right) \\ &= \mathbf{T} [\mathbf{h}(l-1) + \frac{b}{l} w(e(l)) e(l) \mathbf{R}_u^{-1} \mathbf{u}(l)] + \mathbf{r}, \end{aligned} \quad (21)$$

where $\mathbf{T} = \mathbf{I} - \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} \mathbf{S}^T$ and $\mathbf{r} = \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} \mathbf{g}$.

$$\begin{aligned} \text{Using } d(l) &= \mathbf{h}_0^T \mathbf{u}(l) + \eta(l) \text{ and assumption (1), we obtain} \\ w(e(l)) d(l) \mathbf{u}(l) &= w(e(l)) \mathbf{u}(l) \mathbf{u}^T(l) \mathbf{h}_0 \\ &+ w(e(l)) \eta(l) \mathbf{u}(l) \\ \Rightarrow \mathbf{p}_w &= \mathbf{R}_w \mathbf{h}_0 \\ \Rightarrow \mathbf{h}_0 &= \mathbf{R}_w^{-1} \mathbf{p}_w. \end{aligned} \quad (22)$$

The following two error measures are introduced to simplify the analysis:

$$\tilde{\mathbf{h}}(l) = \mathbf{h}(l) - \mathbf{h}_{opt}, \quad e^c(l) = (\mathbf{h}_0 - \mathbf{h}(l))^T \mathbf{u}(l), \quad (23)$$

where $e^c(l)$ and $\tilde{\mathbf{h}}(l)$ represent prior error and weighted error vectors, respectively. We also define

$$\zeta_{\mathbf{h}} = \mathbf{h}_0 - \mathbf{h}_{opt}. \quad (24)$$

Based on the above analysis, $\zeta_{\mathbf{h}}$ in (24) can be written as

$$\zeta_{\mathbf{h}} = \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} (\mathbf{S}^T \mathbf{h}_0 - \mathbf{g}). \quad (25)$$

Subtracting \mathbf{h}_{opt} from both sides of (21) and using $e(l) = d(l) - \mathbf{u}^T(l) \mathbf{h}(l-1)$, we obtain

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \mathbf{T} [\mathbf{h}(l-1) + \frac{b}{l} d(l) w(e(l)) \mathbf{R}_u^{-1} \mathbf{u}(l) \\ &- \frac{b}{l} w(e(l)) \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \mathbf{h}(l-1)] + \mathbf{r} - \mathbf{h}_{opt}, \end{aligned} \quad (26)$$

and due to $\mathbf{T} \mathbf{h}_{opt} - \mathbf{h}_{opt} + \mathbf{r} = \mathbf{0}$, (26) is modified to

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \mathbf{T} \left[\mathbf{I} - \frac{b}{l} w(e(l)) \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \right] \tilde{\mathbf{h}}(l-1) \\ &+ \frac{b}{l} w(e(l)) \mathbf{T} (\mathbf{R}_u^{-1} \mathbf{u}(l) \eta(l) + \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \zeta_{\mathbf{h}}). \end{aligned} \quad (27)$$

Owning to the definition of \mathbf{T} , we have

$$\mathbf{T} \tilde{\mathbf{h}}(l) = \tilde{\mathbf{h}}(l). \quad (28)$$

Then, (27) can be rewritten as

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \left[\mathbf{I} - \frac{b}{l} w(e(l)) \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \right] \tilde{\mathbf{h}}(l-1) \\ &+ \frac{b}{l} w(e(l)) (\mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \eta(l) + \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \zeta_{\mathbf{h}}). \end{aligned} \quad (29)$$

Exerting square expectation on both-sides of (29), and utilizing assumptions (1), (2) and (3), we get

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|^2 \right] &= \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l-1) \right\|_{\mathbf{G}(l)}^2 \right] + \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \times \\ &\mathbb{E} [v^2(l)] \mathbb{E} [\mathbf{u}^T(l) \mathbf{R}_u^{-1} \mathbf{T}^T \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l)] + \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \zeta_{\mathbf{h}} \\ &\times \mathbb{E} [\mathbf{u}(l) \mathbf{u}^T(l) \mathbf{R}_u^{-1} \mathbf{T}^T \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l)] \zeta_{\mathbf{h}}^T, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \mathbf{G}(l) &= \left[\mathbf{I} - \frac{b}{l} w(e(l)) \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \right]^T \\ &\times \left[\mathbf{I} - \frac{b}{l} w(e(l)) \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l) \right]. \end{aligned} \quad (31)$$

From (30), utilizing Isserlis' theory [20], we get

$$\begin{aligned} \mathbb{E} [\mathbf{u}(l) \mathbf{u}^T(l) \mathbf{R}_u^{-1} \mathbf{T}^T \mathbf{T} \mathbf{R}_u^{-1} \mathbf{u}(l) \mathbf{u}^T(l)] \\ = 2\mathbf{T}^T \mathbf{T} + tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \mathbf{R}_u. \end{aligned} \quad (32)$$

Then, (31) is modified to

$$\begin{aligned} \mathbf{G}(l) &= \mathbf{I} - \frac{b}{l} w(e(l)) (\mathbf{T}^T + \mathbf{T}) + \frac{b^2}{l^2} w^2(e(l)) \\ &\times (2\mathbf{T}^T \mathbf{T} + tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \mathbf{R}_u). \end{aligned} \quad (33)$$

Since

$$\begin{aligned} \mathbf{T} \mathbf{R}_u \zeta_{\mathbf{h}} &= \left(\mathbf{I} - \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} \mathbf{S}^T \right) \\ &\times \mathbf{R}_u \mathbf{R}_u^{-1} \mathbf{S} (\mathbf{S}^T \mathbf{R}_u^{-1} \mathbf{S})^{-1} (\mathbf{S}^T \mathbf{h}_0 - \mathbf{g}) = \mathbf{0}, \end{aligned} \quad (34)$$

we get

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|^2 \right] &= \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l-1) \right\|_{\mathbf{G}(l)}^2 \right] \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \mathbb{E} [v^2(l)] tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \zeta_{\mathbf{h}}^T \mathbb{E} [tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \mathbf{R}_u] \zeta_{\mathbf{h}}. \end{aligned} \quad (35)$$

Let t_j and r_j ($1 \leq j \leq M$) be eigenvalues of \mathbf{T} and \mathbf{R}_u , respectively. Based on the above analysis, the mean square stability condition is satisfied if

$$\begin{aligned} \left| 1 - \frac{2b}{l} w(e(l)) t_j + \frac{b^2}{l^2} w^2(e(l)) \right. \\ \left. \times (2t_j^2 + tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} r_j) \right| < 1. \end{aligned} \quad (36)$$

Therefore, the stability condition is to be

$$l > \max_j \frac{b \mathbb{E} [w^2(e(l))] (2t_j^2 + tr \{ \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} r_j)}{2t_j \mathbb{E} [w(e(l))]} \quad (37)$$

C. MSD analysis

Consider (34), and assume that \mathbf{Y} is an arbitrary $M \times M$ -symmetric non-negative definite-matrix, then we get

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|_{\mathbf{Y}}^2 \right] &= \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l-1) \right\|_{\mathbf{Q}(l)}^2 \right] \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \mathbb{E} [v^2(l)] tr \{ \mathbf{Y} \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \zeta_{\mathbf{h}}^T \mathbb{E} [tr \{ \mathbf{Y} \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \mathbf{R}_u] \zeta_{\mathbf{h}}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \mathbf{Q}(l) &= \left(\mathbf{I} - \frac{b}{l} \mathbb{E} [w(e(l))] \mathbf{T} \right)^T \mathbf{Y} \left(\mathbf{I} - \frac{b}{l} \mathbb{E} [w(e(l))] \mathbf{T} \right) \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] (2\mathbf{T}^T \mathbf{Y} \mathbf{T} + tr \{ \mathbf{Y} \mathbf{T} \mathbf{R}_u^{-1} \mathbf{T}^T \} \mathbf{R}_u) \\ &- \frac{b^2}{l^2} \mathbb{E}^2 [w(e(l))] \mathbf{T}^T \mathbf{Y} \mathbf{T}. \end{aligned} \quad (39)$$

Then, we use the vector correlation property $\text{vec}(\mathbf{XYZ}) = (\mathbf{Z}^T \otimes \mathbf{X}) \text{vec}(\mathbf{Y})$ and $tr(\mathbf{X}^T \mathbf{Y}) = \text{vec}^T(\mathbf{Y}) \text{vec}(\mathbf{X})$, where $\text{vec}(\cdot)$ is the vectorization operating and \otimes is Kronecker-product. Utilizing these properties, we get

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|_{\mathbf{y}}^2 \right] &= \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l-1) \right\|_{\mathbf{F}(l)\mathbf{y}}^2 \right] \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \mathbb{E} [v^2(l)] \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T) \mathbf{y} \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T) \mathbf{y}, \end{aligned} \quad (40)$$

where $\mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|_{\mathbf{y}}^2 \right] = \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|_{\mathbf{y}}^2 \right]$, and

$$\begin{aligned} \mathbf{F}(l) &= (\mathbf{I} - \frac{1}{l} \mathbb{E} [w(e(l))] \mathbf{T})^T \otimes (\mathbf{I} - \frac{1}{l} \mathbb{E} [w(e(l))] \mathbf{T}) \\ &+ \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \\ &\times (2(\mathbf{T}^T \otimes \mathbf{T}) + \mathbf{vec}(\mathbf{R}_{\mathbf{u}}) \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T)) \\ &- \frac{b^2}{l^2} \mathbb{E}^2 [w(e(l))] (\mathbf{T}^T \otimes \mathbf{T}). \end{aligned} \quad (41)$$

From [21], MSD(l) is written as $\mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|^2 \right]$, and the relationship between MSD($l-1$) and MSD(l) is given by [14]

$$\frac{\mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|^2 \right]}{\mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l-1) \right\|^2 \right]} = \frac{l-1}{l}. \quad (42)$$

Utilizing (41), we rewrite (39) as

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|_{(\mathbf{I} - \frac{1}{l-1} \mathbf{F}(l))\mathbf{y}}^2 \right] &= \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \mathbb{E} [v^2(l)] \\ &\times \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T) \mathbf{y} + \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} \\ &\times \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T) \mathbf{y}. \end{aligned} \quad (43)$$

With $\mathbf{y} = (\mathbf{I} - \frac{1}{l-1} \mathbf{F}(l))^{-1} \mathbf{vec}(\mathbf{I})$, the MSD is given by

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{\mathbf{h}}(l) \right\|^2 \right] &= \frac{b^2}{l^2} \mathbb{E} [w^2(e(l))] (\mathbb{E} [v^2(l)] + \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}}) \\ &\times \mathbf{vec}^T (\mathbf{TR}_{\mathbf{u}}^{-1} \mathbf{T}^T) (\mathbf{I} - \frac{1}{l-1} \mathbf{F}(l))^{-1} \mathbf{vec}(\mathbf{I}). \end{aligned} \quad (44)$$

In the Gaussian-noise case, let $e(l) = e^c(l) + v(l)$. If $v(l)$ is a ZM Gaussian random number with variance σ_v^2 , then $e(l)$ is also a ZM Gaussian with a variance of σ_e^2 . Using approximation $\mathbf{h}(l) \approx \mathbf{h}_{opt}$ in steady state, we obtain

$$e^c(l) \approx (\mathbf{h}_0 - \mathbf{h}_{opt})^T \mathbf{u}(l) = \zeta_{\mathbf{h}}^T \mathbf{u}(l). \quad (45)$$

Based on (44), we have

$$\sigma_e^2 \approx \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} + \sigma_v^2. \quad (46)$$

Under the steady-state, $w(e(l))$ is approximated as $w(e(l)) \approx \frac{2}{c^2} - \frac{e^2(l)}{3c^4}$ using the Taylor expansion. Therefore, $\mathbb{E} [w(e(l))]$ and $\mathbb{E} [w^2(e(l))]$ are respectively represented by

$$\mathbb{E} [w(e(l))] = \frac{2}{c^2} - \frac{1}{3c^4} (\zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} + \sigma_v^2), \quad (47)$$

and

$$\begin{aligned} \mathbb{E} [w^2(e(l))] &= \frac{4}{c^4} + \frac{e^4(l)}{9c^8} - \frac{4}{3c^6} e^2(l) \\ &= \frac{4}{c^4} + \frac{1}{9c^8} (\zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} + \sigma_v^2)^2 - \frac{4}{3c^6} (\zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} + \sigma_v^2). \end{aligned} \quad (48)$$

In a non-Gaussian noise case, considering Taylor expansions of $w(e(l))$ about $v(l)$ and $e^c(l)$, we obtain

$$\begin{aligned} w(e(l)) &= w(e^c(l) + v(l)) = w(v(l)) + w'(v(l))e^c(l) \\ &+ \frac{1}{2}w''(v(l))(e^c(l))^2 + o((e^c(l))^2). \end{aligned} \quad (49)$$

Accordingly, we have

$$\mathbb{E} [w(e(l))] = \frac{2}{c^2} - \frac{1}{3c^4} \mathbb{E} [v^2(l)] - \frac{1}{3c^4} \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}}, \quad (50)$$

$$\begin{aligned} \mathbb{E} [w^2(e(l))] &= \frac{4}{c^4} + \frac{1}{9c^8} \mathbb{E} [v^4(l)] - \frac{4}{3c^6} \mathbb{E} [v^2(l)] \\ &+ \zeta_{\mathbf{h}}^T \mathbf{R}_{\mathbf{u}} \zeta_{\mathbf{h}} \left[-\left(\frac{2}{c^2} - \frac{1}{3c^4} \mathbb{E} [v^2(l)]\right) \times \frac{2}{3c^4} + \frac{4}{9c^8} \mathbb{E} [v^2(l)] \right]. \end{aligned} \quad (51)$$

IV. SIMULATION RESULTS

In Fig. 1(a), three different non-Gaussian noise distributions, namely, uniform-noise, binary-noise and impulsive-noise, are considered to verify the analysis under $c = 1.4$. It is seen that theoretical analysis and simulation results match well, which shows the accuracy of the analysis. Here, the probability for the uniform noise is $f(a) = \begin{cases} 0.5x, & -x \leq a \leq x \\ 0, & \text{others} \end{cases}$ with a zero mean and variance of 0.86, while it for the binary noise is $\Pr\{a = x\} = \Pr\{a = -x\} = 0.5$ with zero mean and variance of 0.9. The impulsive-noise model is given by $v_j(l) = p(l) + m(l)\phi(l)$, where $p(l)$ represents white-Gaussian-noise, and its mean equals to zero, and its variance is σ_p^2 , respectively. $m(l)\phi(l)$ is a Bernoulli Gaussian process that has a success probability $\Pr[m(l) = 1] = \Pr$, $\Pr[m(l) = 0] = 1 - \Pr$, and

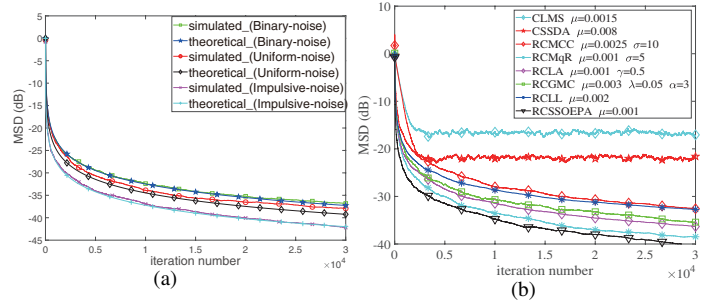


Fig. 1: (a) Theoretical and simulated transient MSD of the proposed RCSOEP algorithm under binary noise, uniform noise and impulsive noise; (b) Convergence of the CSSDA, RCSOEP and mentioned recursive algorithms under Laplace noise.

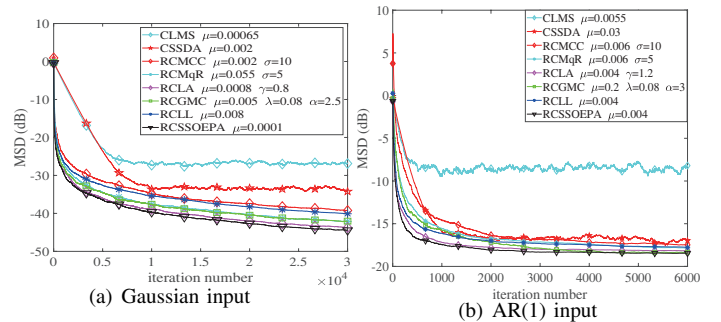


Fig. 2: Convergence of the CSSDA, RCSOEP and mentioned recursive algorithms under impulsive-noise with two different input signals: (a) Gaussian; (b) AR(1).

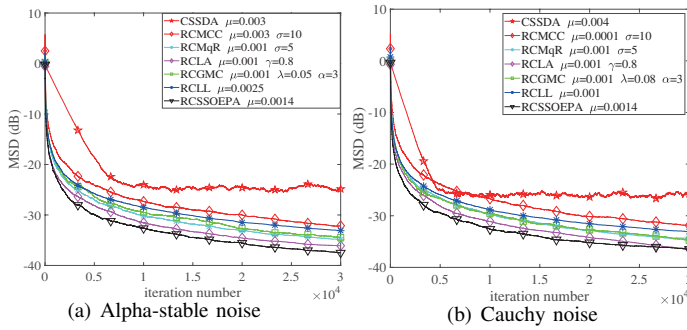


Fig. 3: Convergence of the CSSDA, RCSOPEA and mentioned recursive algorithms under alpha-stable and Cauchy noises.

$\phi(l)$ is a ZM-Gaussian process with $\sigma_\phi^2 = d\sigma_p^2 (d \gg 1)$ [22], [23]. The parameters \mathbf{S} , \mathbf{g} and \mathbf{h}_0 are obtained by referring to [6]. Herein, Pr equals to 0.3, and variances of the additive Gaussian noise and $\phi(l)$ are 0.08 and 1, respectively. From the simulation results, the theoretical analysis and simulation results match very well thus verifying this analysis. In Fig.1(b), the convergence of the proposed RCSOPEA algorithm under Laplace noise [1] is given, where we can see that the proposed RCSOPEA is still superior to mentioned algorithms.

Fig. 2 compares the performance of RCSOPEA and RCMCC [14], RCLL [16], RCMqR [17], RCLA [18], RCGMC [19] and CSSDA (CSSDA is obtained by minimizing (4)) under impulsive-noise with Gaussian and AR(1) input signals. Variances of the additive Gaussian noise and $\phi(l)$ are 0.2 and 9, respectively, and $\text{Pr} = 0.3$. Here, AR(1) signal is obtained from a first-order process $Q(Z) = \frac{1}{1-0.8Z^{-1}}$, and the kernel-width for RCMCC and CMCC are 10. Adjust the step-sizes to make the initial convergence for the CMCC, CLMS and CSSDA the same. Simulation results show that the RCSOPEA algorithm can get faster convergence and achieve lower MSD than mentioned algorithms.

We now consider two nonGaussian noise (Cauchy and Alpha-stable) with the probability density functions of $f(v) = \frac{1}{\pi(1+v^2)}$ and $q(v) = \exp\{j\delta v - \rho|v|^\alpha [1 + j\beta \text{sgn}(v) U(v, \alpha)]\}$, respectively [7], [8], where $U(v, \alpha) = \begin{cases} \tan(\frac{\alpha\pi}{2}), & \alpha \neq 1, \\ \frac{2}{\pi} \log|v|, & \alpha = 1. \end{cases}$, and we define the four parameters α, β, ρ and δ as parameter vector $\tau = (\alpha, \beta, \rho, \delta)$, where α is characteristics factor, β is symmetry factor, ρ is dispersion factor and δ is location parameter. Since the CLMS algorithm is not suitable for processing Cauchy and Alpha-stable noises, only CMCC, CSSDA, RCMCC and RCSOPEA algorithms are compared in Fig. 3. In this simulation, the Cauchy noise parameter v is 0.5, Alpha-stable noise parameter vector is written as $\tau = (1, 0, 0.5, 0)$, and $c = 2.5$ for CSSDA and RCSOPEA. From the simulated results, the proposed RCSOPEA algorithm provides the lowest MSD.

V. CONCLUSION

This brief has proposed a recursive constrained sine second-order error promoting adaptive (RCSOPEA) algorithm by

constructing the SSOE cost function. The convergence and MSD have been theoretically analyzed and numerically investigated to verify the analysis. The simulated results have shown that the analysis matches very well to simulations, and the devised RCSOPEA has a smaller MSD and faster convergence compared to other algorithms under various noises.

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