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Chapter 1

No speedup for geometric theories

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Geometric theories based on classical logic are conservative over their intuitionistic counterparts for geometric implications. The latter result (sometimes referred to as Barr's theorem) is squarely a consequence of Gentzen's Hauptsatz. Prima facie though, cut elimination can result in superexponentially longer proofs. In this paper it is shown that the transformation of a classical proof of a geometric implication in a geometric theory into an intuitionistic proof can be achieved in a feasible way.*

Contents

1. Introduction

For geometric theories it is known that the existence of a classical proof of a geometric implication yields the existence of an intuitionistic proof. Existing effective proofs of this fact use cut elimination (see [1–5]) and thus are liable to effect a superexponential blow-up.

When I visited Stanford in November 2013, Grisha Mints told me that

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*Paper written in honor of Grisha Mints and Erik Palmgren.

he was aiming to show that the transformation can be achieved via a polynomial time algorithm. Sadly, this was the last time I saw him. Some of what he told me is reflected in his article [6], which presents a partial result in that a merely polynomially longer proof is achievable in an augmentation of the intutitionistic theory via relational Skolem axioms. The article, though, does not achieve what he had originally intended.[a] At the time, I did not start to work on the problem but the conversation with Grisha lingered in my mind over the years and for some reason I became convinced that there should be a simple argument based on an interpretation akin to the Friedman–Dragalin A-translation. The short paper before the reader is an elaboration of that idea.[b]

As I was working on such a translation it dawned on me that similar ideas must have been considered before. Indeed, Leivant in [8] used interpretations of classical logic into intuitionistic logic to obtain (partial) conservativity results for classical theories over their intuitionistic versions. He explained that the Friedman–Dragalin interpretation "is derived naturally from a trivial translation of I into M" ([8, p. 683]), where I and M signify intuitionistic and minimal logic, respectively, and "trivial translation" refers to Kolmogorov's 1925 translation [9]. The observation that the latter is actually an interpretation into Johanssons' 1937 minimal logic [10] emerges as the fulcrum for obtaining conservativity results in that in minimal logic falsum $\perp$ just acts as placeholder for an arbitrary formula. Crucial in the machinery of [8] are the definitions of three syntactical classes, namely the *spreading, wiping and isolating* schemata and formulas with regard to the Kolmogorov interpretation (also see [11], Ch. 2, Sect. 3 for an exposition). The ideas underlying these classes have informed Definition 3.5 and Proposition 3.7. But alas, I couldn't see how to directly infer the conservativity of classical over intuitionistic geometric theories by reassembling results from [8]. However, Ishihara's article [12] actually furnishes what is needed. A nice move in [12] is to introduce a new propositional constant into the language which can act as a placeholder for arbitrary formulas not only in minimal logic but also in intuitionistic logic.[c]

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[a] See also the comments on and an evaluation of [6] in [7].

[b] Barr's theorem is sometimes alleged to have miraculous powers that allow one to remove the axiom of choice from proofs of geometric implications. For a skeptical investigation of such claims see [4].

[c] The translations of Kolmogorov, Gentzen, Gödel and Johansson belong to the oldest and simplest syntactic transformations in proof theory. They have been modified and mulled over many times and are still being discussed in relatively recent contemporary research, some of which is related to the theme of this article, e.g. [13–17].

1.1. *Postscriptum*

A preprint version of this paper was deposited on arXiv in May 2021. I gave a talk emanating from it at the meeting Dagstuhl meeting "Geometric Logic, Constructivisation and Automated Theorem Proving" in November 2021. Only a couple of days before my talk, however, I chanced upon the 2002 article *An Intuitionistic Axiomatisation of Real Closed Fields* [18] by Erik Palmgren in which he also uses the Friedman–Dragalin to show the conservativity of classical over intuitionistic geometric theories. The procedure in [18] is somewhat different to the one in this paper in that [18] uses certain normal forms for geometric implications. Furthermore, the current paper shows conservativity for a larger syntactic class $\mathcal{Q}$ and the techniques could lend themselves to further generalizations.[d] It also elaborates on details as to why the transformation yields a polynomial time algorithm.

More importantly, though, I am glad that this paper gives me a chance to dedicate it to the memory of two friends, Grisha Mints and Erik Palmgren, who are keenly missed.

2. Geometric theories

Definition 2.1. The *positive formulas* are constructed from atomic formulas and falsum $\perp$ by $\wedge$, $\vee$, and $\exists$.

Geometric implications are made up of the positive formulas, implications of positive formulas and the result of prefixing universal quantifiers to positive formulas and implications of positive formulas.

$\neg\varphi$ is defined as $\varphi \to\perp$. Thus if φ is a positive formula then $\neg\varphi$ is a geometric implication.

A theory is *geometric* if all its axioms are geometric implications.

Below we shall give several examples of geometric theories.

Examples 2.2.

(i) Robinson arithmetic formulated in the language with a constant 0, a unary successor function symbol suc, binary function symbols $+$ and $\cdot$, and a binary predicate symbol $<$.

(ii) The theories of groups, rings, local rings and division rings have geometric axiomatizations. Local rings are commutative rings with $0 \neq 1$ having just one maximal ideal. On the face of it, the latter property appears to be second order but it can be rendered

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[d] Besides, the current paper is much indebted to Ishihara's article [12].

geometrically as follows:

$$\forall x\,(\exists y\, x \cdot y = 1 \;\vee\; \exists y\,(1-x)\cdot y = 1).$$

(iii) The theories of fields, ordered fields, algebraically closed fields and real closed fields have geometric axiomatizations. To express invertibility of non-zero elements one uses $\forall x\,(x = 0 \vee \exists y\, x \cdot y = 1)$ rather than the non-geometric axiom $\forall x\,(x \neq 0 \to \exists y\, x \cdot y = 1)$. To express algebraic closure replace axioms

$$s \neq 0 \;\to\; \exists x\, sx^n + t_1x^{n-1} + \ldots + t_{n-1}x + t_n = 0$$

by

$$s = 0 \vee \exists x\, sx^n + t_1x^{n-1} + \ldots + t_{n-1}x + t_n = 0$$

where sx^k is short for $s \cdot x \cdot \ldots \cdot x$ with k many x.

Also the theory of *differential fields* has a geometric axiomatization. This theory is written in the language of rings with an additional unary function symbol δ. The axioms are the field axioms plus $\forall x \forall y\, \delta(x+y) = \delta(x) + \delta(y)$ and $\forall x \forall y\, \delta(x \cdot y) = x \cdot \delta(y) + y \cdot \delta(x)$.

(iv) The theory of projective geometry has a geometric axiomatization.

(v) The theories of equivalence relations, dense linear orders, infinite sets and graphs also have geometric axiomatizations.

It is also interesting that Kant's logic in his *Critique of Pure Reason* [19] and the *Jäsche Logik* [20] can be identified with geometric logic as shown by T. Achourioti and M. van Lambalgen in [21, 22].

3. Conservativity

The best and most elegant proof system for proof-theoretic investigations is Gentzen's sequent calculus. With minor notational variations, this article will follow the presentation in Takeuti's book *Proof Theory* [23]. We will deviate, though, a bit from the setup in chapter 1 of [23] in that we

- use $\perp$ as a propositional constant (or 0-ary predicate symbol) and define $\neg\varphi$ to be $\varphi \to\perp$;
- use the symbol $\to$ rather than $\supset$ for the implication symbol;
- use $\Rightarrow$ to separate the left and right part of a sequent, i.e., $\Gamma \Rightarrow \Delta$ rather than $\Gamma \to \Delta$;
- add sequents $\Gamma, \perp\Rightarrow \Delta$ as axioms[e] and omit the rules for $\neg$ (this axiom scheme for $\perp$ will be refereed to as $\mathsf{Ax}_\perp$).

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[e] In [23], axioms are called *initial sequents*.

Definition 3.1. Intuitionistic sequents $\Gamma \Rightarrow \Delta$ satisfy the extra requirement that the succedent Δ contains at most one formula. In the **intuitionistic** version of this sequent calculus only intuitionistic sequents are allowed. In the **minimal logic** version only intuitionistic sequents are permitted and the scheme $\mathsf{Ax}_\perp$ is omitted.

We convey derivability of a sequent $\Gamma \Rightarrow \Delta$ in classical, intuitionistic, and minimal logic by writing $\vdash_c \Gamma \Rightarrow \Delta$, $\vdash_i \Gamma \Rightarrow \Delta$, and $\vdash_m \Gamma \Rightarrow \Delta$, respectively.

A theory T is a set of sentences. In derivability in T one can use any sequent $\Gamma \Rightarrow \varphi$ with $\varphi \in T$ as an axiom (initial sequent). $T \vdash_c \Gamma \Rightarrow \Delta$, $T \vdash_i \Gamma \Rightarrow \Delta$, and $T \vdash_m \Gamma \Rightarrow \Delta$ are defined accordingly.

For a formula φ, we shall write $\vdash_c \varphi$, $\vdash_i \varphi$, and $\vdash_m \varphi$ to convey that $\vdash_c \emptyset \Rightarrow \varphi$, $\vdash_c \emptyset \Rightarrow \varphi$, and $\vdash_c \emptyset \Rightarrow \varphi$, respectively, where $\emptyset$ stands for the empty sequence of formulas.

Definition 3.2. We shall use E as a symbol for a new propositional constant (or predicate symbol of arity 0). Its purpose will be to serve as a placeholder for an arbitrary formula. Let $\neg_{\mathsf{E}}\varphi$ be an abbreviation for $\varphi \to \mathsf{E}$. The E-negative translation ${}^{\mathsf{E}}$ is defined as follows:

$$P^{\mathsf{E}} := \neg_{\mathsf{E}}\neg_{\mathsf{E}} P \text{ for } P \text{ prime and } P \not\equiv \perp; \;\; \perp^{\mathsf{E}} := \mathsf{E}$$
$$(\varphi \circ \psi)^{\mathsf{E}} := \varphi^{\mathsf{E}} \circ \psi^{\mathsf{E}} \text{ for } \circ \in \{\wedge, \to\}; \;\; (\varphi \vee \psi)^{\mathsf{E}} := \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\varphi^{\mathsf{E}} \vee \psi^{\mathsf{E}})$$
$$(\forall x\, \varphi)^{\mathsf{E}} := \forall x\, \varphi^{\mathsf{E}}; \;\; (\exists x\, \varphi)^{\mathsf{E}} := \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x\, \varphi^{\mathsf{E}}.$$

The foregoing translation is basically the Gentzen–Gödel negative translation (see [11, 3.4, 3.5]), which engineers an interpretation of classical logic in minimal logic.

Corollary 3.3. *Given a theory T, the theory T^{E} has as axioms all formulas ψ^{E} with ψ an axiom of T.*

(i) $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi^{\mathsf{E}} \leftrightarrow \varphi^{\mathsf{E}}$.
(ii) $T \vdash_c \varphi \;\Rightarrow\; T^{\mathsf{E}} \vdash_m \varphi^{\mathsf{E}}$.

Proof. Since $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\theta_0 \vee \theta_1) \leftrightarrow \neg_{\mathsf{E}}(\neg_{\mathsf{E}}\theta_0 \wedge \neg_{\mathsf{E}}\theta_1)$ and $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x\theta(x) \leftrightarrow \neg_{\mathsf{E}}\forall x \neg_{\mathsf{E}}\theta(x)$ hold it follows that the ${}^{\mathsf{E}}$-translation amounts to the same as the Gentzen–Gödel translation (the ^{g}-translation in [11, 3.4, 3.5]), taking into account that in minimal logic $\perp$ is an arbitrary propositional constant for which we can substitute E. Hence (i) follows from [11, 2.3.3] and (ii) from [11, 2.3.5]. □

Below we shall frequently adopt the convention that a string of implications $\to$ is considered to be bracketed to the right, i.e., $\varphi_1 \to \varphi_2 \to \ldots \to \varphi_{n-1} \to \varphi_n$ is an abbreviation for $\varphi_1 \to (\varphi_2 \to (\ldots(\varphi_{n-1} \to \varphi_n)\ldots))$.

Lemma 3.4. *(1)* $\vdash_m \varphi \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi$;
(2) $\vdash_m (\varphi \to \psi) \to (\neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\psi)$;
(3) $\vdash_m \neg_{\mathsf{E}}\neg(\varphi \wedge \psi) \to \neg_{\mathsf{E}}\neg\varphi \wedge \neg_{\mathsf{E}}\neg\psi$;
(4) $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi \wedge \neg_{\mathsf{E}}\neg_{\mathsf{E}}\psi \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\varphi \wedge \psi)$;
(5) $\vdash_m \neg_{\mathsf{E}}\neg(\varphi \vee \psi) \to \neg_{\mathsf{E}}(\neg\varphi \wedge \neg\psi)$;
(6) $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi \vee \neg_{\mathsf{E}}\neg_{\mathsf{E}}\psi) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\varphi \vee \psi)$;
(7) $\vdash_m \neg_{\mathsf{E}}\neg(\varphi \to \psi) \to (\neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi \to \neg_{\mathsf{E}}\neg\psi)$;
(8) $\vdash_i (\neg_{\mathsf{E}}\neg\varphi \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\psi) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\varphi \to \psi)$;
(9) $\vdash_m \neg_{\mathsf{E}}\neg\forall x\varphi(x) \to \forall x\neg_{\mathsf{E}}\neg\varphi(x)$;
(10) $\vdash_m \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x\neg_{\mathsf{E}}\neg_{\mathsf{E}}\varphi(x) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x\varphi(x)$.

Proof. These claims are stated in [12, Lemma 2] without proofs. (1), (2), (4), (6), and (10) are well-known with E replaced by $\bot$ (see e.g. [8, 1.2]), so it's clear that they hold in minimal logic. We now turn to the interesting cases that mix $\neg$ and $\neg_{\mathsf{E}}$.

For (3), notice that $\vdash_m \neg\varphi \to \neg(\varphi \wedge \psi)$ and $\vdash_m \neg\psi \to \neg(\varphi \wedge \psi)$, and therefore

$$\vdash_m (\neg(\varphi \wedge \psi) \to \mathsf{E}) \to (\neg\varphi \to \mathsf{E}) \wedge (\neg\psi \to \mathsf{E}).$$

(5): We have $\vdash_m [(\varphi \to\bot) \wedge (\psi \to\bot)] \to (\varphi \vee \psi) \to\bot$, yielding

$$\vdash_m [((\varphi \vee \psi) \to\bot) \to \mathsf{E}] \to ([(\varphi \to\bot) \wedge (\psi \to\bot)] \to \mathsf{E}).$$

(7): Successively we see that:

$$\vdash_m \neg\psi \to (\varphi \to \neg(\varphi \to \psi))$$
$$\vdash_m \neg\psi \to \varphi \to (\neg(\varphi \to \psi) \to \mathsf{E}) \to \mathsf{E}$$
$$\vdash_m \neg\psi \to (\neg(\varphi \to \psi) \to \mathsf{E}) \to \varphi \to \mathsf{E}$$
$$\vdash_m \neg\psi \to (\neg(\varphi \to \psi) \to \mathsf{E}) \to ((\varphi \to \mathsf{E}) \to \mathsf{E}) \to \mathsf{E}$$
$$\vdash_m (\neg(\varphi \to \psi) \to \mathsf{E}) \to ((\varphi \to \mathsf{E}) \to \mathsf{E}) \to \neg\psi \to \mathsf{E}$$

(8): $\vdash_i \neg\varphi \to \varphi \to \psi$ and $\vdash_i ((\varphi \to \psi) \to \mathsf{E}) \to \neg\varphi \to \mathsf{E}$, so

(a) $\vdash_i ((\varphi \to \psi) \to \mathsf{E}) \to [(\neg\varphi \to \mathsf{E}) \to (\psi \to \mathsf{E}) \to \psi] \to (\psi \to \mathsf{E}) \to \mathsf{E}$

(b) $\vdash_i ((\varphi \to \psi) \to \mathsf{E}) \to \psi \to \mathsf{E}$

From (a) and (b) we obtain the desired

$$\vdash_i ((\varphi \to \psi) \to \mathsf{E}) \to [(\neg\varphi \to \mathsf{E}) \to (\psi \to \mathsf{E}) \to \psi] \to \mathsf{E}.$$

(9): We have $\vdash_m (\varphi(a) \to\perp) \to \forall x\varphi(x) \to\perp$, and hence

$$\vdash_m [(\forall x\varphi(x) \to\perp) \to \mathsf{E}] \to (\varphi(a) \to\perp) \to \mathsf{E}, \quad \text{whence}$$
$$\vdash_m [(\forall x\varphi(x) \to\perp) \to \mathsf{E}] \to \forall x[(\varphi(x) \to\perp) \to \mathsf{E}].$$

□

The following syntactic classes bear some resemblance to the *spreading, wiping and isolating* schemata and formulas in [8] but are actually singled out in [12].

Definition 3.5. We define syntactic classes of formulas $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{J}$ simultaneously by the following clauses:

(1) $\perp$ and every atomic formula $Pt_1 \dots t_n$ belong to $\mathcal{Q}$. If $Q, Q', \tilde{Q}(a) \in \mathcal{Q}$ then so are $Q \wedge Q'$, $Q \vee Q'$, $\exists x\tilde{Q}(x)$ and $\forall x\tilde{Q}(x)$. If $Q \in \mathcal{Q}$ and $J \in \mathcal{J}$ then $J \to Q \in \mathcal{Q}$.
(2) $\perp\in \mathcal{R}$. If $R, R', \tilde{R}(a) \in \mathcal{R}$ then so are $R \wedge R'$, and $\forall x\tilde{R}(x)$. If $R \in \mathcal{R}$ and $J \in \mathcal{J}$ then $J \to R \in \mathcal{R}$.
(3) $\perp$ and every atomic formula $Pt_1 \dots t_n$ belong to $\mathcal{J}$. If $J, J', \tilde{J} \in \mathcal{J}$ then so are $J \wedge J'$, $J \vee J'$ and $\exists x\tilde{J}(x)$. If $J \in \mathcal{J}$ and $R \in \mathcal{R}$ then $R \to J \in \mathcal{J}$.

Corollary 3.6. *(i) All positive formulas are in both, $\mathcal{Q}$ and $\mathcal{J}$.*
(ii) All geometric implications are in $\mathcal{Q}$.

Proof. Obvious. □

The following proposition is due to Ishihara [12].

Proposition 3.7. *(i) For $\varphi \in \mathcal{Q}$, $\vdash_i \varphi \to \varphi^{\mathsf{E}}$.*
(ii) For $\psi \in \mathcal{R}$, $\vdash_i \neg_{\mathsf{E}}\neg\psi \to \psi^{\mathsf{E}}$.
(iii) For $\theta \in \mathcal{J}$, $\vdash_i \theta^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\theta$.

Proof. We prove these derivabilities simultaneously by induction on the generation of the classes $\mathcal{Q}, \mathcal{R}, \mathcal{J}$. The proof given here is more detailed than the one for [12, Proposition 7].

(i): Obviously we have $\vdash_i\perp\to \mathsf{E}$ and $\vdash_i \psi \to (\psi \to \mathsf{E}) \to \mathsf{E}$, which yields $\vdash_i A \to A^{\mathsf{E}}$ for atomic formulas A.

Now suppose $\vdash_i Q_i \to Q_i^{\mathsf{E}}$ for $i \in \{0, 1\}$. Then

$$\vdash_i Q_0 \wedge Q_1 \to Q_0^{\mathsf{E}} \wedge Q_1^{\mathsf{E}},$$

thus $\vdash_i Q_0 \wedge Q_1 \to (Q_0 \wedge Q_1)^{\mathsf{E}}$. Likewise, one has $\vdash_i Q_0 \vee Q_1 \to Q_0^{\mathsf{E}} \vee Q_1^{\mathsf{E}}$ and hence $\vdash_i Q_0 \vee Q_1 \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(Q_0^{\mathsf{E}} \vee Q_1^{\mathsf{E}})$, i.e., $\vdash_i Q_0 \vee Q_1 \to (Q_0 \vee Q_1)^{\mathsf{E}}$.

Next assume $\vdash_i Q(a) \to Q(a)^{\mathsf{E}}$. Then

$$\vdash_i \forall x Q(x) \to \forall x Q(x)^{\mathsf{E}},$$

thus, $\vdash_i \forall x Q(x) \to (\forall x Q(x))^{\mathsf{E}}$. Likewise, we have $\vdash_i \exists x Q(x) \to \exists x Q(x)^{\mathsf{E}}$, and so $\vdash_i \exists x Q(x) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x Q(x)^{\mathsf{E}}$, which is $\vdash_i \exists x Q(x) \to (\exists x Q(x))^{\mathsf{E}}$.

Finally assume $\vdash_i J^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J$ and $\vdash_i Q \to Q^{\mathsf{E}}$. Then, as $\vdash_i (J \to Q) \to (\neg_{\mathsf{E}}\neg_{\mathsf{E}} J \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q)$ holds by Lemma 3.4(2),

$$(*) \quad (J \to Q) \to (J^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q).$$

We also obtain $\vdash_i \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q^{\mathsf{E}} \to Q^{\mathsf{E}}$ from Corollary 3.3(i). As $\vdash_i Q \to Q^{\mathsf{E}}$ yields $\vdash_i \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q^{\mathsf{E}}$, we have $\vdash_i \neg_{\mathsf{E}}\neg_{\mathsf{E}} Q \to Q^{\mathsf{E}}$, which yields $\vdash_i (J \to Q) \to (J^{\mathsf{E}} \to Q^{\mathsf{E}})$ by $(*)$, thus $\vdash_i (J \to Q) \to (J \to Q)^{\mathsf{E}}$.

(ii): Since $\vdash_i \neg \perp$ we have $\vdash_i \neg_{\mathsf{E}}\neg \perp \to \mathsf{E}$.

Now suppose that $\vdash_i \neg_{\mathsf{E}}\neg R_j \to R_j^{\mathsf{E}}$ holds for $j \in \{0,1\}$. According to Lemma 3.4(3) we have

$$\vdash_i \neg_{\mathsf{E}}\neg(R_0 \wedge R_1) \to \neg_{\mathsf{E}}\neg R_0 \wedge \neg_{\mathsf{E}}\neg R_1$$

and thus $\vdash_i \neg_{\mathsf{E}}\neg(R_0 \wedge R_1) \to R_0^{\mathsf{E}} \wedge R_1^{\mathsf{E}}$, i.e., $\vdash_i \neg_{\mathsf{E}}\neg(R_0 \wedge R_1) \to (R_0 \wedge R_1)^{\mathsf{E}}$.

Next assume that $\vdash_i \neg_{\mathsf{E}}\neg R(a) \to R(a)^{\mathsf{E}}$. By Lemma 3.4(9) we have $\vdash_i \neg_{\mathsf{E}}\neg\forall x R(x) \to \forall x \neg_{\mathsf{E}}\neg R(x)$. Therefore, $\vdash_i \neg_{\mathsf{E}}\neg\forall x R(x) \to \forall x R(x)^{\mathsf{E}}$, i.e., $\vdash_i \neg_{\mathsf{E}}\neg\forall x R(x) \to (\forall x R(x))^{\mathsf{E}}$.

Finally suppose that $\vdash_i J^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J$ and $\vdash_i \neg_{\mathsf{E}}\neg R \to R^{\mathsf{E}}$. Then,

$$\vdash_i (\neg_{\mathsf{E}}\neg_{\mathsf{E}} J \to \neg_{\mathsf{E}}\neg R) \to (J^{\mathsf{E}} \to R^{\mathsf{E}})$$

and hence, by Lemma 3.4(7), $\vdash_i \neg_{\mathsf{E}}\neg(J \to R) \to (J^{\mathsf{E}} \to R^{\mathsf{E}})$, i.e., $\vdash_i \neg_{\mathsf{E}}\neg(J \to R) \to (J \to R)^{\mathsf{E}}$.

(iii): We have $\vdash_i \perp^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} \perp$ and $\vdash_i A^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} A$ for atomic A since $A^{\mathsf{E}} \equiv \neg_{\mathsf{E}}\neg_{\mathsf{E}} A$.

Now assume that $\vdash_i J_i^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J_i$ for $i \in \{0,1\}$. Then, $\vdash_i (J_0 \wedge J_1)^{\mathsf{E}} \to (\neg_{\mathsf{E}}\neg_{\mathsf{E}} J_0 \wedge \neg_{\mathsf{E}}\neg_{\mathsf{E}} J_1)$, thus $\vdash_i (J_0 \wedge J_1)^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(J_0 \wedge J_1)$ follows by Lemma 3.4(4).

We also have $\vdash_i J_0^{\mathsf{E}} \vee J_1^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J_0 \vee \neg_{\mathsf{E}}\neg_{\mathsf{E}} J_1$ and hence

$$\vdash_i \neg_{\mathsf{E}}\neg_{\mathsf{E}}(J_0^{\mathsf{E}} \vee J_1^{\mathsf{E}}) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(\neg_{\mathsf{E}}\neg_{\mathsf{E}} J_0 \vee \neg_{\mathsf{E}}\neg_{\mathsf{E}} J_1),$$

from which $\vdash_i (J_0 \vee J_1)^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(J_0 \vee J_1)$ follows by Lemma 3.4(6).

Assuming $\vdash_i J(a)^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J(a)$, we have $\vdash_i \exists x J(x)^{\mathsf{E}} \to \exists x \neg_{\mathsf{E}}\neg_{\mathsf{E}} J(x)$, and therefore $\vdash_i \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x J(x)^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x J(x)$ by Lemma 3.4(10), i.e., $\vdash_i (\exists x J(x))^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\exists x J(x)$.

Finally, assume $\vdash_i \neg_{\mathsf{E}}\neg R \to R^{\mathsf{E}}$ and $\vdash_i J^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J$. Then,

$$\vdash_i (R^{\mathsf{E}} \to J^{\mathsf{E}}) \to \neg_{\mathsf{E}}\neg R \to \neg_{\mathsf{E}}\neg_{\mathsf{E}} J,$$

and hence, by Lemma 3.4(8), $\vdash_i (R^{\mathsf{E}} \to J^{\mathsf{E}}) \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(R \to J)$, i.e., $\vdash_i (R \to J)^{\mathsf{E}} \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}(R \to J)$. ☐

We will make use of substitutions for variables and for the placeholder E. In the sequent calculi à la Gentzen and Takeuti [23] and in the Schütte calculi [24, 25] one distinguishes syntactically between free $a, b, c, \ldots$ and bound $x, y, z, \ldots$ variables. As terms can contain only free variables there will never be a problem of substitutability of terms for variables.

We use $\varphi(a/t)$ for the result of replacing every occurrence of the free variable a in φ by the term t. Similarly, for a sequent $\Gamma \Rightarrow \Delta$ and a derivation $\mathcal{D}$ we use $\Gamma(a/t) \Rightarrow \Delta(a/t)$ and $\mathcal{D}(a/t)$, respectively, for the result of replacing every occurrence of the free variable a by the term t. Note, however, that while $\varphi(t/a)$ will be a formula, too, $\mathcal{D}(a/t)$ may no longer be a derivation.

In a similar vein, for a propositional constant E we convey the result of replacing each of its occurrences in a formula, sequent and derivation via $\varphi(\mathsf{E}/\psi)$, $\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta(\mathsf{E}/\psi)$ and $\mathcal{D}(\mathsf{E}/\psi)$, respectively. However, we have to add a caveat here. The formula formation rules in calculi that have different symbols for free and bound variables (such as [23]) allow to go from a formula $\varphi(a)$ to $\forall x\varphi(x)$ only if x does not already occur in $\varphi(a)$. Thus, if one substitutes a formula ψ for E in a formula the resulting syntactic object may no longer be a formula. As a result, we tacitly require that before substitutions are made, bound variables in ψ have to replaced by ones that avoid this clash.

Now, by obeying this additional requirement, $\varphi(\mathsf{E}/\psi)$ will be again a formula. However, $\mathcal{D}(\mathsf{E}/\psi)$ may no longer be a derivation as some eigenvariable conditions may have become violated in the process. So we have to take care of that as well.

Two substitutability results will be useful.

Lemma 3.8. *Let T be a theory. If $\mathcal{D}(a)$ is a T-derivation of $\Gamma \Rightarrow \Delta$ and c is a variable that doesn't occur in $\mathcal{D}$ then $\mathcal{D}(a/c)$ is a T-derivation of $\Gamma(a/c) \Rightarrow \Delta(a/c)$.*

Proof. This is obvious as c is a completely new variable as far as $\mathcal{D}$ is concerned, so no eigenvariable conditions are affected by this substitution. Formally one proves this by induction on the number of inferences of $\mathcal{D}$ (see [23, Ch. 1, Lemma 2.10] □

Proposition 3.9. *Let T be a theory whose axioms do not contain E. If $\mathcal{D}$ is a T-derivation of $\Gamma \Rightarrow \Delta$ and ψ is an arbitrary formula then there is a T-derivation $\mathcal{D}'$ of $\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta(\mathsf{E}/\psi)$.*

Proof. Use induction on the number of inferences of $\mathcal{D}$. The only kinds of inferences we need to look at are $\forall$: right and $\exists$: left. So suppose the last inference of $\mathcal{D}$ was $\forall$: right with premise $\Gamma \Rightarrow \Delta', \varphi$ and conclusion $\Gamma \Rightarrow \Delta_0, \forall x\, \varphi(a/x)$, where Δ is $\Delta_0, \forall x\, \varphi(a/x)$ and a does not occur in $\Gamma \Rightarrow \Delta$. Let $\mathcal{D}_0$ be the immediate subderivation of $\mathcal{D}$ with end sequent $\Gamma \Rightarrow \Delta_0, \varphi$. Let c be a free variable that neither occurs in $\mathcal{D}$ nor in ψ. By Lemma 3.8, $\mathcal{D}_1 := \mathcal{D}_0(a/c)$ is a derivation, too. Note that $\mathcal{D}_1$ is a derivation of

$$\Gamma \Rightarrow \Delta_0, \varphi(a/c)$$

owing to the eigenvariable condition satisfied by a.

Since $\mathcal{D}_1$ has fewer inferences than $\mathcal{D}$ we can apply the induction hypothesis to arrive at a derivation $\mathcal{D}_2$ of

$$\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta_0(\mathsf{E}/\psi), (\varphi(a/c))(\mathsf{E}/\psi).$$

As c does not occur in $\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta_0(\mathsf{E}/\psi)$ and ψ, we can apply an inference $\forall$: right to obtain a derivation $\mathcal{D}'$ of

$$\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta_0(\mathsf{E}/\psi), \forall x\, (((\varphi(a/c))(\mathsf{E}/\psi))(c/x))$$

which is the same as $\Gamma(\mathsf{E}/\psi) \Rightarrow \Delta_0(\mathsf{E}/\psi)$ since $\forall x\, (((\varphi(a/c))(\mathsf{E}/\psi))(c/x)) \equiv (\forall x\, \varphi(a/x))(\mathsf{E}/\psi)$.

$\exists$: left is dealt with in a similar fashion. □

We still haven't strictly shown that the proof length increases at most polynomially. This will be addressed in the next section.

Theorem 3.10. *If T is a geometric theory, i.e. the axioms of T are geometric implications, and φ is a geometric implication, then*

$$T \vdash_c \varphi \ \ \textit{yields} \ \ T \vdash_i \varphi.$$

Moreover, if $\mathcal{D}$ is a classical deduction of φ in T, then the size of the intuitionistic deduction of φ in T increases at most polynomially in the size of $\mathcal{D}$.

Proof. Suppose $T \vdash_c \varphi$. By Corollary 3.3(ii) we conclude that

$$T^{\mathsf{E}} \vdash_i \varphi^{\mathsf{E}}. \tag{1}$$

As $T \subseteq \mathcal{Q}$ it follows from (1) and Proposition 3.7(i) that

$$T \vdash_i \varphi^{\mathsf{E}}. \tag{2}$$

Now φ is of the form $\forall x_1 \ldots \forall x_r(\psi \to \theta)$ with $\psi, \theta \in \mathcal{Q} \cap \mathcal{J}$. Thus, by Proposition 3.7(i), we can conclude from (2) that

$$T \vdash_i \psi \to \theta^{\mathsf{E}}. \tag{3}$$

Fact (3) in conjunction with Proposition 3.7(iii) yields

$$T \vdash_i \psi \to \neg_{\mathsf{E}}\neg_{\mathsf{E}}\theta. \tag{4}$$

Now, since E is just a placeholder we may substitute θ for E everywhere in the derivation of (4) by Proposition 3.9, yielding a derivation showing that

$$T \vdash_i \psi \to ((\theta \to \theta) \to \theta). \tag{5}$$

As a result of (5) we have $T \vdash_i \psi \to \theta$ and hence $T \vdash_i \varphi$, as desired. □

From the proof of the foregoing theorem it's clear that conservativity obtains for a wider collection of theories than just geometric ones.

Corollary 3.11. *If T is a theory whose axioms are in $\mathcal{Q}$ and φ is a geometric implication, then*

$$T \vdash_c \varphi \ \text{ yields } \ T \vdash_i \varphi.$$

Moreover, if $\mathcal{D}$ is a classical deduction of φ in T, then the size of the intuitionistic deduction of φ in T increases at most polynomially in the size of $\mathcal{D}$.

Proof. This follows from the proof of Theorem 3.10. □

4. Polynomial time bounds

In view of the foregoing results, it might be rather obvious that the transformation of a classical proof of a geometric implication in a geometric theory into an intuitionistic proof can be carried out in polynomial time. It might be in order, though, to be a bit more precise. The plan, however, is not to do this in detail but rather from a "higher" point of view.

It is a fact that the syntax of first-order logic can be recognized and manipulated by polynomial time algorithms in Buss' theory S^1_2. One place where the arithmetization of metamathematics for the sequent calculus is carried out in detail is [26, Ch. 7]. Among other things, we require functions for the arithmetization of substitutions in Lemma 3.8 and Proposition 3.9. They give rise to Σ^b_1-defined functions of S^1_2 (see [26, p. 130], where this is carried for substitution of a term into a formula). Moreover, all Σ^b_1-definable functions of S^1_2 are polynomial time computable functions (see [26, Corollary 8]). As all manipulation of proofs in this paper can be carried out by Σ^b_1-definable functions of S^1_2 we have achieved our goal.

5. ∞-geometric theories

5.1. *The infinite geometric case*

Much more powerful notions of geometricity are available in infinitary logics. $\mathcal{L}_{\infty\omega}$-logic allows for the formation of infinite disjunctions $\bigvee\Phi$ and conjunctions $\bigwedge\Phi$, where Φ is an arbitrary set of (infinitary) formulae. In this richer syntax a formula is said to be an ∞-*positive* formula, if it can be generated from atoms and $\bot$ via $\vee, \wedge, \exists$ and $\bigvee$. More precisely, the latter means that the infinite disjunction $\bigvee\Phi$ is an ∞-positive formula whenever Φ is a set of ∞-positive formula.

The ∞-geometric implications are obtained in the same way from the ∞-positive formula as the geometric implications are obtained from the positive formulas. An ∞-geometric theory is one whose axioms are ∞-geometric implications.

Examples of such theories are the theories of *flat modules* over a ring (see [27]), *torsion groups*, fields of prime characteristic, archimedean ordered fields and connected graphs. Even Peano arithmetic has an ∞-geometric axiomatization (see [4, 2.4]).

∞-geometric classical theories are also conservative over their intuitionistic version for ∞-geometric formulas. This can be proved in Constructive Zermelo-Fraenkel set theory **CZF** (see [4, Theorem 7.9]) via cut elimination for $\mathcal{L}_{\infty\omega}$. The techniques of this paper can also be extended to the $\mathcal{L}_{\infty\omega}$ context. As a result one can prove conservativity already in a much weaker fragment of **CZF**, namely intuitionistic Kripke-Platek set theory with elementhood induction restricted to Σ-formulas, $\mathbf{IKP}^r$.

As the theories **CZF** and $\mathbf{IKP}^r$ allow for witness extraction from proofs of existential statements (see [28, 6.1], [29, 2.35], [30]) this offers the exciting prospect of extracting bounds from proofs in classical ∞-geometric theories. However, note that for this the classical proof must exist as an object in the constructive background theory. So it's not enough to know of its existence by appealing to principles such as the axiom of choice or Zorn's lemma.

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