



UNIVERSITY OF LEEDS

This is a repository copy of *Infinite friezes and triangulations of annuli*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/211616/>

Version: Accepted Version

Article:

Baur, K. orcid.org/0000-0002-7665-476X, Çanakçı, İ, Jacobsen, K.M. et al. (2 more authors) (2023) Infinite friezes and triangulations of annuli. *Journal of Algebra and Its Applications*. ISSN 0219-4988

<https://doi.org/10.1142/s0219498824502074>

This item is protected by copyright. This is an author produced version of an article published in *Journal of Algebra and Its Applications*. Uploaded in accordance with the publisher's self-archiving policy.

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

Journal of Algebra and Its Applications
(2024) 2450207 (31 pages)
© World Scientific Publishing Company
DOI: 10.1142/S0219498824502074



Infinite friezes and triangulations of annuli

Karin Baur*

*School of Mathematics
University of Leeds, Leeds, UK
k.u.baur@leeds.ac.uk*

İlke Çanakçı

*Department of Mathematics
VU Amsterdam, 1081 HV, Amsterdam
The Netherlands
i.canakci@vu.nl*

Karin M. Jacobsen

*Department of Mathematics
Aarhus University, Aarhus C, Denmark
karin.jacobsen@au.dk*

Maitreyee C. Kulkarni[†]

*Mathematisches Institut
Universität Bonn, Bonn, Germany
kulkarni@math.uni-bonn.de*

Gordana Todorov

*Department of Mathematics
Northeastern University, Boston, USA
g.todorov@northeastern.edu*

Received 30 May 2022

Accepted 24 August 2022

Published

Communicated by A. Srivastava

It is known that any infinite periodic frieze comes from a triangulation of an annulus by Theorem 4.6 of [K. Baur, M. J. Parsons and M. Tschabold, Infinite friezes, *European J. Combin.* **54** (2016) 220–237]. In this paper, we show that each infinite periodic frieze determines a triangulation of an annulus in essentially a unique way. Since each triangulation of an annulus determines a pair of friezes, we study such pairs and show how they determine each other. We study associated module categories and determine

*On leave from the University of Graz, Graz, Austria.

[†]Corresponding author.

K. Baur et al.

the growth coefficient of the pair of friezes in terms of modules as well as their quiddity sequences.

Keywords: Conway–Coxeter friezes; frieze patterns; infinite friezes; triangulation; annulus; cluster categories; growth coefficients.

Mathematics Subject Classification 2020: 16G20, 05E10

1. Introduction

The notion of *finite friezes* was defined in 1971 by Coxeter [14]. A finite frieze is a grid consisting of a finite number of rows of positive integers satisfying a determinant rule for each diamond in the grid. Each row is of infinite length. The first two and the last two rows are fixed, with entries 0 and 1 as shown in Fig. 1. Because of the determinant rule, the first nontrivial row completely determines the frieze and is called the *quiddity row*. Finite friezes and their properties are well understood. It was proven by Conway and Coxeter [12, 13] that every finite frieze has periodic quiddity row and they showed that:

Theorem ([12; 13, Question 27]). There is a bijection between periodic finite friezes and triangulations of regular polygons.

Caldero and Chapoton showed that finite friezes are connected to the theory of cluster algebras via the Caldero–Chapoton (CC) map and through the association with triangulations of regular polygons [10].

In this paper, we consider another type of friezes called *infinite periodic friezes* (see Fig. 2 and Definition 2.1). Similar to finite friezes, infinite periodic friezes start with a row of 0’s and a row of 1’s; their quiddity row is also periodic. But the number of rows in this case is infinite. In 2017, it was shown in [6] (also see Definition 2.17) the following.

Theorem ([6, Theorem 4.6]). Each infinite periodic frieze corresponds to a triangulation of an annulus.

We will see later that each triangulation of an annulus defines a pair of infinite

$$\begin{array}{cccccccc}
 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
 \cdots & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 \cdots & 1 & 3 & 1 & 3 & 1 & \cdots \\
 \cdots & 2 & 2 & 2 & 2 & 2 & 2 & \cdots \\
 \cdots & 3 & 1 & 3 & 1 & 3 & \cdots \\
 \cdots & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots
 \end{array}$$

Fig. 1. A finite frieze.

periodic friezes; one associated with each boundary component. One of the goals of this paper is to establish a one-to-one correspondence between such pairs of infinite periodic friezes and triangulations of an annulus, see Proposition 3.10. In order to get such a correspondence, we define skeletal triangulations and skeletal friezes. A *skeletal triangulation* is a triangulation with only bridging arcs in it (arcs connecting the two boundary components). A *skeletal frieze* is a frieze whose quiddity sequence has no 1's in it and which is different from $q = (2, \dots, 2)$. With this in mind, we prove the following theorem.

Theorem A (Theorem 3.22). *Given a skeletal frieze, there is exactly one other skeletal frieze such that this pair of friezes corresponds to triangulation of an annulus.*

Every skeletal frieze (or skeletal triangulation of an annulus) represents an infinite family of infinite periodic friezes (or triangulations of an annulus) (see Theorem 3.22). Every skeletal frieze corresponds to a non-oriented cyclic quiver and each such quiver will recover the corresponding pair of skeletal friezes (Theorem 3.20).

An interesting phenomenon of infinite periodic friezes is that for a frieze of period n , the difference between an entry in its n th row and the entry right above it in row $n - 2$ is constant. This difference is called the *growth coefficient* of the frieze. It is shown in [4, Theorem 3.4] that for a pair of friezes coming from a triangulation of an annulus, the growth coefficients of the two friezes are equal. We show that:

Theorem B (Corollary 4.10). *Consider an infinite periodic frieze with quiddity sequence $q = (a_1, a_2, \dots, a_n)$. Then its growth coefficient is*

$$s_q = \left(\sum_I (-1)^{\ell_I} \prod_{k \in I} a_k \right) + \delta_n,$$

where I 's are certain subsets of $\{1, 2, \dots, n\}$. The integers ℓ_I and δ_n are defined as in Sec. 4.2.

Infinite periodic friezes are also related to cluster algebras through the corresponding triangulations. This gives us a relation to cluster categories of type \tilde{A} as defined by [9], through the CC map [10]. In particular, we explain in Secs. 4.3 and 4.4 how infinite periodic friezes are related to the non-homogeneous tubes in the Auslander–Reiten quivers of cluster-tilted algebras of type \tilde{A} . Hence we get a representation theoretic interpretation of infinite periodic frieze patterns and their growth coefficients.

Theorem C (Corollary 4.14). *Let $q = (a_1, \dots, a_n)$ be a quiddity sequence, let \mathcal{B} be the associated rank n tube. Let M be any indecomposable in \mathcal{B} at level n and \tilde{M} be the indecomposable right below it in the Auslander–Reiten quiver, at level $n + 2$. Then we have*

$$s(M) - s(\tilde{M}) = s_q,$$

where $s(M)$ is a generalized version of the number of submodules of M as in Sec. 4.3.

K. Baur et al.

This paper is organized as follows. In Sec. 2, we recall the notion of friezes and triangulations of annuli. Moreover, we introduce skeletal friezes and skeletal triangulations and show how these are related to each other. In Sec. 3, we describe how to associate a pair of infinite periodic friezes to a triangulation, and study the uniqueness of this correspondence when restricted to skeletal. Furthermore, we give a direct map between non-oriented cyclic quivers and skeletal friezes by associating a pair of quiddity sequences to every non-oriented cycle, see Sec. 3.2. This association has indirectly been known before, via triangulations. To our knowledge, the direct map is new. In Sec. 4, we recall the growth coefficient of an infinite periodic frieze and some results from [4]. We then state our result (Corollary 4.10) which computes the growth coefficient using only the quiddity sequence. This leads us to the representation theoretic interpretation of the differences of entries in the infinite periodic friezes in terms of certain modules, see Theorem 4.13, and hence of the growth coefficient, see Corollary 4.14.

2. Triangulations, Friezes and Quivers

We have three players here: triangulations, friezes and quivers. There are clear correspondences between them, some of which are already well known and some which we describe in the paper and prove their properties. In order to have uniqueness in such correspondences (up to certain factors) we consider certain reduced versions for each of these notions, namely when they are “skeletal”, so that we can go easily between the three players.

2.1. Finite and infinite friezes

Friezes were introduced by Coxeter in 1971 [14]. A *frieze* consists of a possibly infinite number of rows of positive integers as shown in Fig. 2. The first and second rows consist of 0’s and 1’s, respectively. The rows of 0’s and 1’s are called *trivial*. Every diamond of adjacent entries

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

in a frieze satisfies the determinant rule $ad - bc = 1$. The *order* of a frieze is the number of nontrivial rows in the frieze. A frieze is called *finite* if it has finite order. The last row of a finite frieze consists of 0’s and the second to last of 1’s. A frieze is *infinite* if it has infinite order. An example of a finite frieze is shown in Fig. 1. That frieze has order 3.

Definition 2.1. A frieze \mathcal{F} is determined by its first nontrivial row $(a_i)_{i \in \mathbb{Z}}$, called the *quiddity row*. A frieze is called *periodic* if the quiddity row is periodic. So for any i , an n -periodic frieze \mathcal{F} is determined completely by an n -tuple $q = (a_i, \dots, a_{i+n-1})$ in its quiddity row. Every such tuple is a *quiddity sequence* of \mathcal{F} .

Infinite friezes and triangulations of annuli

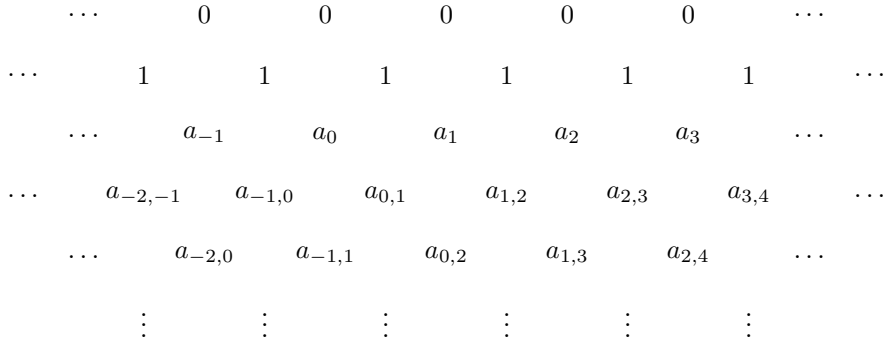


Fig. 2. The layout of an infinite frieze.

Throughout the paper, we will consider infinite periodic friezes, which we will sometimes simply call *infinite friezes*. We always consider quiddity sequences up to cyclic equivalence, i.e. $(a_1, \dots, a_n) \sim (a_2, \dots, a_n, a_1)$.

2.2. Skeletal friezes

Quiddity sequences and hence friezes can be simplified through the process of *reduction*, as introduced in [12, 13] and described below. This will be a key ingredient for the results in Sec. 3. These reduced friezes help us to establish a bijective correspondence between infinite periodic friezes and triangulations of annuli.

Definition 2.2. A *reduction (at a 1)* of a quiddity sequence q of a frieze is obtained by subtracting 1 from both neighboring entries in the quiddity sequence q provided that the neighboring entries are > 1 .

More precisely, a *reduction ρ* of a quiddity sequence $q = (a_1, \dots, a_r)$ at $a_i = 1$ is obtained by:

- (a) reducing both a_{i-1} and a_{i+1} by 1 and deleting $a_i = 1$ in q if $r \geq 3$ and $a_{i-1}, a_{i+1} \geq 2$, or
- (b) if $q = (1, k)$ then deleting 1 and replacing k by $k - 2$, i.e. $\rho(q) = (k - 2)$ if $k \geq 3$.

We note that reduction primarily makes sense when considering quiddity sequences where no two consecutive entries are equal to one. The full set of cases is discussed below.

Example 2.3. For instance the reduction of the quiddity sequence $q = (1, 5)$ is $q' = (3)$, i.e. $\rho((1, 5)) = (3)$.

Remark 2.4. Given a quiddity sequence, we can also perform a reverse reduction by inserting a new entry 1 into the sequence and increasing the neighboring entries by 1. A reverse reduction of the quiddity sequence $q = (2)$ is then $q = (1, 4)$, and a reverse reduction of $q = (2, 3)$ is $q = (3, 1, 4)$.

K. Baur et al.

Definition 2.5. A *peripheral triangle* or *ear* of a triangulation is a triangle whose sides are: two boundary segments and a peripheral arc (Definition 2.13) (so, all three vertices are on the same boundary component).

Remark 2.6. (a) It is well known that finite (infinite) friezes correspond to triangulations of polygons (annuli). We will describe this correspondence for infinite periodic friezes in Sec. 2.5.

(b) In case of both finite and infinite friezes, reduction of a quiddity sequence at a 1 corresponds to deletion of the corresponding peripheral triangle (ear) in the corresponding triangulation (see Questions 23, 26, 27 of [12, 13] for finite friezes).

Remark 2.7. (1) If we repeatedly apply reduction to the quiddity sequence of a *finite* frieze, we will eventually obtain a quiddity sequence where all entries are 1. This can be proved using: (i) all finite friezes correspond to triangulations of polygons, (ii) each reduction of the quiddity sequence reduces the number of vertices of the polygon by Remark 2.6 and (iii) the quiddity sequence for the triangle is $(1, 1, 1)$. Also a short reference is [12; Question 23; 13, for answer].

(2) However, if we repeatedly apply reduction to the quiddity sequence of an *infinite* frieze, all 1's will eventually disappear since otherwise, repeated reductions would give the quiddity sequence (1) . If the period is 1, the quiddity sequence is (a) with $a \geq 2$. For period 2, we can have $(1, a)$ with $a \geq 4$, which we can reduce to (2) and (a, b) with $a, b \geq 2$. If the period is at least 3 and if there is an entry 1 in the quiddity sequence, we reduce and get a sequence with one entry less, it is still a quiddity sequence of an infinite periodic frieze [20, Theorem 2.7]. We iterate until all entries are ≥ 2 . Observe that there can never be two entries $a_i = a_{i+1} = 1$ as by the diamond rule, the entry below them would be 0, so the frieze would not be infinite.

(3) If we start with a quiddity sequence of an infinite periodic frieze, the final quiddity sequence where all entries are $\neq 1$ is independent of the sequence of reductions. Indeed, since any reduction on the quiddity sequence corresponds to removing a peripheral triangle (ear) in the surface by Remark 2.6(b), and two peripheral triangles may share at most a common point, the order in which we remove peripheral triangles does not matter. As a consequence the notions reduced quiddity sequence and reduced frieze (Definitions 2.8 and 2.9) are well defined.

Definition 2.8. Given a quiddity sequence q of an infinite periodic frieze, the *reduced quiddity sequence* q^s is obtained from q by successively applying reduction to q until there are no occurrences of 1's left.

Definition 2.9. Let \mathcal{F} be an infinite periodic frieze with quiddity sequence q . Then we define the *reduced frieze* \mathcal{F}^s of \mathcal{F} to be the infinite periodic frieze of the quiddity sequence q^s .

Infinite friezes and triangulations of annuli

$$\begin{array}{cccccccc}
 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \\
 \cdots & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 \cdots & 5 & 4 & 1 & 2 & 5 & 4 & \cdots \\
 \cdots & 9 & 19 & 3 & 1 & 9 & 19 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & &
 \end{array}$$

Fig. 3. Frieze \mathcal{F} for the quiddity sequence $q = (4, 1, 2, 5)$.

$$\begin{array}{cccccccc}
 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \\
 \cdots & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 \cdots & 2 & 4 & 2 & 4 & 2 & \cdots & \\
 \cdots & 7 & 7 & 7 & 7 & 7 & 7 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & &
 \end{array}$$

Fig. 4. The skeletal frieze \mathcal{F}^s for the quiddity sequence $q^s = (2, 4)$.

Definition 2.10. If q is a quiddity sequence with $q = q^s$ and if q is different from $(2, 2, \dots, 2)$, we will call q a *skeletal* quiddity sequence. A *skeletal frieze* is a frieze obtained from a skeletal quiddity sequence.

Example 2.11. If $q = (4, 1, 2, 5)$ then $q^s = (2, 4)$. The frieze and its skeletal frieze are shown in Figs. 3 and 4, respectively.

We end this section with the special frieze associated to the so called “trivial quiddity sequence” $q = (2, 2, \dots, 2)$. This is an infinite periodic frieze which arises from a triangulation of a punctured disc as well as from a triangulation of an annulus using spiraling arcs. See Remark 2.16. We will later exclude these types of friezes as they form a class of their own which is not relevant for our work, and have been well studied elsewhere.

Example 2.12. Consider the quiddity sequence $q = (2, 2, \dots, 2)$. It defines an infinite periodic frieze where the entries in the k th nontrivial row are all equal to $k + 1$. See Fig. 5. The entries in each diagonal row grow linearly, this is an example of an *arithmetic infinite frieze* as studied by Tschabold [20]. We will call this the *trivial quiddity sequence*.

2.3. *Triangulations of surfaces and their associated quivers*

Let S be a connected oriented surface with boundary. We denote by M , a finite set of marked points on the boundary. The pair (S, M) is called a bordered surface with

K. Baur et al.

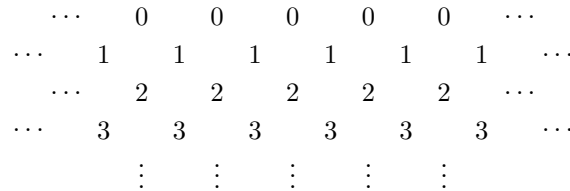


Fig. 5. The infinite periodic frieze with quiddity sequence $q = (2, 2, \dots, 2)$.

marked points if S is non-empty and each connected component of the boundary of S has at least one marked point, see [15, Sec. 2] for details.

An *arc* γ in (S, M) is a class of curves equivalent up to homotopy in S which start and end at the marked points in M . We require that there exists a curve γ in (S, M) for which the following conditions hold (we use the same notation for the arc and its representative):

- γ has no self-intersections,
- γ does not intersect the set M and the boundary of the surface S , except at its endpoints,
- γ is not contractible to a subset of the boundary of S .

Two arcs are compatible if they have representatives which do not intersect each other inside S . A maximal collection of such pairwise compatible arcs is called a *triangulation* of (S, M) . A triangulation of (S, M) has two types of arcs.

Definition 2.13. An arc that connects two marked points on the same boundary component is called a *peripheral arc*. An arc that connects two marked points on different boundary components is a *bridging arc*.

We recall the definition of the quiver of a triangulation of a surface, as illustrated in Fig. 6.

Definition 2.14. Let \mathcal{T} be a triangulation of a surface (S, M) . We associate a *quiver* $Q_{\mathcal{T}}$ to \mathcal{T} as follows:

Vertices: For every arc in \mathcal{T} we have a vertex in $Q_{\mathcal{T}}$,

Arrows: Suppose that i and j are arcs in the triangulation sharing one endpoint such that i is a direct predecessor to j with respect to anti-clockwise rotation at their shared endpoint. Then there is an arrow from i to j in $Q_{\mathcal{T}}$.

Remark 2.15. We observe that (isotopy classes of) infinite curves which start at one marked point and spiral around infinitely many times have also been used in triangulations of surfaces, as introduced in [3, Sec. 3.2]. Such arcs are called *spiraling arcs*. See Fig. 8 for examples. However such arcs do not appear in the results of this paper.

Infinite friezes and triangulations of annuli

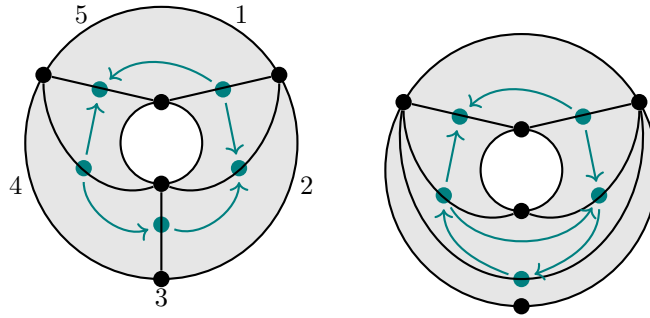


Fig. 6. Examples of quivers $Q_{\mathcal{T}}$ and $Q_{\mathcal{T}'}$ from two triangulations of $C_{3,2}$.

2.4. *Triangulations and friezes*

Now let us recall the correspondence between triangulations of regular polygons and finite friezes [12; 13, Sec. 3]. First, note that every finite frieze of order n has period $(n + 3)$ [12, 13, Question 26]. In fact, every finite frieze of order n comes from a triangulation of a regular $(n + 3)$ -gon [12; 13, Questions 28 and 29]. The correspondence can be described as follows: let \mathcal{F} be a frieze of period $(n + 3)$ with quiddity sequence q . Then the $(n + 3)$ entries of q count the number of triangles at the vertices in a triangulation of an $(n + 3)$ -gon.

Every triangulation of an annulus gives rise to an infinite periodic frieze and every infinite periodic frieze can be realized by a triangulation of an annulus [6, Theorems 3.7 and 4.6]. For this reason, the surface S we consider will always be an annulus. We denote by $C_{m,n}$ an annulus with m marked points on one boundary component B_1 and n marked points on the other boundary component B_2 . We will refer to them as inner and outer boundary components of annulus if we use a geometric presentation of the annulus in figures. We will label the marked points on B_1 anti-clockwise around the outer boundary and the marked points on B_2 anti-clockwise around the inner boundary.

Remark 2.16. It is worth pointing out that there are infinite periodic friezes which can also be given by triangulations of a punctured disk. These are the arithmetic infinite friezes from Examples 2.12 and 2.19.

2.5. *Triangulations of annuli and friezes*

We recall from [6, Theorem 4.6] that for every infinite periodic frieze we can construct a triangulation of an annulus such that the number of triangles at the marked points on one boundary give the quiddity sequence of the frieze (as in Definition 2.17). While finite friezes are in bijection with triangulations of polygons, infinite periodic friezes arise from triangulations of annuli, but not in a unique way. In fact, each infinite periodic frieze gives rise to a family of triangulations of an

K. Baur et al.

annulus. If we restrict to skeletal friezes, the triangulation is essentially unique as we will see in Proposition 3.10.

As a consequence we show in Theorem 3.22 that if \mathcal{F}_1 is skeletal, it determines another skeletal frieze \mathcal{F}_2 essentially uniquely such that the pair arises from a triangulation of an annulus. We start with a triangulation of an annulus and associate two infinite periodic friezes to it.

Definition 2.17. Let \mathcal{T} be a triangulation on an annulus $C_{m,n}$. For each of the two boundaries B_1 and B_2 define the *quiddity sequences* $q_1 = (a_1, \dots, a_m)$ and $q_2 = (b_1, \dots, b_n)$ as the number of triangles at marked points on B_1 and B_2 , respectively. The friezes corresponding to q_1 and q_2 will be denoted by \mathcal{F}_1 and \mathcal{F}_2 , respectively.

Note that we will not distinguish between the two annuli $C_{m,n}$ and $C_{n,m}$. Let \mathcal{T} be a triangulation of $C_{m,n}$ and \mathcal{T}' be the triangulation of $C_{n,m}$ obtained by exchanging the boundaries of $C_{m,n}$, then $Q_{\mathcal{T}}$ and $Q_{\mathcal{T}'}$ are isomorphic quivers.

Example 2.18. Consider the triangulation \mathcal{T} of an annulus $C_{3,2}$ shown in Fig. 7, the corresponding quiddity sequences are $q_1 = (2, 3, 3)$ and $q_2 = (3, 4)$. The frieze \mathcal{F}_1 corresponding to the boundary B_1 is defined to be the frieze which has quiddity sequence $q_1 = (2, 3, 3)$ and therefore is as follows:

$$\begin{array}{cccccccc} \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ & \dots & 1 & 1 & 1 & 1 & 1 & \dots \\ \dots & 3 & 3 & 2 & 3 & 3 & 2 & \dots \\ & \dots & 8 & 5 & 5 & 8 & 5 & \dots \\ & & & \ddots & & \ddots & & \end{array}$$

The frieze \mathcal{F}_2 corresponding to the boundary B_2 with quiddity sequence $q_2 = (3, 4)$ is as follows:

$$\begin{array}{cccccccc} \dots & 0 & 0 & 0 & 0 & \dots \\ & \dots & 1 & 1 & 1 & 1 & \dots \\ \dots & 3 & 4 & 3 & 4 & \dots \\ & \dots & 11 & 11 & 11 & 11 & \dots \\ & & & \ddots & & \ddots & & \end{array}$$

Example 2.19. The triangulation of an annulus giving rise to the trivial quiddity sequence $(2, 2, 2, 2)$ is given by arcs spiraling around a non-contractible curve in the annulus as shown in Fig. 8.

Remark 2.20. We will from now on always exclude asymptotic triangulations, the trivial quiddity sequences $(2, 2, \dots, 2)$ and their infinite periodic friezes (as in Fig. 5).

Infinite friezes and triangulations of annuli

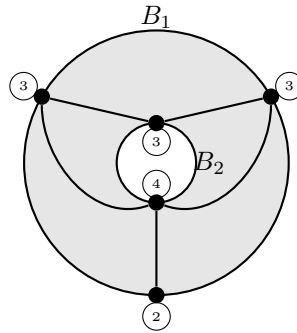


Fig. 7. A triangulation of an annulus $C_{3,2}$ with number of triangles at each marked point.

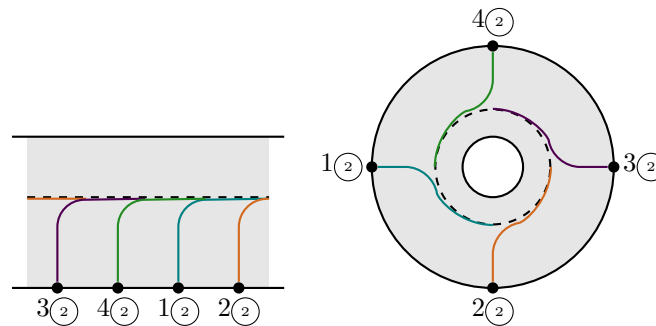


Fig. 8. The asymptotic triangulation of an annulus corresponding to $q = (2, 2, 2, 2)$. The figure on the left shows a fundamental domain for the surface, where the two vertical ends of the shaded region are identified. It is often convenient to draw arcs in this way.

3. Pairs of Friezes

In this section, we characterize the relation between the two infinite periodic friezes associated to a triangulation.

3.1. Skeletal triangulations

In Sec. 2.2, we studied skeletal friezes. Similarly we can define skeletal triangulations. To a triangulation \mathcal{T} of an annulus with marked points, we associate a skeletal triangulation \mathcal{T}^s of a new annulus with marked points. We show that the essential properties that we are concerned about are preserved when going from a triangulation/frieze to the corresponding skeletal triangulation/frieze. Since skeletal triangulations/friezes are easier to deal with, we will work with them. Recall that we only allow finite arcs in triangulations.

K. Baur et al.

Definition 3.1. A *flip* of an arc in a triangulation \mathcal{T} is the replacement of the arc by the unique other arc in the quadrilateral formed by the two triangles incident with the arc.

An example of a flip is shown in Fig. 6. The vertical arc in the annulus on the left hand is flipped to a peripheral arc in the right hand.

Definition 3.2. A peripheral arc in a triangulation \mathcal{T} is a *bounding arc* (for \mathcal{T}) if its flip is a bridging arc as defined in Definition 2.13.

Remark 3.3. Every bounding arc of a triangulation splits the annulus into two parts. One part is a triangulated polygon and the other is a triangulated annulus.

Lemma 3.4. *Every triangulation with a peripheral arc has a bounding arc.*

Proof. Assume that \mathcal{T} has a peripheral arc γ . Let γ' be maximal and peripheral above γ (γ lies between γ' and the boundary it is attached to and there is no other peripheral arc γ'' such that γ' lies between the boundary and γ''). Denote the endpoints of γ by P_1 and P_2 . Then γ' is incident with two triangles of \mathcal{T} : one consists only of peripheral arcs, say with vertices P_1, P_2, P_3 along the boundary and the other one has vertices P_1, Q, P_2 with Q on the other boundary. Flipping γ results in the arc connecting P_3 with Q , a bridging arc. Therefore, γ is bounding. \square

Definition 3.5. Let \mathcal{T} be a triangulation of an annulus. The *skeletal triangulation* \mathcal{T}^s of \mathcal{T} is obtained by cutting along all bounding arcs and removing the triangulated polygon attached with them.

Example 3.6. In Fig. 9, we give an example of a triangulation \mathcal{T} of an annulus and the corresponding skeletal triangulation \mathcal{T}^s .

Lemma 3.7. *Let \mathcal{T} be a triangulation of an annulus. Then:*

- (1) *The process of constructing the skeletal triangulation \mathcal{T}^s is well defined.*
- (2) *The skeletal triangulation \mathcal{T}^s has only bridging arcs.*

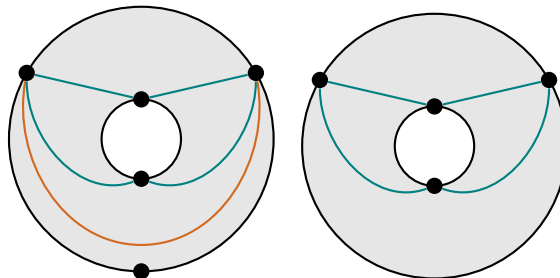


Fig. 9. A triangulation of an annulus $C_{3,2}$ and the corresponding skeletal triangulation of $C_{2,2}$.

Proof. (1) This is clear since bounding arcs do not intersect except possibly at end points, and no new bounding arcs are created after cutting along existing bounding arcs. Thus the order of removing them does not matter.

(2) Every non-bounding peripheral arc lies in a polygon given by some bounding arc and the corresponding boundary of the annulus between the two endpoints of the bounding arc. In particular, every peripheral arc gets removed when going from \mathcal{T} to \mathcal{T}^s . \square

Note that the skeletal triangulation \mathcal{T}^s of a triangulation \mathcal{T} of an annulus, is actually a triangulation of a different annulus (as in Fig. 9).

Lemma 3.8. *If a triangulation of $C_{m,n}$ has k peripheral arcs, then its skeletal triangulation is of an annulus $C_{m-k_1, n-k_2}$ where k_1 is the number of peripheral arcs on the outer boundary and k_2 the number of peripheral arcs on the inner boundary (in particular, $k_1 + k_2 = k$).*

Proof. The proof follows since removing one peripheral arc of a triangle with two boundary segments corresponds to removing one marked point on that boundary. \square

By Lemma 3.7(2), all arcs of a skeletal triangulation are bridging. We have the following.

Lemma 3.9. *Let \mathcal{T} be a triangulation of an annulus. Then, $\mathcal{T} = \mathcal{T}^s$ if and only if the quiver $Q_{\mathcal{T}}$ of \mathcal{T} is a non-oriented cycle.*

Proof. Assume first that $\mathcal{T} = \mathcal{T}^s$. Then the only arcs in the annulus are bridging. In particular, every triangle of \mathcal{T} has a boundary segment as one of its edges and two bridging arcs. So every triangle gives rise to exactly one arrow in $Q_{\mathcal{T}}$ and every vertex of $Q_{\mathcal{T}}$ has exactly two arrows incident with it. Thus the graph underlying $Q_{\mathcal{T}}$ is a cycle. The arcs in \mathcal{T} form a collection of fans from each boundary (some of these fans only consist of two arcs). Look at the outer boundary. Take any two neighbored fans f_1 and f_2 : the right most arc of f_1 together with the left most arc of f_2 form a V . Together with the next arc to the left, they form an N , together with the next arc on the right they form a \mathcal{N} . This means that $Q_{\mathcal{T}}$ has at least one sink and at least one source, respectively.

Assume now that $Q_{\mathcal{T}}$ is a non-oriented cyclic quiver, and suppose that the corresponding triangulation \mathcal{T} has a peripheral arc γ' ; then \mathcal{T} also must have at least one bounding arc γ by Lemma 3.4. For \mathcal{T} to be a triangulation, we need γ to be part of an internal triangle. This internal triangle gives rise to an oriented 3-cycle in $Q_{\mathcal{T}}$, leading to a contradiction. \square

The following proposition is a consequence of the construction from [6, Sec. 4]. It uses a construction of a triangulation for a skeletal quiddity sequence.

K. Baur et al.

Proposition 3.10. *Let $q = q^s$ be a skeletal quiddity sequence. Then up to rotating the inner boundary and its marked points, there is a unique skeletal triangulation of an annulus $C_{m,n}$ such that the infinite periodic frieze associated to the outer boundary has q as its quiddity sequence.*

Proof. This follows from the algorithm described in the proof of [6, Corollary 4.5]: Let $q = q^s = (a_1, \dots, a_m)$. Recall that $q \neq (2, 2, \dots, 2)$. Draw an annulus with m vertices on the outer boundary, labeled $1, 2, \dots, m$ clockwise. For every a_i , draw $a_i - 1$ arc segments at vertex i . See the first picture in Fig. 10 for an illustration.

Extend the leftmost arc segment at vertex 1 and connect it with the inner boundary, thus creating a marked point at the inner boundary. See second picture in Fig. 10.

If $m = 1$, continue connecting the arc segments (from left to right) at vertex 1 to the inner boundary, until one arc is left, creating $a_1 - 2$ marked points at the inner boundary. Then connect the remaining segment to the first marked point on the inner boundary.

If $m > 1$, continue connecting the arc segments (from left to right) at vertex 1 to the inner boundary, creating $a_1 - 1$ marked points at the inner boundary.

Now assume the arc segments at vertices $1, 2, \dots, i - 1$ have been connected to the inner boundary. Connect the left most arc segment at vertex i to the last of the marked points created for vertex $i - 1$. If $a_i = 2$, this is the only segment to connect. If $i < m$ and if there is an index j with $i < j \leq m$ such that $a_j > 2$, create new vertices for the next $a_i - 2$ arc segments at i . See third picture in Fig. 10. If $i = m$ or if $a_j = 2$ for all $j = i + 1, \dots, m$, create new vertices for the next $a_i - 3$ arc segments at i and connect the last arc segment to the first marked point created for vertex 1, as in the last picture of Fig. 10. In the latter case, connect any remaining vertices j with $j > i$ and $a_j = 2$ with this same marked point.

As we see from Fig. 11, the triangulation is only dependent on the first arc: once the endpoint of the first arc on the inner boundary is fixed, the rest is determined. Thus we have the uniqueness of the triangulation up to this first choice. \square

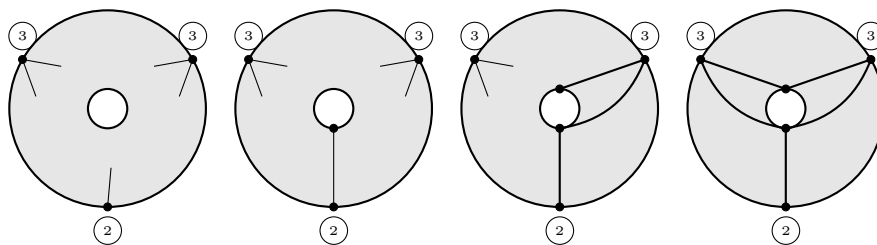


Fig. 10. Triangulation from the quiddity sequence $(2, 3, 3)$.

Infinite friezes and triangulations of annuli

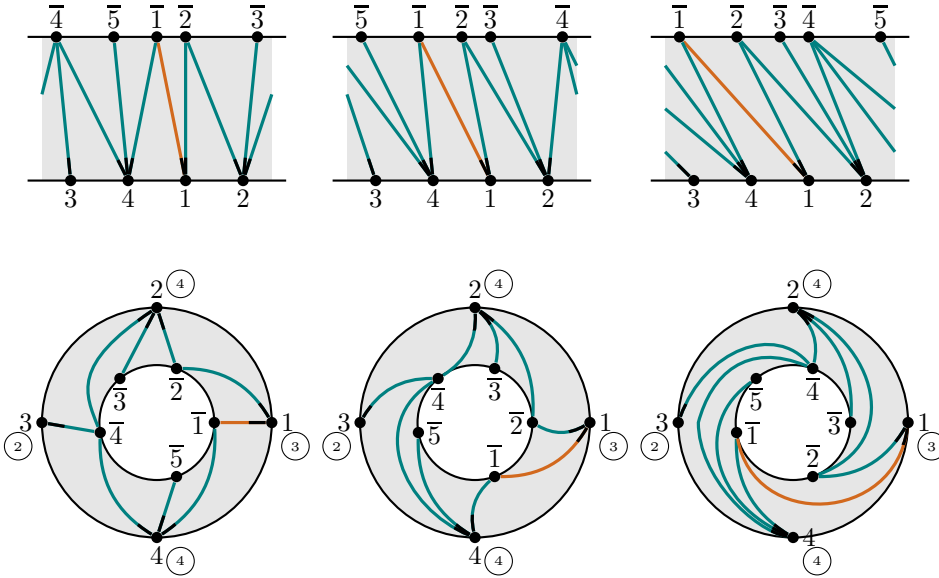


Fig. 11. (Color online) An illustration of the uniqueness of the triangulation up to the choice of the first bridging arc (in orange), for the quiddity sequence $q = (3, 4, 2, 4)$.

The above proposition also suggests that if q_1 is a skeletal quiddity sequence corresponding to the outer boundary in the skeletal triangulation \mathcal{T} , then it determines the skeletal quiddity sequence q_2 for the inner boundary as well. We state the formula in the next corollary. For this, we write the entries of a skeletal quiddity sequence q as follows:

$$q = (a_{i_1}, \underbrace{2, \dots, 2}_{k_1}, a_{i_2}, \underbrace{2, \dots, 2}_{k_2}, \dots, a_{i_r}, \underbrace{2, \dots, 2}_{k_r}),$$

or $(a_{i_1}, 2^{(k_1)}, a_{i_2}, 2^{(k_2)}, \dots, a_{i_r}, 2^{(k_r)})$ for short, where $a_{i_j} > 2$ and $k_j \geq 0$ for all j .

Corollary 3.11. *Let $q_1 = (a_{i_1}, 2^{(k_1)}, a_{i_2}, 2^{(k_2)}, \dots, a_{i_r}, 2^{(k_r)})$ be a skeletal quiddity sequence, where $a_{i_j} > 2$ and $k_j \geq 0$ for all j . Let \mathcal{T} be the triangulation associated to q_1 by Proposition 3.10. Then the quiddity sequence given by the inner boundary of \mathcal{T} is $q_2 = (2^{(a_{i_1}-3)}, k_1 + 3, 2^{(a_{i_2}-3)}, k_2 + 3, \dots, 2^{(a_{i_r}-3)}, k_r + 3)$.*

The following proposition describes the correspondence between triangulations and infinite periodic friezes, and their respective skeletal versions. Let \mathcal{T} be a triangulation of an annulus. Let f be the map that sends \mathcal{T} to the corresponding pair of infinite periodic friezes $f(\mathcal{T}) = (\mathcal{F}_1, \mathcal{F}_2)$. Let $s_{\mathcal{T}}$ be the map that sends a triangulation to its skeletal triangulation. Let $s_{\mathcal{F}}$ be the map sending a pair of infinite periodic friezes to the pair of their skeletal versions, so that $s_{\mathcal{F}}((\mathcal{F}_1, \mathcal{F}_2)) = (\mathcal{F}_1^s, \mathcal{F}_2^s)$.

K. Baur et al.

Proposition 3.12. *We have that $f \circ s_T = s_F \circ f$, i.e. $f(\mathcal{T}^s) = s_F \circ f(\mathcal{T})$.*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\quad} & (\mathcal{F}_1, \mathcal{F}_2) \\ \downarrow & & \downarrow \quad \downarrow \\ \mathcal{T}^s & \xrightarrow{\quad} & (\mathcal{F}_1^s, \mathcal{F}_2^s) \end{array}$$

Proof. First notice that the reduction s_T of a triangulation \mathcal{T} to its skeletal triangulation \mathcal{T}^s consists of cutting along all bounding arcs and removing the triangulated polygons attached with them by Definition 3.5. This process is well defined by Lemma 3.7. So it is enough to show that removing one polygon from a triangulation of an annulus produces the same pair of infinite periodic friezes as are obtained by applying reduction process to the friezes as in Definition 2.2.

Consider now the deletion of a triangulated n -gon from a triangulation of an annulus, where $n \geq 3$. We will show that such a deletion corresponds to a sequence of reductions on the corresponding quiddity sequence. If $n = 3$, we use the fact that reduction of a quiddity sequence at a 1 corresponds to deletion of a peripheral triangle (as shown in Fig. 9) in the corresponding triangulation of an annulus by Remark 2.6. So in this case, deletion trivially corresponds to a sequence of reductions. For higher n we can show the statement by induction on n .

Suppose the statement holds for $n = k$. Suppose we remove a triangulated $(k+1)$ -gon. The triangulated $(k+2)$ -gon must have at least two peripheral triangles, and at least one of them must also be on the boundary of the original triangulated annulus. If we delete this peripheral triangle from the annulus, our original $(k+1)$ -gon is now a k -gon. Looking at the quiddity sequence side, we perform one reduction when deleting the peripheral triangle, and then a series of reductions when deleting the k -gon. Hence the deletion of a $(k+1)$ -gon corresponds to a series of reductions. \square

Lemma 3.13. *Let \mathcal{T} be a triangulation of an annulus, let (q_1, q_2) the pair of quiddity sequences associated to \mathcal{T} . The following are equivalent:*

- (1) $\mathcal{T} = \mathcal{T}^s$.
- (2) *There are no 1's in q_1 and q_2 .*

Proof. Suppose $\mathcal{T} = \mathcal{T}^s$, then there are no peripheral arcs in the triangulation and hence all entries in q_1 and q_2 are ≥ 2 .

Suppose conversely that q_1 and q_2 have no 1's. When we do the construction of Proposition 3.10, this means that every marked point on each boundary will have at least one arc incident to it. Hence there can be no peripheral arcs. \square

3.2. Quivers to quiddity sequences and back

In this section, we explain how any non-oriented cyclic quiver gives rise to a pair of quiddity sequences of an infinite periodic frieze and hence to a triangulation of

an annulus. Let Q be a non-oriented cyclic quiver, with vertices $1, 2, \dots, n$ labeled anti-clockwise around the cycle and with $n \geq 2$. We can assume $n \geq 2$ as we are interested in quivers arising from (skeletal) triangulations of annuli and the associated quivers always have at least two vertices. We associate a pair of quiddity sequences to Q as follows.

First observe that in any non-oriented cycle there is at least one source and at least one sink. Without loss of generality we can assume that the vertex 1 is a source. To define the quiddity sequences we distinguish between *increasing arrows* (arrows $i \rightarrow i + 1$, including $n \rightarrow 1$) and *decreasing arrows* (arrows $i \leftarrow i + 1$, including $1 \rightarrow n$) in Q . Let p be the number of decreasing arrows in Q and let $t = n - p$ be the number of increasing arrows in Q . In case $n = 2$, we have exactly one increasing arrow and one decreasing arrow, so $p = t = 1$. We define an increasing path as a path of increasing arrows. It is maximal when it starts at a source and ends at a sink. Similarly, a decreasing path is a path of decreasing arrows.

Definition 3.14 (Quiddity sequences from a non-oriented cyclic quiver Q).

Let Q be a non-oriented cyclic quiver with vertices $\{1, 2, \dots, n\}$. Assume that vertex 1 is a source.

Let $j_1 = 1$ and $\{j_2, \dots, j_p\}$ be the set of tails of the decreasing arrows in Q for $1 \leq p < n$, with $1 < j_2 < \dots < j_p \leq n$. Let c_1 be the length of the maximal (increasing) linear path starting with $1 \rightarrow 2$. For $k = 2, \dots, p$, let $c_{j_k} \geq 0$ be the length of the maximal increasing linear path starting at j_k . Define

$$\sigma(Q) := (a_1, \dots, a_p) \quad \text{where } a_k := c_{j_k} + 2, \quad \text{for } k = 1, \dots, p.$$

Similarly, let $\{m_1, m_2, \dots, m_t\}$ be the set of heads of the increasing arrows in Q , $1 \leq t < n$, with $2 = m_1 < m_2 < \dots < m_t \leq n$. For $k = 1, \dots, t$, let $d_{j_k} \geq 0$ be the length of the maximal decreasing path ending at m_{j_k} . Define

$$\tilde{\sigma}(Q) := (b_1, \dots, b_t) \quad \text{where } b_k := d_{j_k} + 2, \quad \text{for } k = 1, \dots, t.$$

Note that since we assume that vertex 1 is a source, $c_1 > 0$ and $d_{j_t} > 0$. In case $n = 2$, the resulting quiddity sequences are $\sigma(Q) = (3) = \tilde{\sigma}(Q)$.

Remark 3.15. Let Q be a non-oriented cyclic quiver and consider $\sigma(Q)$ and $\tilde{\sigma}(Q)$ from Definition 3.14. We have $a_j \geq 2$ for all j and $b_i \geq 2$ for all i . Furthermore, $a_1 > 2$ and $b_t > 2$. So by result [6, Corollary 2.2], both are quiddity sequences of infinite friezes.

Example 3.16. Consider the quiver from Fig. 12. Here, $\{j_1, \dots, j_p\} = \{1, 4, 6, 7, 8\}$ and $\{m_1, \dots, m_t\} = \{2, 3, 5, 9\}$. From this, we get $c_{j_1} = c_1 = 2$, $c_{j_2} = 1$, $c_{j_3} = 0$, $c_{j_4} = 0$, $c_{j_5} = 1$ and $d_{m_1} = 0$, $d_{m_2} = 1$, $d_{m_3} = 3$, $d_{m_4} = 1$. And so $\sigma(Q) = (4, 3, 2, 2, 3)$ and $\tilde{\sigma}(Q) = (2, 3, 5, 3)$.

Lemma 3.17. *Each vertex in a non-oriented cyclic quiver is either the head of an increasing arrow or the tail of a decreasing arrow (it cannot be both).*

K. Baur et al.

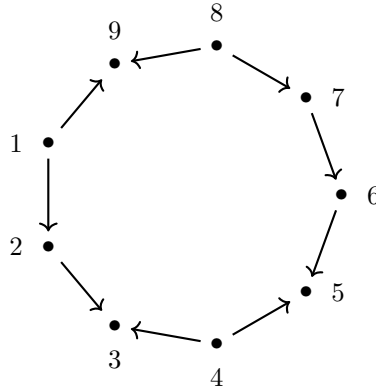


Fig. 12. A non-oriented cycle on nine vertices.

Proof. If a vertex k in a non-oriented cyclic quiver Q is a sink or a source, then it is clear that it is the head of an increasing arrow or the tail of a decreasing arrow (respectively). If the vertex k is neither a sink nor a source, then it is one of the following:

$$k - 1 \longrightarrow k \longrightarrow k + 1,$$

$$k - 1 \longleftarrow k \longleftarrow k + 1.$$

In the first case k is the head of an increasing arrow, and in the second case, it is the tail of a decreasing arrow. To see that it cannot be both, notice that if it is both, we get a two-cycle at that vertex. This contradicts the assumption that Q is a non-oriented cyclic quiver. \square

Lemma 3.18. *Let Q be a non-oriented cyclic quiver with vertices $\{1, 2, \dots, n\}$. Assume that vertex 1 is a source. Let $\sigma(Q) = (a_1, \dots, a_p)$ and $\tilde{\sigma}(Q) = (b_1, \dots, b_t)$ be as in Definition 3.14. Then:*

- (1) $\sigma(Q)$ and $\tilde{\sigma}(Q)$ are skeletal, nontrivial quiddity sequences, i.e. $a_i > 1$ and $b_j > 1$ for all i, j and not all elements of $\sigma(Q)$ and $\tilde{\sigma}(Q)$ are equal to 2,
- (2) $\sum a_i + \sum b_j = 3n$.

Proof. By Remark 3.15, $\sigma(Q) = (a_1, \dots, a_p)$ and $\tilde{\sigma}(Q) = (b_1, \dots, b_t)$ are quiddity sequences giving rise to infinite periodic friezes. By construction, the a_i and the b_j are all greater than or equal to 2. Since Q is not cyclic, at least one c_{j_k} and at least one d_{j_k} is positive, so (1) holds.

To see that (2) holds: The c_{j_i} count the decreasing arrows and the d_{m_k} the increasing arrows in Q . Since we add 2 for every arrow to get a_i and b_j , the total sum is $3n$. \square

Definition 3.19 (A non-oriented cyclic quiver from a quiddity sequence).
Let

$$q = (a_{j_1}, 2^{(k_1)}, a_{j_2}, 2^{(k_2)}, \dots, a_{j_r}, 2^{(k_r)}),$$

be a skeletal quiddity sequence where $a_{j_i} > 2$ and $k_i \geq 0$ for all $i = 1, \dots, r$. To q , we associate a cyclic quiver $Q = \mu(q)$ as follows: Starting with a source at vertex 1, Q has $a_{j_1} - 2$ increasing arrows, then a sink, then $k_1 + 1$ decreasing arrows, then a source, followed by $a_{j_2} - 2$ increasing arrows, another sink, then $k_2 + 1$ decreasing arrows, etc. This ends with $k_r + 1$ decreasing arrows, the last of them being $n \leftarrow 1$.

Theorem 3.20. *Let $n \geq 2$. There is a bijection between the following classes:*

- (1) *Unlabeled non-oriented cyclic quivers with n vertices.*
- (2) *Skeletal quiddity sequences $q = (a_1, \dots, a_p)$ with*

$$n = p + \sum_{k=1}^p (a_k - 2),$$

for some $1 \leq p < n$.

Proof. We use the maps σ and μ to get the bijection. The map σ associates to every non-oriented cyclic quiver Q a skeletal quiddity sequence. We show that the image $\sigma(Q)$ is of the form described in (2). Assume that Q has a source at vertex 1 and consider $\sigma(Q) = (a_1, \dots, a_p)$. Recall from Definition 3.14 that $a_k = c_{j_k} + 2$, where c_{j_k} counts the length of the maximal increasing path starting at j_k , $a_i \geq 2$ for all i , and $a_1 > 2$. So (a_1, \dots, a_p) is a skeletal nontrivial quiddity sequence. Also, $a_k = c_{j_k} + 2$ implies that the right-hand side of the equality in (2) becomes

$$p + \sum_{k=1}^p (a_k - 2) = p + \sum_{k=1}^p c_{j_k}.$$

Let us denote the maximal increasing path starting at j_k by P_k (this can be of length 0). All these paths are disjoint (they do not share vertices). By Lemma 3.17, each vertex in Q is either the head of an increasing arrow or the tail of a decreasing arrow (it cannot be both). If it is the tail of a decreasing arrow, then it is the tail of some P_k . If it is the head of an increasing arrow, then it is either the head of a path P_k or an internal vertex of P_k , for some k . In any case, the vertex belongs to a maximal increasing path. This implies that each vertex of Q appears exactly once in one P_k . The number of vertices in P_k is $c_{j_k} + 1$. Hence we have

$$\sum_{k=1}^p (c_{j_k} + 1) = p + \sum_{k=1}^p c_{j_k} = n.$$

Now we argue that $\mu(\sigma(Q)) = Q$. Let Q be a non-oriented cyclic quiver described as in the beginning of Sec. 3.2 such that $\{j_1 = 1, j_2, \dots, j_p\}$ is the set of tails of its decreasing arrows. Then $\sigma(Q) = (a_1, \dots, a_p)$ is the quiddity sequence from Definition 3.14.

K. Baur et al.

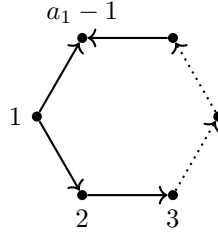


Fig. 13. The quiver $\mu(\sigma(Q))$ where $\sigma(Q) = (a_1)$.

To prove that $\mu(\sigma(Q))$ has n vertices, we use induction on the length of the quiddity sequence. Suppose $\sigma(Q) = (a_1)$, where $a_1 > 2$. We claim that $n = a_1 - 1$. The quiver $\mu(\sigma(Q))$ has a source at vertex 1, then an increasing path of length $a_1 - 2$, then a sink and a decreasing arrow from 1 to that sink. So the number of vertices of $\mu(\sigma(Q))$ is $a_1 - 1$. This is shown in Fig. 13.

Let $\sigma(Q) = (a_1, \dots, a_{\ell+1})$. By induction, the quiddity sequence $q = (a_1, \dots, a_{\ell})$ is such that $\mu(q)$ is a quiver with n vertices fulfilling

$$n = \ell + \sum_{k=1}^{\ell} (a_k - 2).$$

If $a_{\ell+1} = 2$, then the algorithm in Definition 3.19 adds one decreasing arrow and hence one more vertex to the quiver $\mu(q)$. That means the total number of vertices is $n + 1$, which is indeed the case since $a_{\ell+1} - 2 = 0$ and

$$\ell + 1 + \sum_{k=1}^{\ell+1} (a_k - 2) = \ell + 1 + \sum_{k=1}^{\ell} (a_k - 2) = n + 1.$$

If $a_{\ell+1} \geq 3$, then by Definition 3.19, we must add $a_{\ell+1} - 2$ increasing arrows and a decreasing arrow from vertex 1 to the last vertex. This implies that we add $a_{\ell+1} - 2$ vertices for the increasing arrows and one more for the decreasing arrow. \square

Corollary 3.21. *Let $n \geq 2$. There is a bijection between the following classes:*

- (1) *Skeletal quiddity sequences $q = (a_1, \dots, a_p)$ with $n = p + \sum_k (a_k - 2)$, for some $1 \leq p < n$.*
- (2) *Skeletal triangulations of annuli $C_{m, n-m}$ with $1 \leq m < n$.*

Proof. By Theorem 3.20, (1) is equivalent to the associated quiver being non-oriented cycle and (by Lemma 3.9) these in turn correspond to triangulations given by bridging arcs, i.e. skeletal triangulations. \square

Theorem 3.22. *Let \mathcal{F}_1^s be the skeletal frieze of an infinite periodic frieze \mathcal{F}_1 . Then*

- (1) *\mathcal{F}_1^s uniquely determines an infinite skeletal frieze \mathcal{F}_2^s such that the two form a pair of infinite periodic friezes associated to a triangulation of an annulus,*

- (2) \mathcal{F}_1^s gives rise to an infinite family of infinite periodic friezes \mathcal{F}_2 for each of which $(\mathcal{F}_1^s, \mathcal{F}_2)$ is a pair of infinite periodic friezes arising from a triangulation of an annulus.

Proof. (1) Let q_1^s be the quiddity sequence of \mathcal{F}_1^s . Then q_1^s is a skeletal quiddity sequence. By Corollary 3.21, the quiddity sequence q_1^s gives rise to a unique skeletal triangulation \mathcal{T}^s of a marked annulus. Then \mathcal{T}^s uniquely gives rise to a pair of infinite skeletal friezes one of which is \mathcal{F}_1^s and the other is \mathcal{F}_2^s .

(2) By Remark 2.4, we can apply reverse reduction process on q_2^s repeatedly to obtain infinitely many friezes \mathcal{F}_2 . □

4. Growth Coefficients

The entries in any infinite (periodic) frieze continue to grow. In fact, there is an invariant of periodic friezes which governs their growth, called the growth coefficient. It is known by [4, Theorem 3.4] that the pair of infinite friezes associated with a triangulation have the same growth coefficient. In this section, we calculate the growth coefficients in terms of quiddity sequences and study a module theoretic interpretation of them.

4.1. Properties of growth coefficients

Let \mathcal{F} be an infinite periodic frieze of (minimal) period n , i.e. \mathcal{F} is n -periodic but there is no $n' < n$ such that \mathcal{F} is n' -periodic. Set $s_r := a_{1, rn} - a_{2, rn-1}$ for any $r > 0$ (we follow the notation from Fig. 2, see also Fig. 14). We define the *growth coefficient of \mathcal{F}* as $s_{\mathcal{F}} := s_1$. Recall that the first nontrivial row of a frieze is the quiddity row and we count rows onwards from there.

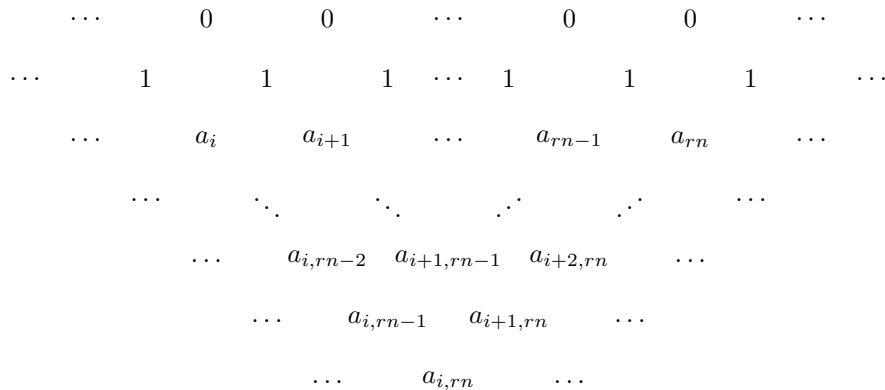


Fig. 14. Indexing of an infinite periodic frieze, see also Fig. 2.

K. Baur et al.

Proposition 4.1. *Let \mathcal{F} be an infinite periodic frieze of (minimal) period n and let s_r and $s_{\mathcal{F}}$ be as in the paragraph above, set $s_0 = 2$. Then the following holds:*

- (1) $a_{i, rn+i-1} - a_{i+1, rn+i-2} = s_r$ for all $i \in \mathbb{Z}$, (it is $a_{i+1, rn+i-2}$ in the second term);
- (2) $s_{r+1} = s_1 s_r - s_{r-1}$ for $r > 0$;
- (3) $s_r = s_{\mathcal{F}}^r + r \sum_{l=1}^{\lfloor r/2 \rfloor} (-1)^l \frac{1}{r-l} \binom{r-l}{l} s_{\mathcal{F}}^{r-2l}$ for $r > 0$.

Proof. Part (1) is [4, Theorem 2.2]. Part (2) is [4, Proposition 2.10]. Part (3) can be obtained by solving the recursion in part (2). \square

By Proposition 4.1(1), the difference between an entry in the n th row and the entry directly above it in row $n - 2$ in any infinite periodic frieze of period n is constant, as is the difference between an entry in row kn and the entry above it in row $kn - 2$, for all $k > 1$. The sequence (s_1, s_2, s_3, \dots) thus determines the growth of the entries in the frieze. By Proposition 4.1(3), these coefficients grow exponentially [4, Proposition 4.7].

We now concentrate on pairs of infinite periodic friezes arising from triangulations of an annulus with marked points.

Definition 4.2. Let \mathcal{T} be a triangulation of $C_{m,n}$ with associated quiddity sequences $q_1 = (a_1, \dots, a_m)$ and $q_2 = (b_1, \dots, b_n)$ and, infinite periodic friezes \mathcal{F}_1 and \mathcal{F}_2 . We denote the entries of \mathcal{F}_1 by a_{ij} and the entries of \mathcal{F}_2 by b_{ij} . Then we define $s_{q_1} := a_{1,m} - a_{2,m-1}$ and $s_{q_2} := b_{1,n} - b_{2,n-1}$. We call s_{q_i} the *growth coefficient* of q_i .

- Remark 4.3.** (1) In order to determine growth coefficients, it is enough to work with skeletal quiddity sequences, see [4, Theorem 3.1].
- (2) In our infinite periodic friezes, we always have $s_q > 2$. There are infinite periodic friezes with $s_q = 2$. In these cases, all the coefficients s_i are 2. Such a quiddity sequence generates an arithmetic frieze which we have excluded from our considerations, see Example 2.19 and Remark 2.20.
- (3) By definition, the growth coefficient s_q of a quiddity sequence $q = (a_1, \dots, a_n)$ is equal to one of the s_r from above, it is not necessarily equal to the growth coefficient $s_1 = s_{\mathcal{F}}$ of the associated frieze, as the quiddity sequence may be symmetric, i.e. the repetition of a shorter subsequence. We have $s_q = s_{\mathcal{F}}$ if and only if the minimal period of \mathcal{F}_i is equal to n , i.e. if and only if the minimal period of \mathcal{F}_i is equal to the length of its quiddity sequence.

Example 4.4. As an example, consider the leftmost triangulation in Fig. 9. The quiddity sequences are $q_1 = (1, 4, 4)$ and $q_2 = (3, 3)$ and one can compute $s_{q_1} = 7$, $s_{q_2} = 7$: For q_1 we have to compute the difference between an entry in the third nontrivial row and the entry above it in the quiddity row. For q_2 , we compute the difference between an entry in the second nontrivial row and an entry above it. However, if one views \mathcal{F}_2 as an infinite 1-periodic frieze, one obtains $s_{\mathcal{F}_2} = 3$.

We observe that $s_{q_1} = s_{q_2}$ in Example 4.4. This is no coincidence, as we will see now.

Proposition 4.5. *Let \mathcal{T} be a triangulation of an annulus, with associated infinite periodic friezes \mathcal{F}_1 and \mathcal{F}_2 and quiddity sequences q_1 and q_2 , respectively. Then $s_{q_i} = s_{q_i^s}$ for $i = 1, 2$ and $s_{q_1} = s_{q_2}$.*

Proof. This is [4, Theorem 3.4]. □

Let us point out that not every pair of infinite periodic friezes with a common growth coefficient can be realized via a triangulation of an annulus. Consider $q_1 = q_2 = (2, 3)$ with growth coefficient $s_{q_1} = s_{q_2} = 4$. But this pair does not correspond to any triangulation of an annulus. Another such example is: $q_1 = (4, 3, 4, 3)$ and $q_2 = (5, 20)$ where $s_{q_1} = s_{q_2} = 98$.

4.2. A formula for calculating growth coefficients

Consider an arbitrary element a_{ij} of the frieze in Fig. 14. By [4, Lemma 3.5], building on work by [8], we can find the entry a_{ij} of the infinite periodic frieze by calculating a determinant

$$a_{ij} = \det \begin{pmatrix} a_i & 1 & 0 & \cdots & \cdots & 0 \\ 1 & a_{i+1} & 1 & \cdots & \cdots & \vdots \\ 0 & 1 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 & 0 \\ \vdots & \cdots & \cdots & \cdots & 1 & a_{j-1} & 1 \\ 0 & \cdots & \cdots & 0 & 1 & a_j \end{pmatrix}. \tag{4.1}$$

Definition 4.6. Consider a finite ordered set of integers $S = \{i, i + 1, \dots, j - 1, j\}$. A *pair-excluding subset* $I \subseteq S$ is a subset obtained by removing zero or more disjoint pairs of consecutive integers from S .

A *cyclic pair-excluding subset* $J \subseteq S$ is a subset obtained by removing zero or more disjoint pairs of consecutive integers from S , when the first and the last element of S are also considered consecutive.

Note that any pair-excluding subset is also cyclic pair-excluding. The empty set is a pair-excluding subset if and only if the cardinality of S is even.

Example 4.7. The pair-excluding subsets of $\{1, 2, 3, 4, 5\}$ are

$$\begin{aligned} & \{1, 2, 3, 4, 5\}, \\ & \{1, 2, 3\}, \quad \{1, 2, 5\}, \quad \{1, 4, 5\}, \quad \{3, 4, 5\}, \\ & \{1\}, \quad \{3\}, \quad \text{and} \quad \{5\}. \end{aligned}$$

K. Baur et al.

The cyclic pair-excluding subsets are the above, together with the sets

$$\{2, 3, 4\}, \{2\} \quad \text{and} \quad \{4\}.$$

Similarly, the pair-excluding subsets of $\{1, 2, 3, 4, 5, 6\}$ are

$$\begin{aligned} &\{1, 2, 3, 4, 5, 6\}, \\ &\{1, 2, 3, 4\}, \quad \{1, 2, 3, 6\}, \quad \{1, 2, 5, 6\}, \quad \{1, 4, 5, 6\}, \quad \{3, 4, 5, 6\}, \\ &\{1, 2\}, \quad \{1, 4\}, \quad \{1, 6\}, \quad \{3, 4\}, \quad \{3, 6\}, \quad \{5, 6\}. \end{aligned}$$

and \emptyset .

The cyclic pair-excluding subsets are the sets above, together with the sets

$$\{2, 3, 4, 5\}, \{2, 3\}, \{2, 5\} \quad \text{and} \quad \{4, 5\}.$$

Theorem 4.8. *With the notation above*

$$a_{ij} = \sum_{\substack{I \subseteq \{i, \dots, j\} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k,$$

where ℓ_I is the number of pairs that were excluded from $\{i, \dots, j\}$ to create I ; in other words $\ell_I = \frac{j-i+1-|I|}{2}$.

Proof. The theorem follows from using [17, §544–546] on formula (4.1). \square

Example 4.9. For $i = j$, the formula simply reads $a_{ii} = a_{ii}$, and for $j = i + 1$, it says $a_{i,i+1} = a_{ii}a_{i+1,i+1} - 1$, where the term -1 appears due to the empty set being a pair-excluding subset when the original set is of even length. For $i = 1, j = 5$ the formula is as follows:

$$\begin{aligned} a_{15} &= a_1 a_2 a_3 a_4 a_5 \\ &\quad - a_1 a_2 a_3 - a_1 a_2 a_5 - a_1 a_4 a_5 - a_3 a_4 a_5 \\ &\quad + a_1 + a_3 + a_5. \end{aligned}$$

For $i = 1, j = 6$ the formula is as follows:

$$\begin{aligned} a_{16} &= a_1 a_2 a_3 a_4 a_5 a_6 \\ &\quad - a_1 a_2 a_3 a_4 - a_1 a_2 a_3 a_6 - a_1 a_2 a_5 a_6 - a_1 a_4 a_5 a_6 - a_3 a_4 a_5 a_6 \\ &\quad + a_1 a_2 + a_1 a_4 + a_1 a_6 + a_3 a_4 + a_3 a_6 + a_5 a_6 \\ &\quad - 1. \end{aligned}$$

Corollary 4.10. *Consider an infinite periodic frieze with quiddity sequence $q = (a_1, \dots, a_n)$. The growth coefficient of this frieze is given by*

$$s_q = \left(\sum_{\substack{I \subseteq \{1, \dots, n\} \\ \text{cyclical} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k \right) + \delta_n.$$

Here ℓ_I is the number of pairs that were excluded from $\{1, \dots, n\}$ to create I ; in other words $\ell_I = \frac{n - |I|}{2}$. Furthermore,

$$\delta_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is divisible by 4,} \\ -1 & \text{if } n \text{ otherwise.} \end{cases}$$

Proof. We know that the growth coefficient is obtained by taking an entry in the n th row of the frieze, and subtracting the entry directly above it, in row $n - 2$. The growth coefficient is independent of the choice of entry, so we may write $s_q = a_{1,n} - a_{2,n-1}$.

Consider the pair-excluding subsets of $\{2, \dots, n - 1\}$. These are precisely the subsets of $\{1, \dots, n\}$ which are cyclic pair-excluding, but not pair-excluding. Hence we get

$$\begin{aligned} s_q &= a_{1,n} - a_{2,n-1} \\ &= \left(\sum_{\substack{I \subseteq \{1, \dots, n\} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k \right) - \left(\sum_{\substack{I \subseteq \{2, \dots, n-1\} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k \right) \\ &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \text{cyclical} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k + \delta_n. \end{aligned}$$

For n even, we have a term ± 1 in the expression for $a_{2,n-1}$ which is recovered by the inclusion of δ_n in the formula; this term does not occur for odd n . \square

Example 4.11. Consider the infinite periodic frieze given by the quiddity sequence $q = (2, 3, 4, 2, 4)$ as illustrated in Fig. 15. By Corollary 4.10, we calculate the growth coefficient to be

$$\begin{aligned} s_q &= 2 \cdot 3 \cdot 4 \cdot 2 \cdot 4 \\ &\quad - (2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 4 - 4 \cdot 2 \cdot 4 - 2 \cdot 3 \cdot 4) \\ &\quad + (2 + 3 + 4 + 2 + 4) \\ &= 87, \end{aligned}$$

which is also the growth coefficient we read off the frieze in Fig. 15.

K. Baur et al.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		3	4	2	4	2	3	4	2	3	4	2	4	2
	5	11	7	7	7	5	11	7	7	5	11	7	7	5
17	18	19	24	12	17	18	19	24	12	17	18	19	24	12
	62	31	65	41	29	62	31	65	41	29	62	31	65	41
104	105	106	111	99	104	105	106	111	99	104	105	106	111	99
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 15. The frieze with quiddity sequence (2, 3, 4, 2, 4).

4.3. Cluster category for an infinite periodic frieze

We consider a cluster category $\mathcal{C} = D^b(\mathbb{C}Q)/\tau^{-1}[1]$ of type \tilde{A} associated to a non-oriented cyclic quiver Q with $n + m$ vertices, where m arrows are of the form $i \rightarrow i + 1$ and n arrows of the form $i \rightarrow i - 1$, as defined in [9]. The Auslander–Reiten quiver (AR quiver) of this cluster category has two tubes, one of rank n and one of rank m , see [7] for details. We use the specialized CC map [10] (see [19] for the general setting) with respect to the algebra $\mathbb{C}Q$ to associate infinite periodic friezes to the tubes of the AR quiver of \mathcal{C} . In this case, the CC map sends the shifted projectives to 1. All other indecomposables can be viewed as $\mathbb{C}Q$ -modules. The CC map sends such a module M to $\sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}M)$ where the sum is over submodules of M with dimension vector \underline{e} and χ is the Euler–Poincaré characteristic of the complex Grassmannian. For rigid modules in the tube, this is the number of submodules. For non-rigid modules, i.e. modules in a tube of rank n , which are at least n steps away from the mouth of the tube, the Euler–Poincaré characteristic can be computed by taking into account simples which occur with higher multiplicities. Indeed, for any module M in a tube there is a representation of M by an arc, see e.g. [1, 5, 21]. In the same setting, the authors in [18] construct a planar graph to each arc, called *snake graph*, and use cardinalities of certain sets in this graph, namely *perfect matchings*, to compute the Euler–Poincaré characteristic $\chi(\text{Gr}_{\underline{e}}M)$ of a fixed dimension vector \underline{e} , see [18, Theorem 13.1]. By establishing a bijection between the lattice structure of perfect matchings of a snake graph and the *extended* submodule structure of M (this corresponds to the submodule lattice of M' where M' is obtained by considering the simple composition factors of M with their multiplicities), [11, Corollary 3.10] establishes a formula for $\chi(\text{Gr}_{\underline{e}}M)$ in terms of submodules of M .

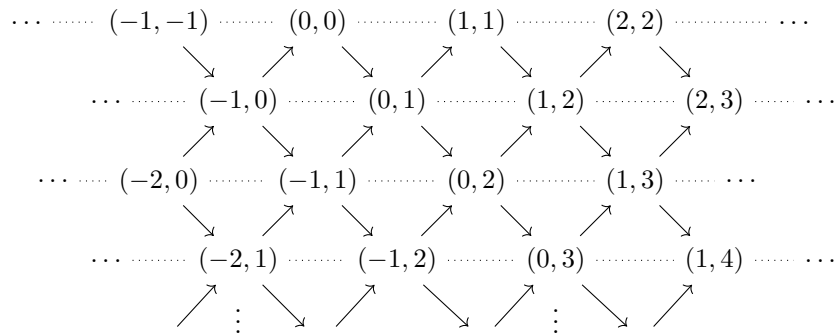
We denote the specialized CC map by s . Thus if M is an indecomposable object of \mathcal{C} which is not a shifted projective and which is at most $n - 2$ steps away from the mouth of the tube (where the mouth itself is considered to be 0 steps away), then $s(M)$ is simply the number of submodules of M and if M is a shifted projective, $s(M) = 1$. If M is at least $n - 1$ steps away from the mouth of a tube of rank n , then s is computed by counting the submodules of M' where M' is obtained by

relabeling the simples that occur with higher multiplicities in M .

In type \tilde{A} almost split sequences have either one or two middle terms. After applying the specialized CC map to the tubes of the AR quiver, the entries satisfy the determinant rule to form two infinite periodic friezes, one for each tube (similar as in [10, Sec. 5]). Namely, if $\tau M \rightarrow B \rightarrow M$ is an Auslander–Reiten triangle in the cluster category \mathcal{C} , then applying the CC map by [2, Proposition 2.2], we obtain $s(\tau M)s(M) - s(B) = 1$ where s is the specialized CC map.

4.4. Module-theoretic interpretation

In this section, we associate a cluster tilting object in a cluster category of type \tilde{A} to an infinite periodic frieze. Let q be the quiddity sequence of a skeletal infinite periodic frieze \mathcal{F} . This gives rise to a skeletal triangulation $\mathcal{T} = \mathcal{T}(q)$ of an annulus with n marked points on one boundary component (say B_1) of the annulus, by Corollary 3.21. Let m be the number of marked points on B_2 (see Sec. 2.5). Then \mathcal{T} is a triangulation of the annulus $C_{n,m}$. Let $\mathcal{C} = \mathcal{C}_{n,m}$ be the category from Sec. 4.3. The indecomposable objects of this category except the ones that come from the homogeneous tubes are known to be in bijection with arcs in $C_{n,m}$, and triangulations of $C_{n,m}$ correspond to cluster tilting objects for \mathcal{C} , as described in [5, 21]. We claim that one of the infinite periodic friezes coming from the tubes is \mathcal{F} . Note that the indecomposable objects in the tubes correspond to the peripheral arcs on the boundaries B_1 and B_2 , respectively. We restrict to the tube arising from B_1 . We use the following notation to label indecomposables in that tube. Note that we follow the frieze notation here and draw the vertices at the mouth of the tube on the top line. Going down in the quiver thus corresponds to injections.



We write M_{ij} for the indecomposable module corresponding to the vertex (i, j) with indices reduced modulo n (Fig. 16). The dotted lines indicate the Auslander–Reiten translation τ in the category, which sends the module M_{ij} to the module $M_{i-1, j-1}$. Let $M_i = M_{ii}$. Note that the indexing of vertices/modules follows the indexing we use for friezes. Under the correspondence between indecomposables and

K. Baur et al.

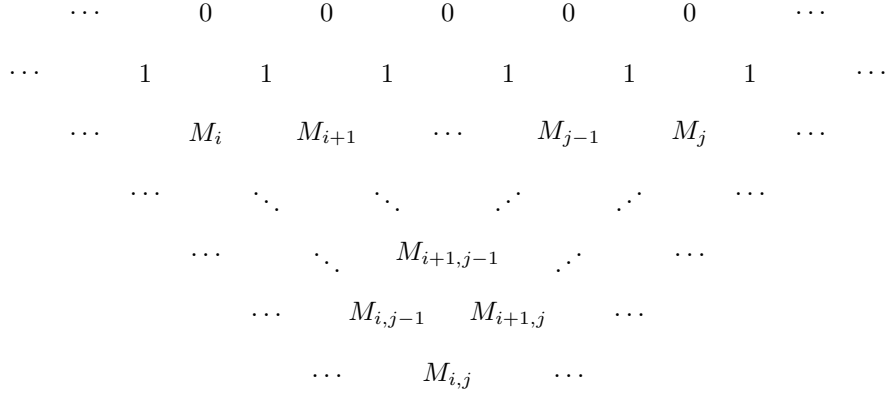


Fig. 16. A tube of the AR quiver with modules $M = M_{i,j}$ and $\widetilde{M} = M_{i+1,j-1}$, and modules M_i, \dots, M_j at its mouth, where $j = i + t - 1$.

arcs in $C_{n,m}$, the indecomposable objects in this tube correspond to the peripheral arcs based at B_1 . In particular, the M_i 's correspond to the peripheral arcs of B_1 joining the marked points i and $i + 2$.

In order to link the indecomposable objects with the friezes, we use the specialized CC map s defined above. We can then use [16, Sec. 0.1] to obtain part (1) of the following.

Proposition 4.12. *Following the notation above and with $M := M_{i,i+t-1}$, for $t \geq 1$,*

- (1) $s(M_i) = a_i$.
- (2) $s(M) = a_{i,i+t-1}$.

Proof. (2) follows from (1), using the fact that s is multiplicative on short exact sequences, see for example [16, Sec. 0.1]. \square

The modules M_i form the *mouth* of the category. All modules in a given τ -orbit are of the form $M_{i,i+n-1}$ for some $n > 0$. We say that module $M_{i,i+n-1}$ is at level n .

Let M be an indecomposable in \mathcal{C} at level n . The *wing* of M , denoted by $\mathcal{W}(M)$, consists of the indecomposable modules which are submodules, quotients or subquotients of M . In the AR quiver, these are the indecomposables positioned in the triangle in Γ whose apex is M . This means that if $M = M_{i,i+n-1}$, then $\mathcal{W}(M)$ is given by $\{M_{u,v} \mid i \leq u \leq v \leq i + n - 1\}$.

Then M_i, \dots, M_{i+n-1} are the modules at the mouth of the wing $\mathcal{W}(M)$. Note that M is obtained through iterated extensions from M_i, \dots, M_{i+n-1} . Now assume $n \geq 3$ and let \widetilde{M} be $M_{i+1,i+n-2}$, so that $\mathcal{W}(\widetilde{M}) = \{M_{u,v} \mid i+1 \leq u \leq v \leq i+n-2\}$. Furthermore, define $N := M_i \oplus M_{i+1} \oplus \dots \oplus M_{i+n-1}$.

In the following, we will quotient out direct sums of cyclic consecutive pairs of

indecomposables from N : for $i \leq j \leq i + n - 1$, let

$$N_j := N/(M_j \oplus M_{j+1}).$$

In particular, $N_{i+n-1} = N/(M_{i+n-1} \oplus M_i)$ since we reduce modulo n . Similarly, for $i \leq j_1 < j_2 - 1 \leq i + n - 2$,

$$N_{j_1, j_2} := N/(M_{j_1} \oplus M_{j_1+1} \oplus M_{j_2} \oplus M_{j_2+1}),$$

and for j_1, \dots, j_k with $k \geq 2$, $j_1 < j_2 - 1 < \dots < j_k - (k - 1) \leq i + n - k$,

$$N_{j_1, j_2, \dots, j_k} := N/(M_{j_1} \oplus M_{j_1+1} \oplus M_{j_2} \oplus M_{j_2+1} \oplus \dots \oplus M_{j_k} \oplus M_{j_k+1}).$$

In particular, for $k = 2$ and $j_2 = i + n - 1$, this gives $N_{j_1, i+n-1} = N/(M_i \oplus M_{j_1} \oplus M_{j_1+1} \oplus M_{i+n-1})$. Analogously, modules N_{i_1, \dots, i_k} are defined, for $k \leq n/2$.

Let $\{r_1, \dots, r_n\} = \{1, 2, \dots, n\}$ (not necessarily ordered). We will use the following two properties in the proof of the theorem below:

$$s(N/(M_{r_1} \oplus \dots \oplus M_{r_m})) = s(M_{r_{m+1}} \oplus \dots \oplus M_{r_n}), \quad (4.2)$$

$$s(M_{r_{m+1}} \oplus \dots \oplus M_{r_t}) = s(M_{r_{m+1}}) \cdots s(M_{r_t}). \quad (4.3)$$

Theorem 4.13. *Let M be an indecomposable of \mathcal{C} at level $t \geq 3$, let \widetilde{M} and M_i, \dots, M_{i+t-1} be as above.*

If $t = 2k + 1$,

$$s(M) - s(\widetilde{M}) = s(N) - \sum_{j=i}^{i+t-1} s(N_j) + \sum_{\substack{j_1, j_2=i \\ j_1 < j_2-1}}^{i+t-1} s(N_{j_1, j_2}) + \dots \pm (-1)^k \sum_{j=i}^{i+t-1} s(M_j).$$

If $t = 2k$,

$$s(M) - s(\widetilde{M}) = s(N) - \sum_{j=i}^{i+t-1} s(N_j) + \sum_{\substack{j_1, j_2=i \\ j_1 < j_2-1}}^{i+t-1} s(N_{j_1, j_2}) + \dots \pm (-1)^k \cdot 2.$$

Note that the $(k + 1)$ th term on the right hand of the expressions is

$$\sum_{\substack{j_1, \dots, j_k=i \\ j_1 < j_2-1 < \dots < j_k-(k-1)}}^{i+t-1} s(N_{j_1, j_2, \dots, j_k}).$$

Proof. We use the formula from Theorem 4.8 for $M = M_{i, i+t-1}$ and $\widetilde{M} = M_{i+1, i+t-2}$ with Proposition 4.12(2)

$$\begin{aligned} s(M) - s(\widetilde{M}) &= a_{i, i+t-1} - a_{i+1, i+t-2} \\ &= \sum_{\substack{I \subseteq \{i, \dots, i+t-1\} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k - \sum_{\substack{I \subseteq \{i+1, \dots, i+t-2\} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k. \end{aligned}$$

K. Baur et al.

Consider the second term on the right-hand side: we can view pair-excluding subsets of $\{i+1, \dots, i+t-2\}$ as the cyclically pair-excluding subsets of $\{i, \dots, i+t-1\}$ which are not pair-excluding subsets of $\{i, \dots, i+t-1\}$. Now observe that for any (cyclically) pair-excluding subset I , the sign of $(-1)^{\ell_I}$ when I is viewed as a subset of $\{i, \dots, i+t-1\}$ is the opposite of the sign of $(-1)^{\ell_I}$ when I is viewed as a subset of $\{i+1, \dots, i+t-2\}$. Given that the second term on the right is subtracted from the first, these signs cancel each other. We can thus add the cyclic pair-excluding subsets from the second term to the first term and get the following:

$$s(M) - s(\widetilde{M}) = \sum_{\substack{I \subseteq \{i, \dots, i+t-1\} \\ \text{cyclic} \\ \text{pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k.$$

The claim then follows using Eqs. (4.2) and (4.3) since for any cyclic pair-excluding subset I of $\{i, \dots, i+t-1\}$,

$$\prod_{k \in I} a_k = \prod_{i \in I} s(M_i) = s\left(\bigoplus_{i \in I} M_i\right) = s\left(N / \bigoplus_{i \notin I} M_i\right),$$

where the last term is equal to $s(N_{j_1, \dots, j_r})$ for $\{i, i+1, \dots, i+t-1\} \setminus I = \{j_1, j_1+1, j_2, j_2+1, \dots, j_r, j_r+1\}$. \square

We can use the theorem to give the growth coefficient of a quiddity sequence a module theoretic interpretation. For a given skeletal quiddity sequence (a_1, \dots, a_n) of an infinite periodic frieze, let \mathcal{T} be the associated triangulation of $C_{n,m}$ as at the beginning of this section. Then take the cluster category $\mathcal{C}_{n,m}$ as in Sec. 4.3 and let s be the specialized CC map.

Corollary 4.14. *Let $q = (a_1, \dots, a_n)$ be a quiddity sequence, let \mathcal{B} be the associated rank n tube as above. Let $M = M_{i, i+n-1}$ be any indecomposable in \mathcal{B} at level n and $\widetilde{M} = M_{i+1, i+n-2}$. Then we have*

$$s(M) - s(\widetilde{M}) = s_q.$$

Acknowledgments

The authors would like to thank Chelsea Walton, Georgia Benkart, Eleonore Faber, Ellen Kirkman and other organizers for organizing WINART2 at the University of Leeds, where this project started.

This work was supported by the FWF (grant numbers P30549-N26, DK1230); a Royal Society Wolfson Fellowship; the Engineering and Physical Sciences Research Council (grant number EP/P016014/1); the Alexander von Humboldt Fellowship; the Institute for Advanced Study (Princeton) and the Max Planck Institute for Mathematics (Bonn).

References

- [1] I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P.-G. Plamondon, Gentle algebras arising from surface triangulations, *Algebra Number Theory* **4**(2) (2010) 201–229.
- [2] I. Assem and G. Dupont, Friezes and a construction of the Euclidean cluster variables, *J. Pure Appl. Algebra* **215**(10) (2011) 2322–2340.
- [3] K. Baur, A. B. Buan and B. R. Marsh, Torsion pairs and rigid objects in tubes, *Algebr. Represent. Theory* **17**(2) (2014) 565–591.
- [4] K. Baur, K. Fellner, M. J. Parsons and M. Tschabold, Growth behaviour of periodic tame friezes, *Rev. Mat. Iberoamericana* **35**(2) (2019) 575–606.
- [5] K. Baur and B. R. Marsh, A geometric model of tube categories, *J. Algebra* **362** (2012) 178–191.
- [6] K. Baur, M. J. Parsons and M. Tschabold, Infinite friezes, *European J. Combin.* **54** (2016) 220–237.
- [7] K. Baur and H. A. Torkildsen, A geometric interpretation of categories of type \tilde{A} and of morphisms in the infinite radical, *Algebr. Represent. Theory* **23** (2019) 657–692.
- [8] F. Bergeron and C. Reutenauer, SL_k -tilings of the plane, *Illinois J. Math.* **54**(1) (2010) 263–300.
- [9] A. B. Buan, B. R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204**(2) (2006) 572–618.
- [10] P. Caldero and F. Chapoton, Cluster algebras as Hall algebras of quiver representations, *Comment. Math. Helv.* **81**(3) (2006) 595–616.
- [11] İ. Çanakçı and S. Schroll, Lattice bijections for string modules, snake graphs and the weak Bruhat order, preprint (2018), arXiv:1811.06064.
- [12] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, *Math. Gaz.* **57**(400) (1973) 87–94.
- [13] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, *Math. Gaz.* **57**(401) (1973) 175–183.
- [14] H. S. M. Coxeter, Frieze patterns, *Acta Arith.* **18** (1971) 297–310.
- [15] S. Fomin, M. Shapiro and D. Thurston, Cluster algebras and triangulated surfaces. I. Cluster complexes, *Acta Math.* **201**(1) (2008) 83–146.
- [16] T. Holm and P. Jørgensen, Generalized friezes and a modified Caldero–Chapoton map depending on a rigid object, *Nagoya Math. J.* **218** (2015) 101–124.
- [17] T. Muir, *A Treatise on The Theory of Determinants* (Dover Publications, New York, 1960), vii+766 pp.
- [18] G. Musiker, R. Schiffler and L. Williams, Positivity for cluster algebras from surfaces, *Adv. Math.* **227**(6) (2011) 2241–2308.
- [19] Y. Palu, Cluster characters for 2-Calabi–Yau triangulated categories, *Ann. Inst. Fourier (Grenoble)* **58**(6) (2008) 2221–2448.
- [20] M. Tschabold, Arithmetic infinite friezes from punctured discs, preprint (2015), arXiv:1503.04352.
- [21] M. Warkentin, Fadenmoduln über \tilde{A}_n und Cluster-Kombinatorik, Diploma Thesis, University of Bonn (2008), <http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-94793>.