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Recognizing Geometric Intersection Graphs Stabbed by a Line

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March 7, 2024

Abstract

In this paper, we determine the computational complexity of recognizing two graph classes, *grounded* L-graphs and *stabtable grid intersection* graphs. An L-shape is made by joining the bottom end-point of a vertical (|) segment to the left end-point of a horizontal (–) segment. The top end-point of the vertical segment is known as the *anchor* of the L-shape. Grounded L-graphs are the intersection graphs of L-shapes such that all the L-shapes’ anchors lie on the same horizontal line. We show that recognizing grounded L-graphs is NP-complete. This answers an open question asked by Jelínek & Töpfer (Electron. J. Comb., 2019).

Grid intersection graphs are the intersection graphs of axis-parallel line segments in which two vertical (similarly, two horizontal) segments cannot intersect. We say that a (not necessarily axis-parallel) straight line ℓ stabs a segment s , if s intersects ℓ . A graph G is a stabtable grid intersection graph (STABGIG) if there is a grid intersection representation of G in which the same line stabs all its segments. We show that recognizing STABGIG graphs is NP-complete, even on a restricted class of graphs. This answers an open question asked by Chaplick *et al.* (Order, 2018).

1 Introduction

Recognizing a graph class means deciding whether a given graph is a member of the graph class. In this paper, we deal with the computational complexity of recognizing intersection graphs of certain types of geometric objects in the plane. These recognition problems stemmed from various fields of active research, like VLSI design [11, 12, 24], map labelling [1], wireless networks [20], computational biology [27], and have now become an indelible part of computational geometry.

Perhaps the most extensively studied class of geometric intersection graphs is the class of *interval graphs* (intersection graphs of intervals on the real line), introduced by Benzer [5] while studying the fine structure of genes. In 1962, Lekkerkerker & Boland [21] proved that the class of interval graphs is precisely the class of graphs without *holes*¹ and *asteroidal triples*². The above elegant result of Lekkerkerker & Boland [21] motivated researchers to study further generalizations of interval graphs and their characterizations. One such generalization was considered by Gyárfás and Lehel [16] who considered the class of *L-intersection graphs*. An *L-shape* [16] is a couple made of a vertical and horizontal segment, whose bottom and left end-points coincide. *L-intersection graphs* are the intersection graphs of L-shapes. Observe that interval graphs can be expressed as intersection graphs of L-shapes as follows: given an interval representation of a graph, replace each interval with an L-shape whose horizontal segment is the same as the interval on the horizontal line $y = 0$. The popularity of L-graphs among graph theorists increased when Gonçalves *et al.* [15] proved that all planar graphs (graphs that can be drawn in the plane in such a way that edges can intersect only at their end-points) are L-graphs. In contrast, recognizing L-graphs is NP-complete [9].

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¹A hole is an induced cycle with 4 or more vertices.

²Three vertices of a graph form an asteroidal triple if the removal of any one of the vertices (along with all its neighbouring vertices) from the graph does not disconnect the other two.

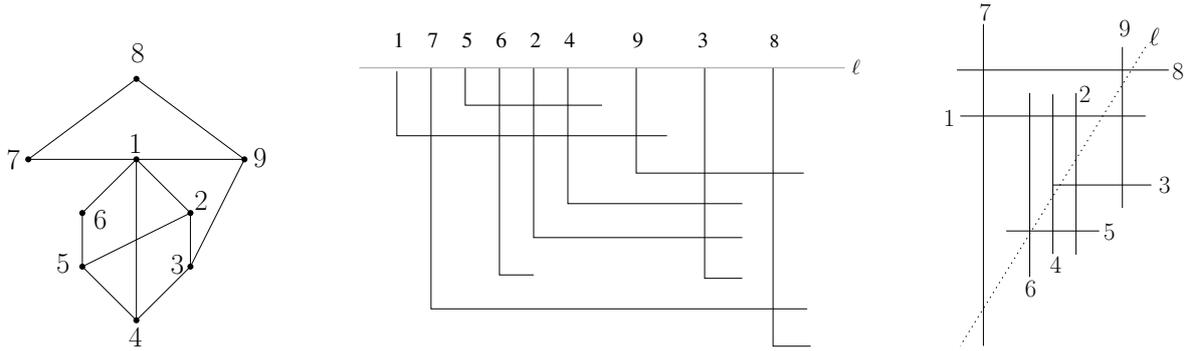


Figure 1: A graph (left) with its grounded L-shape representation (middle) and its stabbable grid intersection representation (right).

An interesting subclass of L-graphs called *infinite-L-graphs* was considered by McGuinness [22]. The top end-point of the vertical segment of an L-shape is called the *anchor* of the L-shape. *Infinite-L-graphs*, also known as *grounded L-graphs* are the intersection graphs of L-shapes whose anchors belong to the same horizontal line, called the *ground line*. See Figure 1 for an example. Interval graphs are (also) grounded L-graphs. (Indeed, given a set of intervals, replace each interval with an L-shape whose horizontal segment is the same as the interval on the horizontal line $y = 0$ and whose anchor lies on the horizontal line $y = 1$.) Other well-studied subclasses of grounded L-graphs are (see [2]) outerplanar graphs, permutation graphs, circle graphs etc. Researchers have studied different aspects (e.g. chromatic number [22, 13], dominating set [8], independent set [7], forbidden patterns [18]) of grounded L-graphs. However, the complexity of recognizing grounded L-graphs remained open. In this paper, we prove that recognizing grounded L-graphs is NP-complete answering a question asked by Jelínek and Töpfer [18].

Theorem 1. *Recognizing grounded L-graphs is NP-complete.*

Rectangle intersection graphs are graphs with boxicity 2 *i.e.* intersection graphs of axis-parallel rectangles in the plane. This graph class was introduced by Asplund & Grünbaum [3] in 1960 (even before the introduction of boxicity), who studied the chromatic number of such graphs. After almost three decades, Kratochvíl [19] proved that recognizing rectangle intersection graphs is NP-complete, even if the input graphs are restricted to bipartite graphs. Interestingly, bipartite rectangle intersection graphs are exactly the class of grid intersection graphs (GIG), *i.e.*, intersection graphs of axis-parallel line segments in the plane where no two segments with the same orientation intersect [17]. This implies that the recognition of GIG is NP-complete, and motivates the study of its subclasses. Chaplick *et al.* [10] introduced the class of *stabbable grid intersection graphs*. A segment s in the plane is *stabbed* by a line ℓ if s and ℓ intersect. A graph G is a *stabbable grid intersection graph* (STABGIG) if it has a grid intersection representation such that there exists a straight line that stabs all the segments of the representation. See Figure 1 for an example. All planar bipartite graphs are STABGIGs [14] and Chaplick *et al.* [10] left the problem of recognizing stabbable grid intersection graphs as open. In this paper, we answer their question.

Theorem 2. *Recognizing STABGIG is NP-complete.*

We note that recently the computational complexities of several geometric intersection graph classes with "stabbed" representations have been settled. Examples include *stick* graphs, *bipartite hook* graphs and *max point-tolerance* graphs [23]. Grounded L-graphs and stabbable grid intersection graphs were the largest classes for which the complexities were still unknown. The recognition problem is still open for some smaller classes of graphs with "stabbed" representations (see Section 4).

Organisation: In Sections 2 and 3, we prove Theorem 1 and Theorem 2, respectively. In Section 4, we conclude.

Notations: For a positive integer $n \geq 1$, $[n]$ denotes the set $\{1, 2, \dots, n\}$. All graphs considered in this paper are simple and undirected. For a graph G , the sets $V(G)$ and $E(G)$ denote the vertex set and edge set of G , respectively.

Convention: Both for grounded L-graphs and StabGIGs, we assume without loss of generality that any two segments in the representation intersect in at most one point. Otherwise, slight changes in the position and length of the segments allow to transform a representation that does not have this property into a representation that has this property.

2 Proof of Theorem 1

In our presentation, the geometrical representation of the grounded L-graphs uses \lrcorner -shapes above the horizontal ground line rather than L-shapes below it. For a grounded L-graph, this representation is its *grounded \lrcorner -representation*. Note that the convention above implies that all the anchors are distinct.

The \lrcorner -shape representing a vertex x is denoted by $\lrcorner(x)$, and the anchor of $\lrcorner(x)$, now located at the bottom of the vertical segment, is denoted by x too. The left to right order of the anchors along the ground line is denoted by \prec . Note that for two intersecting \lrcorner -shapes $\lrcorner(x)$ and $\lrcorner(y)$, we have $x \prec y$ if and only if the vertical segment of $\lrcorner(x)$ intersects the horizontal segment of $\lrcorner(y)$. A \lrcorner -shape that has intersections only on its vertical (respectively horizontal) segment is called a \lrcorner^v -*shape* (respectively a \lrcorner^h -*shape*). Then the problem we are interested in may be restated as follows:

GROUNDED L-GRAPHS RECOGNITION (GROUNDED L-REC)

Input: A graph H .

Question: Is there a grounded \lrcorner -representation for H ?

Theorem 1 is equivalent to the statement that GROUNDED L-REC is NP-complete. In order to show it, we use another class of geometric intersection graphs, namely stick graphs. They have been defined by Chaplick et al. [10] as the intersection graphs of a set A of vertical segments in the plane and a set B of horizontal segments in the plane, such that the bottom end-point of the segments in A and the left end-point of the segments in B belong to a ground straight line with slope -1. Again, all the endpoints may be considered as distinct. Stick graphs are bipartite graphs, and the aforementioned geometrical representation is called their *stick representation*. We prove Theorem 1 using a reduction from the problem below, which is NP-complete [23]:

STICK GRAPHS RECOGNITION (STICKREC)

Input: A bipartite graph $G = (A \cup B, E)$.

Question: Is there a Stick representation of G with sets A and B ?

Grounded L-graphs and stick graphs are related by the relationships given in Propositions 3 and 4 below, which are essential to our construction. A grounded \lrcorner -representation of a bipartite graph $G = (A \cup B, E)$ is called *nice* if the vertices in A are represented by \lrcorner^h -shapes, and those in B by \lrcorner^v -shapes. We use the notations $H(a)$ and $V(c)$ for the horizontal and vertical segments of $\lrcorner(c)$ respectively, where $c \in A \cup B$.

Proposition 3. *All stick graphs are grounded L-graphs. Moreover, each stick graph has a nice grounded \lrcorner -representation.*

Proof. Let G be a stick graph. Figure 2 illustrates the steps of the proof. Consider a stick representation of G , and replace first each vertical (respectively horizontal) segment with a \lrcorner -shape having the same anchor as the initial segment, whose horizontal (respectively vertical) segment is not used for intersections. Then extend the horizontal segments of the \lrcorner -shapes towards left so that to place all the anchors on the

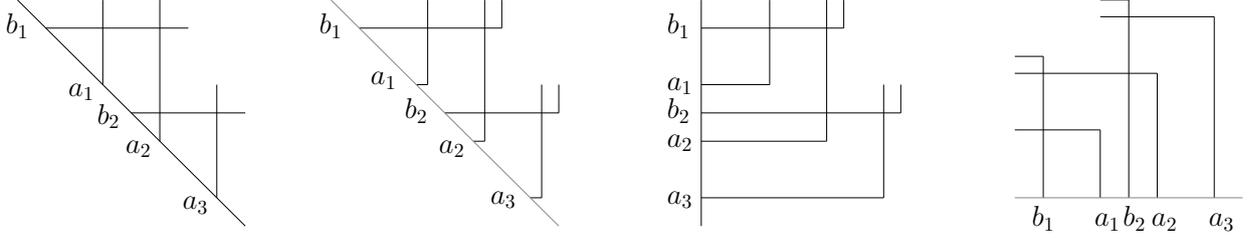


Figure 2: Transforming a stick representation into a grounded Υ -representation. Here, the stick graph is the induced path $a_1b_1a_2b_2a_3$.

same vertical line, that becomes the new ground line. Finally, perform a 90 degrees counterclockwise rotation of the whole representation to obtain a grounded Υ -representation of G . In this representation, the vertices in A are represented by Υ^h -shapes, and those in B by Υ^v -shapes. \square

Remark 1. *Not all the bipartite grounded \perp -graphs are stick graphs. The bipartite graph in Figure 3 is not a stick graph [10], but is a grounded \perp -graph as shown by the Υ -representation we provide.*

Proposition 4. *All bipartite grounded \perp -graphs which have a nice grounded Υ -representation are stick graphs.*

Proof. Let G be a bipartite graph with a nice grounded Υ -representation R . We transform this representation into a stick representation of G as follows. Draw a line ℓ with slope 1 that intersects the grounded line to the left of all the anchors, and such that all the segments $H(a)$, for $a \in A$, lie in the lower half-plane defined by ℓ . Such a line always exists, it is sufficient to consider the intersection point sufficiently far to the left. Consider a Cartesian coordinate system whose origin is the intersection point of the two lines, the x -axis is the grounded line and the y -axis is oriented upwards.

For each $c \in A \cup B$, denote by $l(c)$ and $r(c)$ the x -coordinate of the left and right endpoint of $H(c)$ respectively, and by $h(c)$ the common y -coordinate of these two endpoints. For $c, c' \in A \cup B$, we say that $H(c)$ is below $H(c')$ (equivalently, $H(c')$ is above $H(c)$) whenever $h(c) < h(c')$ and the projections of $H(c)$ and $H(c')$ on the x -axis intersect.

Since ℓ is above $H(a)$, for $a \in A$, we deduce that $h(a) < l(a) < r(a)$. We show that the heights of the Υ -shapes can be modified so as to place the right endpoint of $H(a)$ on the line ℓ , for each $a \in A$. Equivalently, we show that:

(P) There exists a grounded Υ -representation R' of G such that $h(a) = r(a)$ for each $a \in A$.

Let $a_1, a_2, \dots, a_{|A|}$ be the vertices in A , in decreasing order of their value $r(a)$ in the grounded Υ -representation R . We use induction to show that, for each $1 \leq i \leq |A|$, there exists a grounded Υ -representation R_i of G such that $h(a_k) = r(a_k)$ for $1 \leq k \leq i$. Note that for each u, v such that $l(a_u) < r(a_v) < r(a_u)$ we necessarily have $h(a_u) > h(a_v)$. In the contrary case, $\Upsilon(a_u)$ and $\Upsilon(a_v)$ would intersect, which is impossible since G is bipartite and A is one of its parts. Then, the order $a_1, a_2, \dots, a_{|A|}$ of the vertices implies that, for $u < v$, either the projections $H(a_v)$ and $H(a_u)$ on the x -axis are disjoint and in this order from left to right, or $H(a_v)$ is below $H(a_u)$.

Let $i = 1$. As noticed above, no $H(a_j)$ with $j \neq 1$ is above $H(a_1)$. We then modify $\Upsilon(a_1)$ so that $h(a_1)$ becomes equal to $r(a_1)$, and appropriately lengthen the vertical segment of $\Upsilon(b)$ for each neighbor b of a_1 so that $\Upsilon(b)$ intersects $\Upsilon(a_1)$ (equivalently, such that $H(b)$ is above $H(a_1)$). Note that this transformation does not modify the intersections between $\Upsilon(b)$ and the other Υ -shapes, since the segment $V(b)$ – which realizes by hypothesis all the intersections of $\Upsilon(b)$ with other Υ -shapes – is neither moved (to left or right) nor shortened. Therefore, this is a new grounded Υ -representation of G , denoted R_1 .

Assume now, by inductive hypothesis, that the grounded Υ -representation R_i of G exists. In R_i , the only segments $H(a_j)$ that are possibly placed above $H(a_{i+1})$ satisfy $j \leq i$, and by the inductive hypothesis they also satisfy $h(a_j) = r(a_j)$. The ordering of the vertices in A implies that $r(a_j) > r(a_{i+1})$.

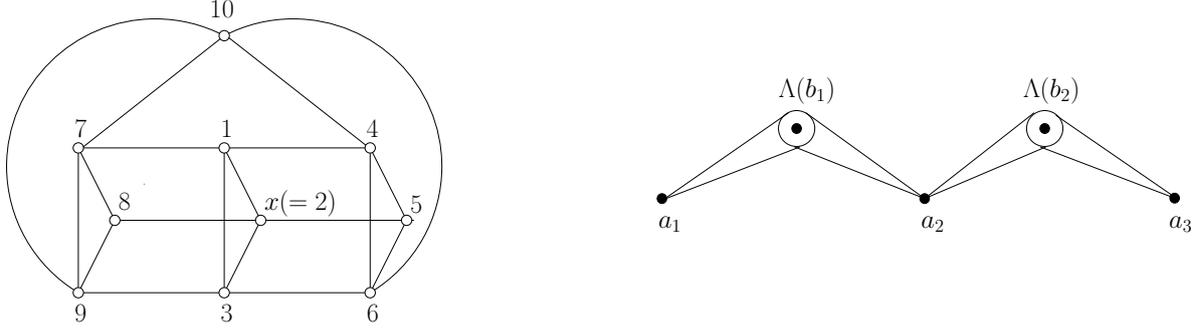


Figure 4: The graph $\Lambda(x)$ (left) and the construction of the graph H when G is the path $a_1b_1a_2b_2a_3$ (right).

We show in the next section that this reduction proves the NP-completeness of GROUNDED L-REC. To this end, we follow the following steps:

- (i) We show in Proposition 5 that $\Lambda(x)$ accepts exactly two grounded Υ -representations, shown in Figure 5(c) (details are given later) and roughly represented as in Figure 5(d).
- (ii) We show in Proposition 6 that adding to $\Lambda(x)$ a universal vertex u yields another grounded L-graph, whose grounded Υ -representations always place the anchor of $\Upsilon(u)$ outside the representation of $\Lambda(x)$, to the right of it. Consequently, x (u respectively) is represented by a Υ^v -shape (Υ^h -shape respectively). Recalling that in our construction of H each vertex $a \in A$ is universal with respect to $\Lambda(b)$, for each of its neighbors b from G , this implies that the grounded Υ -representations of H mimic those of G and vice-versa, as shown in Figure 6.
- (iii) We show in Proposition 7 that G is a stick graph if and only if H is a grounded L-graph.
- (iv) We conclude the proof of Theorem 1.

2.2 The proofs

The graph $\Lambda(x)$ has been chosen so as to have very few and convenient grounded Υ -representations. In any grounded Υ -representation of $\Lambda(x)$, the left to right order of the anchors 1, 2, 3 defines a permutation def on $\{1, 2, 3\}$. For $i \in \{1, 2, 3\}$, let v_i^1 be the neighbor of i from $\{1, 8, 9\}$ and v_i^2 the neighbor of i from $\{4, 5, 6\}$.

Proposition 5. *The graph $\Lambda(x)$ is a grounded L-graph. It admits exactly two grounded Υ -representations, given in Figure 5(c), where we have either $d = 1$ and $f = 3$, or $d = 3$ and $f = 1$.*

Proof. When $\Upsilon(d)$, $\Upsilon(e)$ and $\Upsilon(f)$ (only) are placed on the ground line such that $d \prec e \prec f$, the Υ -shapes of the two neighbors v_e^1 and v_e^2 of e may be placed (independently of each other) in exactly two locations:

- either to the left of the anchor d ,
- or to the right of the anchor f .

Indeed, we cannot have $e \prec v_e^i \prec f$, for some $i \in \{1, 2\}$, as explained hereafter. If, for instance, $e \prec v_e^1 \prec f$, then we must have $d \prec v_d^1, v_f^1, 10 \prec e$ in order to satisfy all the adjacencies and non-adjacencies. Furthermore, $d \prec v_f^2 \prec e$ is not possible, since then $\Upsilon(v_f^2)$ intersects either $\Upsilon(v_d^1)$ or $\Upsilon(v_e^1)$ before it intersects $\Upsilon(f)$, a contradiction. We then have $v_f^2 \prec d$ or $e \prec v_f^2$, which implies that v_e^2 is also outside the interval defined by d and e . But then 10 cannot have two neighbors among v_d^2, v_e^2 and v_f^2 , since $\Upsilon(10)$ intersects neither $\Upsilon(d)$ nor $\Upsilon(e)$, a contradiction.

Two cases appear, for which the proofs are similar. In each case, we first deduce the possible grounded Υ -representations of the subgraph Λ^9 induced by the vertices 1 to 9, using the variables i, v_i^1, v_i^2 for $1 \leq i \leq 3$. For each of these representations, we then test the possibility to add $\Upsilon(10)$.

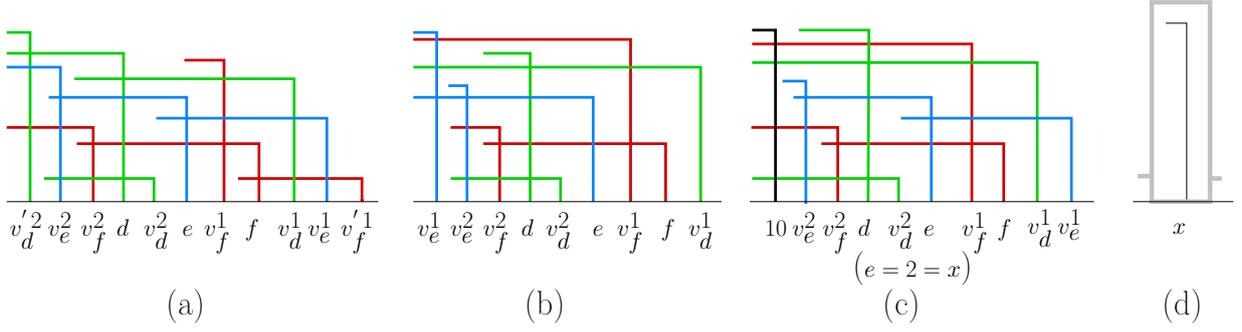


Figure 5: Representations of $\Lambda(x)$: (a) Case 1 for Λ^9 . (b) Case 2 for Λ^9 . (c) Possible grounded Υ -representations for $\Lambda(x)$. (d) The drawing replacing any representation, when only $\Upsilon(x)$ and the level of the lowest horizontal segment are needed.

Case 1. v_e^1 and v_e^2 belong to different locations (see Figure 5(a)). Assume w.l.o.g. that $v_e^2 \prec d$ and $f \prec v_e^1$. Then, there is exactly one possible position for v_f^2 and two possible positions for v_f^1 (the anchors are denoted by v_f^1 and v_f^1) so as to obtain the sought intersections with f and v_e^2, v_e^1 respectively. Similarly, only one position is possible for v_d^1 and two positions for v_d^2 (the anchors are denoted by v_d^2 and v_d^2). Figure 5(a) thus records all the possible representations of the graph induced by the vertices 1 to 9: any possible representation is obtained by choosing one of v_f^1, v_f^1 and one of v_d^2, v_d^2 .

The vertex 10 is adjacent to $v_i^1, v_i^2, v_j^1, v_j^2$, with $i \neq j$ and $i, j \in \{d, e, f\}$. As d separates v_e^1 and v_e^2 , and 10 is not adjacent to d , we deduce that $i, j \neq e$. Moreover, $\{i, j\} = \{d, f\}$ is possible, as shown in Figure 5(c) where an appropriate position is proposed for $\Upsilon(10)$. This position is unique and implies that the anchor v_d^2 (and not v_d^2) has to be used, as well as the anchor v_f^1 (and not v_f^1).

Since e is the unique vertex among $\{1, 2, 3\}$ none of whose neighbors is adjacent to 10, we deduce that $e = 2$, and thus $\{d, f\} = \{1, 3\}$. In conclusion, we have two possible Υ -representations for $\Lambda(x)$, with $d = 1$ and $f = 3$, and vice-versa.

Case 2. v_e^1 and v_e^2 belong to the same location. If they are both on the right side of f , with (for instance) $f \prec v_e^1 \prec v_e^2$, then there is no possible place for $\Upsilon(v_d^1)$ so as to intersect $\Upsilon(v_e^1)$ and $\Upsilon(d)$ but not $\Upsilon(v_e^2)$ nor e . Thus we may assume w.l.o.g that $v_e^1 \prec v_e^2 \prec d$. We cannot have $v_d^1 \prec v_e^1$, since then $\Upsilon(v_f^1)$ is impossible to place so as to intersect $\Upsilon(f)$, $\Upsilon(v_d^1)$ and $\Upsilon(v_e^1)$, but no other Υ -shape. We deduce that $f \prec v_d^1$, and further that $e \prec v_f^1 \prec f$, $v_e^2 \prec v_f^2 \prec d$ and $d \prec v_d^2 \prec e$. The resulting order of the anchors for the vertices of Λ^9 is given in Figure 5(b).

Looking for the neighbors $v_i^1, v_i^2, v_j^1, v_j^2$ with $i \neq j$ and $i, j \in \{d, e, f\}$ of the vertex 10, we notice that e separates v_d^1 and v_d^2 , as well as v_f^1 and v_f^2 . None of these pairs of vertices can be adjacent to 10 (since 10 is not adjacent to e), and thus there is no possible way to place $\Upsilon(10)$ so as to intersect the vertices it needs to intersect. No Υ -representation for $\Lambda(x)$ can be found in this case.

Consequently, all the grounded Υ -representations of $\Lambda(x)$ are obtained in Case 1, and are drawn in Figure 5(c). \square

Convention. In the remainder of the paper, the grounded Υ -representation in Figure 5(a) is drawn as a grey box around $\Upsilon(x)$, together with two lateral segments indicating the level of the lowest horizontal segment in the representation. See Figure 5(d).

Let $\Lambda(x) + u$ be the graph obtained by adding to $\Lambda(x)$ an universal vertex u .

Proposition 6. *The graph $\Lambda(x) + u$ is a grounded L-graph. In its grounded Υ -representations, u is always the rightmost anchor, and thus $\Upsilon(u)$ is a Υ^h -shape.*

Proof. A grounded Υ -representation of $\Lambda(x) + u$ is obtained by placing the anchor of u to the extreme right of a grounded Υ -representation of $\Lambda(x)$, and letting the horizontal segment of $\Upsilon(u)$ intersect the

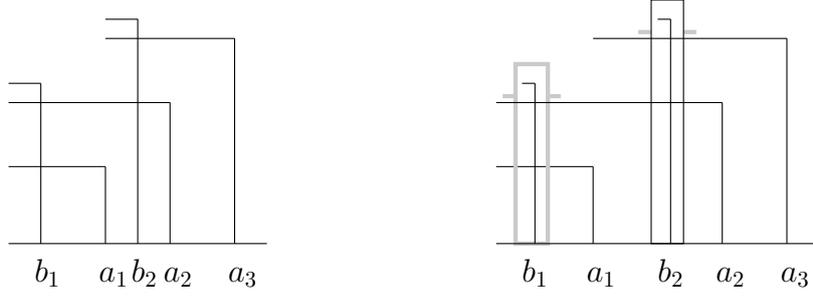


Figure 6: The grounded Υ -representation of G (left) and the grounded Υ -representation of H (right), when G is the induced path $a_1b_1a_2b_2a_3$.

vertical segments of all the other Υ -shapes at a very low height (lower than the horizontal segment of $\Upsilon(v_d^2)$). Then $\Upsilon(u)$ is a Υ^h -shape. The conclusion of Proposition 6 is reached by noticing that no other position is possible for u . Indeed, if u was to the left of v_e^1 , then $\Upsilon(u)$ could intersect both $\Upsilon(v_d^2)$ and $\Upsilon(v_e^1)$ only if $d \prec u \prec v_d^2$. But then the horizontal segment $H(u)$ of $\Upsilon(u)$ should be both below $H(v_f^2)$ (in order to intersect $\Upsilon(v_f^2)$) and above $H(v_f^1)$ (in order to intersect $\Upsilon(v_f^1)$) and this is not possible. \square

Proposition 7. *The graph G is a yes-instance of STICKREC if and only if the graph H is a yes-instance of GROUNDED L-REC.*

Proof. For the forward direction, Proposition 3 implies that the stick graph G has a grounded Υ -representation. We transform this representation of G into a grounded Υ -representation of H as follows. For each $b \in B$, build a grounded Υ -representation of $\Lambda(b)$ (whose existence is ensured by Proposition 5) as a block, in the immediate neighborhood of $\Upsilon(b)$. See Figure 6. The lowest horizontal segment of a Υ -shape in this grounded Υ -representation of $\Lambda(b)$, indicated in the figure by the two grey lateral segments of the grey box, must be placed above all the horizontal segments of Υ -shapes $\Upsilon(a)$ with $a \in A$ and $ab \in E$. This is always possible, by increasing the lengths of the vertical segments as needed. Furthermore, for each $a \in A$ with $ab \in E$, extend the horizontal segment of $\Upsilon(a)$ such that it intersects all the Υ -shapes representing vertices in $\Lambda(b)$. The resulting grounded Υ -representation is a grounded Υ -representation of H .

We now consider the backward direction. Let $a \in A, b \in B$ such that $ab \in E$. By the construction of H , a is universal for $\Lambda(b)$. By Proposition 6 for the subgraph $\Lambda(b) + a$ of H , we deduce that the anchor of $\Upsilon(a)$ is placed outside the representation of $\Lambda(b)$, to the right, so that $\Upsilon(a)$ is a Υ^h -shape. It follows that the intersection between $\Upsilon(b)$ and $\Upsilon(a)$ holds on $V(b)$ and on $H(a)$, for each pair $a \in A, b \in B$ such that $ab \in E$.

Then, when we focus only on the vertices in G , the resulting grounded Υ -representation of G is nice. We deduce by Proposition 4 that G is a stick graph. \square

We are ready to prove that GROUNDED L-REC is NP-complete.

Proof. (of Theorem 1). To show that GROUNDED L-REC is in NP, we first note that a graph has a grounded Υ -representation if and only if it has a grounded Υ -representation whose segments have integer coordinates. For the forward direction, consider a graph that has a grounded Υ -representation with real coordinates for the segments. Order the $2n$ endpoints of the horizontal segments in increasing order of their x -coordinate (equal values may appear, that yield the same rank). Remember the rank of each point (called x -rank). Do the same for the y -coordinate, and the y -rank. Then replace the initial coordinates of each endpoint with its x -rank and its y -rank, and draw the Υ -shapes accordingly. The backward direction is immediate.

In order to show that GROUNDED L-REC is in NP, it is sufficient to test if a set of n given Υ -shapes with integer coordinates defines a grounded Υ -representation for the input graph, and this is done in

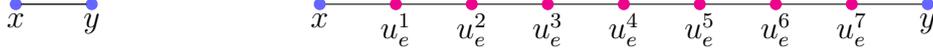


Figure 7: An edge $e = xy$ (left) and a 7-subdivision of e (right).

polynomial time. Proposition 7 shows that our reduction is correct. Moreover, it takes linear time to build H knowing G , so that the reduction is polynomial. \square

3 Proof of Theorem 2

In this section, we focus on the following problem:

STABBABLE GRID INTERSECTION GRAPHS RECOGNITION (STABGIGREC)

Input: A bipartite graph G .

Question: Is there a grid intersection representation of G in which all segments are stabbed by the same straight line?

We perform a reduction from the following problem. A *Hamiltonian path* in a graph is a path that visits each vertex of the graph exactly once.

PLANAR HAMILTONIAN PATH COMPLETION PROBLEM (PHPC)

Input: A planar graph G .

Question: Is G a subgraph of a planar graph with a Hamiltonian path?

3.1 The reduction

Convention. Throughout our proof, we will assume that G is a connected graph with at least five vertices, as every graph on four vertices is planar and the planarity of a non-connected graph is reduced to that of its components.

Let k be a fixed positive odd number. Given a planar graph P on $p \geq 5$ vertices, we construct a bipartite apex graph $P_{k\text{-div}}^*$ in $\text{poly}(p)$ time as follows.

Let $P_{k\text{-div}}$ be the full k -subdivision of P , *i.e.*, $P_{k\text{-div}}$ is the graph obtained by replacing each edge of P with an induced path of length $k + 1$. Formally, we replace each $e = xy \in E(P)$ with the path $(x, u_e^1, u_e^2, \dots, u_e^k, y)$ (see Figure 7).

$$\begin{aligned} V(P_{k\text{-div}}) &= V(P) \cup \{u_e^1, u_e^2, \dots, u_e^k \mid e \in E(P)\}; \\ E(P_{k\text{-div}}) &= \{xu_e^1, u_e^1u_e^2, \dots, u_e^ky \mid e = xy \in E(P)\}. \end{aligned}$$

We call the vertices of $V(P) \subseteq V(P_{k\text{-div}})$ as the *original vertices* of $P_{k\text{-div}}$ and the remaining vertices as the *subdivision vertices* of $P_{k\text{-div}}$. Finally, we construct $P_{k\text{-div}}^*$ by adding a new vertex a (called the ‘apex’) to $P_{k\text{-div}}$ and making it adjacent to all the original vertices of $P_{k\text{-div}}$. Formally, $P_{k\text{-div}}^*$ is defined as follows.

$$\begin{aligned} V(P_{k\text{-div}}^*) &= V(P_{k\text{-div}}) \cup \{a\}; \\ E(P_{k\text{-div}}^*) &= E(P_{k\text{-div}}) \cup \{av \mid v \in V(P)\}. \end{aligned}$$

The reduction from PHPC to STABGIGREC associates to the planar graph G , the input of PHPC, the graph $G_{k\text{-div}}^*$ built as above, with $P = G$ and an arbitrary odd integer k such that $k \geq 7$. See Figure 8 for an example of the reduction.

We need to prove that a planar graph G is a yes-instance of PHPC if and only if $G_{k\text{-div}}^*$ is a yes-instance of STABGIGREC. The forward direction is Proposition 9 of the next section. For the backward

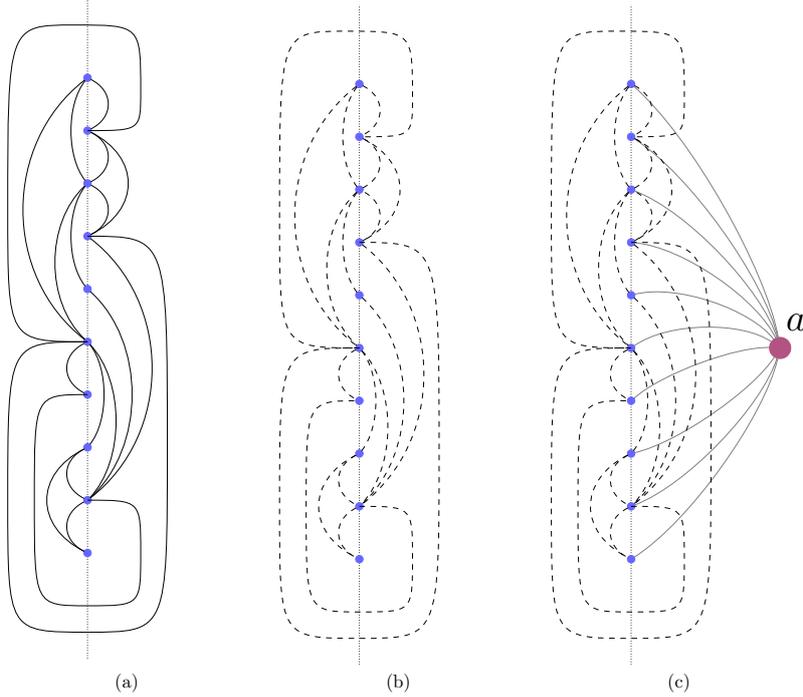


Figure 8: (a) An example graph G which is a yes-instance of PHPC. (b) Each edge of the graph is replaced with a path with $k + 1$ edges. (A dashed curve indicates a path of length $k + 1$.) (c) The graph $G_{k\text{-div}}^*$, constructed by adding one extra vertex (the apex vertex a) in $G_{k\text{-div}}$ which adjacent to all the vertices of G .

direction, we will need the following result. A graph is a *1-string graph* if it is the intersection graph of simple curves in the plane, also called *strings*, such that any two strings intersect at most once, and whenever they intersect they cross each other.

Proposition 8 ([9]). *For a planar graph G , if k is odd and $G_{k\text{-div}}^*$ is a 1-string graph, then G is a yes-instance of PHPC.*

3.2 The proofs

We start by the forward direction:

Proposition 9. *If G is a yes-instance of PHPC, then $G_{k\text{-div}}^*$ is a STABGIG.*

The proof of this result is presented in Sections 3.2.1 to 3.2.3. Section 3.2.4 contains the proof of Theorem 2.

We use the notation (x, y) , where x and y are real values, for the Cartesian coordinates of a point, and the notation $[(x_1, y_1), (x_2, y_2)]$ for the segment whose endpoints are (x_1, y_1) and (x_2, y_2) .

3.2.1 Overview of the construction

In view of proving Proposition 9, let G be a yes-instance of PHPC, and let $V(G) = \{v_1, v_2, \dots, v_n\}$. We start with a plane drawing of G , *i.e.*, a drawing of G in the plane, in which the vertices are points and the edges are strings (that is, simple curves). Then we modify the drawing in a step-by-step manner to end with a STABGIG representation of $G_{k\text{-div}}^*$. Let us now provide an overview of these steps.

- (a) Consider a drawing of G in the plane in which the vertex v_i (for every $i \in [n]$) has the Cartesian coordinate $(0, i - 1)$, each edge of G crosses the y -axis (the line $x = 0$) at most once, and no edge of G crosses the y -axis between $y = 0$ and $y = n - 1$. (Auer & Gleißner [4] show that such a drawing

exists for every graph that is a yes-instance of PHPC.) See Figure 8(a) for an illustration, where v_1 is located at $(0, 0)$ and v_{10} is located at $(0, 9)$.

- (b) In this drawing, though no edge of G crosses the y -axis between the points $(0, 0)$ and $(0, n - 1)$, there might be edges of G that cross the y -axis below $y = 0$ or above $y = n - 1$. We place new vertices on the intersection points of those edges with the y -axis, to create a new graph H . Let n_H be the number of vertices in H . See Figure 9 for an illustration, where there are three such vertices, indicated by square nodes.
- (c) The graph H has some nice properties. Firstly, $n \leq n_H \leq 4n$. Secondly, H is a subdivision of G . As subdivision does not affect planarity, H is also planar. Thirdly, the drawing of H ensures that no edge of H crosses the y -axis. We fix the line $y = x$ as our stab line ℓ (recall that ℓ does not need to be axis-parallel), and use the current plane drawing of H to devise a STABGIG representation of the graph $H_{3\text{-div}}^*$. Recall that the graph $H_{3\text{-div}}^*$ is obtained as described in Section 3.1, by considering $P = H$ and $k = 3$. Recall that the apex vertex is only adjacent to the original vertices of H and not to the See Figure 9. Note that the graph $G_{k\text{-div}}^*$ is obtained from $H_{3\text{-div}}^*$ by further subdividing some of its edges.
- (d) In the STABGIG representation of $H_{3\text{-div}}^*$, the apex segment of $H_{3\text{-div}}^*$ is represented by a long vertical segment coinciding with the y -axis. The n_H original vertices of $H_{3\text{-div}}^*$ are represented by n_H horizontal segments, each one crossing the y -axis at the point defined by its corresponding vertex in the plane representation of H . Each edge $e = xy$ of H (which is a string in the plane drawing of H) becomes a path with three intermediate vertices u_e^1, u_e^2, u_e^3 in $H_{3\text{-div}}^*$. These three vertices are represented as three segments that form a \sqcup -shape or a \sqcap -shape, whose two vertical segments appropriately intersect the horizontal segments representing respectively x and y . See Figure 10 for an illustration (formal details are given below).
- (e) For each edge e of H , we need to very precisely and carefully describe the positions of the three segments corresponding to its three subdivision vertices (which will form a \sqcup -shape or a \sqcap -shape) u_e^1, u_e^2, u_e^3 in $H_{3\text{-div}}^*$, in such a way that all the three segments intersect the stab line ℓ . This is the most non-trivial part of our proof. See Figure 10 for an illustration.
- (f) Lastly, we need to convert $H_{3\text{-div}}^*$ to $G_{k\text{-div}}^*$. Note that $H_{3\text{-div}}^* \setminus \{a\}$ is a subdivision of G in which each edge of G is subdivided either 3 or 7 times. An overview of this construction is illustrated by Figure 11. Finally, with the apex segment $[(0, 0), (0, n - 1)]$, we have a STABGIG representation of $G_{k\text{-div}}^*$.

3.2.2 Construction of $H_{3\text{-div}}^*$ and its STABGIG representation

Now we commence with the formal proof of Proposition 9. Let us start with item (a). Since G is a yes-instance of PHPC, according to Auer & Gleißner [4], the edge set $E(G)$ can be partitioned into three sets $E_{\text{left}}(G)$, $E_{\text{right}}(G)$, and $E_{\text{cross}}(G)$, such that G admits a plane drawing satisfying the following properties.

- For every $i \in [n]$, the vertex v_i has the Cartesian coordinate $(0, i - 1)$.
- Every edge $e \in E_{\text{left}}(G)$ lies in the half-plane $x \leq 0$.
- Every edge $e \in E_{\text{right}}(G)$ lies in the half-plane $x \geq 0$.
- Every edge $e \in E_{\text{cross}}(G)$ crosses the line $x = 0$ exactly once, at $(0, Y_e)$, where $Y_e < 0$ or $Y_e > n - 1$.

Figure 8(a) presents an example of a graph, and of a plane drawing of it which satisfies the above properties. Note that the above plane drawing crucially uses the fact that the line $x = 0$ partitions the plane into two half-planes ($x \leq 0$ and $x \geq 0$). Now, let us move on to item (b).

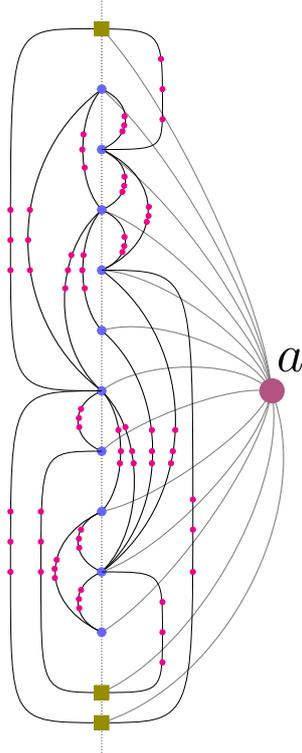


Figure 9: The graph $H_{3\text{-div}}^*$, constructed from the the graph G shown in Figure 8(a) using the intermediate graph H . Note that H is obtained from G by introducing three new vertices (indicated by square nodes) that perform a first subdivision of the three edges that cross the y-axis, namely v_2v_4 , v_5v_7 and v_5v_9 . The graph $G_{k\text{-div}}^*$ is then obtained from $H_{3\text{-div}}^*$ by introducing an even number of additional subdivision vertices for all edges v_iv_j of G that are insufficiently subdivided.

The construction of H from G is done as follows. For each edge $e = v_iv_j \in E_{\text{cross}}(G)$, we add a new vertex new_e to the graph, represented by the point $(0, Y_e)$. Consequently, the edge $e = v_iv_j$ is replaced by two edges $v_i\text{new}_e$ and new_e, v_j . One of those edges is in $E_{\text{left}}(H)$ and the other is in $E_{\text{right}}(H)$. Therefore, the vertex new_e is a subdivision vertex of the edge e . More formally,

$$\begin{aligned} V(H) &= V(G) \cup \{\text{new}_e \mid e \in E_{\text{cross}}(G)\}; \\ E(H) &= E_{\text{left}}(G) \cup E_{\text{right}}(G) \cup \{x\text{new}_e, \text{new}_ey \mid e = xy \in E_{\text{cross}}(G)\}. \end{aligned}$$

What did this achieve for us? The graph H has some nice properties, which we mentioned in item (c), and will elaborate now.

- H is a subdivision of G . Edges of $E_{\text{left}}(G)$ and $E_{\text{right}}(G)$ are retained as-is in H , and edges of $E_{\text{cross}}(G)$ are subdivided once.
- H is a planar graph. This is because G is a planar graph, and subdivision does not affect planarity.
- No edge of H crosses the y-axis ($E_{\text{cross}}(H) = \emptyset$). This is a straightforward feature of our construction.
- $n \leq n_H \leq 4n$, where $n_H = |V(H)|$. This is because G has at most $3n$ edges (as G is planar), and each edge of G is subdivided at most once in H . Since H also retains the vertices of G , $n_H \leq n + 3n$.

From H , we construct $H_{3\text{-div}}^*$, as described in Section 3.1. Then we progressively build its STABGIG representation (item (d)). We first define the positions of the segments corresponding to the apex vertex of $H_{3\text{-div}}^*$ and the original vertices of H in our STABGIG representation of $H_{3\text{-div}}^*$. Note that the apex

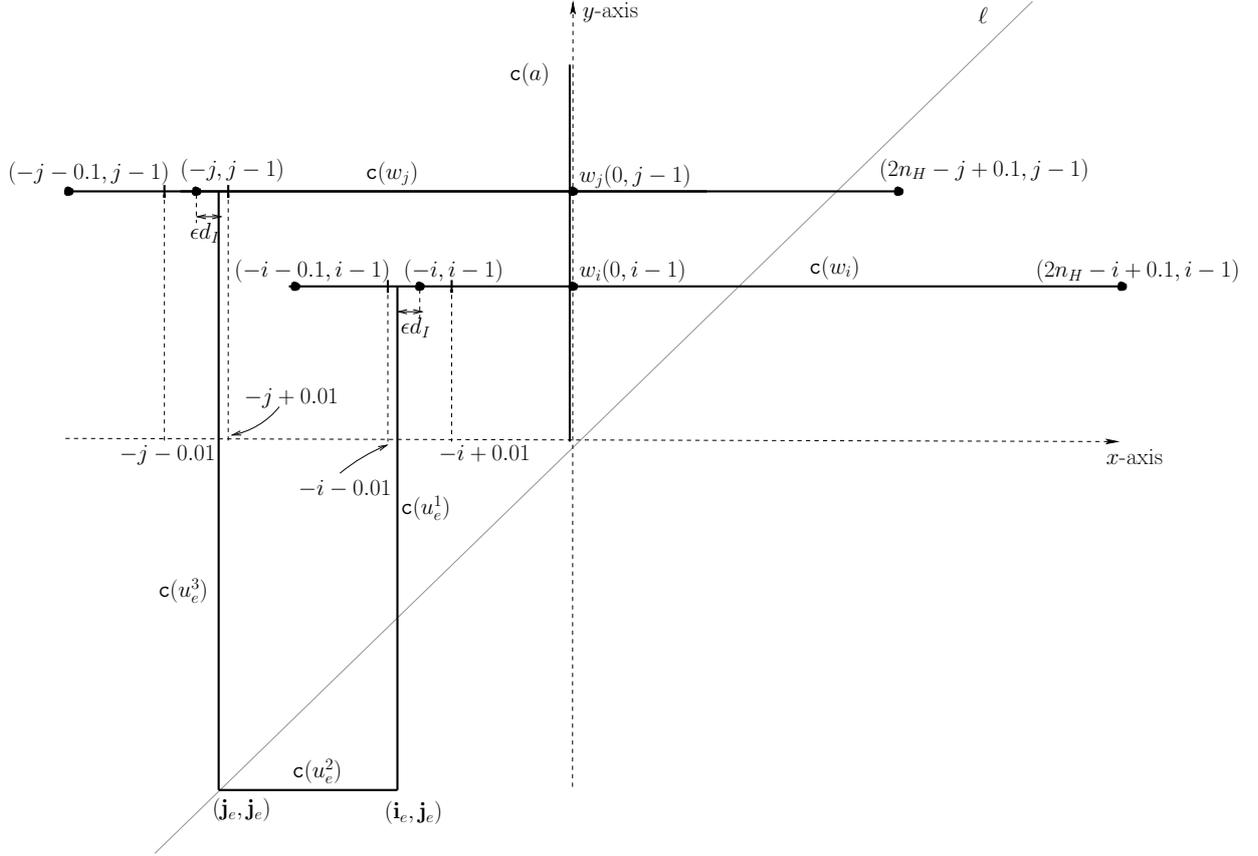


Figure 10: STABGIG representation of the path $(w_i, u_e^1, u_e^2, u_e^3, w_j)$ of $H_{3^*}^*_{-div}$ corresponding to the edge $e = w_i w_j$ of H , with $i < j$.

vertex in $H_{3^*}^*_{-div}$ is called a , just as the apex vertex in $G_k^*_{-div}$, since these vertices finally coincide in our construction.

The vertices with the minimum and maximum y -coordinates in H are at $(0, Y_{\min})$ and $(0, Y_{\max})$, where

$$Y_{\min} = \min(\{0\} \cup \{Y_e \mid e \in E_{\text{cross}}(G)\});$$

$$Y_{\max} = \max(\{n-1\} \cup \{Y_e \mid e \in E_{\text{cross}}(G)\}).$$

We may assume that $Y_{\min} = 0$ and $Y_{\max} = n_H - 1$, and that the other vertices of H can be shifted accordingly so that their coordinates are $(0, 0), (0, 1), (0, 2), \dots, (0, n_H - 1)$.³ Furthermore, the vertices are suitably renamed as $w_1, w_2, w_3, \dots, w_{n_H}$ so that each w_i has the coordinate $(0, i - 1)$. For each vertex $x \in V(H_{3^*}^*_{-div})$, let $c(x)$ be its corresponding segment. See Figure 10.

The apex vertex a of $H_{3^*}^*_{-div}$: $c(a) = [(0, 0), (0, n_H - 1)]$

The original vertices of H : $c(w_i) = [(-i - 0.1, i - 1), (2n_H - i + 0.1, i - 1)] \quad \forall i \in [n_H]$.

We fix the line $y = x$ as our stab line ℓ (recall that ℓ does not need to be axis-parallel). It is easy to see that all the segments described so far intersect the stab line. This brings us to item (e), where we need to place the segments corresponding to the three subdivision vertices of e for each edge e of H .

Let us consider edges of $E_{\text{left}}(H)$. Later, we will deal with edges of $E_{\text{right}}(H)$. We construct a set of intervals $\mathcal{J}_{\text{left}}$ from $E_{\text{left}}(H)$ as follows. For each edge $e = w_i w_j \in E_{\text{left}}(H)$ (where $i < j$), the set $\mathcal{J}_{\text{left}}$ contains an open interval $(i - 1, j - 1)$. Thus, the number of intervals in $\mathcal{J}_{\text{left}}$ is the number of edges in

³This can be done, for example, by shifting the x -axis downwards and making the gap between consecutive points as 1.

$E_{\text{left}}(H)$. Now, note that $\mathcal{J}_{\text{left}}$ has the following interesting property.

Observation 10. *If two intervals of $\mathcal{J}_{\text{left}}$ have an overlap (i.e., the intervals are not disjoint), then one of them must be contained in the other.*

Observation 10 is well known and easy to see; it is the basic underlying principle of *book thickness* or *book embeddings* of graphs (for a proof, see [6]). We will use this observation crucially in our construction of the three subdivision segments for each edge e of $E_{\text{left}}(H)$.

Given an interval $I \in \mathcal{J}_{\text{left}}$, let its *depth*, denoted by \mathbf{d}_I , be the number of intervals $I' \in E_{\text{left}}(H)$ such that $I \subseteq I'$. In other words, \mathbf{d}_I is the number of intervals that contain I (including I itself). Thus, $1 \leq \mathbf{d}_I \leq n_H - 1$. Now we are ready to describe our construction.

Fix $\varepsilon = 0.01/n_H$. Let $e = w_i w_j \in E_{\text{left}}(H)$ such that $i < j$. Let $I = (i - 1, j - 1)$ be the interval corresponding to e in $\mathcal{J}_{\text{left}}$. After the 3-subdivision, the edge e becomes the path $(w_i, u_e^1, u_e^2, u_e^3, w_j)$ in $H_{3\text{-div}}^*$. We already know the coordinates of the segments for w_i and w_j . As for the other three segments, we define \mathbf{j}_e and \mathbf{i}_e as follows.

$$\begin{aligned}\mathbf{j}_e &= -j + \varepsilon \mathbf{d}_I; \\ \mathbf{i}_e &= -i - \varepsilon \mathbf{d}_I.\end{aligned}$$

Now, we are set to define the coordinates of the other three subdivision segments of e , which draw a \sqcup -shape. See Figure 10.

$$\begin{aligned}\mathbf{c}(u_e^3) &= [(\mathbf{j}_e, j - 1), (\mathbf{j}_e, \mathbf{j}_e)]; \\ \mathbf{c}(u_e^2) &= [(\mathbf{j}_e, \mathbf{j}_e), (\mathbf{i}_e, \mathbf{j}_e)]; \\ \mathbf{c}(u_e^1) &= [(\mathbf{i}_e, \mathbf{j}_e), (\mathbf{i}_e, i - 1)].\end{aligned}$$

Before proving that the 3 segments do not intersect any segments apart from w_i , w_j , and each other, let us prove that all 3 segments intersect the stab line. Consider the point of intersection of $\mathbf{c}(u_e^3)$ and $\mathbf{c}(u_e^2)$, namely $(\mathbf{j}_e, \mathbf{j}_e)$. This point has the same x- and y-coordinate, and so it lies on the stab line $y = x$. Thus, all that remains to be shown is that $\mathbf{c}(u_e^1)$ also intersects the stab line. Since $i \leq j - 1$ and $\varepsilon \mathbf{d}_I < 0.01$,

$$\mathbf{j}_e \leq \mathbf{i}_e \leq i - 1.$$

Hence, the point $(\mathbf{i}_e, \mathbf{i}_e)$ lies on the segment $\mathbf{c}(u_e^1) = ((\mathbf{i}_e, \mathbf{j}_e), (\mathbf{i}_e, i - 1))$, and on the stab line $y = x$ also.

Now, we will show that for two edges $e = w_i w_j$ (where $i < j$) and $e' = w_{i'} w_{j'}$ (where $i' < j'$) of $E_{\text{left}}(H)$, the segments corresponding to the three subdivision vertices of e do not intersect the segments corresponding to the three subdivision vertices of e' . Let their corresponding intervals in $\mathcal{J}_{\text{left}}$ be I and I' , respectively. Due to Observation 10, we have only two cases.

1. Case 1: I and I' are disjoint. Suppose that $i < j \leq i' < j'$ (the other sub-case, $i' < j' \leq i < j$, is similar). We have the following.

$$\begin{aligned}i &< j \leq i' < j' \\ -j' &< -i' \leq -j < -i \\ -j' + \varepsilon \mathbf{d}_{I'} &< -i' - \varepsilon \mathbf{d}_{I'} < -j + \varepsilon \mathbf{d}_I < -i - \varepsilon \mathbf{d}_I \quad (\text{since } 2\varepsilon \mathbf{d}_{I'} < 1 \leq j' - i' \text{ and } 2\varepsilon \mathbf{d}_I < 1 \leq j - i) \\ \mathbf{j}_{e'} &< \mathbf{i}_{e'} < \mathbf{j}_e < \mathbf{i}_e\end{aligned}$$

Thus, the x-coordinate of each of the 3 subdivision segments of e' is at most $\mathbf{i}_{e'}$, and the x-coordinate of each of the 3 subdivision segments of e is at least \mathbf{j}_e . This completes the proof of Case 1.

2. Case 2: One of I or I' contains the other. Suppose that $I \subseteq I'$ (the other sub-case, $I' \subseteq I$, is similar). Then, note that $\mathbf{d}_{I'} < \mathbf{d}_I$. We have the following.

$$\begin{aligned}
i' &\leq i < j \leq j' && \text{(since } I \subseteq I') \\
-j' &\leq -j < -i \leq -i' \\
-j' + \varepsilon \mathbf{d}_{I'} &< -j + \varepsilon \mathbf{d}_I < -i - \varepsilon \mathbf{d}_I < -i' - \varepsilon \mathbf{d}_{I'} && \text{(since } \mathbf{d}_{I'} < \mathbf{d}_I \text{ and } 2\varepsilon \mathbf{d}_I < 1 \leq j - i) \\
\mathbf{j}_{e'} &< \mathbf{j}_e < \mathbf{i}_e < \mathbf{i}_{e'}
\end{aligned}$$

Thus, the x-coordinate of $\mathbf{c}(u_{e'}^3)$ (namely $\mathbf{j}_{e'}$) is less than the x-coordinate of each of the 3 subdivision segments of e , and the x-coordinate of $\mathbf{c}(u_{e'}^1)$ (namely $\mathbf{i}_{e'}$) is greater than the x-coordinate of each of the 3 subdivision segments of e . Finally, the y-coordinate of $\mathbf{c}(u_{e'}^2)$ (namely $\mathbf{j}_{e'}$) is less than the y-coordinate of each of the 3 subdivision segments of e . This completes the proof of Case 2.

This completes the construction of the STABGIG representation for the subgraph of $H_{3\text{-div}}^*$ induced by the original vertices of H , the apex vertex a and the subdivision vertices located in the left half-plane. For the vertices and edges located in the right half-plane, the construction is similar but replaces the \sqcup -shapes by \sqcap -shapes. The correctness of our construction follows from Cases 1 and 2 above.

3.2.3 STABGIG representation of $G_{k\text{-div}}^*$

In order to conclude our proof of Proposition 9, we need – according to item (f) – to convert the STABGIG representation of $H_{3\text{-div}}^*$ into a STABGIG representation of $G_{k\text{-div}}^*$. This operation reduces to further subdividing one edge of $H_{3\text{-div}}^*$, that may be appropriately chosen, by transforming it into a path of odd length $2h + 1$, for an appropriate integer $h \geq 1$. We always choose an edge represented by a horizontal segment, and subdivide it (that is, we replace it with a sequence of segments), as illustrated in Figure 11.

Consider the STABGIG representation of $H_{3\text{-div}}^*$. For each edge $e \in E_{\text{cross}}(G)$, a 7-subdivision is performed in $H_{3\text{-div}}^*$. But for each edge $e \in E_{\text{left}}(G) \cup E_{\text{right}}(G)$, only a 3-subdivision is performed in $H_{3\text{-div}}^*$. When $k = 7$, the latter edges are insufficiently subdivided in $H_{3\text{-div}}^*$ with respect to $G_{k\text{-div}}^*$. Moreover, when $k > 7$ all the original edges in G are insufficiently subdivided. In all cases, let $e \in E(G)$ be one of the edges that are insufficiently subdivided, and assume that $e \in E_{\text{left}}(H)$. (Otherwise, when $e \in E_{\text{right}}(H)$ the approach is similar, whereas when $e \in E_{\text{cross}}(G)$ we have to perform the same approach on the 'half-edge' of e that belongs to $E_{\text{left}}(H)$.) In $H_{3\text{-div}}^*$, e is 3-subdivided. Assume we need a k -subdivision of e , with odd $k \geq 7$, and let $h = (k - 3)/2$. Then we perform a $2h$ -subdivision of the edge $u_e^1 u_e^2$ of $H_{3\text{-div}}^*$, and describe below the new segments representing u_e^1, u_e^2 and the $2h$ supplementary vertices.

In $H_{3\text{-div}}^*$, the intersection point between the segments $\mathbf{c}(u_e^3)$ and $\mathbf{c}(u_e^2)$ is $(\mathbf{j}_e, \mathbf{j}_e)$. The closest possible similar point located on the stab line towards right is the point $(\mathbf{j}_e + \varepsilon, \mathbf{j}_e + \varepsilon)$. We then replace the segment $\mathbf{c}(u_e^2)$ with a succession of $2h + 1$ alternating horizontal and vertical segments whose intersections define the path required by the $2h$ -subdivision of (u_e^1, u_e^2) . See Figure 11.

Formally, let $\varepsilon_2 = \varepsilon/(h + 1)$, and define the following segments. The horizontal segment obtained for $t = 0$ is the new segment $\mathbf{c}(u_e^2)$.

For $0 \leq t \leq h - 1$:

$$\begin{aligned}
&[(\mathbf{j}_e + t\varepsilon_2, \mathbf{j}_e + t\varepsilon_2), (\mathbf{j}_e + (t + 1)\varepsilon_2, \mathbf{j}_e + t\varepsilon_2)] \\
&[(\mathbf{j}_e + (t + 1)\varepsilon_2, \mathbf{j}_e + t\varepsilon_2), (\mathbf{j}_e + (t + 1)\varepsilon_2, \mathbf{j}_e + (t + 1)\varepsilon_2)]
\end{aligned}$$

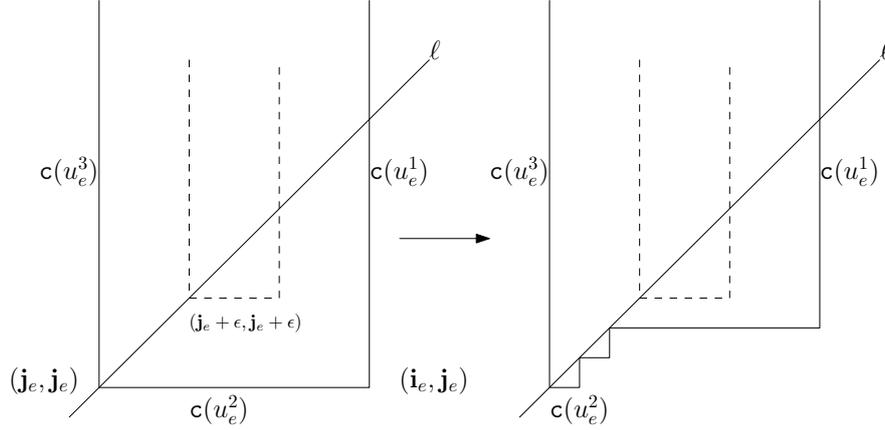


Figure 11: Replacement of $c(u_e^2)$ (left), for some edge e which is not sufficiently subdivided in $H_{3\text{-div}}^*$, with a sequence of segments. The dotted \sqcup -shape indicates the closest possible location of an \sqcup -shape located above the one we defined for e .

Complete this set of segments with:

$$[(\mathbf{j}_e + h\varepsilon_2, \mathbf{j}_e + h\varepsilon_2), (\mathbf{i}_e, \mathbf{j}_e + h\varepsilon_2)]$$

$$c(u_e^1) = [(\mathbf{i}_e, \mathbf{j}_e + h\varepsilon_2), (\mathbf{i}_e, i - 1)]$$

All the segments corresponding to the $2h$ new subdivision vertices lie strictly below the closest upper segment not involved in the subdivision of e (whose y -coordinate is at least $\mathbf{j}_e + \varepsilon$), and (not strictly) above the former segment $c(u_e^2)$. They also lie between the x -coordinates of $c(u_e^3)$ and $c(u_e^1)$. Thus they cannot intersect other segments. So we have a representation of the path with $(2h + 1)$ edges required by the subdivision.

This ends the proof of Proposition 9.

3.2.4 Proof of Theorem 2

We argue that recognizing STABGIG is in NP. Note that a graph is a STABGIG if and only if it has a stabbable grid intersection representation where the stab line has slope 1, and the intersection points of the segments with the stab line as well as with the other segments have integer coordinates. Such a representation is called *good*. The backward direction is immediate. The forward direction is proved as follows. Let G be a STABGIG, and consider an arbitrary stabbed grid intersection representation of it. First, appropriately rotate and then scale the representation in the horizontal direction, in order to obtain a grounded line with slope 1. Second, order the segments according to the bottom to top order of their intersection points with the stab line. Given a segment s , let $r(s)$ be its rank in this order. Third, build a new stabbed representation, as follows. For each horizontal segment s of the initial representation, its new y -coordinate is $r(s)$, and the new x -coordinates of its left and right endpoints are $\min\{r(s') \mid s' \text{ intersects } s\} \cup \{r(s)\}$ and $\max\{r(s') \mid s' \text{ intersects } s\} \cup \{r(s)\}$ respectively. For each vertical segment s of the initial representation, its new x -coordinate is $r(s)$ and the new y -coordinates of its bottom and top endpoints are $\min\{r(s') \mid s' \text{ intersects } s\} \cup \{r(s)\}$ and $\max\{r(s') \mid s' \text{ intersects } s\} \cup \{r(s)\}$ respectively. Then, the segment s intersects the stab line at $(r(s), r(s))$ and any other segment s' it must intersect at $(r(s), r(s'))$ when s is vertical, respectively at $(r(s'), r(s))$ when s is horizontal.

Therefore, it is sufficient to prove that the problem of deciding whether an input graph has a good representation is in NP. Due to the integer coordinates, the task of checking whether a given set of segments, one for each vertex, is a good representation of the input graph is polynomial. Furthermore, $G_{k\text{-div}}^*$ is obtained in polynomial time from G . It remains to show that G is a yes-instance of PHPC if and only if $G_{k\text{-div}}^*$ is a yes-instance of STABGIGREC. Proposition 9 shows the forward direction. For

the backward direction, we remark – seeking to use Proposition 8 – that if $G_{k\text{-div}}^*$ is a STABGIG, then $G_{k\text{-div}}^*$ is a 1-string graph. To this end, it is sufficient to note that each segment in a grid representation of $G_{k\text{-div}}^*$ is a 1-string, and that two segments that intersect without crossing (because an endpoint of one segment lies on the second segment) may be easily transformed into two crossing segments. Now, using Proposition 8, we deduce that G is a yes-instance of PHPC. This completes the proof of Theorem 2.

Observe that the graph $G_{k\text{-div}}^*$ used in our proof is ‘almost’ planar, in the sense that $G_{k\text{-div}}^* \setminus \{a\}$ is planar. A graph that can be made planar by removing one of its vertices is called an *apex* graph [26, 25]. Moreover, $G_{k\text{-div}}^*$ has girth $g = k + 3 \geq 10$ so that we may deduce the following result.

Corollary 11. *For each integer $g \geq 10$, STABGIGREC remains NP-complete even if the inputs are restricted to apex graphs of girth exactly g .*

4 Conclusion and Open problems

In this paper, we proved that recognizing grounded L-graphs is NP-complete. One direction of research would be to consider natural super classes of grounded L-graphs and study the computational complexity of recognizing those classes.

Assume as before that the ground line is horizontal. A *skew L-shape* is an L-like shape where the angle α between the two segments satisfies $0 < \alpha < \pi$. Thus an L-shape is a skew L-shape with $\alpha = \pi/2$. Similarly to grounded L-shapes, grounded skew L-shapes are placed *above* the ground line, and thus become *grounded skew Υ -shapes*. The segment of a skew Υ -shape containing the anchor is the *anchored segment* (or *A-segment*), whereas its other segment is the *bent segment* (or *B-segment*). The angle between the A-segment of a grounded skew Υ -shape and the negative x -axis (that we imagine along the ground line) is denoted β , and satisfies $0 < \beta < \pi$. A skew Υ -shape with angle α that is anchored to the ground line using an angle β is said to be of (α, β) -*type*.

Let S be a set of pairs (α, β) with $0 < \alpha, \beta < \pi$. The class of intersection graphs of grounded skew Υ -shapes whose types belong to S is denoted $Skew_S$. Then $Skew_{\{(\pi/2, \pi/2)\}}$ is the class of grounded L-graphs. Moreover, $Skew_{\{(\pi/2, \pi/2)\}} = Skew_{\{(\pi/2, \beta)\}}$, for each $0 < \beta < \pi$. To see this, consider the representation of a graph belonging to $Skew_{\{(\pi/2, \beta)\}}$, and *readjust* it by performing a rotation of the ground line (alone) with an appropriate angle until this line becomes orthogonal to the A-segments of the Υ -shapes, and then another rotation of the entire representation (as a block) so as to get an horizontal ground line.

We focus now on $Skew_S$ with $S = \{(\pi/2, \beta_1), (\alpha_2, \beta_2)\}$, where $0 < \beta_1, \alpha_2, \beta_2 < \pi$. The following classes are defined as immediate generalizations of grounded L-graphs, and we provide examples showing that they are indeed strictly larger than grounded L-graphs.

- I. $S = \{(\pi/2, \beta_1), (\pi/2, \beta_2)\}$ (same shape, different ground angles)
- II. $S = \{(\pi/2, \beta_1), (\alpha_2, \beta_1)\}$ (different shapes, same ground angles)

Lemma 12. *The graph shown in Figure 12(a) is not a grounded L-graph.*

Proof. Let G denote the graph shown in Figure 12(a) and let G' be the subgraph induced by $V(G) \setminus \{11\}$. Observe that G' is exactly the gadget we used in the NP-completeness proof for grounded L-graphs. See Figure 4(left).

Assume by contradiction that G has a Υ -representation R . Consider the Υ -representation of G' in R and let def be the permutation of $\{1, 2, 3\}$ in a left to right ordering of the anchors in R . Due to Proposition 5, G' admits exactly two grounded Υ -representations (given in Figure 5(c)), where we have either $d = 1$ and $f = 3$, or $d = 3$ and $f = 1$. When $(d, f) = (1, 3)$ the set $\{8, 9, 6, 4\}$, containing the neighbors of 11, satisfies $\{8, 9, 6, 4\} = \{v_e^1, v_f^1, v_f^2, v_d^2\}$, and it is easy to check from Figure 5(c) that there cannot be a Υ -shape which only intersects $\{8, 9, 6, 4\}$. When $(d, f) = (3, 1)$ we have $\{8, 9, 6, 4\} = \{v_e^1, v_d^1, v_d^2, v_f^2\}$ and again it is easy to check that there cannot be a Υ -shape which only intersects $\{8, 9, 6, 4\}$. Hence there is no possible place for $\Upsilon(11)$ and G is not a grounded L-graph. \square

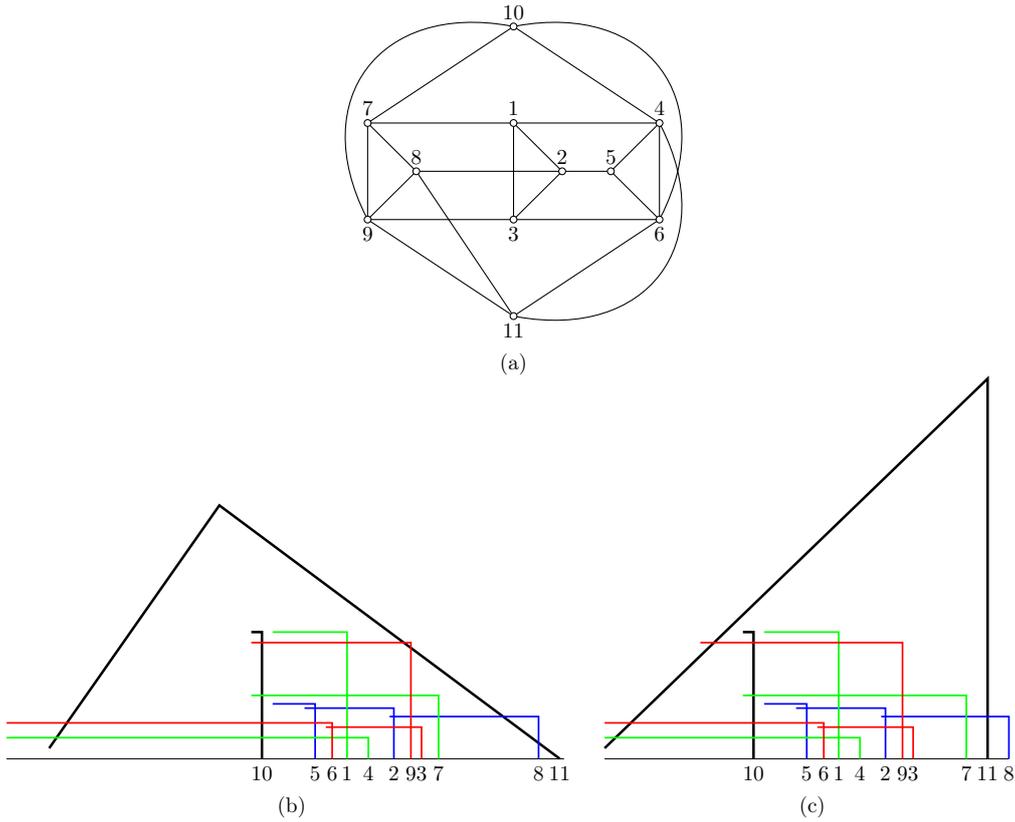


Figure 12: (a) A graph G that is not a grounded L-graph but lies in both class I and class II with $\alpha_2 < \pi/2$. (b) shows a $Skew_S$ representation of G with $S = \{(\pi/2, \beta_1), (\pi/2, \beta_2)\}$ assuming w.l.o.g. that $\beta_1 > \beta_2$ (the representation is readjusted such that β_1 becomes $\pi/2$, and thus β_2 becomes $\beta'_2 < \pi/2$). (c) shows a $Skew_S$ representation of G with $S = \{(\pi/2, \beta_1), (\alpha_2, \beta_1)\}$ and $\alpha_2 < \pi/2$ (the representation is readjusted such that β_1 becomes $\pi/2$).

Figure 12(b) and Figure 12(c) imply that Figure 12(a) lies in both class I and class II with $\alpha_2 < \pi/2$. Indeed, assume without loss of generality that $\beta_1 > \beta_2$ (in class I), and β_1 is arbitrary (in class II). Consider one of the classes, and readjust the representation such that the angle β_1 becomes an angle of $\pi/2$, similarly to the readjustment we performed above for $Skew_{\{(\pi/2, \beta)\}}$. Then in class I the new angle β'_2 satisfies $\beta'_2 = \beta_2 + \pi/2 - \beta_1$ and, except in the particular case where $\beta_1 = \beta_2 + \pi/2$, we have that $0 < \beta_2 < \pi/2$. Thus Figure 12(b) is a representation of G with respect to class I. (Note that different angles β'_2 are obtained by scaling the figure.) In class II, Figure 12(c) shows a representation of the graph G when $\alpha_2 < \pi/2$. (Similarly, different angles α_2 are obtained by scaling the figure.) When $\alpha_2 > \pi/2$, a similar example may be found, where the vertex 11 is adjacent with the set $\{1, 3, 5, 7, 8, 9\}$ of vertices in the gadget (see Figure 13(a)), that is, with $\{d, f, v_e^1, v_e^2, v_d^1, v_f^1\}$ (see Figure 5(c)). The new graph has no grounded Υ -representation, but has a representation as in class II with a grounded skew Υ -shape anchored between v_f^1 and f (see Figure 13(b)).

We deduce that class I and class II are strict super-classes of grounded L-graphs. The following question is open:

Question 1. *What are the computational complexities of recognizing classes I and II?*

Since grounded L-graphs, and thus stick graphs, belong to both classes I and II, a reduction similar to that in Section 2 could be attempted to prove the NP-hardness of recognizing classes I and II. However, the gadget $\Lambda(x)$ is not efficient for either of these classes, as its representations in classes I and II are not sufficiently restricted to deduce a result similar to Proposition 6. Figure 14 shows counterexamples. More sophisticated gadgets, that have more restricted representations in classes I or II, are needed to

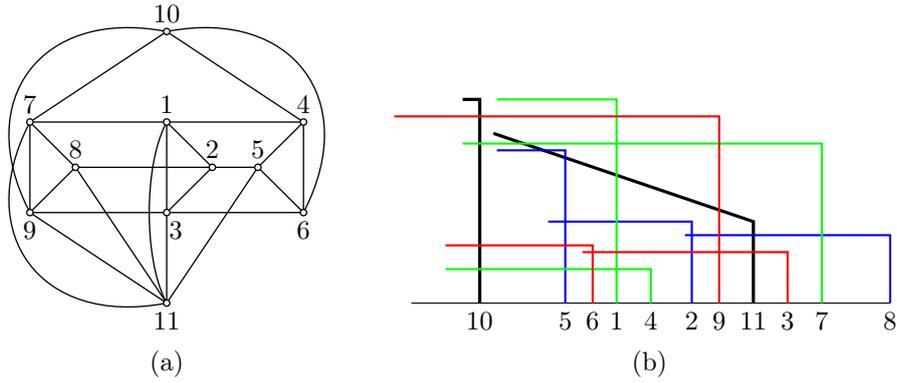


Figure 13: A graph G that is not a grounded L-graph but lies in class II with $\alpha_2 > \pi/2$. (b) shows a $Skew_S$ representation of G with $S = \{(\pi/2, \beta_1), (\alpha_2, \beta_1)\}$ and $\alpha_2 > \pi/2$.

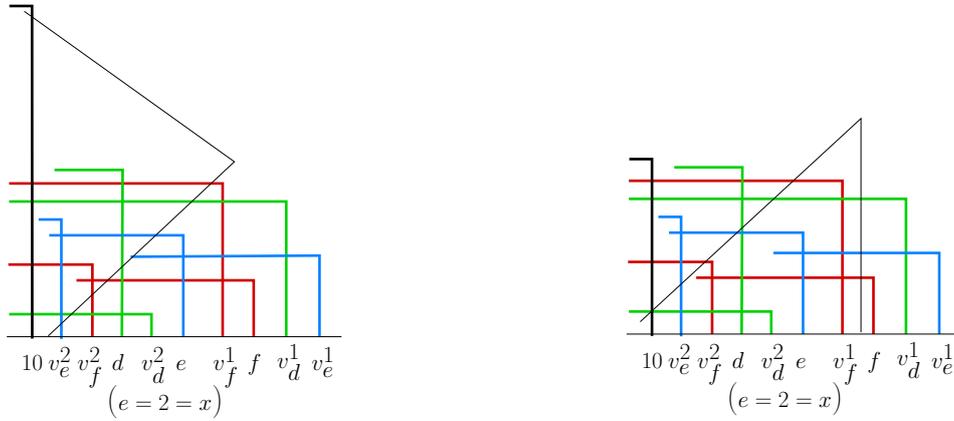


Figure 14: Representations of $\Lambda(x) + u$ in classes I and II respectively, where u is not the rightmost anchor. For $\Lambda(x)$, the notations are those in Figure 5(c). The shape $\Upsilon(u)$ is the one with $\alpha \neq \pi/2$ and $\beta \neq \pi/2$ respectively.

prove that recognizing these classes are NP-hard.

Another direction of research would be to find interesting subclasses of grounded L-graphs with polynomial time recognition algorithms. One such candidate class could be the *grounded unit L-graphs*. A unit L-shape is made by joining the bottom end-point of a vertical ($|$) segment (of arbitrary length) to the left end-point of a horizontal ($-$) segment of *unit* length. Grounded unit L-graphs are the intersection graphs of unit L-shapes such that all the L-shapes' anchors lie on the same horizontal line. Grounded unit L-graphs is a subclass of *cocomparability* graphs and contains *unit interval* graphs and *permutation graphs* as its subclasses. The above discussion motivates the following question.

Question 2. *What is the computational complexity of recognizing grounded unit L-graphs?*

Another open question concerns grounded square L-graphs i.e. intersection graphs of grounded L-shapes whose horizontal and vertical segments have the same length [7].

Question 3. *What is the computational complexity of recognizing grounded square L-graphs?*

As noticed in Corollary 11, recognizing STABGIG remains NP-complete even if the inputs are restricted to apex graphs. Apex graphs do not contain K_6 as a minor. In contrast, recognizing K_4 -minor free stabbable grid intersection graphs is trivial (since K_4 -minor free graphs are planar and all planar bipartite graphs are stabbable grid intersection graphs). This motivates the following question.

Question 4. *What is the computational complexity of recognizing K_5 -minor free stabbable grid intersection graphs?*

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