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Mathematical structuralism and bundle theory

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Abstract

According to the realist rendering of mathematical structuralism, mathematical structures are ontologically prior to individual mathematical objects such as numbers and sets. Mathematical objects are merely positions in structures: their nature entirely consists in having the properties arising from the structure to which they belong. In this paper, I offer a bundle-theoretic account of this structuralist conception of mathematical objects: what we normally describe as an individual mathematical object is the mereological bundle of its structural properties. An emerging picture is a version of mereological essentialism: the structural properties of a mathematical object, as a bundle, are the mereological parts of the bundle, which are possessed by it essentially.

KEYWORDS
bundle theory, individuation, mathematical structuralism, mereological essentialism, ontological dependence

1 | INTRODUCTION

Mathematical structuralism is the thesis that mathematics is about *structures*. In a sense, this claim is incontrovertible: working mathematicians are just likely to say that this is what mathematics is about: it is about various structures of numbers, functions, sets, spaces, etc. But structuralists are not just recording their agreement with mathematicians.

To appreciate the point, let us, following Shapiro (1997), make a distinction between *systems* and *abstract structures* (structure, hereafter). A system is a set or a plurality of objects of some kind, with some relations and functions defined on it. For instance, the members of a department of philosophy, in order of increasing

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age, is a system. The *elements* of a system are typically objects identifiable independently of the system to which they belong. Two systems may have the same *form*, when there is a one-to-one correspondence between the elements of one and those of the other, under which the elements of the first system stand in certain relations just in case the corresponding elements of the second system stand in the corresponding relations. In this case, the systems are said to be *isomorphic*. A structure is what isomorphic systems have in common. For example, the system of the odd and even natural numbers are isomorphic, so both are instances of a common structure.

The realist versions of mathematical structuralism, which is the focus of this paper, take our talk of structures at face-value. Any *categorical* mathematical theory—that is, a theory all of whose systems are pairwise isomorphic—is about the structure that its isomorphic systems have in common. For example, second-order arithmetic is about the structure that all ω-sequences have in common. Individual natural numbers are, on this picture, *positions* in structures: the natural number 1 is the second position in the structure of the natural numbers. The crucial point is that positions of structures, as opposed to *elements* of systems, are not identifiable independently of their home structure. They are, rather, purely structural entities: their nature entirely consists in their bearing certain relations to one another in the structure to which they belong. The central structuralist thesis is that there is *no more* to mathematical objects, as positions in structures, than their structural properties. For example, there is no more to the natural number 1 than being the successor of 0, the predecessor of 2, and so on.[1]

My aim in this paper is to sketch a novel structuralist account of the nature of positions in structures as *bundles* of their structural properties. The structuralist thesis that there is no more to mathematical objects than their structural properties will thus be rendered as the thesis that what we normally describe as individual or particular mathematical objects are nothing than the bundle of their structural properties. This contrasts with Shapiro's view in which positions of mathematical structures are 'simple' entities: they are 'atoms' with no proper parts, so they are not bundles:

> Places [or positions] within structures are not, as MacBride [2006] suggests, bundles of universals. Places are *components* of universals. Each *ante rem* structure consists of some places and some relations. [...] The 'places' are indeed simple, or atomic, in the sense that they do not themselves have places or other components. (Shapiro, 2008, pp. 302–303; 305)

Bundle theories typically identify ordinary material objects with bundles of their qualitative properties; or to put it more neutrally, they describe *fundamental* facts about material objects in terms of their qualitative properties. Similarly, the mathematical structuralist who employs the resources of bundle theory holds that what we normally describe as individual mathematical objects are just their structural properties that are somehow bundled together.[2]

The structure of this paper is as follows. In Section 2, I discuss some of the natural attractions of bundle theory for mathematical structuralism. I explore how the bundle theorist can accommodate some of the key metaphysical theses of structuralism. Section 3 discusses MacBride's (2005, 2006) objection against structuralism, which is reminiscent of Black's (1952) famous argument against bundle theory. I argue that MacBride's objection rests on a conception of bundles to which the structuralist is not necessarily committed. On this conception, bundles are *extensional* entities: bundles with the same properties as their 'parts' (in the sense to be discussed below) are numerically identical.

[1] See Korbmacher and Schiemer (2018) and Schiemer and Wigglesworth (2019) for various characterizations of the notion of structural property.

[2] For various modern versions of bundle theory, see Barker and Jago (2018), Benocci (2018), Jago (2021), Keinanen and Tahko (2019), McDaniel (2001), O'Leary-Hawthorne (1995), Paul (2012, 2017), Rodriguez-Pereyra (2004), and Simons (1994).

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As an alternative, I propose, in Section 4, a *non-extensional* bundle theory, according to which bundles can be numerically distinct even if they are constituted by the same properties. I also argue that the structuralist has independent motivations for adopting this conception of bundles.[3]

In Section 5, I offer a bundle-theoretic account of mathematical structuralism, according to which the structural properties of what we normally describe as a mathematical object are mereological parts of the bundle, and are essentially so. I thereby defend a version of *mereological essentialism* for mathematical structuralism. The natural number 2 is the bundle of its structural properties; it will not, figuratively put, 'remain' what it is if it is not the predecessor of 3, the successor of 1, etc. Mereological essentialism sits well with the bundle-theoretic characterization of mathematical objects as positions in structures.

Section 6 deals with the problem of *individuation*: how are facts about the numerical identity and distinctness of bundles to be grounded? No appeal can be made to their parts given that, as the non-extensional bundle theory states, there could be numerically distinct bundles having the same parts. I defend a *primitivist* account of individuation: identity and distinctness facts are fundamental or primitive, which are not to be grounded in further facts. This view has been defended in the context of mathematical and scientific structuralism—for example, by Leitgeb and Ladyman (2008) and Shapiro (2008). I locate it in the context of bundle theory, and defend it against some recent objections.

2 | BUNDLE THEORY FOR MATHEMATICAL STRUCTURALISM

What are the attractions of bundle theory for the structuralist conception of mathematical objects as positions in structures? In this section, I provide the mathematical structuralist with some motivations from bundle theory.

The first motivation comes from the *ontological priority* of structures over objects. According to the structuralist, the identity of mathematical objects depends on the structure to which they belong. For example, the natural number 2 has its identity merely in virtue of its being a position in the structure of the natural numbers; that is what 2 *is*. As Shapiro writes:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (Shapiro, 2000, p. 258)[4]

According to bundle theory, an individual or a particular is merely a bundle of its qualitative properties. What is ontologically primary is the network of properties possessed by what we normally describe as individuals; the individuals themselves are secondary. This view about the ontological priority of structures over individuals naturally suggests a bundle-theoretic reading. The number 2 is nothing other than certain 'compresence' of structural properties: where we would ordinarily say that there is an object that possesses both *being the predecessor of 3* and *being the successor of 1*, the bundle theorist structuralist will say that these properties are compresent.

According to the second motivation, if mathematical objects are nothing but positions in structures, then the structuralist does not need anything like *substance theory*, which takes individuals to be fundamental or irreducible substances of different sorts. In addition, the thesis that mathematical objects are nothing but positions in structures dispenses with the ontologies of *substratum theory* in which there is a non-qualitative core—a bare particular or a

[3]For various forms of non-extensional bundle theories, see Rodriguez-Pereyra (2004), Paul (2017), and Benocci (2018).

[4]See Linnebo (2008) for an elaboration and partial defense of the 'ontological dependence' of positions on structures.

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haecceity—lurking behind the façade of qualitative properties. As a consequence, there will be no need to explain facts about their numerical identity and distinctness in terms of such metaphysically suspicious posits. Bundle theory provides the structuralist with resources to give an account of the nature and identity of mathematical objects that does not depend on substances or substrata.

The third motivation concerns the structuralist's rejection of *merely permutational differences*. At the heart of structuralism lies the thesis that isomorphic structures are numerically identical: '[W]e stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one' (Shapiro, 1997, p. 93). Some other variants of mathematical structuralism also accommodate this principle. For example, Leitgeb (2021a, 2021b) identifies structures with unlabelled graphs whose criterion of identity is specified in terms of isomorphism: for any unlabelled graphs G_1 and G_2, G_1 is identical to G_2 if and only if G_1 and G_2 are isomorphic.[5] This is what fundamentally distinguishes structures from systems.

As we mentioned in Section 1, there are isomorphic and yet numerically distinct *systems*; whereas isomorphism between structures is sufficient for their numerical identity. If you permute the positions of the natural number structure, you do not get a new structure. But that the mere permutation of positions does not correspond to genuinely distinct structures is (partly) because of what positions *are*: there is no more to them than their structural properties. Bundle theory is a way to substantiate the status of mathematical objects as such purely structural entities.

The fourth motivation: the central plank in the structuralist conception of mathematical objects is that there is no more to mathematical objects than their structural properties. So, it is natural to think of structural properties as *essential properties*: for an entity to be the natural number 2 just is for it to be the successor of 1, the predecessor of 3, and so on. Bundle theory provides the structuralist with a helpful metaphysical handle here. A bundle is what it is in virtue of the properties it has as its parts: remove a property from a bundle, and you will have a different bundle. There is, thus, a sense in which some version of *mereological essentialism* fits into the characterization of mathematical objects as positions in structures. (See Section 5.) This suggests a fundamental difference between mathematical objects and ordinary concrete objects about which mereological essentialism is hard to digest.

3 | EITHER BAD NEWS OR WORSE NEWS

In a number of writings, MacBride (2005, 2006) raises a dilemma against mathematical structuralism. As he formulates it, structuralism is either old news or bad news. The news is bad if the structuralist is committed to the claim that structurally indiscernible objects, such as geometrical points in the Euclidean space or the imaginary numbers *i* and −*i* in complex analysis, are numerically identical. (More generally, structures admitting of non-trivial automorphisms—i.e. automorphisms other than the identity mapping—have structurally indiscernible objects, where *x* is structurally indiscernible from *y* if and only if *x* and *y* possess the same structural properties.) And the news is old if the structuralist tries to remedy this difficulty by appealing to a non-structural core to ground the numerical distinctness of otherwise indiscernible objects.

It is not clear, though, what is really *old* about the old news. Traditional mathematical platonism is not committed to anything like bare particulars or substrata, either. For example, according to the Frege-Russell non-structuralist conception of platonism, mathematical objects do indeed enjoy a rich nature that goes beyond their structural properties. The natural numbers, on this conception, are not just what satisfy the axioms of the Dedekind-Peano Arithmetic; they are, rather, fundamentally cardinal numbers in terms of which we count various

[5]See also Leitgeb and Ladyman (2008) and Awodey (2014).

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collections of objects. Nowhere in this picture, however, does lie a thesis concerning substrata or anything like them. The emphasis on old platonism seems to be misleading.

A more apt way to frame the dilemma is to say that structuralism is either bad news or worse news! It is bad news if, as MacBride rightly points out, structuralism identifies numerically distinct mathematical objects. It is worse news if the structuralist posits non-qualitative cores, which fly in the face of the central claim of structuralism to the effect that there is no more to mathematical objects than their structural properties.[6]

All the same, what matters for our purposes is MacBride's argument for the bad-news horn of his dilemma. The argument is of this general shape: (1) There is no more to mathematical objects than their structural properties. (2) If there is no more to mathematical objects than their structural properties, then structurally indiscernible mathematical objects are numerically identical. Therefore, (3) structurally indiscernible mathematical objects are numerically identical.

As we discussed above, (1) is the central thesis of structuralism. The consequent of (2) rests on the principle of the identity of indiscernibles (PII)—the thesis that entities that are indiscernible in a specified way are identical. Thus, the main idea behind (2) is that if mathematical objects *x* and *y* are structurally indiscernible, then any structural property of *x* is a structural property of *y*, and vice versa. Hence, by PII, *x* and *y* are identical. However, (3) is plainly false, since there are numerically distinct and yet structurally indiscernible mathematical objects.

From the perspective of the mathematical bundle theorist, the problem with MacBride's challenge is not primarily in its commitment to PII; it, rather, lies in its presupposition of a questionable variation of bundle theory, according to which bundles are extensional entities: bundles with the same properties are numerically identical—in the same way in which sets with the same elements are numerically identical. This conception of bundles, however, is not forced on the bundle theorist. For she could invoke a version of bundle theory which would allow for numerically distinct bundles having the same properties as their parts. This motivates a *non-extensional bundle theory*, which is the subject of the next section.[7]

4 | THE NON-EXTENSIONAL BUNDLE THEORY

What is the relation between a bundle and its properties? The traditional variations of bundle theory have left this relation somewhat unaccounted for. A promising account has been offered, in various forms, by Simons (1994), McDaniel (2001), and Paul (2012, 2017): a bundle is the *mereological sum* of its properties, where *x* is a mereological sum of the *y*s if and only if *x* has all the *y*s as parts, and no parts distinct from the *y*s. Taking bundles to be mereological sums of properties gives us *mereological bundle theory*. This construal of bundle theory may be inessential for our purposes. Its advantage mainly lies in providing us with an informative account of the relation between a bundle and its properties. The central thesis of the mereological bundle theory is the following principle:

> *The principle of decomposition*: For any bundle (i.e., mereological sum) *x*, there are properties $F_1,\ldots,F_n$ such that *x* is the mereological sum of $F_1,\ldots,F_n$.

The principle of decomposition merely states that every bundle is a mereological sum of certain properties. Thus, as Benocci (2018) observes, it does not, as such, rule out the possibility that there are numerically distinct bundles composed of the same properties. This is the main difference between extensional and non-extensional bundle theories.

[6]But see Menzel (2018) for an attempt to reconcile structuralism with haecceitism.
[7]The terminology of extensional and non-extensional bundle theory is due to Benocci (2018).

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According to the former, bundles with the same properties are numerically identical. The latter, however, rejects this, thus allowing for the possibility of numerically distinct bundles having the same properties as their parts.[8]

The main difficulty with the extensional bundle theory is that it cannot rule out certain situations, the most famous instance of which is Black's (1952) symmetrical universe, in which there are only two spheres, two miles apart from each other, having the same diameter, temperature, color, shape, size, etc. They have the same intrinsic and extrinsic properties, and stand in the same relations to each other. Given that the spheres of Black's world instantiate exactly the same properties, the extensional bundle theorist cannot account for the possibility of such situations. As we saw above, this line of argument against bundle theory is central to MacBride's dilemma against structuralism: if mathematical objects are just bundles of their structural properties, then bundles with same structural properties must be identical. The background assumption of MacBride's argument is that bundles are extensional entities.

As Benocci (2018) observes, the bundle theorist may adopt a non-extensional version of bundle theory with the following core principle:

> *The principle of non-extensionality*: It is possible that there are bundles *x* and *y* and properties $F_1,\ldots,F_n$ such that *x* is the mereological sum of $F_1,\ldots,F_n$; *y* is the mereological sum of $F_1,\ldots,F_n$; and *x* is numerically distinct from *y*.

The principle of non-extensionality thus rules out the thesis that necessarily, for any bundles *x* and *y*, if *x* and *y* have exactly the same properties as their parts, then *x* is numerically identical to *y*. Precisely for this reason, the principle of non-extensionality is not compatible with the standard mereological principle of extensionality, which does not allow for numerically distinct sums with the same proper parts. But would that be a problem for the non-extensional bundle theorist?

The standard mereological composition can be rejected for independent reasons. For example, as Wiggins (1968) argues, the composition of material objects does not obey the laws of standard mereology: the statue and the clay are distinct, since they have different modal properties and persistence-conditions; nevertheless, they have the same proper parts. It is thus possible that there are numerically distinct material objects which are constituted by exactly the same proper parts.

The mathematical structuralist who employs the resources of the non-extensional bundle theory could say that the positions of a structure are entities that allow for the possibility of distinct sums with the same make-up. In fact, this is precisely what we see in the case of structures with structurally indiscernible positions. The positions of such structures are bundles of the same structural properties, and yet they are numerically distinct. The structuralist has thus motivations and resources to retain the principle of non-extensionality.

There are further structuralist motivations. According to Shapiro (2008, §4), mathematical structures are *structural universals*. They are complex universals 'constituted' by simpler universals. For example, *being a butane molecule* is a structural universal that has *being a hydrogen atom* as one of its sub-universals. Similarly, *being the natural number structure* is a structural universal that has positions as its sub-universals. Lewis (1986) famously discusses three ways of understanding the constitution or composition relation between a structural universal and its sub-universals, and rejects them as incoherent or unintelligible. In particular, according to what he calls the 'pictorial' conception, a structural universal is a mereological sum of its sub-universals. Lewis then observes that there are numerically distinct structural universals that are composed by the same universals. For example, *being a butane molecule* and *being an iso-butane molecule* are both composed by four carbon atoms and 10 hydrogen atoms, with different bonding configurations. However, according to classical mereology, there are no numerically distinct entities which are composed by exactly the same proper parts. Thus, *being a butane molecule* and *being an iso-butane molecule* must be identical. This, in Lewis's view (1986, p. 38), is a *reductio* of the mereological conception of structural universals.

[8]For various forms of non-extensional bundle theory, see Rodriguez-Pereyra (2004), Paul (2017), and Benocci (2018).

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All the same, the proponent of structural universals in mathematics does not need to be committed to the extensionalist construal of mereological composition. The mathematical structuralist may adopt a non-classical mereological account of structural universals, and argues that the relation between mathematical structures, as structural universals, and their positions, as sub-universals, do not obey the classical laws of mereology.[9]

The issues about individuation remain: how are facts about the numerical identity and distinctness of bundles to be grounded or explained? We cannot appeal to their properties (parts), given that, as the non-extensional bundle theory has it, there could be bundles having the same parts. We address this issue in Section 6.

5 | THE STRUCTURALIST MEREOLOGICAL ESSENTIALISM

In this section, I explore the implications of the above picture of bundle theory for mathematical structuralism. I defend the thesis that if the structural properties of what we normally describe as an individual mathematical object are mereological parts of the bundle, then they are essentially so.

The bundle-theoretic account of mathematical objects consists of two central theses: that the bundles of what we normally describe as individual mathematical objects merely include their structural properties; and that their structural and essential properties are co-extensional. Thus, we have:

> *Essentialist Structuralism*: Given that the property *F* is part of the bundle *x*, *F* is structural if and only if *F* is essential.

The left-to-right direction of the biconditional does not seem to be contentious. The thesis that all the structural properties of mathematical objects are essential to them is not a *distinctively* structuralist thesis. There is no reason why non-structuralists such as Fregeans and the neo-Fregeans should object to this claim—given, of course, that they avail themselves of the essence-talk in the first place.

The distinctively structuralist thesis is the other direction of Essentialist Structuralism: all the essential properties of mathematical objects are structural. This thesis can be captured within a real-definitional account of essence: 'For an entity to be ψ just is for it to be φ'. For example, for an entity to be 2 just is for it to be the successor of 1. On this view, what is essential to an entity pertains to its essence, identity, or nature. It is essential to *x*'s being the natural number 2 that *x* be the successor of 1; it lies in the nature of 2 to be the successor of 1; or for an entity to be 2 just is for it to be the successor of 1.

An advantage of this account is that it distinguishes essential properties from merely *necessary* ones. Fine's (1994) famous example is Socrates's property of *belonging to Socrates's singleton*, which is a necessary property of Socrates, but not an essential one. On the real-definitional account of essence, an object has a property essentially just in case it is true in virtue of its identity that it has that property. In the structuralist's view, there is no more to a natural number than the structural properties it possesses, and if this is what a number *is*, then properties such as *being abstract* or *being a necessary existent* do not pose any threat to the central structuralist thesis. Mathematical objects may have such properties necessarily, but not essentially.[10]

Thus, mathematical objects do possess non-structural—so, non-essential—properties. But such non-essential properties are not included in their bundles. For bundles are not supposed to specify the full qualitative profile of the object. They are, rather, supposed to tell us what an object *is*. Thus construed, when we identify a

[9]See Bennett (2013, pp. 101–102) and Hawley (2010, pp. 124–125) for different accounts of structural universals in terms of non-classical mereology.

[10]These alleged counterexamples have been first discussed by Hellman (2001). For an essentialist treatment of these counterexamples, see Assadian (2022).

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mathematical object with the bundle of its properties, those properties tell us something about the nature or the essence of the object.[11]

Since we have taken bundles to be mereological sums, we will have the following version of *mereological essentialism* for mathematical objects:

> *Structuralist Mereological Essentialism*: If the structural property *F* is a mereological part of the bundle *x*, then *F* is an essential property of *x*.

Mathematical objects are mereological sums of their structural properties, and such properties are essential to them. Mereological essentialism sits well with the bundle-theoretic characterization of *mathematical* objects as positions in structures. This suggests a sharp contrast with the bundle theory of *material* objects for which mereological essentialism is hard to believe.

There is another thesis in this vicinity: for any bundle *x* and every property *F*, if *F* is part of *x*, then necessarily, *F* is part of *x*. Let us call it *modal rigidity*. Sets are often taken to be modally rigid: if a given thing is an element of a set, it is necessarily so. Likewise, mere pluralities are modally rigid: if a given thing is one of some given things, then it is necessarily so, given the existence of the relevant things. The claim is *not* that if Irene is one of the translators, she is necessarily a translator. It is, rather, that the plurality of translators is not allowed to vary in its 'members' across various possibilities. By contrast, groups such as teams or committees do not seem to be modally rigid. For instance, the Supreme Court would not change its identity if one of its members resigns.[12]

What is the status of bundles? If bundles are merely pluralities of properties, they are rigid, because pluralities are. If bundles are, as we have assumed above, mereological sums, then their rigidity depends on the rigidity of mereological sums. I leave this question for another occasion.[13]

6 | INDIVIDUATION

In Section 4, we introduced non-extensional bundle theory which allows for the possibility of numerically distinct bundles having the same properties as their parts. But all bundle theories, extensional or otherwise, must account for the problem of the individuation of bundles. What facts ground the numerical distinctness of two bundles if there is nothing but their properties to tell them apart?

In answering this question, bundle theorists divide into two camps. According to the first camp, bundles constituted by the same properties are multi-located objects. For example, O'Leary-Hawthorne (1995) argues that bundles of properties, just like properties themselves, can be in different locations at the same time. The bundle theorist could then say that Black's world contains only one entity, i.e., one bundle of properties located in two different places. (Even if we accept this strategy for concrete objects like Black's spheres, it is far from clear how it could be applied to abstract, mathematical, cases.) The bundle theorists of the second camp introduce different ways of individuation, such as instances of bundles (Rodriguez-Pereyra, 2004) or irreducible but ontologically dependent objects (Benocci, 2018).

What inspires the main theme of this section is Paul's (2017) position, which is different from the above two approaches. She maintains that it is possible for bundles with exactly the same properties to be *just* distinct: they

[11]The essential bundle theory is a version of bundle theory in which the properties included in the *X*-bundle are essential to *X*. See Barker and Jago (2018) and Jago (2021) for a defense of this view in the case of material objects.

[12]See Florio and Linnebo (2021, ch.10) for a systematic defense of the modal rigidity of pluralities.

[13]The conception of bundles as mere pluralities of properties has been presupposed in Keinanen and Tahko (2019). See Uzquiano (2014) for a detailed discussion on the modal rigidity of mereological sums. Let me just mention that the modal rigidity of bundles does not commit one to their extensionality: if an entity is extensional, it is modally rigid; but the other direction does not hold. See Florio and Linnebo (2021, §10.5). So, even if we accept the modal rigidity of bundles, their non-extensional character, sketched in Section 3, remains untouched.

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are *primitively individuated*. In fact, Paul recommends this move also to the traditional bundle theories, which have primitive bundling relations such as compresence. In the rest of this section, I critically examine three objections against the bundle theorist's use of primitive individuation.

The first objection, advanced by Keinanen and Tahko (2019, pp. 848–850), is directed at a bundle theorist like Paul who adopts a *one-category ontology*, according to which the whole world is only built from one ontological category: properties. The objection rests on the thesis that if there ends up being a primitive distinction between two candidates for ontological categories, then there are two distinct ontological categories. In this context, the idea is that if bundles of properties can be numerically distinct even if their constituent properties are the same, then properties and bundles belong to two distinct ontological categories, precisely like the categorial difference between particulars and properties to which the traditional substance theories were committed. Thus, a bundle theory with primitive individuation reintroduces a two-category ontology.

This is a powerful argument against a bundle theory with a one-category ontology. But it does not have so much force on a mereological bundle theorist who already adopts a two-category ontology in which properties and bundles of properties belong to two different ontological categories. The mereological bundle theorist can propose a two-category ontology in terms of properties and their mereological sums. The bundle of properties is the mereological sum of those properties, which may not belong to the same ontological category to which properties do—just as the *set* of Russell and Whitehead does not belong to the same ontological category to which Russell and Whitehead do.

The second objection has it that if the bundle theorist would like to embrace primitive individuation, she could have lived all along with something like substance or substratum theory. What is the point of bundle theory if one appeals to primitive individuation? What motivates an ontology of bundles, as opposed to an ontology of substances or substrata, if they are primitively individuated?

The answer lies in a broadly *structuralist* conception of individuals. As is explained in Section 2, according to bundle theory, an individual is, or is constructed out of, the bundle of its qualitative properties. What is ontologically primary is thus the network of properties possessed by what we normally describe as individuals (mathematical or otherwise); the individuals themselves are secondary. This picture of ontological priority is absent in the case of substance theory which gives ontological priority to substances. Therefore, in order to retain this kind of ontological priority, an ontology of primitively individuated bundles is superior to the ontologies of primitively individuated substances and substrata.

The third objection concerns the relation between bundle theory, primitive individuation, and the notion of an individual. Following Dasgupta (2009), let us define *individualism* as the thesis that the most fundamental facts about the universe are facts about how individuals possess properties and stand in relations. *Anti-individualism*, by contrast, is the thesis that facts about individuals are grounded in purely qualitative facts. It should be clear that bundle theory is a version of anti-individualism. For it either identifies individuals with properties, or grounds facts about individuals in facts concerning which qualitative properties are compresent. According to the objection we are considering, if a bundle theory is committed to primitively distinct entities, it will not live up to the motivations behind anti-individualist theories such as bundle theory itself.[14]

To spell out the objection further, let us introduce Dasgupta's (2009) notion of a 'dangler'. In his view, if there were individualistic facts with no purely qualitative grounds, then individuals would be *epistemically undetectable* and *empirically redundant*. The idea is that there would be no genuine difference between two qualitatively duplicate physical systems, which differ only in which individuals lie behind the relevant qualities: mere differences in individualistic facts do not give rise to differences in purely qualitative or general facts. The laws of physics are insensitive to individuals, and 'cares' only about non-individualistic facts that the two systems share. In this sense, individuals are empirically redundant. In addition, since we can never tell the difference between qualitatively duplicate physical systems that differ only in their individualistic facts, individuals are epistemically undetectable.

[14]See Builes (2021, §3.1) for a similar line of thought.

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So, the objection has it that there could be two numerically distinct situations containing the same number of *bundles* related in exactly the same way; and since the two situations are qualitatively duplicates, they must differ in an undetectable and redundant way: just as mere differences in duplicate individuals do not give rise to differences in purely qualitative facts, so mere differences in duplicate bundles do not give rise to such differences.

What this objection shows is that bundles, just like individuals, are danglers. But what matters for our purposes is not whether bundles (or individuals) are danglers, but is *whether the status of bundles (or individuals) as danglers has anything to do with the thesis that they are primitively individuated*. And the point I want to emphasize on here is that the undetectability and redundancy of bundles (or individuals) has nothing to do with facts concerning their individuation. For example, the entities making up qualitatively duplicate physical systems may be identified and re-identified in different situations in virtue of certain properties that one possesses but the other does not. Yet, given Dasgupta's argument, it does not follow that those entities are *not* danglers. If they are danglers, that is only because the two situations differing only in their individualistic facts are qualitatively indiscernible, and therefore, the individuals involved would be empirically undetectable.

In short, the argument for the status of bundles as danglers rests on the thesis that mere differences in facts involving duplicate bundles do not give rise to differences in purely qualitative or general facts. As a result, whether or not bundles are primitively individuated neither vindicates that thesis nor rejects it. A bundle theory with primitively distinct entities does not undercut its own anti-individualist motivations.

7 | CONCLUSION

The central metaphysical thesis of mathematical structuralism is that there is no more to mathematical objects than their structural properties. In this paper, I have tried to accommodate this thesis within bundle theory: what we normally describe as individual mathematical objects are just the mereological bundles of their structural properties. Since such properties are essential to them, a version of mereological essentialism for mathematical structuralism will be vindicated.

The background conception of bundles is captured in terms of a non-extensional bundle theory, which allows for the numerical diversity of bundles composed by the same structural properties. This non-extensional account of bundles has distinctively structuralist motivations. In addition, the thesis that bundles with the same (structural) properties could be numerically distinct is supported by a primitivist account of the individuation of bundles: facts concerning the numerical identity and distinctness of bundles are fundamental, not grounded in the distribution and composition of their constituent properties.[15]

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CONFLICT OF INTEREST STATEMENT

None.

[15]A difficulty that I have not explored in this paper is to explain how *relations* between two or more objects can fit into the relevant bundles. A bundle of properties has its properties as its parts. But a relation is not part of any of its relata; it is spread across all of them, so to speak. Given that structural relations seem to be indispensable to the individuation of mathematical objects as positions in structures, this objection seems to be pressing for mathematical structuralism. See Sider (2020: §3.9) for more discussion.

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