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Constructing the Constructible Universe Constructively

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Abstract

We study the properties of the constructible universe, L , over intuitionistic theories. We give an extended set of fundamental operations which is sufficient to generate the universe over Intuitionistic Kripke-Platek set theory without Infinity. Following this, we investigate when L can fail to be an inner model in the traditional sense. Namely, we show that over Constructive Zermelo-Fraenkel (even with the Power Set axiom) one cannot prove that the Axiom of Exponentiation holds in L .

Keywords: Constructive Mathematics, Constructible Universe, Intuitionistic Zermelo-Fraenkel, Constructive Zermelo-Fraenkel, Intuitionistic Kripke-Platek

2000 MSC: primary 03E45, 03E70, secondary 03D65, 03E10, 03F55

1. Introduction

In general, working with an arbitrary “*model of set theory*” can be complicated. For example; the continuum could be arbitrarily large; there may be no definable global well-order or there may not be a clear, combinatorial structure to the universe. Instead, we may want to work with a “*nice*” universe which has a clear fine structure that is well understood and permits a detailed analysis. Such a structure will usually also satisfy additional, useful axioms, for example the existence of a definable global well-order. The general framework for this is the concept of an *inner model*. Over the standard set-theoretic framework of Zermelo-Fraenkel, ZF, M is said to be an inner model of $N \models \text{ZF}$ if:

- $M \subseteq N$,
- $M \models \text{ZF}$,
- $M \cap \text{ORD} = N \cap \text{ORD}$

and the simplest such example of this is Gödel’s *Constructible Universe*, L .

The Constructible Universe was developed by Gödel in two influential papers, [12] and [13], in the late 1930s in order to prove the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis relative to ZF. As with the standard V_α hierarchy, one builds L in stages where in place of the usual power set operation one instead uses a “*definable*” version. The idea being that this thins out the universe one

constructs by only accepting new sets which can be built from the previous sets in some definable way. Therefore, the main question one has to tackle is what is meant by “definable”. That is, given L_α , what is $L_{\alpha+1}$.

There are three main ways to define this operation, all of which can be undertaken in the weak system of Kripke-Platek set theory:

1. Syntactically; using the notion of a “*definability operator*” so that $L_{\alpha+1}$ is the collection of definable subsets of L_α . This is the original approach taken by Gödel in [12] and was formalised in Kripke-Platek by Devlin in [9].
2. By closure under what Barwise calls “*fundamental operations*” or “*Gödel operations*”. This is the approach taken by Gödel in [13] and further studied by Barwise in [4] where he considered the theory of Kripke-Platek with urelements.
3. Using “*rudimentary*” functions. This is a modified version of using fundamental operations which was developed by Jensen and further explored by Mathias, leading to his weak system of Provi, the weakest known system in which one can do both set forcing and build L . The details of this theory can be found in [19].

In this paper we address the question of inner models in intuitionistic set theories. As part of this, we first investigate a weak theory which is sufficient to construct L . After this we question whether L satisfies the conditions of being an inner model.

The intuitionistic approach to constructing L was first undertaken by Lubarsky in [16] under the assumption that V satisfied Intuitionistic Zermelo-Fraenkel, IZF. His approach was to show that the syntactic definition of the constructible universe still goes through in intuitionistic logic, with some minor modifications. The main obstacle one has to overcome is that the ordinals are no longer linearly ordered so one has to be more careful as to how one finds witnesses for the collection of definable subsets of some given set X . This also adds complications to proving the Axiom of Constructibility, that is proving that $V = L$ holds in L . The main reason for this added difficulty is that, because it is unclear if L should contain every ordinal under IZF, on the face of it there is no reason to assume that

$$\bigcup_{\alpha \in \text{ORD} \cap V} L_\alpha = \bigcup_{\alpha \in \text{ORD} \cap L} L_\alpha.$$

In order to circumvent this issue, Lubarsky proves the following lemma, which we shall reprove later in our weaker context of Intuitionistic Kripke-Platek set theory without infinity, $\text{IKP}^{-\text{Inf}}$.

Lemma 5.9 (Lubarsky). *For every ordinal α in V there is an ordinal α^* in L such that $L_\alpha = L_{\alpha^*}$.*

The syntactic approach has been further studied by Crosilla, and appears in the appendix to her PhD thesis, [8]. Here she shows that the construction can be carried out in a fragment of constructive set theory, which is equivalent to what we have defined as IKP, by essentially the same proof as found in [9].

The third approach, via rudimentary functions, has also been explored in constructive contexts by Aczel [1]. Here he defines the weak system of *Rudimentary Constructive Set Theory* and shows that many of Jensen's techniques can be applied in this theory.

In this paper, we shall be interested in which axioms are sufficient to construct the constructible universe. Because the syntactic approach requires essential use of ω in order to work with arbitrarily long finite sequences, it is not the appropriate method to use in IKP without infinity. Therefore, we shall adapt the second approach and use the fundamental operations. Adapting Barwise's method, we shall show that if one expands the collection of fundamental operations, then one can indeed construct L over the weak system of IKP without infinity. It should be noted that one could also consider urelements, as Barwise does. We have chosen not to undertake this study but this could be done without a significant amount of additional work.

Following the structure of Barwise's original argument in Chapter II of [4], the main difference will be that we need to consider some additional logical operations. These are the operations of conjunction, implication and bounded universal quantification which are not treated in the classical case as distinct cases due to their equivalent definitions using disjunction, negation and bounded existential quantification.

Having constructed L , we consider what happens when the universe is a model of Constructive Zermelo-Fraenkel set theory with Power Set, $CZF_{\mathcal{P}}$. By combining a proof-theoretic ordinal analysis of Power Kripke-Platek with a notion of realizability from [25] that implies truth, we shall see that Exponentiation need not hold in L . Since Exponentiation follows from the axioms of Constructive Zermelo-Fraenkel set theory this means that CZF cannot prove that L is an inner model in the traditional sense.

Theorem 7.12. $CZF_{\mathcal{P}} \not\vdash \text{Exp}^L$. Thus, $CZF_{\mathcal{P}} \not\vdash (CZF)^L$.

This result should be appreciated with the context of the analogous result over Power Kripke-Platek. In Proposition 6.47 of [17], Mathias proves that Power Set, which is classically equivalent to Exponentiation, need not hold in L if the background theory only satisfies Power Kripke-Platek.

So, where does this leave us in the search for intuitionistic inner models? We have that they need not satisfy every axiom of the base theory, it is far from clear if they contain every ordinal and it is even unclear if they contain a nice, combinatorial structure. Classically, there is an easy necessary and sufficient condition for a transitive class to be an inner model of ZF. Moreover, this has the benefit of being expressible by a single sentence:

Theorem 1.1 ([14] 13.9). *A transitive class M is an inner model of ZF if and only if it is closed under the fundamental operations and is almost universal. That is, for every set x which is a subset of M , there is some $y \in M$ for which $x \subseteq y$.*

We are unable to get such a strong connection in our intuitionistic context, due to the possible ill behaviour of our ordinals. On the other hand, we are able to get a slightly more modest result which still contains the essence of the original theorem. This is done by defining the notion of an *external cumulative hierarchy* which is a notion encapsulating the idea that our model is iteratively constructed in the external universe via a class length sequence of set-sized approximations. For example,

$$\bigcup_{\alpha \in \text{ORD} \cap V} L_\alpha.$$

Theorem 6.8. *Suppose that V is a model of IZF and $M \subseteq V$ is a definable, transitive proper class with an external cumulative hierarchy. Then M is a model of IZF if and only if M is closed under the fundamental operations and is almost universal.*

2. Intuitionism

When working intuitionistically, it is important to specify the axioms correctly in order to avoid inadvertently being able to deduce the Law of Excluded Middle. For example, we have the following classical results which can be shown to go through over very weak theories.

Theorem 2.1 ([3] 10.4.1). *The Foundation Scheme implies the Law of Excluded Middle.*

Theorem 2.2 (Diaconescu, [10]). *The Axiom of Choice implies the Law of Excluded Middle.*

Avoiding these principles leads to the following axiomatisations of *Intuitionistic Zermelo-Fraenkel*, IZF, and *Intuitionistic Kripke-Platek*, IKP. It is proven in [11] that, intuitionistically, the Replacement Scheme does not imply the Collection Scheme, even if one assumes Dependent Choice holds along with every classically true Σ_1 -sentence in the language of ZF. Moreover, Replacement alone is insufficient to do many of the constructions we need, so we shall formulate IZF with the Collection Scheme. Secondly, in theories with restricted Separation it is beneficial to strengthen the Axiom of Infinity to asserting that there is an inductive set which is contained in every other inductive set, leading to the Axiom of Strong Infinity. Even over IKP this is known to be equivalent to the Axiom of Infinity (for example see Proposition 4.7 of [2] where this equivalence is proven over CZF) however this is non-trivial, so we include the stronger version here for simplicity.

Definition 2.3 (Axiom of Strong Infinity). $\exists a (\text{Ind}(a) \wedge \forall b (\text{Ind}(b) \rightarrow \forall x \in a (x \in b)))$ where $\text{Ind}(a)$ is an abbreviation for $\emptyset \in a \wedge \forall x \in a (x \cup \{x\}) \in a$.

Definition 2.4. Let IZF denote the theory whose underlying logic is intuitionistic and whose axioms are; Extensionality, Empty Set, Pairing, Unions, Power Set, Strong Infinity, Set Induction Scheme, Separation Scheme and Collection Scheme.

Definition 2.5. Let IKP denote the theory whose underlying logic is intuitionistic and whose axioms are; Extensionality, Empty Set, Pairing, Unions, Strong Infinity, Set Induction Scheme, Σ_0 -Separation Scheme and Σ_0 -Collection Scheme.

Because we are interested in which axioms are necessary to construct the constructible universe, in Section 4 we shall take care to differentiate between IKP *without infinity*, which we will call $\text{IKP}^{-\text{Inf}}$, and IKP.¹ We shall therefore regularly refer to IKP as $\text{IKP}^{-\text{Inf}} + \text{Strong Infinity}$ just to make it clear when Strong Infinity is being assumed.

While Diaconescu's theorem tells us that the Axiom of Choice² results in instances of Excluded Middle, this is not the case for all Choice Principles. For example, one can add Dependent Choice (for example to Bishop's constructive mathematics [6]) and, by a result from Chapter 8 of [5], a version of Zorn's Lemma holds in every Heyting-valued model over a model of ZFC. As a second remark, one can see that many of the basic properties of $\text{KP}^{-\text{Inf}}$ can still be deduced in $\text{IKP}^{-\text{Inf}}$. We note a few such theorems here, the proofs of which are given in Chapter 19 of [3].

Proposition 2.6. *In $\text{IKP}^{-\text{Inf}}$ one can prove that:*

- *Let $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$. Then $\forall a, b, c, d (\langle a, b \rangle = \langle c, d \rangle \rightarrow (a = c \wedge b = d))$.*
- $\forall a, b \exists c (c = a \times b)$.

Another important fact about $\text{IKP}^{-\text{Inf}}$ is that we can still deduce the Reflection and Strong Collection Principles for a larger class of formulae, namely the class of Σ -formulae where

Definition 2.7. The class of Σ -formulae is the smallest class containing the atomic formulae which is closed under $\wedge, \vee, \rightarrow$, the bounded quantifiers $\forall x \in a$ and $\exists x \in a$ and the unbounded existential quantifier $\exists x$.

Theorem 2.8 (Σ -Reflection Principle). *For every Σ -formula $\varphi(v)$ we have*

$$\text{IKP}^{-\text{Inf}} \vdash \varphi(v) \leftrightarrow \exists a \varphi^{(a)}(v),$$

where $\varphi^{(a)}(v)$ is the result of replacing each unbounded quantifier in φ with bounded quantification over a .

¹It is worth noting that some authors, for instance Barwise in [4], do not include the Axiom of Infinity in their formulation of KP but instead use either KPI or KP^ω to denote the theory with Infinity. Moreover, other authors, such as Mathias in [18] formulate KP with only Π_1 -Foundation. All the results in this paper would go through with this weakening but we choose to include full Foundation for simplicity.

²in the form that for every set A and function F with domain A satisfying $\forall x \in A \exists y \in F(x)$ there is a function f with domain A such that $\forall x \in A f(x) \in F(x)$

Theorem 2.9 (Σ -Strong Collection Principle). *For every Σ -formula $\varphi(x, y, z)$ the following is a theorem of $\text{IKP}^{-\text{Inf}}$: For every set z , if $\forall x \in a \exists y \varphi(x, y, z)$ then there is a set b such that*

$$\forall x \in a \exists y \in b \varphi(x, y, z) \wedge \forall y \in b \exists x \in a \varphi(x, y, z).$$

One final, particularly powerful tool that is provable in $\text{IKP}^{-\text{Inf}}$ is definitions by Σ -recursion. This allows one to prove that Σ -definable operations, such as the transitive closure of a set, are provably total.

Theorem 2.10 (Σ -Recursion). *If G is a total $(n + 2)$ -ary Σ -definable class function, then there exists a total $(n + 1)$ -ary Σ -definable class function F such that*

$$\text{IKP}^{-\text{Inf}} \vdash \forall \vec{x} \forall y (F(\vec{x}, y) = G(\vec{x}, y, \{z, F(\vec{x}, z)\} \mid z \in y)).$$

However we are not able to deduce the Δ -Separation Principle. This is because the proof makes essential use of the fact that classically $\forall x \in a (\varphi \rightarrow \psi)$ holds if and only if $\forall x \in a (\psi(x) \vee \neg\varphi(x))$ holds, which is not intuitionistically provable.

An important concept is the notion of an ordinal. This is traditionally defined as a transitive set which is well-ordered by the \in relation, however this is not appropriate in our setting. Therefore, we shall follow the definition given in I.3.6 of Barwise [4], which is equivalent to the standard one over KP:

Definition 2.11. An *ordinal* is a transitive set of transitive sets.

Let 0 denote the ordinal \emptyset , which is the unique set given by the Axiom of Empty Set, and 1 the ordinal $\{0\}$.

An important class of ordinals is the class of *Truth Values*

$$\Omega := \{x \mid x \subseteq 1\}.$$

$\Omega = \{0, 1\}$ is the assertion that every statement is either true or false, which entails the Law of Excluded Middle. Therefore, in general we may have other truth values even if they can't necessarily be determined. For example, these can be ordinals of the form

$$\{0 \in 1 \mid \varphi\}$$

where φ is a formula for which we can neither determine φ nor $\neg\varphi$. The complexity introduced by having more than 2 truth values means that the natural ordering assigned to the ordinals becomes tree-like rather than a simple linear order, as can be seen by the following proposition.

Proposition 2.12.

$$\text{IKP}^{-Inf} \vdash \forall \alpha \in \text{ORD}(0 \in \alpha + 1) \longrightarrow \text{Law of Excluded Middle for } \Sigma_0\text{-formulae.}$$

Proof. Let φ be a Σ_0 -formula and consider the ordinal truth value $\alpha = \{0 \in 1 \mid \varphi\}$. If 0 were an element of $\alpha + 1 = \alpha \cup \{\alpha\}$ this would imply that either $0 \in \alpha$ or $0 = \alpha$. The first of these scenarios implies that φ holds while the second one gives us $\neg\varphi$, and thus $\varphi \vee \neg\varphi$ must be true. \square

3. Intermediate Theories

In this section we include several more intuitionistic theories intermediate between IKP and IZF. The first one extends IKP by adding power sets.

In order to do this, we begin by defining the class of $\Sigma_0^{\mathcal{P}}$ -formulae. This extends the class Σ_0 by adding in bounded universal and existential subset quantifiers. Let $\exists x \subseteq y$ and $\forall x \subseteq y$ be abbreviations for $\exists x(x \subseteq y \wedge \dots)$ and $\forall x(x \subseteq y \rightarrow \dots)$ respectively.

Definition 3.1. The class of $\Sigma_0^{\mathcal{P}}$ -formulae is the smallest class which contains the atomic formulae and is closed under $\wedge, \vee, \rightarrow$ and the quantifiers

$$\forall x \in a, \exists x \in a, \forall x \subseteq a, \exists x \subseteq a.$$

Definition 3.2. Let $\text{IKP}(\mathcal{P})$ denote the theory whose underlying logic is intuitionistic and whose axioms are; Extensionality, Empty Set, Pairing, Unions, Power Set, Strong Infinity, Set Induction Scheme, $\Sigma_0^{\mathcal{P}}$ -Separation Scheme and $\Sigma_0^{\mathcal{P}}$ -Collection Scheme.

Remark 3.3 (Mathias). We observe here that $\text{KP}(\mathcal{P})$ is different from KP plus Power Set. For example, $\text{L}_{\aleph_\omega}^{\text{NL}} \models \text{KP} + \text{Power Set}$ but does not model $\text{KP}(\mathcal{P})$.

The next theories we shall discuss are variants of Aczel's *Constructive Zermelo-Fraenkel*, an in-depth study of this theory can be found in [3]. For these we need the axioms of *Strong Collection* and *Subset Collection*:

Definition 3.4.

(Strong Collection). For any formula φ and set a ,

$$\forall x \in a \exists y \varphi(x, y) \longrightarrow \exists b \left(\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y) \right).$$

(Subset Collection). For any formula φ and sets a, b ,

$$\exists c \forall u \left(\forall x \in a \exists y \in b \varphi(x, y, u) \rightarrow \exists d \in c \left(\forall x \in a \exists y \in d \varphi(x, y, u) \wedge \forall y \in d \exists x \in a \varphi(x, y, u) \right) \right).$$

Definition 3.5.

- Let CZF^- denote IKP plus Strong Collection,
- Let CZF denote CZF^- plus Subset Collection,
- Let $\text{CZF}_{\mathcal{P}}$ denote CZF^- plus Power Set.

Remarks 3.6.

- Clearly CZF is a subtheory of $\text{CZF}_{\mathcal{P}}$ and the former is known to prove the Axiom of Exponentiation (see Theorem 5.1.2 of [3]). Therefore, in this context, Subset Collection can be seen as saying that a weak form of Power Set holds.
- One can easily show, by induction on the complexity of $\Sigma_0^{\mathcal{P}}$ -formulae, that $\text{CZF}_{\mathcal{P}}$ proves every instance of $\Sigma_0^{\mathcal{P}}$ -Separation. Therefore, $\text{IKP}(\mathcal{P})$ is a subtheory of $\text{CZF}_{\mathcal{P}}$.
- An alternative way to view $\text{CZF}_{\mathcal{P}}$ is as IZF with Full Separation weakened to only $\Sigma_0^{\mathcal{P}}$ -Separation.
- The theories CZF^- and CZF are known to be equiconsistent with KP. However, CZF plus Full Excluded Middle results in ZF. On the other hand, as proven in Theorem 15.1 of [24], $\text{CZF}_{\mathcal{P}}$ is much stronger than KP and even stronger than Zermelo set theory.

4. The Fundamental Operations

We now proceed to construct the constructible universe, L . In order to do this, given L_α we would like $L_{\alpha+1}$ to be the collection of “*definable subsets*” of L_α for some appropriate definition of definable. Following Barwise, we will choose to do this by defining a series of Σ -operations, each of which will comprise of two arguments. These operations will then be used to generate the constructible sets. The main benefit to this approach over the syntactic one is that it will avoid assuming any form of the Axiom of Infinity.

The operations we will take are the same as in [4] except for the addition of \mathcal{F}_\cap , \mathcal{F}_\rightarrow and \mathcal{F}_\forall . This is to allow us to prove Lemma 4.4 that the Axiom of Separation for a Σ_0 -formula can be obtained by a sequence of such operations. This will be achieved by an induction on the construction of the class of Σ_0 formulae and in order to generalise this to the intuitionistic setting we will need to consider the three additional cases of conjunction, implication and bounded universals that are not treated by Barwise. Conjunctions will be relatively straight forward and we observe here that, over classical logic, intersections are deducible from the original operations via the identity

$$x \cap y = x \setminus (x \setminus y),$$

but this equivalence does not hold intuitionistically. Since implication cannot be deduced from the other logical connectives we need an operation that specifically does this. To be more specific, suppose that we have sequences of fundamental operations \mathcal{F}_φ and \mathcal{F}_ψ such that $\mathcal{F}_\varphi(a) = \{z \in a \mid \varphi(z)\}$ and $\mathcal{F}_\psi(a) = \{z \in a \mid \psi(z)\}$ and we wanted a sequence \mathcal{G} such that $\mathcal{G}(a) = \{z \in a \mid \varphi(z) \rightarrow \psi(z)\}$. This set is equal to $\{z \in a \mid z \in \mathcal{F}_\varphi(a) \rightarrow z \in \mathcal{F}_\psi(a)\}$ and therefore we need an operation that, given sets a , b and c , returns $\{z \in c \mid z \in a \rightarrow z \in b\}$ (where c is needed to ensure this operation results in a set) which is what \mathcal{F}_\rightarrow does.

The final operation is to allow us to find fundamental sequences giving bounded universal quantification. This will be done by adding the operation \mathcal{F}_\forall which essentially says that, given a set relation x the collection of sets of images, $\{\{u \mid \langle z, u \rangle \in x\} \mid z \in \text{dom}(x)\}$ forms a set. We remark here that the same operation is included in the axioms of Provi by Mathias, [19]. We begin by first fixing some notation:

Definition 4.1. For x an ordered pair, y a set of ordered pairs and z a set, define

- $1^{st}(x) = a$ iff $\exists u \in x \exists b \in u (x = \langle a, b \rangle)$,
- $2^{nd}(x) = b$ iff $\exists u \in x \exists a \in u (x = \langle a, b \rangle)$,
- $y^{\langle\langle z \rangle\rangle} := \{u \mid \langle z, u \rangle \in y\}$.

Definition 4.2. The *fundamental operations* are as follows:

- (\mathcal{F}_p) $\mathcal{F}_p(x, y) := \{x, y\}$,
- (\mathcal{F}_\cap) $\mathcal{F}_\cap(x, y) := x \cap \bigcap y$,
- (\mathcal{F}_\cup) $\mathcal{F}_\cup(x, y) := \bigcup x$,
- (\mathcal{F}_\setminus) $\mathcal{F}_\setminus(x, y) := x \setminus y$,
- (\mathcal{F}_\times) $\mathcal{F}_\times(x, y) := x \times y$,
- (\mathcal{F}_\rightarrow) $\mathcal{F}_\rightarrow(x, y) := x \cap \{z \mid y \text{ is an ordered pair} \wedge (z \in 1^{st}(y) \rightarrow z \in 2^{nd}(y))\}$,
- (\mathcal{F}_\forall) $\mathcal{F}_\forall(x, y) := \{x^{\langle\langle z \rangle\rangle} \mid z \in y\} = \{\{u \mid \langle z, u \rangle \in x\} \mid z \in y\}$,
- (\mathcal{F}_d) $\mathcal{F}_d(x, y) := \text{dom}(x) = \{1^{st}(z) \mid z \in x \wedge z \text{ is an ordered pair}\}$,
- (\mathcal{F}_r) $\mathcal{F}_r(x, y) := \text{ran}(x) = \{2^{nd}(z) \mid z \in x \wedge z \text{ is an ordered pair}\}$,
- (\mathcal{F}_{123}) $\mathcal{F}_{123}(x, y) := \{\langle u, v, w \rangle \mid \langle u, v \rangle \in x \wedge w \in y\}$,
- (\mathcal{F}_{132}) $\mathcal{F}_{132}(x, y) := \{\langle u, w, v \rangle \mid \langle u, v \rangle \in x \wedge w \in y\}$,
- ($\mathcal{F}_=$) $\mathcal{F}_=(x, y) := \{\langle v, u \rangle \in y \times x \mid u = v\}$,
- (\mathcal{F}_\in) $\mathcal{F}_\in(x, y) := \{\langle v, u \rangle \in y \times x \mid u \in v\}$.

Remark 4.3. In order to simplify later notation, we shall let \mathcal{I} be the obvious finite set indexing the above operations.

Note that we form n-tuples inductively as

$$\langle x_3, x_2, x_1 \rangle := \langle x_3, \langle x_2, x_1 \rangle \rangle,$$

and therefore $\text{ran}(\{\langle x_3, x_2, x_1 \rangle\}) = \{\langle x_2, x_1 \rangle\}$. The next lemma is adapted from Lemma II.6.1 of [4] which will give us that any instance of Bounded Separation can be written as a sequence of fundamental operations. Note that in the lemma the order of the variables has been inverted because this turns out to simplify the arguments.

Lemma 4.4. *For every Σ_0 -formula $\varphi(x_1, \dots, x_n)$ with free variables among x_1, \dots, x_n , there is a term \mathcal{F}_φ built up from the operations in Definition 4.2 such that*

$$\text{IKP}^{-\text{Inf}} \vdash \mathcal{F}_\varphi(a_1, \dots, a_n) = \{\langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \varphi(x_1, \dots, x_n)\}.$$

Proof. As in Barwise, we will call a formula $\varphi(x_1, \dots, x_n)$ a *termed-formula*, or *t-formula*, if there is a term \mathcal{F}_φ built from the fundamental operations such that the conclusion of the lemma holds. We shall then proceed by induction on Σ_0 -formulae to show that every such formula is a t-formula. Using the proof of [4], Lemma II.6.1, it only remains to consider the following cases:

- (i.) If $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are t-formulae then so is $\varphi(x_1, \dots, x_n) \wedge \psi(x_1, \dots, x_n)$.
- (ii.) If $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are t-formulae then so is $\varphi(x_1, \dots, x_n) \rightarrow \psi(x_1, \dots, x_n)$.
- (iii.) If $\psi(x_1, \dots, x_{n+1})$ is a t-formula and $\varphi(x_1, \dots, x_n, b)$ is $\forall x_{n+1} \in b \psi(x_1, \dots, x_{n+1})$, where b is an arbitrary set that does not appear in $\{x_1, \dots, x_n\}$, then φ is a t-formula.
- (iv.) If $\psi(x_1, \dots, x_{n+1})$ is a t-formula and $\varphi(x_1, \dots, x_n)$ is $\forall x_{n+1} \in x_j \psi(x_1, \dots, x_{n+1})$, then φ is a t-formula.

Case (i.): Let \mathcal{F}_φ and \mathcal{F}_ψ witness that φ and ψ are t-formulae. First note that for a set z , $\mathcal{F}_p(z, z) = \{z\}$. Then,

$$\begin{aligned} & \mathcal{F}_\cap \left(\mathcal{F}_\varphi(a_1, \dots, a_n), \mathcal{F}_p \left(\mathcal{F}_\psi(a_1, \dots, a_n), \mathcal{F}_\psi(a_1, \dots, a_n) \right) \right) \\ &= \mathcal{F}_\varphi(a_1, \dots, a_n) \cap \bigcap \{ \mathcal{F}_\psi(a_1, \dots, a_n) \} \\ &= \mathcal{F}_\varphi(a_1, \dots, a_n) \cap \mathcal{F}_\psi(a_1, \dots, a_n). \end{aligned}$$

Therefore, we can define $\mathcal{F}_{\varphi \wedge \psi}(a_1, \dots, a_n)$ as

$$\mathcal{F}_{\cap} \left(\mathcal{F}_{\varphi}(a_1, \dots, a_n), \mathcal{F}_{\psi}(a_1, \dots, a_n) \right).$$

Case (ii.): Let \mathcal{F}_{φ} and \mathcal{F}_{ψ} witness that φ and ψ are t-formulae. Then

$$\begin{aligned} & \{ \langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \varphi(x_1, \dots, x_n) \rightarrow \psi(x_1, \dots, x_n) \} \\ &= (a_n \times \dots \times a_1) \cap \{ z \mid z \in \mathcal{F}_{\varphi}(a_1, \dots, a_n) \rightarrow z \in \mathcal{F}_{\psi}(a_1, \dots, a_n) \}. \end{aligned}$$

Also,

$$\langle x, y \rangle = \{ \{x\}, \{x, y\} \} = \mathcal{F}_p(\mathcal{F}_p(x, x), \mathcal{F}_p(x, y))$$

and $a_n \times \dots \times a_1$ can be defined by repeated use of \mathcal{F}_x so we can use these constructions. Thus, the above can be expressed as

$$\mathcal{F}_{\rightarrow} \left(a_n \times \dots \times a_1, \langle \mathcal{F}_{\varphi}(a_1, \dots, a_n), \mathcal{F}_{\psi}(a_1, \dots, a_n) \rangle \right),$$

giving the required construction of $\mathcal{F}_{\varphi \rightarrow \psi}$.

Case (iii.): Let $\varphi(x_1, \dots, x_n, b) \equiv \forall x_{n+1} \in b \psi(x_1, \dots, x_{n+1})$ and let \mathcal{F}_{ψ} witness that ψ is a t-formula. Then

$$\begin{aligned} \mathcal{F}_{\forall}(\mathcal{F}_{\psi}(a_1, \dots, a_n, b), b) &= \{ \mathcal{F}_{\psi}(a_1, \dots, a_n, b) \langle \{z\} \mid z \in b \rangle \} \\ &= \left\{ \{ w \mid \langle z, w \rangle \in \mathcal{F}_{\psi}(a_1, \dots, a_n, b) \} \mid z \in b \right\} \\ &= \left\{ \{ \langle x_n, \dots, x_1 \rangle \mid \langle z, x_n, \dots, x_1 \rangle \in \mathcal{F}_{\psi}(a_1, \dots, a_n, b) \} \mid z \in b \right\} \\ &= \{ \text{ran}(\mathcal{F}_{\psi}(a_1, \dots, a_n, \{z\})) \mid z \in b \}. \end{aligned}$$

Therefore $\mathcal{F}_{\varphi}(a_1, \dots, a_n, b)$ can be expressed as

$$\begin{aligned} & \{ \langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \forall x_{n+1} \in b \psi(x_1, \dots, x_{n+1}) \} \\ &= (a_n \times \dots \times a_1) \cap \left\{ w \mid \forall x_{n+1} \in b \langle x_{n+1}, w \rangle \in \mathcal{F}_{\psi}(a_1, \dots, a_n, \{x_{n+1}\}) \right\} \\ &= (a_n \times \dots \times a_1) \cap \bigcap \{ \text{ran}(\mathcal{F}_{\psi}(a_1, \dots, a_n, \{x_{n+1}\})) \mid x_{n+1} \in b \} \\ &= \mathcal{F}_{\cap} \left(a_n \times \dots \times a_1, \mathcal{F}_{\forall}(\mathcal{F}_{\psi}(a_1, \dots, a_n, b), b) \right). \end{aligned}$$

Case (iv.): Let $\varphi(x_1, \dots, x_n) \equiv \forall x_{n+1} \in x_j \psi(x_1, \dots, x_{n+1})$. Then

$$\{ \langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \forall x_{n+1} \in x_j \psi(x_1, \dots, x_{n+1}) \}$$

is equal to the following set;

$$\{\langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \forall x_{n+1} \in \bigcup a_j (x_{n+1} \in x_j \rightarrow \psi(x_1, \dots, x_{n+1}))\}.$$

So, taking $\vartheta(x_1, \dots, x_n, \bigcup a_j) \equiv \forall x_{n+1} \in \bigcup a_j (x_{n+1} \in x_j \rightarrow \psi(x_1, \dots, x_{n+1}))$, φ is a t-formula by cases (ii.) and (iii.) and the fact that if two formulae are provably equivalent in IKP and one is a t-formula then so is the other.³ \square

Theorem 4.5. *For any Σ_0 -formula $\varphi(x_1, \dots, x_n)$ with free variables among x_1, \dots, x_n , there is a term \mathcal{G}_φ of n arguments built from the operations defined in Definition 4.2 such that:*

$$\text{IKP}^{-Inf} \vdash \mathcal{G}_\varphi(a, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \{x_i \in a \mid \varphi(x_1, \dots, x_n)\}.$$

Proof. This follows easily from our lemma since if \mathcal{F}_φ is the term built in the previous lemma such that

$$\text{IKP}^{-Inf} \vdash \mathcal{F}_\varphi(a_1, \dots, a_n) = \{\langle x_n, \dots, x_1 \rangle \in a_n \times \dots \times a_1 \mid \varphi(x_1, \dots, x_n)\}$$

then our required set can be built from $\mathcal{F}_\varphi(\{x_1\}, \dots, \{x_{i-1}\}, a_i, \{x_{i+1}\}, \dots, \{x_n\})$ by using \mathcal{F}_r $n - i$ times and then \mathcal{F}_d . \square

5. Defining Definability

Having defined the fundamental operations, in this section we shall give a *definability operator*. The idea will be that the definable subsets of b are those sets which can be constructed from b using the fundamental operations. We shall then discuss some of the basic properties one can deduce from this definition and show that the model which one constructs satisfies IKP^{-Inf} . To conclude, we will end this section by mentioning other definability operators.

Definition 5.1. For a set b :

- $\mathfrak{E}(b) := b \cup \{\mathcal{F}_i(x, y) \mid x, y \in b \wedge i \in \mathcal{I}\},$
- $\mathfrak{D}(b) := \mathfrak{E}(b \cup \{b\}).$

The following proposition is then provable using Σ -Collection and the Axiom of Unions in IKP^{-Inf} .

Proposition 5.2. $\text{IKP}^{-Inf} \vdash \forall b \exists x (x = \mathfrak{E}(b)).$

³This is statement (b) in the proof of Lemma II.6.1 of [4]

Observe that, with this construction, $\mathfrak{D}(b)$ need not be transitive even if b is a transitive set. However, following Theorem II.6.4 of [4], one could easily extend the list of fundamental functions to do this. Namely, one could add the following functions, which we state with three variables for simplicity:

- $\mathcal{G}_0(x, y, z) := \langle x, y \rangle,$
- $\mathcal{G}_1(x, y, z) := \{u \mid \langle y, u \rangle \in x\},$
- $\mathcal{G}_2(x, y, z) := \langle x, y, z \rangle,$
- $\mathcal{G}_3(x, y, z) := \langle x, z, y \rangle,$
- $\mathcal{G}_4(x, y, z) := \{x, \langle y, z \rangle\},$
- $\mathcal{G}_5(x, y, z) := \{x, \langle z, y \rangle\}.$

Proposition 5.3. *For any set b , if b is transitive then so is*

$$b \cup \{\mathcal{F}_i(x, y) \mid x, y \in b \wedge i \in \mathcal{I}\} \cup \{\mathcal{G}_i(x, y, z) \mid x, y, z \in b \wedge i \in 6\}.$$

However, apart from making the definable closure into a transitive set, there is no obvious benefit to using this expanded list. Moreover, each of the \mathcal{G}_i can be easily built from at most five applications of the original operations, for example,

$$\{u \mid \langle y, u \rangle \in x\} = \mathcal{F}_r \left(\mathcal{F}_\cap \left(x, \mathcal{F}_p \left(\mathcal{F}_\times \left(\mathcal{F}_p(y, y), \mathcal{F}_r(x) \right), \mathcal{F}_\times \left(\mathcal{F}_p(y, y), \mathcal{F}_r(x) \right) \right) \right) \right).$$

Therefore, we choose to just use the original list as these are the crucial operations.

We would now want to define another operation $\text{Def}(b)$ to be the closure of b under our fundamental operations, that is

$$\text{Def}(b) := \bigcup_{n \in \omega} \mathfrak{D}^n(b).$$

This would have the added benefit that if we defined $L_\alpha := \bigcup_{\beta \in \alpha} \text{Def}(L_\beta)$, then for each ordinal α , L_α would be a transitive model of Bounded Separation. However, this definition requires the Axiom of Infinity, which we are not initially assuming. Therefore, to begin with, we will just use \mathfrak{D} to define our universe and use the different script \mathbb{L} to differentiate between the two notions. The relationship between \mathbb{L} and L in the presence of Strong Infinity will be discussed in Lemma 5.13.

Definition 5.4. For α an ordinal, $\mathbb{L}_\alpha := \bigcup_{\beta \in \alpha} \mathfrak{D}(\mathbb{L}_\beta)$ and $\mathbb{L} := \bigcup_{\alpha \in \text{ORD}} \mathbb{L}_\alpha$.

Remark 5.5. It is worth mentioning that the hierarchy as defined here does not look like the “*standard*” one obtained through the syntactic approach. Firstly, $\mathfrak{D}(b)$ does not close under the fundamental operations, which is an approach we have chosen to take due to a lack of Strong Infinity in the background universe. Secondly, the \mathbb{L} hierarchy does not stratify nicely by rank because $\mathfrak{D}(b)$ may potentially add sets that are

not subsets of b , for example $x \times y$. Moreover, the ordinals of \mathbb{L}_α may not be α , for example in the classical theories of KP or ZF it is possible to see that, for all $n > 0$, $\mathbb{L}_{2n} \cap \text{ORD} = n$. These seem to be useful deficiencies in our weak context because they simplify some of the proofs and this is the approach taken by Barwise. However, we will address alternatives to this approach in Lemma 5.13 and Theorem 5.16.

We start the analysis by noting some of the basic properties of \mathbb{L}_α , which are all easy to see. The final property follows from the fact that each of the fundamental operations is Σ -definable and therefore \mathfrak{D} is also a Σ -definable operation. The required statement then follows by Σ -recursion.

Lemma 5.6. (IKP^{-Inf}) *For all ordinals α, β :*

1. *If $\beta \in \alpha$ then $\mathbb{L}_\beta \subseteq \mathbb{L}_\alpha$,*
2. *If $\beta \subseteq \alpha$ then $\mathbb{L}_\beta \subseteq \mathbb{L}_\alpha$,*
3. $\mathbb{L}_\alpha \in \mathbb{L}_{\alpha+1}$,
4. *If $x, y \in \mathbb{L}_\alpha$ then for any $i \in I$, $\mathcal{F}_i(x, y) \in \mathbb{L}_{\alpha+1}$,*
5. *If for all $\beta \in \alpha$, $\beta + 1 \in \alpha$ then \mathbb{L}_α is transitive,*
6. \mathbb{L} is transitive,
7. *The operation $\alpha \mapsto \mathbb{L}_\alpha$ is Σ -definable.*

Theorem 5.7. *For every axiom φ of IKP^{-Inf}, IKP^{-Inf} \vdash $\varphi^\mathbb{L}$. Moreover,*

$$\text{IKP}^{-\text{Inf}} + \text{“Strong Infinity”} \vdash (\text{Strong Infinity})^\mathbb{L}.$$

Proof. The axioms of Extensionality and \in -induction follow from the fact that \mathbb{L} is a transitive class. Pairing follows from \mathcal{F}_p and Unions from \mathcal{F}_\cup . The Axiom of Empty Set holds because

$$\emptyset = \{y \mid y \neq y\} = \mathbb{L}_0 \in \mathbb{L}_1.$$

Bounded Separation follows from Theorem 4.5. Therefore it remains to prove Bounded Collection.

Suppose that $\varphi(x, y, z)$ is a Σ_0 -formula and, working in IKP^{-Inf}, assume that $a, z \in \mathbb{L}$ and $\forall x \in a \exists y \in \mathbb{L} \varphi(x, y, z)$. Since φ is Σ_0 , it is absolute between \mathbb{L} and the background universe. Now, by definition of \mathbb{L} , we get that

$$\forall x \in a \exists \beta \exists y \in \mathbb{L}_\beta \varphi(x, y, z).$$

Then, using the Σ -Collection principle in the background universe and the fact that the \mathbb{L}_α hierarchy is constructed by a Σ -definable operation, there exists α such that

$$\forall x \in a \exists \beta \in \alpha \exists y \in \mathbb{L}_\beta \varphi(x, y, z)$$

which, by property 1 of Lemma 5.6, yields that $\forall x \in a \exists y \in \mathbb{L}_\alpha \varphi(x, y, z)$. So, $b = \mathbb{L}_\alpha$ is our required witness for Bounded Collection.

Now we work in the theory $\text{IKP}^{-\text{Inf}} + \text{“Strong Infinity”}$. We first prove that $\omega \subseteq \mathbb{L}_\omega$. This is done by induction by showing that for all $n > 0$, $n \in \mathbb{L}_{2n}$. Note that this holds because $1 \in \mathbb{L}_2$ and if $n \in \mathbb{L}_{2n}$ then

$$n + 1 = n \cup \{n\} = \mathcal{F}_\cup(n, \mathcal{F}_p(n, n)) \in \mathbb{L}_{2n+2}.$$

Then $\omega = \{n \in \mathbb{L}_\omega \mid n = \emptyset \vee \exists m \in n (n = m \cup \{m\})\}$ will be in \mathbb{L} by Bounded Separation. \square

It is worth noting here that we will frequently claim that sets of the form

$$\{z \in \mathbb{L}_\delta \mid \varphi(u, z)\}$$

are in $\mathbb{L}_{\delta+k}$ for some $k \in \omega$, where $u \in \mathbb{L}_\delta$ and φ is a Σ_0 -formula, without computing the required k . This k could be computed by breaking down how φ was built up using the fundamental operations, however this is often an unnecessarily tedious computation. Also, we will also often use formulae of the form $\exists i \in \mathcal{I} \varphi(i)$ and these should be seen as an abbreviation for $\bigvee_{i \in \mathcal{I}} \varphi(i)$.

An important property of the constructible universe is the viability of the Axiom of Constructibility; the axiom asserting that $V = \mathbb{L}$. We shall next prove that this axiom does indeed hold in \mathbb{L} , that is:

Theorem 5.8. $\text{IKP}^{-\text{Inf}} \vdash (V = \mathbb{L})^{\mathbb{L}}$.

Our method of proving this will closely follow the corresponding proof over IZF by Lubarsky in [16]. In particular, to prove the theorem it suffices to prove the following lemma which mirrors the version in [16] using the syntactic version of the definability operator.

Lemma 5.9. $(\text{IKP}^{-\text{Inf}})$ For every ordinal α in V there is an ordinal α^* in \mathbb{L} such that $\mathbb{L}_\alpha = \mathbb{L}_{\alpha^*}$.

In order to do this we define the operation of *hereditary addition* on ordinals. This is necessary because in general it will not be true that $\beta \in \alpha$ implies that $\beta + 1 \in \alpha + 1$:

Definition 5.10 (Lubarsky). For ordinals α and γ , hereditary addition is defined inductively on α as

$$\alpha +_H \gamma := \left(\bigcup \{ \beta +_H \gamma \mid \beta \in \alpha \} \cup \{ \alpha \} \right) + \gamma,$$

where “+” is the usual ordinal addition. We will also use the notation

$$(\alpha +_H \gamma)^- := \left(\bigcup \{ \beta +_H \gamma \mid \beta \in \alpha \} \cup \{ \alpha \} \right).$$

Proof. First note that, by using the fundamental operations and Theorem 4.5, there is a fixed natural number k such that for any ordinal τ and set $x \in \mathbb{L}_\tau$,

$$\{\gamma \in \mathbb{L}_\tau \mid \gamma \in \text{ORD} \wedge \mathfrak{D}(\mathbb{L}_\gamma) \subseteq x\} \in \mathbb{L}_{\tau+k}.$$

We prove the lemma by induction on α . So, suppose that $\forall \beta \in \alpha \exists \beta^* \in \mathbb{L}(\mathbb{L}_\beta = \mathbb{L}_{\beta^*})$. Since the \mathbb{L}_α hierarchy is constructed by a Σ -definable operation, by Σ -Collection, we can fix some ordinal δ in V such that $\forall \beta \in \alpha \exists \beta^* \in \mathbb{L}_\delta(\mathbb{L}_\beta = \mathbb{L}_{\beta^*})$. Then, we can define α^* to be the ordinal

$$\alpha^* := \bigcup \{\gamma \in \mathbb{L}_{((\alpha \cup \delta) + Hk)^-} \mid \gamma \in \text{ORD} \wedge \mathfrak{D}(\mathbb{L}_\gamma) \subseteq \mathbb{L}_\alpha\} \in \mathfrak{D}(\mathbb{L}_{(\alpha \cup \delta) + Hk}).$$

Note that we needed to take the union of the above set to ensure that α^* is an ordinal because we are not assuming \mathbb{L}_γ is a transitive set for arbitrary ordinals γ . Now clearly, for any ordinal $\alpha \in V$, $\alpha^* \in \mathbb{L}$. Moreover, if $x \in \mathbb{L}_{\alpha^*}$ then there is some ordinal $\gamma \in \mathbb{L}_{((\alpha \cup \delta) + Hk)^-}$ for which $\mathfrak{D}(\mathbb{L}_\gamma) \subseteq \mathbb{L}_\alpha$ and $x \in \mathfrak{D}(\mathbb{L}_\gamma)$. Thus, $\mathbb{L}_{\alpha^*} \subseteq \mathbb{L}_\alpha$. For the reverse implication, we observe that by the definition of δ , for any $\beta \in \alpha$ there is some $\beta^* \in \mathbb{L}_\delta$ such that $\mathbb{L}_\beta = \mathbb{L}_{\beta^*}$. Thus $\beta^* \in \mathbb{L}_{((\alpha \cup \delta) + Hk)^-}$ and $\mathfrak{D}(\mathbb{L}_{\beta^*}) \subseteq \mathbb{L}_\alpha$, so $\beta^* \in \alpha^*$. Hence,

$$\mathbb{L}_\alpha = \bigcup_{\beta \in \alpha} \mathfrak{D}(\mathbb{L}_\beta) = \bigcup_{\beta \in \alpha} \mathfrak{D}(\mathbb{L}_{\beta^*}) \subseteq \bigcup_{\gamma \in \alpha^*} \mathfrak{D}(\mathbb{L}_\gamma) = \mathbb{L}_{\alpha^*}.$$

□

As mentioned at the beginning of this section, in the presence of Strong Infinity we can define the constructible universe using a different definability operator, Def , where $\text{Def}(b) := \bigcup_{n \in \omega} \mathfrak{D}^n(b)$. This gives us an alternative way to construct the constructible universe, which we now show is equivalent as long as ω exists.

Definition 5.11. For α an ordinal, $L_\alpha := \bigcup_{\beta \in \alpha} \text{Def}(\mathbb{L}_\beta)$ and

$$L := \bigcup_{\alpha \in \text{ORD}} L_\alpha.$$

As before, we can easily observe a few basic properties of this hierarchy:

Proposition 5.12. (IKP^{-Inf} + “Strong Infinity”) For all ordinals α, β :

1. If $\beta \in \alpha$ then $L_\beta \subseteq L_\alpha$,
2. $L_\alpha \in L_{\alpha+1}$,
3. L_α is transitive,
4. L_α is a model of Bounded Separation.

It is possible to compare the hierarchies \mathbb{L}_α and L_α via the following correspondence:

Lemma 5.13. (IKP^{-Inf} + “Strong Infinity”) *For any ordinal α , $L_\alpha = \mathbb{L}_{\omega \cdot \alpha}$.*

Proof. We proceed by induction on α . So assume that our claim holds for all $\beta \in \alpha$. First note that for any ordinal α , $\omega \cdot \alpha = \{\omega \cdot \gamma + n \mid \gamma \in \alpha \wedge n \in \omega\}$ is an ordinal which is closed under successors. Then

$$\begin{aligned} L_\alpha &= \bigcup_{\beta \in \alpha} \text{Def}(L_\beta) = \bigcup_{\beta \in \alpha} \bigcup_{n \in \omega} \mathfrak{D}^n(L_\beta) \\ &= \bigcup_{\beta \in \alpha} \bigcup_{n \in \omega} \mathfrak{D}^n(\mathbb{L}_{\omega \cdot \beta}) = \bigcup_{\beta \in \alpha} \bigcup_{n \in \omega} \mathbb{L}_{\omega \cdot \beta + n} \\ &= \mathbb{L}_{\omega \cdot \alpha}. \end{aligned}$$

□

Corollary 5.14. (IKP^{-Inf} + “Strong Infinity”) $L = \mathbb{L}$.

For completeness we briefly discuss how this method relates to the first, syntactic, approach we mentioned at the beginning of the chapter. We shall be sloppy in our presentation of the syntactic definability operator by using the “ \models ” symbol in our definition of definability instead of the more formal way this is presented in the previously mentioned references. We then defer to [8] for the formal way to do this in IKP^{-Inf} + “Strong Infinity”. We remark here that the syntactic operator is the standard operator we shall use when taking the collection of definable subsets of a given set.

Definition 5.15. Say that a set x is definable over a model $\langle M, \in \rangle$ if there exists a formula φ and $a_1, \dots, a_n \in M$ such that

$$x = \{y \in M \mid \langle M, \in \rangle \models \varphi[y, a_1, \dots, a_n]\}.$$

We can then define the collection of definable subsets of M as

$$\text{def}(M) := \{x \subseteq M \mid x \text{ is definable over } \langle M, \in \rangle\}.$$

The constructible hierarchy can then be defined iteratively as

$$\mathbf{L}_\alpha := \bigcup_{\beta \in \alpha} \text{def}(\mathbf{L}_\beta).$$

Clearly, given $x, y \in M$ and $i \in \mathcal{I}$, $\mathcal{F}_i(x, y)$ is definable from M (even if it may take a few steps when $\mathcal{F}_i(x, y)$ is not a subset of M , for example $x \times y \in \text{def}^3(M)$). Moreover, one can define the notion of being definable over $\langle M, \in \rangle$ using only the fundamental operations, so the two universes they produce will be the same. To see the relationship between the two hierarchies, one can perform a careful analysis of the standard proof, for example Lemma VI.1.17 of [9], which yields;

Theorem 5.16. *For every transitive set M:*

$$\begin{aligned} \text{def}(M) &= \text{Def}(M) \cap \mathcal{P}(M) \\ &= \bigcup_{n \in \omega} \mathfrak{D}^n(M) \cap \mathcal{P}(M). \end{aligned}$$

Therefore, in the theory $\text{IKP}^{-\text{Inf}}$ + “*Strong Infinity*” the two standard formulations of the constructible hierarchy are equivalent. One of the main benefits for using the formulation in terms of the fundamental operations is to avoid the use of Strong Infinity in the construction. The second benefit for using our formulation is because of the versatility of these operations over ZF. A notable example is that it allows us to define when an inner model satisfies ZF. This occurs when the inner model is closed under fundamental operations and it satisfies a property known as *almost universality*. We shall see in the next section that an analogous result holds in IZF.

6. External Cumulative Hierarchies

In this section we shall show that if V satisfies IZF then so does L . This could be done by a very similar repetition of the analysis in the IKP case however we will take a different approach here in order to derive further axiomatic properties under IZF.

The main theorem of this section is adapted from Theorem 13.9 of [14]. The essence of that theorem is that if V is a model of ZF and M contains all of the ordinals then M being a model of ZF can be expressed by a single first-order sentence. On the face of it, the theorem we present here will be slightly weaker than this because it will require the additional assumption that M has an *external cumulative hierarchy*.

Definition 6.1. Let $M \subseteq N$. We say that M has an *external cumulative hierarchy* (e.c.h.) in N if there exists a sequence of sets $\langle M_\alpha \mid \alpha \in \text{ORD} \cap N \rangle$, definable in N , such that:

- For every $\alpha \in \text{ORD} \cap N$, $M_\alpha \in M$,
- $M = \bigcup \{M_\alpha \mid \alpha \in \text{ORD} \cap N\}$,
- If $\beta \in \alpha$ then $M_\beta \subseteq M_\alpha$.

We say that M has an e.c.h. when $N = V$. It is worth remarking that if M is an inner model of IZF, that is a model of IZF containing all of the ordinals, then M will have an external cumulative hierarchy given by the standard rank hierarchy which can be defined as follows:

Definition 6.2. Define the rank of a , $\text{rank}(a)$, recursively as follows:

$$\text{rank}(a) := \bigcup \{\text{rank}(x) + 1 \mid x \in a\},$$

where $z + 1 := z \cup \{z\}$.

Note that one can easily prove that for any set a , $\text{rank}(a)$ is an ordinal and for any ordinal α , $\text{rank}(\alpha) = \alpha$.

Definition 6.3. For α an ordinal, $V_\alpha := \bigcup_{\beta \in \alpha} \mathcal{P}(V_\beta)$.

Proposition 6.4. For any set a , $a \subseteq V_{\text{rank}(a)}$.

Proof. This is formally proved by induction on rank noting that for any $x \in a$, if $x \subseteq V_{\text{rank}(x)}$ then $x \in \mathcal{P}(V_{\text{rank}(x)}) \subseteq V_{\text{rank}(a)}$. \square

Therefore, if M is a definable class which is an inner model of IZF, then $\langle V_\alpha^M \mid \alpha \in \text{ORD} \rangle$ defines an e.c.h. which is moreover uniformly definable. Also, by construction, we have that $\langle L_\alpha \mid \alpha \in \text{ORD} \rangle$ is an e.c.h. for L even though it is unlikely to be the case that L contains all of the ordinals of V .

Definition 6.5. Let $M \subseteq N$. We say that M is *almost universal* in N if for any $x \in N$, if $x \subseteq M$ then there exists some $y \in M$ such that $x \subseteq y$.

Furthermore, it is worth pointing out that classically, for almost universal models, having an e.c.h. implies that the two models have the same ordinals. This will be stated in ZF for simplicity, but in reality only requires very basic set theory and some amount of separation and replacement with regards to the hierarchy.

Proposition 6.6. Suppose that $M \subseteq N$ are transitive models of ZF and M is almost universal in N . If M has an e.c.h. in N then $\text{ORD} \cap M = \text{ORD} \cap N$.

Proof. Let $\langle M_\alpha \mid \alpha \in \text{ORD} \cap N \rangle$ be an external cumulative hierarchy. We shall prove inductively that for any ordinal $\gamma \in N$ there is an ordinal $\beta \in N$ such that $\gamma \subseteq M_\beta$. Then, since $M_\beta \in M$, either $\gamma = M_\beta \cap \text{ORD}$ or $\gamma \in M_\beta \cap \text{ORD}$. Almost universality will then allow us to take some transitive set $y \in M$ covering the set $M_\beta \cap \text{ORD}$ and then Bounded Separation in M yields that $\gamma \in \text{ORD} \cap M$.

Working in N , to prove the claim first note that, by induction,

$$\forall \alpha \in \gamma \exists \tau_\alpha \in N \alpha \in M_{\tau_\alpha}.$$

So, by Collection and the assumption that the hierarchy is cumulative, there is some ordinal β such that for all $\alpha \in \gamma$, $\alpha \in M_\beta$ and therefore $\gamma \subseteq M_\beta$. \square

Remark 6.7. For clarity, the point where we used excluded middle in the above proof was when we asserted that $\gamma \subseteq M_\beta$ implies that $\gamma \in M$. If $\gamma = M_\beta \cap \text{ORD}$ then γ will be in $\text{ORD} \cap M$ as

$$\gamma = \{\alpha \in M_\beta \mid \alpha \text{ is a transitive set of transitive sets}\}.$$

However, if γ is a proper subset of $M_\beta \cap \text{ORD}$ then we require linearity of the ordinals to conclude that γ is an element of this set. To see an example of this, consider an arbitrary truth value $x \subseteq 1$. It is quite plausible for such an ordinal to be a proper subset of some $M_\beta \cap \text{ORD}$ since this set could contain 1 but there is no reason to assume that x itself is a member of this set.

We now present our adaptation of the theorem from Jech. The assumption of an e.c.h. is not necessary for the right-to-left implication of this theorem but it is needed in order to prove that our model M is almost universal in V . This is because, while we can use the rank hierarchy of M to show that if a set $a \in V$ is a subset of M there is some $\beta \in \text{ORD}$ such that $a \subseteq \bigcup_{\alpha \in \beta} V_\alpha^M$, it does not seem possible to show that this union is in fact a set in M because there is no reason why β should be in M .

However, having an e.c.h. seems to be a reasonable additional assumption since in most cases our model M will be built up iteratively over the ordinals in V , which gives a very natural hierarchy. Notably, we have that $L := \bigcup_{\alpha \in \text{ORD} \cap V} L_\alpha$.

Theorem 6.8. *Suppose that V is a model of IZF and $M \subseteq V$ is a definable, transitive proper class with an external cumulative hierarchy. Then M is a model of IZF if and only if M is closed under the fundamental operations and is almost universal.*

Proof. For the left-to-right implication, first see that, if M is a model of IZF, then M is certainly closed under the fundamental operations because they are Σ -definable. For almost universality, let $\langle M_\alpha \mid \alpha \in \text{ORD} \rangle$ be an e.c.h. and suppose that $a \in V$ with $a \subseteq M$. Then we have that

$$\forall x \in a \exists \alpha \in \text{ORD} (x \in M_\alpha).$$

So, by Collection in V , there is some set $b \in V$ such that

$$\forall x \in a \exists \alpha \in b (x \in M_\alpha).$$

Taking $\beta = \text{trcl}(b) \cap \text{ORD}$ we can then conclude that $a \subseteq M_\beta$, and M_β is a set in M by the assumption that the M_α 's form an external cumulative hierarchy.

For the reverse implication, since M is transitive it is a model of Extensionality and \in -induction. Also, Pairing and Unions follow from M being closed under \mathcal{F}_p and \mathcal{F}_\cup while the Axiom of Infinity will follow from

the proof of Theorem 5.7 along with an instance of Bounded Separation. We now proceed to prove the other axioms.

Bounded Separation: This follows from the same argument as given in Theorem 4.5 because any Σ_0 -formula can be expressed using the fundamental operations.

Power Set: Let a be a set and note that $\mathcal{P}(a) \cap M$ is a set in V , and a subset of M . So, by almost universality, we can fix some $b \in M$ such that $\mathcal{P}(a) \cap M \subseteq b$. But then

$$\mathcal{P}^M(a) = \{x \in b \mid x \subseteq a\},$$

which is a set in M by Bounded Separation in M .

Collection: Let a be in M and suppose that, in M , $\forall x \in a \exists y \in M \varphi(x, y, u)$. By Collection in V we can fix some set b' such that

$$V \models \forall x \in a \exists y \in b' \varphi^M(x, y, u)$$

and, by almost universality, we can fix $b \in M$ such that $b \supseteq b'$ yielding, in M ,

$$M \models \forall x \in a \exists y \in b \varphi(x, y, u).$$

Separation: This is shown by induction on the complexity of the formula φ . Σ_0 -formulae and all cases except for those involving quantifiers follow immediately from the consequences of the fundamental operations.

So suppose that $\varphi(x, u) \equiv \exists v \psi(x, u, v)$. Using Separation in V , define a' to be

$$a' := \{x \in a \mid \varphi^M(x, u)\}.$$

Then,

$$V \models \forall x \in a' \exists v \in M \psi^M(x, u, v).$$

So, by Collection in V , there exists some set $b' \subseteq M$ such that

$$V \models \forall x \in a' \exists v \in b' \psi^M(x, u, v).$$

By almost universality, take $b \in M$ such that $b' \subseteq b$. Then

$$V \models \forall x \in a' \exists v \in b \psi^M(x, u, v).$$

Now, by the inductive hypothesis, we have that

$$y := \{\langle x, v \rangle \in a \times b \mid \psi^M(x, u, v)\} \in M,$$

and thus, using \mathcal{F}_d ,

$$z = \text{dom}(y) = \{x \in a \mid \exists v \in b \psi^M(x, u, v)\} \in M.$$

For the final case, suppose that $\varphi(x, u) \equiv \forall v \psi(x, u, v)$. For $r \in M$ let

$$y_r := \{x \in a \mid \forall v \in r \psi^M(x, u, v)\} \in V.$$

Using the inductive hypothesis and the proof of case (iii.) of Lemma 4.4 we shall show that $y_r \in M$ for every $r \in M$. Firstly, by the inductive hypothesis, $c := \{\langle z, y, x \rangle \in r \times \{u\} \times a \mid \psi^M(x, y, z)\}$ is in M . Next, $\mathcal{F}_v(c, r) = \{\{\langle u, x \rangle \mid \langle z, u, x \rangle \in c\} \mid z \in r\}$. Thus, $\mathcal{F}_\cap(\{u\} \times a, \mathcal{F}_v(c, r)) = \{\langle u, x \rangle \mid x \in a \wedge \forall z \in r \psi^M(x, u, z)\}$ from which it follows that $y_r = \mathcal{F}_r(\mathcal{F}_\cap(\{u\} \times a, \mathcal{F}_v(c, r)))$ is in M since M is closed under the fundamental operations.

Furthermore, it is obvious that if $s \subseteq r$ then $y_r \subseteq y_s$. We aim to show that y_M , which is defined in the same way by taking a universal quantifier over M , is in M by showing that it is equal to y_r for some $r \in M$. To do this, we begin by defining

$$Y := \{z \in \mathcal{P}(a) \mid \exists r \in M (z = y_r)\}.$$

So

$$V \models \forall z \in Y \exists r \in M (z = y_r).$$

Therefore, by Collection in V , there is some set $d' \subseteq M$ such that

$$V \models \forall z \in Y \exists r \in d' (z = y_r).$$

Taking the transitive closure if necessary, by almost universality fix a transitive set $d \in M$ such that $d' \subseteq d$. We claim that $y_d = y_M$. Firstly, since $d \subseteq M$, $y_M \subseteq y_d$. For the reverse direction, let $x \in y_d$ and let $v \in M$. Then $y_{\{v\}} \in Y$ so we can fix $r \in d$ such that $y_{\{v\}} = y_r$. Therefore, since $r \subseteq d$, $y_d \subseteq y_r = y_{\{v\}}$ so $x \in y_{\{v\}}$ and, by construction, $\psi^M(x, v, u)$. Finally, since v was arbitrary, $x \in y_M$ as required. \square

Corollary 6.9. *For every axiom φ of IZF, $\text{IZF} \vdash \varphi^L$.*

7. Exponentiation in the Constructible Universe

In order to show that $\text{CZF}_{\mathcal{P}}$ cannot prove that the Axiom of Exponentiation holds in L we shall make use of a specific realizability model. This model is based upon general set recursive functions where a realizer for an existential statement will provide a set of witnesses for the existential quantifier, rather than just a single witness. When combined with truth (for implications) this will then produce an effective procedure for constructing what are known as E_{φ} -recursive functions which witness sentences provable in $\text{CZF}_{\mathcal{P}}$. This construction was originally derived in [25] and we refer to that paper for all necessary details.

In order to give the proof it is necessary to first briefly outline the notions of E -recursive and E_{φ} -recursive functions and the realizability construction that we shall use. This shall be done over the next two subsections.

7.1. E -recursive functions

E -recursion is a way to extend Kleene's notion of computability, $\{e\}(x)$, to apply to arbitrary sets. This generalised notion of recursion, also known as set recursion, is also discussed in detail in sources such as [23] or chapter 10 of [27]. The idea is that, for arbitrary sets a and x , a should be thought of as some form of Turing Machine which takes as input x . In order to avoid any possible confusion with the singleton set, we shall instead use $[a](c)$ for the result of this computation. E_{φ} -recursion will be an extension of this generalised notion to also treat power sets as an initial function. This latter notion has also been previously studied by authors such as Moschovakis [21] and Moss [22].

Here we shall use Kleene's notion of *strong equality*, \simeq , which is that for two partial functions f and g , $f(x) \simeq g(x)$ if and only if neither $f(x)$ nor $g(x)$ are defined, or they are both defined and equal.

Definition 7.1 ([25] 2.9). (IKP) First, we select distinct non-zero natural numbers $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{s}_N, \mathbf{p}_N, \mathbf{d}_N, \bar{\mathbf{0}}, \bar{\omega}, \pi, \nu, \gamma, \rho, \mathbf{i}_1, \mathbf{i}_2$ and \mathbf{i}_3 which will provide indices for special E -recursive class functions. Let $\mathbb{N} := \omega$.

We shall inductively define a class \mathbb{E} of triples $\langle e, x, y \rangle$. Instead of saying " $\langle e, x, y \rangle \in \mathbb{E}$ " we shall write " $[e](x) \simeq y$ ". Next, we shall use $[e](x_1, \dots, x_n) \simeq y$ to indicate that

$$[e](x_1) \simeq \langle e, x_1 \rangle \wedge [e, x_1](x_2) \simeq \langle e, x_1, x_2 \rangle \wedge \dots \wedge [e, x_1, \dots, x_{n-1}](x_n) \simeq y.$$

Finally, we shall say that $[e](x)$ is defined, written $[e](x) \downarrow$, iff $[e](x) \simeq y$ for some y . The relation $[e](x) \simeq y$ is then specified by the following clauses:

Indices for applicative structures (APP):

$$\begin{array}{ll}
[\mathbf{k}](x, y) \simeq x & [\mathbf{s}](x, y, z) \simeq [[x](z)]([y](z)) \\
[\mathbf{p}](x, y) \simeq \langle x, y \rangle & [\mathbf{s}_{\mathbb{N}}](n) \simeq n + 1 \text{ if } n \in \mathbb{N} \\
[\mathbf{p}_{\mathbb{N}}](0) \simeq 0 & [\mathbf{p}_{\mathbb{N}}](n + 1) \simeq n \text{ if } n \in \mathbb{N} \\
[\mathbf{p}_0](x) \simeq 1^{st}(x) & [\mathbf{p}_1](x) \simeq 2^{nd}(x) \\
[\mathbf{d}_{\mathbb{N}}](n, m, x, y) \simeq x \text{ if } n, m \in \mathbb{N} \text{ and } n = m & [\mathbf{d}_{\mathbb{N}}](n, m, x, y) \simeq y \text{ if } n, m \in \mathbb{N} \text{ and } n \neq m \\
[\bar{\mathbf{0}}](x) \simeq 0 & [\bar{\omega}](x) \simeq \omega
\end{array}$$

Indices for set-theoretic axioms:

$$\begin{array}{ll}
[\pi](x, y) \simeq \{x, y\} & [\nu](x) \simeq \bigcup x \\
[\gamma](x, y) \simeq x \cap \bigcap y & [\rho](x, y) \simeq \{[x](u) \mid u \in y\} \\
& \text{if } [x](u) \text{ is defined for all } u \in y
\end{array}$$

Indices for equality axioms:

$$\begin{array}{ll}
[\mathbf{i}_1](x, y, z) \simeq \{u \in x \mid y \in z\} & [\mathbf{i}_2](x, y, z) \simeq \{u \in x \mid u \in y \rightarrow u \in z\} \\
[\mathbf{i}_3](x, y, z) \simeq \{u \in x \mid u \in y \rightarrow z \in u\} &
\end{array}$$

Definition 7.2 ([25] 2.11). *Application terms* are defined inductively as:

1. The distinguished indices indicated in Definition 7.1 are all application terms,
2. variables are application terms,
3. if s and t are application terms then so is $[s](t)$.

An application term is then called *closed* if it does not contain any variables.

Definition 7.3 ([25] 2.13). A partial n -place class function Υ is called an *E-recursive partial function* if there exists a closed application term t_{Υ} such that

$$\text{dom}(\Upsilon) = \{(a_1, \dots, a_n) \mid [t_{\Upsilon}](a_1, \dots, a_n) \downarrow\}$$

and for all sets $(a_1, \dots, a_n) \in \text{dom}(\Upsilon)$,

$$[t_{\Upsilon}](a_1, \dots, a_n) \simeq \Upsilon(a_1, \dots, a_n).$$

When this holds, t_{Υ} is said to be an *index* for Υ and we will freely confuse the two.

When working with $\text{IKP}(\mathcal{P})$ we can extend the above notion of recursive functions to E_φ -recursive functions, which will make the function $\mathcal{P}(x) = \{u \mid u \subseteq x\}$ “computable”. This is done by specifying an additional non-zero natural number $\bar{\wp}$ and adding the clause

$$[\bar{\wp}](x) \simeq \mathcal{P}(x)$$

to Definition 7.1, giving rise to the class \mathbb{E}_φ . The notions of \wp -application terms and closed \wp -application terms are then defined analogously.

Finally, we can state the main technical result for these functions, which is Lemmas 2.20 and 2.21 of [25].

Lemma 7.4 ([25] 2.20, 2.21).

1. For each Σ_0 -formula $\varphi(a, y)$ there is a closed application term t_φ such that

$$\text{IKP} \vdash [t_\varphi](a, b) \downarrow \wedge \forall u (u \in [t_\varphi](a, b) \longleftrightarrow (u \in b \wedge \varphi(a, u))).$$

2. For each $\Sigma_0^{\mathcal{P}}$ -formula $\varphi(a, y)$ there is a closed \wp -application term t_φ such that

$$\text{IKP}(\mathcal{P}) \vdash [t_\varphi](a, b) \downarrow \wedge \forall u (u \in [t_\varphi](a, b) \longleftrightarrow (u \in b \wedge \varphi(a, u))).$$

7.2. Realizability Preliminaries

We can now give the definition of the notion of realizability that we shall use, $\Vdash_{\mathfrak{wt}}$, which will be a relation between sets and formulae. Following the notation of the second author in [25] we shall use the subscript \mathfrak{w} to indicate *weakening* the realizer for an existential statement to only be a set of witnesses for the existential quantifier and the subscript \mathfrak{t} to denote the addition of *truth* to the realizability interpretation. In this relation, we will treat bounded quantifiers as quantifiers in their own right. In the definition we shall use the following notation:

- $\mathbf{p}_0 e$ and $\mathbf{p}_1 e$ for $1^{st}(e)$ and $2^{nd}(e)$ respectively (instead of $[\mathbf{p}_0](e)$ and $[\mathbf{p}_1](e)$),
- $[a](b) \Vdash_{\mathfrak{wt}} \varphi$ for $\exists x ([a](b) \simeq x \wedge x \Vdash_{\mathfrak{wt}} \varphi)$, where $[a](b) \simeq x$ denotes $\langle a, b, x \rangle \in \mathbb{E}$.

Definition 7.5 ([25] 3.1).

$$\begin{array}{ll} a \Vdash_{\mathfrak{wt}} \varphi & \text{iff } \varphi \text{ is true, whenever } \varphi \text{ is an atomic formula,} \\ a \Vdash_{\mathfrak{wt}} \varphi \wedge \psi & \text{iff } \mathbf{p}_0 a \Vdash_{\mathfrak{wt}} \varphi \wedge \mathbf{p}_1 a \Vdash_{\mathfrak{wt}} \psi, \\ a \Vdash_{\mathfrak{wt}} \varphi \vee \psi & \text{iff } \exists u (u \in a) \wedge \forall d \in a ((\mathbf{p}_0 d = 0 \wedge \mathbf{p}_1 d \Vdash_{\mathfrak{wt}} \varphi) \vee (\mathbf{p}_0 d = 1 \wedge \mathbf{p}_1 d \Vdash_{\mathfrak{wt}} \psi)), \\ a \Vdash_{\mathfrak{wt}} \varphi \rightarrow \psi & \text{iff } (\varphi \rightarrow \psi) \wedge \forall c (c \Vdash_{\mathfrak{wt}} \varphi \rightarrow [a](c) \Vdash_{\mathfrak{wt}} \psi), \end{array}$$

$$\begin{aligned}
a \Vdash_{\mathbf{wt}} \forall x \in b \varphi(x) & \quad \text{iff} \quad \forall c \in b ([a](c) \Vdash_{\mathbf{wt}} \varphi[x/c]), \\
a \Vdash_{\mathbf{wt}} \exists x \in b \varphi(x) & \quad \text{iff} \quad \exists u(u \in a) \wedge \forall d \in a (\mathbf{p}_0 d \in b \wedge \mathbf{p}_1 d \Vdash_{\mathbf{wt}} \varphi[x/\mathbf{p}_0 d]), \\
a \Vdash_{\mathbf{wt}} \forall x \varphi(x) & \quad \text{iff} \quad \forall c [a](c) \Vdash_{\mathbf{wt}} \varphi[x/c], \\
a \Vdash_{\mathbf{wt}} \exists x \varphi(x) & \quad \text{iff} \quad \exists u(u \in a) \wedge \forall d \in a (\mathbf{p}_1 d \Vdash_{\mathbf{wt}} \varphi[x/\mathbf{p}_0 d]), \\
\Vdash_{\mathbf{wt}} \varphi & \quad \text{iff} \quad \exists a a \Vdash_{\mathbf{wt}} \varphi.
\end{aligned}$$

Terminology 7.6. In [25] the abbreviation “ $a \neq \emptyset$ ” is used in Definition 3.1 to express the positive statement $\exists u(u \in a)$. We have chosen to write this statement explicitly in order to avoid any confusion.

Note that the above realizability notion was based on E -computability, and this will be used to provide a translation of CZF^- . In order to provide a translation of $\text{CZF}_{\mathcal{P}}$ we will instead use the corresponding realizability notions based on E_{φ} -computability.

Definition 7.7. $a \Vdash_{\mathbf{wt}}^{\varphi} \varphi$ is defined recursively by the same recursion as in Definition 7.5 but using E_{φ} -computability where $[a](b) \simeq c$ now stands for $\langle a, b, c \rangle \in \mathbb{E}_{\varphi}$.

By induction on the construction of formulae, one can show that this notion of realizability implies truth over both of our constructive theories.

Proposition 7.8 ([25] 3.3). *Let $\varphi(x_1, \dots, x_n)$ be a formula with free variables among x_1, \dots, x_n . Then*

1. $\text{CZF}^- \vdash (\exists e e \Vdash_{\mathbf{wt}} \varphi(x_1, \dots, x_n)) \rightarrow \varphi(x_1, \dots, x_n)$.
2. $\text{CZF}_{\mathcal{P}} \vdash (\exists e e \Vdash_{\mathbf{wt}}^{\varphi} \varphi(x_1, \dots, x_n)) \rightarrow \varphi(x_1, \dots, x_n)$.

Next, by induction on the complexity of formulae, one can show that within our constructive theories our realizability structure realizes every statement provable from the given theory. This is done by first ensuring that the interpretation preserves logical inferences and then that we are able to realize every axiom of CZF^- ($\text{CZF}_{\mathcal{P}}$) via an E -recursive (E_{φ} -recursive) function.

Theorem 7.9 ([25] 3.7, 3.9). *Let $\varphi(x_1, \dots, x_n)$ be a formula with free variables among x_1, \dots, x_n .*

1. *If $\text{CZF}^- \vdash \varphi(x_1, \dots, x_n)$ then one can effectively construct an index of an E -recursive function f such that $\text{CZF}^- \vdash \forall a_1 \dots \forall a_n f(a_1, \dots, a_n) \Vdash_{\mathbf{wt}} \varphi(a_1, \dots, a_n)$.*
2. *If $\text{CZF}_{\mathcal{P}} \vdash \varphi(x_1, \dots, x_n)$ then one can effectively construct an index of an E_{φ} -recursive function f such that $\text{CZF}_{\mathcal{P}} \vdash \forall a_1 \dots \forall a_n f(a_1, \dots, a_n) \Vdash_{\mathbf{wt}}^{\varphi} \varphi(a_1, \dots, a_n)$.*

Moreover, if one removes the “*truth*” requirement, that is by defining the interpretation $\Vdash_{\mathbf{w}}$ using the same conditions as in Definition 7.5 except that the statement for implication is replaced by

$$a \Vdash_{\mathbf{w}} \varphi \rightarrow \psi \quad \text{iff} \quad \forall c (c \Vdash_{\mathbf{w}} \varphi \rightarrow [a](c) \Vdash_{\mathbf{w}} \psi),$$

then one can perform a similar analysis to give a realizability interpretation of CZF^- and $\text{CZF}_{\mathcal{P}}$ over the background theories of IKP and $\text{IKP}(\mathcal{P})$. Moreover, this interpretation will imply truth for bounded formulae. These translations can then be used to show that CZF^- proves the same Π_2 assertions as IKP . Note that a formula is said to be Π_2 if it is of the form $\forall x \exists y \varphi(x, y, z)$ where $\varphi(x, y, z)$ is Σ_0 . The class $\Pi_2^{\mathcal{P}}$ is defined analogously by allowing $\varphi(x, y, z)$ to be of $\Sigma_0^{\mathcal{P}}$ form.

Theorem 7.10 ([25] 4.8).

1. CZF^- is conservative over IKP for Π_2 -sentences.
2. $\text{CZF}_{\mathcal{P}}$ is conservative over $\text{IKP}(\mathcal{P})$ for $\Pi_2^{\mathcal{P}}$ -sentences.

Via an ordinal analysis using infinitary cut-free derivations, a strong form of reflection for $\Sigma^{\mathcal{P}}$ -formulae is obtained in Theorem 8.2 of [26]. Namely, from any proof of a $\Sigma^{\mathcal{P}}$ -sentence φ one can effectively extract an ordinal (representation) σ below the Bachmann-Howard ordinal, $\psi_{\Omega}(\varepsilon_{\Omega+1})$, such that $\text{KP}(\mathcal{P})$ proves that the von Neumann hierarchy exists up to level σ and φ holds in V_{σ} . It was further shown in [7] that this technique can be transferred to the intuitionistic counterpart of $\text{KP}(\mathcal{P})$, $\text{IKP}(\mathcal{P})$.

Theorem 7.11 ([7] 3.27). *Let φ be a $\Sigma^{\mathcal{P}}$ -sentence. If $\text{IKP}(\mathcal{P}) \vdash \varphi$ then there exists some ordinal term $\sigma < \psi_{\Omega}(\varepsilon_{\Omega+1})$, which can be computed from the derivation, such that $V_{\sigma} \models \varphi$.*

7.3. Exponentiation in $\text{CZF}(\mathcal{P})$

Using this machinery we are now able to show that $\text{CZF}_{\mathcal{P}}$ is unable to prove that the Axiom of Exponentiation holds in L . In particular, this means that CZF is unable to show that L satisfies every axiom of CZF .

Theorem 7.12. $\text{CZF}_{\mathcal{P}} \not\vdash \text{Exp}^L$. Thus, $\text{CZF}_{\mathcal{P}} \not\vdash (\text{CZF})^L$.

Proof. It will suffice to show that $\text{CZF}_{\mathcal{P}}$ does not prove the following: *The statement “the collection of all functions from ω to ω forms a set” relativizes to L .* Aiming for a contradiction, suppose that $\text{CZF}_{\mathcal{P}}$ was able to prove this statement.

Let $\varphi_{\text{nat}}(u)$ be a Σ_0 formula which uniquely defines ω , that is to say $\text{IKP} \vdash \exists! u \varphi_{\text{nat}}(u)$. For example:

$$\varphi_{\text{nat}}(u) \equiv \exists x \in u \forall y \in x (y \neq y) \wedge \forall x \in u (x \cup \{x\} \in u) \wedge \forall x \in u (x = \emptyset \vee \exists y \in u (x = y \cup \{y\})).$$

Now, by our assumption,

$$\text{CZF}_{\mathcal{P}} \vdash \exists u, z \in L \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right).$$

Therefore,

$$\text{CZF}_{\mathcal{P}} \vdash \exists a \in \text{ORD} \exists u, z \in L_a \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right).$$

By Theorem 7.9 (2), it follows that we are able to effectively construct an index for an E_φ -recursive function g such that $\text{CZF}_{\mathcal{P}} \vdash \forall x g(x)$ is defined and

$$\text{CZF}_{\mathcal{P}} \vdash \forall x g(x) \Vdash_{\text{wt}}^{\varphi} \exists a \in \text{ORD} \exists u, z \in L_a \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right).$$

Note that x is a dummy variable and the fact that our statement in $\text{CZF}_{\mathcal{P}}$ can be realized by the E_φ -recursive function g will allow us to find a bound for the witnessing ordinal a . So, setting $x = 0$ we have,

$$\text{CZF}_{\mathcal{P}} \vdash g(0) \Vdash_{\text{wt}}^{\varphi} \exists a \in \text{ORD} \exists u, z \in L_a \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right).$$

Unpacking this by using our definition of realizability with truth, we have that $\text{CZF}_{\mathcal{P}} \vdash \exists w (w \in g(0))$ and

$$\text{CZF}_{\mathcal{P}} \vdash \forall y \in g(0) \mathbf{p}_1 y \Vdash_{\text{wt}}^{\varphi} \mathbf{p}_0 y \in \text{ORD} \wedge \exists u, z \in L_{\mathbf{p}_0 y} \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right).$$

Since realizability implies truth, we have that

$$\text{CZF}_{\mathcal{P}} \vdash \forall y \in g(0) \left(\mathbf{p}_0 y \in \text{ORD} \wedge \exists u, z \in L_{\mathbf{p}_0 y} \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right) \right). \quad (*)$$

Now, the important thing to observe is that g was an E_φ recursive function, which means that the statement “ $g(0)$ is defined” is definable by a $\Sigma^{\mathcal{P}}$ -sentence. Therefore, by Theorem 7.10,

$$\text{IKP}(\mathcal{P}) \vdash g(0) \text{ is defined.}$$

Next, by Theorem 7.11, we can effectively construct an ordinal $\sigma < \psi_\Omega(\varepsilon_{\Omega+1})$ such that the von Neumann hierarchy exists up to level σ and $g(0) \in V_\sigma$ holds provably in $\text{IKP}(\mathcal{P})$ and hence in $\text{CZF}_{\mathcal{P}}$. Thus, for any $y \in g(0)$, $\mathbf{p}_0 y \in V_\sigma$. Plugging this into (*) and using that $g(0)$ is inhabited, we obtain,

$$\text{CZF}_{\mathcal{P}} \vdash \exists a \in V_\sigma \left(a \in \text{ORD} \wedge \exists u, z \in L_a \left(\varphi_{\text{nat}}(u) \wedge \forall f \in {}^u u (f \in L \rightarrow f \in z) \right) \right).$$

Thus, there is some $a \in L_\sigma$ for which ${}^\omega \omega \cap L$ is an element of L_a . But from this it follows that ${}^\omega \omega \cap L_a = {}^\omega \omega \cap L_\sigma$, so

$$\text{CZF}_{\mathcal{P}} \vdash \exists a \in L_\sigma \forall f \in {}^\omega \omega (f \in L_\sigma \leftrightarrow f \in L_a).$$

However, this is now a Σ -sentence, since the L_α hierarchy is defined by Σ -recursion, which means that

the same sentence holds in $\text{IKP}(\mathcal{P})$ by Theorem 7.10. Therefore, the result will also hold in the theory $\text{KP}(\mathcal{P})$. We say that an ordinal α is a *gap ordinal* if $(L_{\alpha+1} \setminus L_\alpha) \cap \mathcal{P}(\omega) = \emptyset$ and clearly the ordinal a above must be a gap ordinal. However, as proven in [15], the first gap ordinal is equal to the smallest ordinal such that $L_\beta \models \text{ZFC}^-$ (where ZFC^- stands for ZFC without powerset) an ordinal much larger than the Bachmann-Howard ordinal. Thus we have obtained our desired contradiction. \square

We will not explicitly state the background theory needed for the proof of the above result but remark here that it will follow from $\text{KP}(\mathcal{P})$ augmented by the statement that the von Neumann hierarchy V_α exists for every $\alpha \leq \psi_\Omega(\varepsilon_{\Omega+1})$. With more analysis this could be significantly weakened.

8. Constructive Ordinals

As mentioned at the beginning of this paper, being an inner model over a classical theory has three defining properties:

- It is a subclass of the universe;
- It models the same base theory as the universe;
- It contains every ordinal in the universe.

While we have proven that L is a definable subclass of the universe and if the universe satisfies one of $\text{IKP}^{-\text{Inf}}$, IKP or IZF then so does L , we have also shown that the same cannot be said for CZF and $\text{CZF}_{\mathcal{P}}$. Moreover, there is one significant, fundamental and structural question that remains open,

Question 8.1 (Lubarsky, [16]). Does L contain every ordinal? That is, is $L \cap \text{ORD} = V \cap \text{ORD}$?

This appears to be a difficult question to resolve. By Lemma 5.9 we know that L always contains a proper class of ordinals and, as a transitive structure, if L contains one ordinal, then it contains every smaller ordinal. However, if Excluded Middle were to fail, then the ordinals can no longer be linearly ordered so while we may have no gaps in the ordinals, our “tree” of ordinals could be missing branches from the ground model. It is worth remarking that L does at least contain every *truth value* in V .

Proposition 8.2. *For every $\alpha \in \Omega = \{x \mid x \subseteq 1\}$, $\alpha = L_\alpha \in L$.*

Proof. Suppose $\alpha \in \Omega$, then $L_\alpha := \bigcup_{\beta \in \alpha} \text{def}(L_\beta) = \{x \mid \exists \beta \in \alpha (x \in \text{def}(L_\beta))\}$. Now, since $\alpha \subseteq 1$, $L_\alpha = \{x \mid \exists \beta \in 1 (\beta \in \alpha \wedge x \in \text{def}(L_\beta))\}$. But if $\beta \in 1$ then $\text{def}(L_\beta) = \{0\}$ and therefore we must have that $L_\alpha = \{x \mid 0 \in \alpha \wedge x = 0\} = \{x \mid x \in \alpha\} = \alpha$. \square

A difficulty arises in trying to prove that L contains infinite ordinals which are somehow produced by functions which are not definable in L . As a final motivating example for future work, suppose that we were

in an intuitionistic setting with $V \neq L$ and $c: \omega \rightarrow 2$ was some function that was not in L . Let $\alpha \in \Omega$ be an arbitrary truth value (which we cannot prove to be either 0 or 1) and let

$$\delta_c := \bigcup_{n \in \omega} ((n+2) \cup \{\alpha\}) + c(n).$$

As a union of ordinals, δ_c is again an ordinal however it does not seem possible to show that $\delta_c \in L$ because to construct it one somehow needs to know about the non-constructible function c . Moreover, when one looks at the ordinal $(\delta_c)^*$ produced in Lemma 5.9 such that $L_{\delta_c} = L_{(\delta_c)^*}$, the construction includes $((n+2) \cup \{\alpha\}) + 1$ for every $n \in \omega$ which erases the function c from the definition. On the other hand, it is difficult to construct an explicit example for which we can prove that we cannot deduce that $\delta_c \in L$. The main strategies we have to build intuitionistic models are via realizability, Kripke model or intuitionistic forcing. In each of these cases one adds many new ordinals and we do not know how these strange new ordinals affect the construction of the Constructible Universe.

Question 8.3. Under the above construction, is it always the case that δ_c is provably in L ?

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