Article:

Nazarov, Maxim (2024) An elliptic solution of the classical Yang-Baxter equation associated with the queer Lie superalgebra. Annales Henri Poincare. AHPO-D-24-00034. 5339–5347. ISSN 1424-0661

https://doi.org/10.1007/s00023-024-01449-8

Ann. Henri Poincaré *Online First*

https://doi.org/10.1007/s00023-024-01449-8

Annales Henri Poincaré

Original Paper

An Elliptic Solution of the Classical Yang–Baxter Equation Associated with the Queer Lie Superalgebra

Maxim Nazarov

Abstract. A solution of the classical Yang–Baxter equation associated with the queer Lie superalgebra is constructed in terms of Hermite theta functions.

1. Introduction

Let $\mathfrak{g}$ be any finite-dimensional Lie superalgebra over a complex field $\mathbb{C}$. Let $r(u,v)$ be a meromorphic function of two complex variables u and v which takes values in $\mathfrak{g} \otimes \mathfrak{g}$. The *classical Yang–Baxter equation* for the function $r(u,v)$ is

$$[\, r_{12}(u,v), r_{13}(u,w)\,] + [\, r_{12}(u,v), r_{23}(v,w)\,] + [\, r_{13}(u,w), r_{23}(v,w)\,] = 0$$

where w is another complex variable and the function of u, v, w at the left-hand side takes values in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. Here we use the standard notation, for example $r_{23}(v,w) = 1 \otimes r(v,w)$. For conventions on the tensor products see Sect. 2.

A solution of the above equation is called *nondegenerate* if not every value of the function $r(u,v)$ is degenerate as a quadratic tensor. For simple Lie algebras $\mathfrak{g}$ the nondegenerate solutions were classified in [4,5]. In particular, it was shown in [5] that for any nondegenerate solution $r(u,v)$ with a simple Lie algebra $\mathfrak{g}$ one can find a domain $D \subset \mathbb{C}$ and two holomorphic maps

$$\psi : D \to \mathbb{C} \quad \text{and} \quad \omega : D \to \operatorname{Aut} \mathfrak{g}$$

where ψ is not constant and the function $(\omega(u) \otimes \omega(v))\, r(\psi(u), \psi(v))$ depends only on the difference $u - v$.

This basic result of [5] does not hold for Lie superalgebras. Solutions $r(u,v)$ which depend not only on the difference $u - v$ were constructed in [11].

Published online: 22 May 2024 Birkhäuser

There $\mathfrak{g}$ is the general linear Lie superalgebra $\mathfrak{gl}_{n|n}$ with any positive integer n. These solutions are *antisymmetric*, that is

$$r_{21}(v,u) = -\,r(u,v). \tag{1}$$

A general phenomenon underlying this construction was independently described in [1]. See [12] for further discussion of this remarkable phenomenon.

The solutions $r(u,v)$ constructed in [11] are rational functions of variables u and v. Here we construct a solution $r(u,v)$ which is expressed in theta functions of $u-v$ and $u+v$. Our $\mathfrak{g}$ is the quotient of the special linear Lie superalgebra $\mathfrak{sl}_{n|n}$ by its one-dimensional centre. This quotient is denoted by $\mathfrak{psl}_{n|n}$.

Elliptic solutions $r(u,v)$ for the Lie superalgebra $\mathfrak{g}=\mathfrak{psl}_{n|n}$ were constructed in [9]. They have the form $r(u,v)=s(u-v)$ where $s(u)$ is a function such that

$$s_{21}(u) = -\,s(-u). \tag{2}$$

Hence these solutions are antisymmetric.

Now let η be the involutive automorphism of $\mathfrak{gl}_{n|n}$ defined in Sect. 2. The fixed point subalgebra of $\mathfrak{gl}_{n|n}$ relative to η is the queer Lie superalgebra $\mathfrak{q}_n$. The automorphism η preserves the subalgebra $\mathfrak{sl}_{n|n}$ and descends to $\mathfrak{psl}_{n|n}$. In our Sect. 5 we show that the function $s(u)$ in [9] can be so chosen that

$$(\eta\otimes\eta)\,s(u) = s(-u) \tag{3}$$

and that

$$r(u,v) = s(u-v) + (\mathrm{id}\otimes\eta)\,s(u+v) \tag{4}$$

is another solution of classical Yang–Baxter equation for $\mathfrak{g}=\mathfrak{psl}_{n|n}$.

It immediately follows from (3) and (4) that

$$r(u,v) = s(u-v) + (\eta\otimes\mathrm{id})\,s(-\,u-v). \tag{5}$$

By comparing the definition (4) with (5) and by using (2) we see that our solution $r(u,v)$ is antisymmetric as well.

2. General Conventions

We shall use the following general conventions. Let A and B be any associative $\mathbb{Z}_2$-graded algebras. Their tensor product A⊗B is also an associative $\mathbb{Z}_2$-graded algebra such that for any homogeneous elements $X,X'\in\mathrm{A}$ and $Y,Y'\in\mathrm{B}$

$$(X\otimes Y)(X'\otimes Y') = XX'\otimes YY'\,(-1)^{\,\deg X'\deg Y},$$
$$\deg\,(X\otimes Y) = \deg X + \deg Y\,.$$

Furthermore, for any two $\mathbb{Z}_2$-graded modules U and V over A and B, respectively, the vector space $U\otimes V$ is a $\mathbb{Z}_2$-graded module over $\mathrm{A}\otimes\mathrm{B}$ such that for any homogeneous elements $x\in U$ and $y\in V$

$$(X\otimes Y)(x\otimes y) = Xx\otimes Yy\,(-1)^{\,\deg x\,\deg Y}, \tag{6}$$
$$\deg(x\otimes y) = \deg x + \deg y\,. \tag{7}$$

Now let the indices i, j, k, l run through $\pm 1, \ldots, \pm n$. Put $\bar{\imath} = 0$ if $i > 0$ and $\bar{\imath} = 1$ if $i < 0$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{n|n}$. Let $e_i \in \mathbb{C}^{n|n}$ be an element of the standard basis. The $\mathbb{Z}_2$-grading on $\mathbb{C}^{n|n}$ is defined by $\deg e_i = \bar{\imath}$.

Let $E_{ij} \in \operatorname{End} \mathbb{C}^{n|n}$ be the standard matrix unit, defined by $E_{ij}\, e_k = \delta_{jk}\, e_i$. The associative algebra $\operatorname{End} \mathbb{C}^{n|n}$ is $\mathbb{Z}_2$-graded by setting $\deg E_{ij} = \bar{\imath} + \bar{\jmath}$. Hence $\mathbb{C}^{n|n}$ is a $\mathbb{Z}_2$-graded module over $\operatorname{End} \mathbb{C}^{n|n}$. For any positive integer m we can also identify the tensor product $(\operatorname{End} \mathbb{C}^{n|n})^{\otimes m}$ with the algebra $\operatorname{End}((\mathbb{C}^{n|n})^{\otimes m})$ acting on the vector space $(\mathbb{C}^{n|n})^{\otimes m}$ by repeatedly using conventions (6) and (7).

The *supertrace* str is a linear function $\operatorname{End} \mathbb{C}^{n|n} \to \mathbb{C}$ defined by setting

$$\operatorname{str}(E_{ij}) = \delta_{ij}\, (-1)^{\bar{\imath}}.$$

This definition implies that for any homogeneous elements $X, Y \in \operatorname{End} \mathbb{C}^{n|n}$

$$\operatorname{str}(YX) = \operatorname{str}(XY)\, (-1)^{\deg X \deg Y}.$$

Further, we can define an involutive automorphism η of $\operatorname{End} \mathbb{C}^{n|n}$ by mapping

$$\eta : \, E_{ij} \mapsto E_{-i,-j}. \tag{8}$$

This automorphism is the conjugation by the involutive odd element of $\operatorname{End} \mathbb{C}^{n|n}$

$$E_{1,-1} + E_{-1,1} + \cdots + E_{n,-n} + E_{-n,n}.$$

We have

$$[\, E_{ij}, E_{kl}\,] = \delta_{jk}\, E_{il} - \delta_{li}\, E_{kj}\, (-1)^{(\bar{\imath} + \bar{\jmath})(\bar{k} + \bar{l})} \tag{9}$$

in $\operatorname{End} \mathbb{C}^{n|n}$. Here the square brackets indicate the supercommutator. We will also consider each E_{ij} as an element of the Lie superalgebra $\mathfrak{gl}_{n|n}$. The special linear Lie superalgebra $\mathfrak{sl}_{n|n}$ is the subalgebra of $\mathfrak{gl}_{n|n}$ defined as the kernel of the function str. The centre of $\mathfrak{gl}_{n|n}$ is spanned by the element

$$E_{11} + E_{-1,-1} + \cdots + E_{nn} + E_{-n,-n} = 1.$$

The subalgebra $\mathfrak{sl}_{n|n}$ contains this element. The quotient of the Lie superalgebra $\mathfrak{sl}_{n|n}$ by the one-dimensional subspace spanned by this element is denoted by $\mathfrak{psl}_{n|n}$. According to [8] the Lie superalgebra $\mathfrak{psl}_{n|n}$ is simple if and only if $n > 1$. By using (9) we obtain that the Lie bracket on $\mathfrak{psl}_{1|1}$ is just zero.

By (9) our η is also an involutive automorphism of the Lie superalgebra $\mathfrak{gl}_{n|n}$. This automorphism preserves its subalgebra $\mathfrak{sl}_{n|n}$. The *queer Lie superalgebra* $\mathfrak{q}_n$ is the fixed point subalgebra of $\mathfrak{gl}_{n|n}$ by η.

3. Theta Functions

Fix any complex number τ with a positive imaginary part. For any real numbers a and b consider the *Hermite theta function*

$$\theta_{a,b}(u) = \sum_{m=-\infty}^{\infty} e^{\pi \mathrm{i} \tau (a+m)^2 + 2\pi \mathrm{i} (a+m)(b+u)}. \tag{10}$$

The above series converges to a holomorphic function of the complex variable u. All zeroes of this function are simple and form the subset

$$(a+\tfrac{1}{2}+\mathbb{Z})\,\tau+(b+\tfrac{1}{2}+\mathbb{Z})\subset\mathbb{C},$$

see [7, pp. 196–199]. The numbers a and b here are called *characteristics*. For $a=b=0$ the series (10) is the *Jacobi theta function*. It follows from (10) that

$$\theta_{a,b}(u+1)=e^{\,2\pi\mathrm{i}a}\,\theta_{a,b}(u)\quad\text{and}\quad\theta_{a,b}(u+\tau)=e^{\,-2\pi\mathrm{i}(u+b+\frac{\tau}{2})}\,\theta_{a,b}(u).\tag{11}$$

By changing m to $m+1$ in (10) and by using the first equation in (11) we obtain

$$\theta_{a+1,b}(u)=\theta_{a,b}(u)\quad\text{and}\quad\theta_{a,b+1}(u)=e^{\,2\pi\mathrm{i}a}\,\theta_{a,b}(u)\tag{12}$$

respectively. Further, by changing m to $-m$ in (10) we obtain the parity relation

$$\theta_{a,b}(-u)=\theta_{-a,-b}(u).\tag{13}$$

Now let g and h be any integers not simultaneously divisible by $2n$ and by n, respectively. Consider the function of the complex variable u

$$\varphi_{g,h}(u)=\frac{\theta_{\frac{g}{2n}+\frac{1}{2},\frac{1}{2}-\frac{h}{n}}(u)\,\theta'_{\frac{1}{2},\frac{1}{2}}(0)}{\theta_{\frac{g}{2n}+\frac{1}{2},\frac{1}{2}-\frac{h}{n}}(0)\,\theta_{\frac{1}{2},\frac{1}{2}}(u)}\,.$$

This function is meromorphic. It has simple poles at every point on the lattice

$$\mathcal{L}=\mathbb{Z}+\mathbb{Z}\,\tau\subset\mathbb{C}.$$

Due to the chosen normalisation the residue of the pole of $\varphi_{g,h}(u)$ at $u=0$ is 1.

It immediately follows from the relations (12) that

$$\varphi_{g+2n,h}(u)=\varphi_{g,h}(u)\quad\text{and}\quad\varphi_{g,h+n}(u)=\varphi_{g,h}(u).$$

Thus $\varphi_{g,h}(u)$ depends on the integers g and h only modulo $2n$ and n, respectively. From now on g and h will run not through $\mathbb{Z}$, but through the additive groups $\mathbb{Z}_{2n}=\mathbb{Z}/2n\,\mathbb{Z}$ and $\mathbb{Z}_n=\mathbb{Z}/n\,\mathbb{Z}$, respectively.

Let $\varepsilon=e^{\,\pi\mathrm{i}/n}$ be a primitive root of unity of the $2n$. Direct calculation using (11) yields the periodicity properties

$$\varphi_{g,h}(u+1)=\varepsilon^{\,g}\,\varphi_{g,h}(u)\quad\text{and}\quad\varphi_{g,h}(u+\tau)=\varepsilon^{\,2h}\,\varphi_{g,h}(u).\tag{14}$$

Another direct calculation using (12) and (13) yields the parity relation

$$\varphi_{g,h}(-u)=-\,\varphi_{-g,-h}(u).\tag{15}$$

4. Commuting Automorphisms

Consider the following two elements of the algebra $\operatorname{End}\mathbb{C}^{n|n}$,

$$\begin{aligned}A=&\,E_{11}+\varepsilon^2E_{22}+\cdots+\varepsilon^{\,2(n-1)}E_{nn}\\&+\varepsilon\,E_{-1,-1}+\varepsilon^{-1}E_{-2,-2}+\cdots+\varepsilon^{\,3-2n}E_{-n,-n}\end{aligned}$$

and

$$B = E_{12} + \cdots + E_{n-1,n} + E_{n1} \\ + E_{-2,-1} + \cdots + E_{-n,1-n} + \cdots + E_{-1,-n}.$$

These elements are invertible and of $\mathbb{Z}_2$-degree zero. They satisfy the relations

$$A^{2n} = B^n = 1 \quad \text{and} \quad BA = \varepsilon^2 AB. \tag{16}$$

By (8) we get

$$\eta(A) = \varepsilon\, A^{-1} \quad \text{and} \quad \eta(B) = B^{-1}. \tag{17}$$

Let us define two automorphisms α and β of the algebra $\mathrm{End}\,\mathbb{C}^{n|n}$ by setting

$$\alpha(X) = A^{-1} X A \quad \text{and} \quad \beta(X) = B^{-1} X B$$

for $X \in \mathrm{End}\,\mathbb{C}^{n|n}$. These automorphisms commute and are of degrees $2n$ and n, respectively. Here $\alpha(X) = \beta(X) = X$ if and only if X is a linear combination of

$$E_{11} + \cdots + E_{nn} \quad \text{and} \quad E_{-1,-1} + \cdots + E_{-n,-n}.$$

Since the elements A and B are of $\mathbb{Z}_2$-degree 0, for each $X \in \mathrm{End}\,\mathbb{C}^{n|n}$ we have

$$\mathrm{str}\,(\alpha(X)) = \mathrm{str}\,(X) \quad \text{and} \quad \mathrm{str}\,(\beta(X)) = \mathrm{str}\,(X). \tag{18}$$

Let us now regard α and β as automorphisms of the Lie superalgebra $\mathfrak{gl}_{\,n|n}$. It follows from the first equation in (16) that $\alpha^{2n} = \beta^{\,n} = 1$. The eigenvalues of α and β are $\varepsilon^{\,g}$ and $\varepsilon^{\,2h}$ where g and h range over $\mathbb{Z}_{2n}$ and $\mathbb{Z}_n$, respectively. Let $\mathfrak{gl}_{\,n|n}^{\,g,h}$ be the joint eigenspace of α and β corresponding to $\varepsilon^{\,g}$ and $\varepsilon^{\,2h}$. By (18)

$$\mathfrak{gl}_{\,n|n}^{\,g,h} \subset \mathfrak{sl}_{\,n|n} \quad \text{for} \quad (g,h) \neq (0,0). \tag{19}$$

Consider the *Casimir element* of the tensor square $\mathfrak{gl}_{\,n|n} \otimes \mathfrak{gl}_{\,n|n}$

$$t = \sum_{i,j} E_{ij} \otimes E_{ji}\,(-1)^{\bar{\jmath}}\,.$$

This element is invariant by $\mathfrak{gl}_{\,n|n}$ as for any indices k,l by using (9) we get

$$[\,t, E_{kl} \otimes 1 + 1 \otimes E_{kl}\,] = 0. \tag{20}$$

Note that

$$(\eta \otimes \eta)\, t = -\,t. \tag{21}$$

It follows from (20) that the Casimir element t is invariant by both $\alpha \otimes \alpha$ and $\beta \otimes \beta$. Therefore t belongs to the direct sum of subspaces

$$\mathfrak{gl}_{\,n|n}^{\,g,h} \otimes \mathfrak{gl}_{\,n|n}^{\,-g,-h} \subset \mathfrak{gl}_{\,n|n} \otimes \mathfrak{gl}_{\,n|n}. \tag{22}$$

Let $t_{g,h}$ be the projection of element t to the direct summand (22). By (17),(21)

$$(\eta \otimes \eta)\, t_{g,h} = -\,t_{-g,-h}. \tag{23}$$

Further, let σ be the linear transformation of $\mathfrak{gl}_{n|n} \otimes \mathfrak{gl}_{n|n}$ defined by setting

$$\sigma\,(X \otimes Y) = Y \otimes X\,(-1)^{\,\deg X \,\deg Y}$$

for homogeneous elements X and Y. By the definition of t we have $\sigma(t) = t$. So

$$\sigma\,(\,t_{g,h}\,) = t_{-g,-h}. \tag{24}$$

We do not need explicit formulas for every projection $t_{g,h}$. We only note that

$$t_{\,0,0} = \frac{1}{2n}\,(J \otimes 1 + J \otimes 1) \tag{25}$$

where

$$J = E_{11} - E_{-1,-1} + \cdots + E_{nn} - E_{-n,-n}.$$

5. Classical Yang–Baxter Equation

The central element 1 of the Lie superalgebra $\mathfrak{gl}_{n|n}$ is contained in the eigenspace $\mathfrak{gl}^{\,0,0}_{n|n}$. Therefore for any $(g,h) \neq (0,0)$ the element $t_{g,h}$ can be identified with its image in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$. By using this identification, introduce a function of u

$$s(u) = \sum_{(g,h)\neq(0,0)} \varphi_{g,h}(u)\, t_{g,h}.$$

Observe that the function $s(u)$ takes all its values in the subspace

$$\mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n} \subset \mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}.$$

Changing the summation indices g,h to $-g,-h$, respectively, and then using (15), (24) proves that $s(u)$ indeed satisfies the condition (2) for $\mathfrak{g} = \mathfrak{psl}_{n|n}$. Using (15), (23) proves that $s(u)$ satisfies (3). Here we regard η as an automorphism of the Lie algebra $\mathfrak{psl}_{n|n}$. The automorphisms α and β of $\mathfrak{gl}_{n|n}$ preserve $\mathfrak{sl}_{n|n}$ and descend to $\mathfrak{psl}_{n|n}$ too. By the periodicity properties (14) of the function $\varphi_{g,h}(u)$

$$s(u+1) = (\alpha \otimes \mathrm{id})\, s(u) = (\mathrm{id} \otimes \alpha^{-1})\, s(u) \tag{26}$$

and

$$s(u+\tau) = (\beta \otimes \mathrm{id})\, s(u) = (\mathrm{id} \otimes \beta^{-1})\, s(u). \tag{27}$$

By (25) the image in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$ of the element $t_{\,0,0}$ is zero. Hence the image of the element t is

$$\sum_{(g,h)\neq(0,0)} t_{g,h}.$$

Denote this image by p. The residue of the function $s(u)$ at $u = 0$ equals p.

Now consider the function $r(u,v)$ as defined by (4). We already observed in Sect. 1 that (2) and (3) imply the antisymmetry property (1). Let us show that $r(u,v)$ is indeed a solution of the classical Yang–Baxter equation for $\mathfrak{g} = \mathfrak{psl}_{n|n}$. We will employ general arguments from [4, Sect. 5].

First observe that due to (4) and to the first equalities in (26) and (27) we get

$$r(u+1,v) = (\alpha \otimes \mathrm{id})\, r(u,v) \quad \text{and} \quad r(u+\tau,v) = (\beta \otimes \mathrm{id})\, r(u,v). \tag{28}$$

Similarly, due to (5) and to the second equalities in (26) and (27) we get

$$r(u,v+1) = (\mathrm{id} \otimes \alpha)\, r(u,v) \quad \text{and} \quad r(u,v+\tau) = (\mathrm{id} \otimes \beta)\, r(u,v).$$

Let $f(u,v,w)$ be the left-hand side of the classical Yang–Baxter equation for our $r(u,v)$ with $\mathfrak{g} = \mathfrak{psl}_{n|n}$. Since α and β are Lie algebra automorphisms, we have

$$f(u+1,v,w) = (\alpha \otimes \mathrm{id} \otimes \mathrm{id})\, f(u,v,w) \tag{29}$$

and

$$f(u+\tau,v,w) = (\beta \otimes \mathrm{id} \otimes \mathrm{id})\, f(u,v,w). \tag{30}$$

Choose any values of v and w such that $v-w, v+w \notin \mathcal{L}$. We will prove that then $f(u,v,w)$ is a holomorphic function of u in the whole $\mathbb{C}$. By (29) and (30) this function is bounded and hence a constant. This constant is then an element of $\mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n}$ invariant by α and β applied in the first tensor factor. However the only element of $\mathfrak{psl}_{n|n}$ invariant by both α and β is zero. Therefore our function must be zero.

If $u \pm v \in \mathcal{L}$, then $u \pm w \notin \mathcal{L}$ since $v \pm w \notin \mathcal{L}$. Then the function $r_{13}(u,w)$ has no pole. Consider the first two of the three summands of $f(u,v,w)$. By the definition (4) their sum can be written as

$$[\, s_{12}(u-v), r_{13}(u,w) + r_{23}(v,w)\,] \; + \tag{31}$$

$$[\,(\mathrm{id} \otimes \eta \otimes \mathrm{id})\, s_{12}(u+v), r_{13}(u,w) + r_{23}(v,w)\,]. \tag{32}$$

By multiplying (31) by $u-v$ and then setting $u=v$ we get

$$[\, p, r_{13}(v,w) + r_{23}(v,w)\,] = 0.$$

Here we employ (20) and the definition of p as the image of t in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$. Therefore (31) has no pole at $u-v=0$. It now follows from (26), (27), (28) that the summand (31) has no pole whenever $u-v \in \mathcal{L}$.

By multiplying the summand (32) by $u+v$ and then setting $u=-v$ we get

$$\begin{aligned}
&[\,(\mathrm{id} \otimes \eta \otimes \mathrm{id})\, p, r_{13}(-v,w) + r_{23}(v,w)\,] \\
&\quad = (\mathrm{id} \otimes \eta \otimes \mathrm{id})\,[\, p, r_{13}(-v,w) + (\mathrm{id} \otimes \eta \otimes \mathrm{id})\, r_{23}(v,w)\,] \\
&\quad = (\mathrm{id} \otimes \eta \otimes \mathrm{id})\,[\, p, r_{13}(-v,w) + r_{23}(-v,w)\,] = 0.
\end{aligned}$$

So (32) has no pole at $u+v=0$. It follows from (26), (27), (28) that (32) has no pole whenever $u+v \in \mathcal{L}$.

Thus the function $f(u,v,w)$ has no pole whenever $u \pm v \in \mathcal{L}$. Further, by using the antisymmetry property (1) the function $-f(u,v,w)$ can be written as

$$[\, r_{13}(u,w), r_{12}(u,v)\,] + [\, r_{13}(u,w), r_{32}(w,v)\,] + [\, r_{12}(u,v), r_{32}(w,v)\,].$$

Similarly to the above argument we can show that the latter function has no pole when $u \pm w \in \mathcal{L}$. So $r(u,v)$ is a solution of the classical Yang–Baxter equation.

Following [2,3,6,10] it would be interesting to find a solution of the quantum Yang–Baxter equation corresponding to our $r(u,v)$. For the rational solutions of the classical equation constructed in [11] this was already done in the same work.

Acknowledgements

I thank Evgeny Sklyanin for helpful discussions on the theory of elliptic functions.

References

[1] Avan, J.: Graded Lie algebras in the Yang–Baxter equation. Phys. Lett. B **245**, 491–496 (1990)

[2] Baxter, R.: One-dimensional anisotropic Heisenberg chain. Ann. Phys. **70**, 323–337 (1972)

[3] Belavin, A.: Dynamical symmetry of integrable quantum systems. Nucl. Phys. B **180**, 189–200 (1981)

[4] Belavin, A., Drinfeld, V.: Solutions of the classical Yang–Baxter equation for simple Lie algebras. Funct. Anal. Appl. **16**, 159–180 (1982)

[5] Belavin, A., Drinfeld, V.: Classical Yang–Baxter equation for simple Lie algebras. Funct. Anal. Appl. **17**, 220–221 (1983)

[6] Cherednik, I.: On the properties of factorized S matrices in elliptic functions. Sov. J. Nucl. Phys. **36**, 320–324 (1983)

[7] Hurwitz, A., Courant, R.: Funktionentheorie. Springer, Berlin (1929)

[8] Kac, V.: Lie superalgebras. Adv. Math. **26**, 8–96 (1977)

[9] Leites, D., Serganova, V.: Solutions of the classical Yang–Baxter equation for simple superalgebras. Theor. Math. Phys. **58**, 16–24 (1984)

[10] Matushko, M., Zotov, A.: Anisotropic spin generalization of elliptic Macdonald-Ruijsenaars operators and R-matrix identities. Ann. Henri Poincaré **24**, 3373–3419 (2023)

[11] Nazarov, M.: Yangians of the strange Lie superalgebras. Lect. Notes Math. **1510**, 90–97 (1992)

[12] Nazarov, M.: Yangian of the queer Lie superalgebra. Commun. Math. Phys. **208**, 195–223 (1999)

Maxim Nazarov
Department of Mathematics
University of York
York YO10 5DD
UK
e-mail: maxim.nazarov@york.ac.uk

Communicated by Nikolai Kitanine.
Received: February 1, 2024.
Accepted: May 4, 2024.