Article:

Zastawniak, Tomasz (2024) Fundamental Theorem of Asset Pricing under fixed and proportional costs in multi-asset setting and finite probability space. Decisions in Economics and Finance. pp. 137-149. ISSN 1129-6569

https://doi.org/10.1007/s10203-024-00439-z

Decisions in Economics and Finance
https://doi.org/10.1007/s10203-024-00439-z

Fundamental Theorem of Asset Pricing under fixed and proportional costs in multi-asset setting and finite probability space

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Received: 4 April 2023 / Accepted: 17 February 2024

Abstract
The Fundamental Theorem of Asset Pricing is extended to a market model over a finite probability space with many assets that can be exchanged into one another under combined fixed and proportional transaction costs. The absence of arbitrage in this setting is shown to be equivalent to the existence of a family of absolutely continuous single-step probability measures and a multi-dimensional martingale with respect to such a family.

Keywords Arbitrage · Transaction costs · Martingale measure

JEL codes C00, C65, G10, G12

1 Introduction

In this paper the Fundamental Theorem of Asset pricing is extended to a market model with many assets which can be exchanged directly into one another in the presence of both proportional and fixed transaction costs, and without necessarily involving a numeraire. The setting is similar to the multi-asset model with proportional costs introduced by Kabanov (1999) and further developed by Kabanov and Stricker (2001), Kabanov et al. (2002), Schachermayer (2004), and others, but includes fixed transaction costs in addition to proportional costs.

In Kabanov's model (Kabanov 1999) proportional transaction costs are implemented as a matrix of exchange rates $\pi_t^{ij} > 0$ between assets i, j at time t. On top of that, we allow for fixed cost $C_t^{ij} > 0$ whenever a positive quantity of asset i is exchanged into asset j at time t. Such a combination of fixed and proportional costs is

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Published online: 12 March 2024

ever present in real-world markets, with multiple assets such as currencies exchanged directly into one another. It is therefore of great importance to characterize the absence of arbitrage in this setting in terms of risk neutral probabilities.

The main difficulties stem form the fact that the set of solvent portfolios becomes non-convex when fixed costs are added on top of proportional ones. Moreover, just like in the case of multiple assets with proportional transaction costs, the absence of arbitrage in a model with many time steps is not equivalent to the condition that every single-step model should be free from arbitrage. This precludes an argument by reduction to a single step or direct application of convex analysis methods to handle the problem in hand.

It turns out that in the presence of both proportional and fixed transaction costs, the absence of arbitrage in a market with multiple assets is equivalent to the existence of a so-called family of absolutely continuous single-step probability measures and a multi-dimensional martingale with respect to such a family. These notions are introduced in Sect. 2, and the result extending the Fundamental Theorem of Asset pricing is established in Theorem 1. The family of single-step probability measures plays a role similar to a risk neutral probability in the classical friction-free setting. This builds on the work of Brown and Zastawniak (2020), who proved the Fundamental Theorem of Asset Pricing under combined, fixed and proportional, transaction costs for a model with a single risky asset and a cash numeraire. To handle the multi-asset case a different proof has been developed to circumvent the complexities of an explicit description of the various non-convex multidimensional geometric objects involved.

The present paper can be seen as a development in a long line of results starting with the seminal paper by Harrison and Pliska (1981), who established the Fundamental Theorem of Asset Pricing for discrete time models on a finite probability space in the absence of market friction. Dalang et al. (1990) extended the theorem to the case of infinite state space, and Delbaen and Schachermayer (1994, 2006) to continuous time. Harrison and Pliska's result was extended to models with proportional transaction costs by Jouini and Kallal (2001), Kabanov and Stricker (2001), Ortu (2001). Roux (2011) allowed for interest rate spreads in addition to proportional transaction costs. Models with proportional transaction costs on an infinite state space were studied by Zhang et al. (2002), Kabanov et al. (2002), Schachermayer (2004). Further results concerning the Fundamental Theorem in the presence of proportional costs include (Kabanov 1999; Grigoriev 2005; Bouchard 2006; Guasoni 2006; Cherny 2007; Kabanov and Safarian 2009; Guasoni et al. 2010, 2012; Denis and Kabanov 2012; Dolinsky and Soner 2014; Rola 2015; Zhao and Lepinette 2018), and others. Under fixed costs alone, the Fundamental Theorem was studied in just one paper by Jouini et al. (2001). Furthermore, no-arbitrage conditions for fixed-cost models in terms of separating risk measures (but not risk neutral measures as in the Fundamental Theorem) can be found in the work of Lepinette and Tran (2016, 2017).

This paper extends the previous work, cited above, on multi-asset models under proportional costs, as well as the results of Jouini et al. (2001) involving fixed costs. The absence of arbitrage under proportional costs was characterised in terms of the existence of an equivalent measure and a process with values within the bid-ask spreads that is a martingale under this measure. Moreover, Jouini et al. (2001) characterised the absence of arbitrage under fixed costs in terms of the existence of a family of absolutely continuous measures on all subtrees such that the asset price process is a martingale under each of these measures. Here we extend both these approaches in the multi-asset case in a setting with both proportional and fixed transaction costs. In doing so, we also simplify the object (a family of measures on all subtrees) used by Jouini et al. (2001), by demonstrating that a significantly simpler object, a family of measures on one-step subtrees is enough to characterise the absence of arbitrage not only under fixed costs, but in fact also under combined, proportional and fixed, costs.

With reference to the work of Lepinette and Tran (2016, 2017), who studied arbitrage in a non-convex market model with friction, including simultaneous proportional and fixed costs, we observe that they characterised the absence of asymptotic arbitrage in terms of the existence of a so-called equivalent separating probability measure, and no link with martingale measures was made in their work.

The paper is organised as follows. The preliminaries cover some notation and a few basic facts and notions. The multi-asset model with fixed and proportional cost is described in the following section, which leads to the definition of an arbitrage strategy in this setting. Next, we have some auxiliary results. The Fundamental Theorem of Asset Pricing for a multi-asset market under fixed and proportional costs is formulated and proved in the final section.

2 Preliminaries

Let d be a positive integer, the number of assets in the market. By $x \cdot y = \sum_{i=1}^{d} x^i y^i$ we denote the scalar product of $x, y \in \mathbb{R}^d$. Let $e^i \in \mathbb{R}^d$, $i = 1, \ldots, d$ be the canonical basis in $\mathbb{R}^d$. That is, for any $i, j = 1, \ldots, d$ we have $(e^i)^j = 1$ if $i = j$ and $(e^i)^j = 0$ if $i \neq j$. For any $x, y \in \mathbb{R}^d$, we shall write $x \leq y$ (resp. $x < y$) whenever $x^i \leq y^i$ (resp. $x^i < y^i$) for each $i = 1, \ldots, d$. For any $c \in \mathbb{R}$, we shall also write c in place of $c \sum_{i=1}^{d} e^i \in \mathbb{R}^d$. For a convex cone $A \subset \mathbb{R}^d$, the dual cone will be denoted by $A^* = \left\{x \in \mathbb{R}^d \,|\, 0 \leq x \cdot y \text{ for each } y \in A\right\}$. If $A, B \subset \mathbb{R}^d$ are closed convex cones, then $(A^*)^* = A$, $(A \cap B)^* = A^* + B^*$ and $(A + B)^* = A^* \cap B^*$.

Let T be a positive integer and $(\Omega, \Sigma, \mathbb{P})$ a finite probability space with filtration $\mathscr{F} = (\mathscr{F}_t)_{t=0}^{T}$ such that $\mathscr{F}_0 = \{\emptyset, \Omega\}$. The probability measure $\mathbb{P}$ will play the role of physical probability.

Definition 1 For any $t = 0, \ldots, T$, we write Λ_t for the set of atoms in $\mathscr{F}_t$, and refer to the elements of Λ_t as the *nodes* at time t. For any $t = 0, \ldots, T-1$ and any node $\lambda \in \Lambda_t$, we write

$$\operatorname{succ}(\lambda) = \{\mu \in \Lambda_{t+1} \,|\, \mu \subset \lambda\},$$

and call it the set of *successor nodes* of λ.

We can regard any $\mathscr{F}_t$-measurable random variable X as a function defined on Λ_t.

Definition 2 By a *family of absolutely continuous single-step probability measures* we understand a collection Q of probability measures $Q_t(\lambda, \cdot)$ defined on the sigma-field

$$\lambda \cap \mathscr{F}_{t+1} = \{\lambda \cap A \,|\, A \in \mathscr{F}_{t+1}\}$$

for each $t = 0, \dots, T-1$ and each $\lambda \in \Lambda_t$ such that $\mathbb{P}(\lambda) > 0$, and satisfying the condition

$$Q_t(\lambda, \mu) = 0 \quad \text{for each } \mu \in \text{succ}(\lambda) \text{ such that } \mathbb{P}(\mu) = 0.$$

Definition 3 Given a family of absolutely continuous single-step probability measures Q, for any $t = 0, \dots, T-1$ and any $\mathscr{F}_{t+1}$-measurable random variable Y, we define $E_Q(Y|\mathscr{F}_t)$ to be an $\mathscr{F}_t$-measurable random variable such that

$$E_Q(Y|\mathscr{F}_t)(\lambda) = \sum_{\mu \in \text{succ}(\lambda)} Q_t(\lambda, \mu) Y(\mu) \quad \text{for each } \lambda \in \Lambda_t \text{ with } \mathbb{P}(\lambda) > 0.$$

Moreover, we say that an adapted $\mathbb{R}^d$-valued process $S = (S_t)_{t=0}^T$ is a *martingale* with respect to Q whenever

$$S_t = \mathbb{E}_Q(S_{t+1}|\mathscr{F}_t) \quad \text{for each } t = 0, \dots, T-1.$$

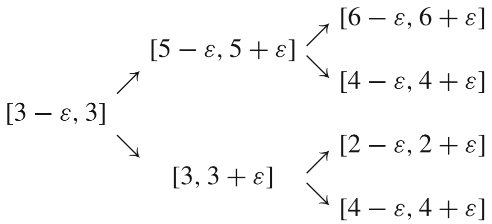
For each family of absolutely continuous single-step measures Q, there is the corresponding probability measure $\mathbb{Q}$ on Λ_T given by

$$\mathbb{Q}(\lambda) = \begin{cases} \prod_{t=0}^{T-1} Q_t(\lambda_t, \lambda_{t+1}) & \text{if } \mathbb{P}(\lambda) > 0 \\ 0 & \text{if } \mathbb{P}(\lambda) = 0 \end{cases} \quad \text{for every } \lambda \in \Lambda_T,$$

where $\lambda_t \in \Lambda_t$, $t = 0, \dots, T$ is the unique sequence of nodes such that $\lambda = \lambda_T \subset \cdots \subset \lambda_0$. Clearly, $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. If S is a martingale with respect the family of measures Q, then it is also a martingale under $\mathbb{Q}$ in the ordinary sense. Generally, Q is a richer structure than $\mathbb{Q}$. The same measure $\mathbb{Q}$ may correspond to several different families Q.

For simplicity, throughout this paper we assume that $\mathbb{P}(\omega) > 0$ for each $\omega \in \Omega$. Otherwise we can remove all $\omega \in \Omega$ such that $\mathbb{P}(\omega) = 0$. Because we work on a finite probability space, this involves no loss of generality.

Example 1 We take a two-step binary tree as the probability space Ω, with nodes u, d at time 1 and uu, ud, du, dd at time 2, and two assets: cash with interest rate 0, and a risky asset with the following bid-ask spreads, where $0 < \varepsilon < 1/2$:



There is a family of absolutely continuous single-step measures Q and a process S with values within the bid ask spreads such that S is a martingale under Q:

$$\begin{array}{llll} Q_0(\Omega, \mathrm{u}) = 0, & & Q_0(\Omega, \mathrm{d}) = 1, & \\ Q_1(\mathrm{u}, \mathrm{uu}) = \frac{1}{2}, & Q_1(\mathrm{u}, \mathrm{ud}) = \frac{1}{2}, & Q_1(\mathrm{d}, \mathrm{du}) = \frac{1}{2}, & Q_1(\mathrm{d}, \mathrm{dd}) = \frac{1}{2} \end{array}$$

and

$$\begin{array}{ccccc} & & & \nearrow & S_2(\mathrm{uu}) = 6 \\ & & S_1(\mathrm{u}) = 5 & & \\ & \nearrow & & \searrow & S_2(\mathrm{ud}) = 4 \\ S_0 = 3 & & & & \\ & \searrow & & \nearrow & S_2(\mathrm{du}) = 2 \\ & & S_1(\mathrm{d}) = 3 & & \\ & & & \searrow & S_2(\mathrm{dd}) = 4 \end{array}$$

The family Q gives rise to the absolutely continuous measure $\mathbb{Q}$ such that

$$\mathbb{Q}(\mathrm{uu}) = \mathbb{Q}(\mathrm{ud}) = 0, \quad \mathbb{Q}(\mathrm{du}) = \mathbb{Q}(\mathrm{dd}) = \frac{1}{2}$$

and S is a martingale under $\mathbb{Q}$.

Observe that there is no equivalent measure making a process with values within the bid-ask spreads a martingale. This is related to the fact that there is an arbitrage opportunity under proportional transaction costs (that is, when the fixed costs are zero): at time 0 buy the risky asset for the ask price 3, and at time 1 sell it for the bid price 3 at node d or $5 - \varepsilon$ at node u.

Nonetheless, there is in fact no arbitrage opportunity when fixed transaction costs are also present: in the above strategy there will be a loss due to the fixed costs when selling the asset for the bid price 3 at node d. We shall see in Theorem 1 that this is related to the existence of the family of measures Q and the martingale S under Q.

It is also interesting to see what happens if we modify the bid-ask spread at node ud to be $[7 - \varepsilon, 7 + \varepsilon]$, leaving the remaining bid-ask spreads as above. This creates an arbitrage opportunity also in the presence of positive fixed costs: at time 1, node u buy $x > 0$ shares of the risky asset for the ask price $5 + \varepsilon$, and at time 2 sell them for the bid price $6 - \varepsilon$ at node uu or $7 - \varepsilon$ at node ud. Without fixed costs the profit would be at least $x((6 - \varepsilon) - (5 + \varepsilon)) = x(1 - 2\varepsilon)$, and taking x large enough so that this amount is bigger than the fixed costs paid, one can achieve arbitrage with the fixed costs present. Observe that in this case there is no family of absolutely continuous single-step measures that can make a process with values within the bid-ask spreads a martingale. Once again, this is Theorem 1 at work. Moreover, observe that the above absolutely continuous measure $\mathbb{Q}$ is such that there is a martingale under $\mathbb{Q}$ with values within the bid-ask spreads, but no equivalent measure can have this property.

Based on these observations, we can conclude that:

– An equivalent martingale measure cannot be used to characterise the existence of arbitrage under combined (that is, proportional and fixed) costs: we have seen an example where there is no arbitrage under combined costs, yet an equivalent martingale measure does not exist. We need to allow nodes of measure zero to characterise arbitrage under combined costs.

– But an absolutely continuous martingale measure cannot be used to characterise the lack of arbitrage in the presence of combined costs either: we have seen an example where an absolutely continuous martingale measure exists, yet there is arbitrage under combined costs. This is because such a measure can't keep track of what happens beyond any node of measure zero (node u in the example), while the reason why arbitrage exists or not may be beyond such a node.
– A family of absolutely continuous single-step measures is capable of keeping track beyond nodes of measure zero. The existence of such a family is exactly what is needed to characterise the absence of arbitrage under combined costs. This is the result proved in Theorem 1 for multi-asset models.

3 Multi-asset model with fixed and proportional transaction costs

On the probability space $(\Omega, \Sigma, \mathbb{P})$ with filtration $\mathscr{F} = (\mathscr{F}_t)_{t=0}^T$ we consider a model with d assets which can be exchanged into one another. The exchange rates representing proportional transaction costs are given by an $\mathbb{R}^{d\times d}$-valued adapted process $\pi = (\pi_t)_{t=0}^T$, and the fixed transaction costs by an $\mathbb{R}^{d\times d}$-valued adapted process $C = (C_t)_{t=0}^T$. We assume that $\pi_t^{ij}(\lambda) > 0$ and $C_t^{ij}(\lambda) > 0$ for each time $t = 0, \ldots, T$, node $\lambda \in \Lambda_t$ and $i, j = 1, \ldots, d$, where $\pi_t^{ij}(\lambda)$ is the number of units of asset i that need to be exchanged to receive a single unit of asset j, and where $C_t^{ij}(\lambda) > 0$ is the fixed cost paid in units of asset i when exchanging a positive quantity of asset i into asset j at time t and node λ.

A portfolio of assets at time $t = 0, \ldots, T$ can be represented by an $\mathbb{R}^d$-valued $\mathscr{F}_t$-measurable random variable x.

Definition 4 We say that the portfolio is *solvent under combined transaction costs* (that is, proportional and fixed transaction costs) when

$$0 \leq x^j + \sum_{i=1}^d z^{ij} - \sum_{i=1}^d z^{ji}\pi_t^{ji} - \sum_{i=1}^d 1_{\{z^{ji}>0\}} C_t^{ji} \quad \text{for each } j = 1, \ldots, d$$

for some $\mathbb{R}^{d\times d}$-valued $\mathscr{F}_t$-measurable random variable $z = (z^{ij})_{i,j=1}^d$ such that $z^{ij} \geq 0$ for each $i, j = 1, \ldots, d$. The set of such solvent portfolios x under combined transaction costs will be denoted by $\mathscr{C}_t$.

Here z^{ij} represents the number of units of asset j received by exchanging asset i, and $z^{ji}\pi_t^{ji}$ the number of units of j changed into i. This solvency condition can equivalently be written as

$$0 \leq x - \sum_{i,j=1}^d z^{ij} f_t^{ij} - c_t(z),$$

where $f_t^{ij} = \pi_t^{ij} e^i - e^j$ for each $i, j = 1, \ldots, d$, with e^i, $i = 1, \ldots, d$ being the canonical basis in $\mathbb{R}^d$, and where

$$c_t(z) = \sum_{i,j=1}^{d} 1_{\{z^{ji}>0\}} C_t^{ji} e^j.$$

Definition 5 For any $t = 0, \ldots, T$, we shall also use the symbol $\mathscr{K}_t$ to denote the set of *solvent portfolios under proportional costs*, which are $\mathbb{R}^d$-valued $\mathscr{F}_t$-measurable random variables x satisfying the condition

$$0 \le x - \sum_{i,j=1}^{d} z^{ij} f_t^{ij}$$

for some $\mathbb{R}^{d \times d}$-valued $\mathscr{F}_t$-measurable random variable $z = (z^{ij})_{i,j=1}^d$ such that $z^{ij} \ge 0$ for each $i, j = 1, \ldots, d$.

Clearly, $\mathscr{K}_t$ is a convex cone generated by the vectors e^i, $i = 1, \ldots, d$ and f_t^{ij}, $i, j = 1, \ldots, d$ in the space $\mathscr{L}_t$ of $\mathbb{R}^d$-valued $\mathscr{F}_t$-measurable random variables.

Definition 6 A *self-financing strategy* under combined costs is an $\mathbb{R}^d$-valued predictable process $X = (X_t)_{t=0}^{T+1}$ such that

$$X_t - X_{t+1} \in \mathscr{C}_t \quad \text{for each } t = 0, \ldots, T.$$

We say that X is an *arbitrage strategy* under combined costs if, additionally,

$$X_0 = 0, \quad \mathbb{P}(0 \le X_{T+1}) = 1, \quad \mathbb{P}(0 \ne X_{T+1}) > 0.$$

4 Auxiliary results

We construct a random family of closed convex cones $\mathscr{Z}_t$ by backward induction:

- $\mathscr{Z}_T = \mathscr{K}_T$;
- If $\mathscr{Z}_t$ has already been constructed for some $t = 1, \ldots, T$, then

$$\mathscr{Z}_{t-1}(\lambda) = \mathscr{K}_{t-1}(\lambda) + \bigcap_{\mu \in \mathrm{succ}(\lambda)} \mathscr{Z}_t(\mu) \quad \text{for each } \lambda \in \Lambda_{t-1}.$$

The last condition can also be written as

$$\mathscr{Z}_{t-1} = \mathscr{K}_{t-1} + \mathscr{Z}_t \cap \mathscr{L}_{t-1},$$

where $\mathscr{L}_{t-1}$ is the set of all $\mathbb{R}^d$-valued $\mathscr{F}_{t-1}$-measurable random variables.

The following lemma means that $\mathscr{Z}_t$ consists of portfolios at time t that hedge the contingent claim with payoff 0 at time T under proportional costs.

Lemma 1 *For any $t = 0, \ldots, T$, the cone $\mathscr{Z}_t$ consists of all $x \in \mathscr{L}_t$ such that there is a sequence $y_s \in \mathscr{K}_s$, $s = t, \ldots, T$ satisfying $x = y_t + \cdots + y_T$.*

Proof We proceed by backward induction on t. For $t = T$, the claim is obvious since $\mathscr{Z}_T = \mathscr{K}_T$. Now suppose that the claim holds for some $t = 1, \ldots, T$. Take any $x \in \mathscr{Z}_{t-1}$. Then $x \in \mathscr{L}_{t-1}$. Since $\mathscr{Z}_{t-1} = \mathscr{K}_{t-1} + \mathscr{Z}_t \cap \mathscr{L}_{t-1}$, we have $x = y_{t-1} + z_{t-1}$ for some $y_{t-1} \in \mathscr{K}_{t-1}$ and $z_{t-1} \in \mathscr{Z}_t \cap \mathscr{L}_{t-1}$. Then, by the induction hypothesis, there is a sequence $y_s \in \mathscr{K}_s$, $s = t, \ldots, T$ such that $z_{t-1} = y_t + \cdots + y_T$. Hence $x = y_{t-1} + y_t + \cdots + y_T$. Conversely, suppose that $x \in \mathscr{L}_{t-1}$ and there is a sequence $y_s \in \mathscr{K}_s$, $s = t-1, \ldots, T$ such that $x = y_{t-1} + y_t + \cdots + y_T$. Since $y_{t-1} \in \mathscr{K}_{t-1} \subset \mathscr{L}_{t-1}$, it follows that $y_t + \cdots + y_T \in \mathscr{L}_{t-1} \subset \mathscr{L}_t$, hence $y_t + \cdots + y_T \in \mathscr{Z}_t$ by the induction hypothesis. As a result, $y_{t-1} + y_t + \cdots + y_T \in \mathscr{K}_{t-1} + \mathscr{Z}_t \cap \mathscr{L}_{t-1} = \mathscr{Z}_{t-1}$, completing the induction argument. □

The next proposition captures the following idea: If $\mathscr{Z}_t(\lambda) = \mathbb{R}^d$, then we could have a portfolio $x \in \mathscr{Z}_t(\lambda)$ with positions in all assets smaller than any given negative constant $-C$. In view of Lemma 1, this portfolio would hedge under proportional costs the contingent claim with payoff 0 at time T. This, in turn, would mean that it is possible to construct a portfolio x' with non-positive positions in all assets at time t that hedges under proportional costs a contingent claim with any given payoff at time T. This means arbitrage under combined (proportional and fixed) costs as the payoff of the hedged contingent claim can be chosen to be high enough to absorb all the fixed transaction costs paid as part of the hedging strategy.

Proposition 1 *If there is no combined-cost arbitrage strategy, then*

$$\mathscr{Z}_t(\lambda) \neq \mathbb{R}^d \quad \textit{for each } t = 0, \ldots, T \textit{ and } \lambda \in \Lambda_t.$$

Proof Suppose that $\mathscr{Z}_t(\lambda) = \mathbb{R}^d$ for some $t = 0, \ldots, T$ and $\lambda \in \Lambda_t$. Hence there is an $x \in \mathscr{Z}_t(\lambda)$ such that

$$C < -x,$$

where

$$C = Td \max\{C_t^{ij}(\lambda) | i, j = 1, \ldots, d, \ t = 0, \ldots, T, \ \lambda \in \Lambda_t\}.$$

Since $1_\lambda x \in \mathscr{Z}_t$, it follows by Lemma 1 that there is a sequence $y_s \in \mathscr{K}_s$, $s = t, \ldots, T$ such that

$$y_t + \cdots + y_T = 1_\lambda x.$$

For each $s = t, \ldots, T$, there is an $\mathbb{R}^{d \times d}$-valued $\mathscr{F}_s$-measurable random variable $z_s = (z_s^{ij})_{i,j=1}^d$ such that $z_s^{ij} \geq 0$ for each $i, j = 1, \ldots, d$ and

$$0 \leq y_s - \sum_{i,j=1}^{d} z_s^{ij} f_s^{ij},$$

so $y_s + c_s(z_s) \in \mathscr{C}_s$. For each $s = 0, \ldots, T+1$, we put

$$X_s = -1_{\{s>t\}\cap\lambda} \sum_{r=t}^{s-1} (y_r + c_r(z_r)) .$$

Then $X = (X_s)_{s=0}^{T+1}$ is a predictable process such that

$$X_s - X_{s+1} = 1_{\{s>t\}\cap\lambda} (y_s + c_s(z_s)) \in \mathscr{C}_s \quad \text{for each } s = 0, \ldots, T,$$

which means that X is a combined-cost self-financing strategy. Moreover,

$$X_{T+1} = -1_\lambda \sum_{r=t}^{T} (y_r + c_r(z_r)) = 1_\lambda \left(-x - \sum_{r=t}^{T} c_r(z_r) \right)$$

and

$$0 \le C - \sum_{r=t}^{T} c_r(z_r) < -x - \sum_{r=t}^{T} c_r(z_r),$$

which means that X is a combined-cost arbitrage strategy. This proves the proposition. □

5 Fundamental theorem

We are ready to state and prove the Fundamental Theorem of Asset Pricing under combined (fixed and proportional) transaction costs in a multi-asset setting.

Theorem 1 *The following conditions are equivalent:*

1) *There is no combined-cost arbitrage strategy.*
2) *There exist an adapted $\mathbb{R}^d$-valued process S and a family of absolutely continuous single-step probability measures Q such that S is a martingale with respect to the family Q and*

$$0 < S_t^j \le S_t^i \pi_t^{ij} \quad \text{for each } t = 0, \ldots, T \text{ and } i, j = 1, \ldots, d. \tag{1}$$

Proof First, observe that (1) is equivalent to $S_t \in \mathscr{K}_t^* \setminus \{0\}$ for each $t = 0, \ldots, T$. This is because the convex cone $\mathscr{K}_t$ is generated by the vectors e^i and $f_t^{ij} = \pi_t^{ij} e^i - e^j$ for $i, j = 0, \ldots, d$, which means that $S_t \in \mathscr{K}_t^* \setminus \{0\}$ is equivalent to $S_t^i = S_t \cdot e^i \ge 0$, $S_t^i \pi_t^{ij} - S^j = S_t \cdot f^{ij} \ge 0$ for all $i, j = 1, \ldots, d$, and $S_t \ne 0$. This, in turn, is equivalent to (1).

1) $\Longrightarrow$ 2). Suppose that there is no combined-cost arbitrage strategy.

Then, by Proposition 1, we have $\mathscr{Z}_t(\lambda) \ne \mathbb{R}^d$, hence $\mathscr{Z}_t^*(\lambda) \ne \{0\}$ for each $t = 0, \ldots, T$ and $\lambda \in \Lambda_t$. We can construct a process S_t and a family of single-step probability measures Q_t by induction on t. For $t = 0$, we take any $S_0 \in \mathscr{Z}_0^* \setminus \{0\}$. Now suppose that an $\mathscr{F}_t$-measurable random variable $S_t \in \mathscr{Z}_t^* \setminus \{0\}$ has already been constructed for some $t = 0, 1, \ldots, T-1$. For any $\lambda \in \Lambda_t$, we have

$$\mathscr{Z}_t^*(\lambda) = \mathscr{K}_t^*(\lambda) \cap \sum_{\mu \in \mathrm{succ}(\lambda)} \mathscr{Z}_{t+1}^*(\mu) = \mathscr{K}_t^*(\lambda) \cap \mathrm{conv}\left\{\mathscr{Z}_{t+1}^*(\mu) \mid \mu \in \mathrm{succ}\,(\lambda)\right\},$$

where conv denotes the convex hull. Hence, there are $q_\mu \geq 0$ and $s_\mu \in \mathscr{Z}_{t+1}^*(\mu)$ for each $\mu \in \mathrm{succ}\,(\lambda)$ such that $\sum_{\mu \in \mathrm{succ}(\lambda)} q_\mu = 1$ and $\sum_{\mu \in \mathrm{succ}(\lambda)} q_\mu s_\mu = S_t(\lambda)$. Moreover, if $s_\nu = 0$ for some $\nu \in \mathrm{succ}\,(\lambda)$, then $q_\nu < 1$, and we can replace q_μ, s_μ by $q'_\mu = q_\mu/(1-q_\nu)$ and $s'_\mu = (1-q_\nu)s_\mu$ for any $\mu \in \mathrm{succ}\,(\lambda)$ such that $\mu \neq \nu$, and q_ν, s_ν by $q'_\nu = 0$ and any $s'_\nu \in \mathscr{Z}_{t+1}^*(\nu) \setminus \{0\}$. In this way we can ensure that $s_\mu \neq 0$ for each $\mu \in \mathrm{succ}\,(\lambda)$. Putting $Q_t(\lambda, \mu) = q_\mu$, we obtain a probability measure $Q_t(\lambda, \cdot)$ on $\lambda \cap \Lambda_{t+1}$. Moreover, for $S_{t+1}(\mu) = s_\mu$, we get $S_{t+1}(\mu) \in \mathscr{Z}_{t+1}^*(\mu) \setminus \{0\}$ for all $\mu \in \mathrm{succ}(\lambda)$ and

$$\sum_{\mu \in \mathrm{succ}(\lambda)} Q_t(\lambda, \mu) S_{t+1}(\mu) = S_t(\lambda).$$

We have constructed an adapted process S and a single-step family of measures Q such that S is a martingale with respect to Q. Because $0 \neq S_t \in \mathscr{Z}_t^* \subset \mathscr{K}_t^*$, it follows that (1) is satisfied, completing this part of the proof.

2) $\Longrightarrow$ 1). Suppose that condition 2) is satisfied, and take any combined-cost self-financing strategy $X = (X_t)_{t=0}^{T+1}$ with $X_0 = 0$ and $X_{T+1} \geq 0$.

To begin with, we verify by backward induction that

$$0 \leq X_{t+1} \cdot S_t \quad \text{for each } t = 0, \ldots, T.$$

For $t = T$, we have $0 \leq X_{T+1} \cdot S_T$ since $0 \leq X_{T+1}$ and $0 \leq S_T$. Now suppose that $0 \leq X_{t+1} \cdot S_t$ for some $t = 1, \ldots, T$. Since $X_t - X_{t+1} \in \mathscr{C}_t$, there exists an $\mathbb{R}^d$-valued $\mathscr{F}_t$-measurable random variable $z = (z^{ij})_{i,j=1}^d$ such that $z^{ij} \geq 0$ for each $i, j = 1, \ldots, d$ and $\sum_{i,j=1}^d z^{ij} f_t^{ij} + c_t(z) \leq X_t - X_{t+1}$, which implies

$$0 \leq \sum_{i,j=1}^{d} z^{ij} f_t^{ij} \cdot S_t + c_t(z) \cdot S_t \leq X_t \cdot S_t - X_{t+1} \cdot S_t$$

because $S_t \in \mathscr{K}_t^*$, $f_t^{ij} \in \mathscr{K}_t$, $0 \leq S_t$ and $0 \leq c_t(z)$, so $0 \leq f_t^{ij} \cdot S_t$ and $0 \leq c_t(z) \cdot S_t$. By the induction hypothesis, it follows that

$$0 \leq E_Q(X_{t+1} \cdot S_t | \mathscr{F}_{t-1}) \leq E_Q(X_t \cdot S_t | \mathscr{F}_{t-1}) = X_t \cdot E_Q(S_t | \mathscr{F}_{t-1}) = X_t \cdot S_{t-1},$$

completing the induction argument.

Next, we put

$$\tau = \max\{s = 0, \ldots, T+1 \mid X_s \leq \ldots \leq X_0\}.$$

Because X is predictable, we have

$$\{\tau = t\} = \{X_t \leq \cdots \leq X_0\} \setminus \{X_{t+1} \leq X_t\} \in \mathscr{F}_t$$

for each $t = 0, \ldots, T+1$, so τ is a stopping time. We claim that $\tau = T+1$. Suppose that this is not the case, that is, $\tau = t$ on λ for some $t = 0, \ldots, T$ and $\lambda \in \Lambda_t$. Hence $X^k_{t+1}(\lambda) > X^k_t(\lambda)$ for some $k = 1, \ldots, d$. Since X is a combined-cost self-financing strategy, we have $X_t - X_{t+1} \in \mathscr{C}_t$, so

$$0 \leq c_t(z) \leq X_t - X_{t+1} - \sum_{i,j=1}^{d} z^{ij} f_t^{ij}$$

for some $\mathbb{R}^d$-valued $\mathscr{F}_t$-measurable random variable $z = (z^{ij})_{i,j=1}^d$ such that $z^{ij} \geq 0$ for each $i, j = 1, \ldots, d$. Because $X^k_{t+1}(\lambda) > X^k_t(\lambda)$, we therefore must have

$$0 < -\sum_{i,j} z^{ij}(\lambda)(f_t^{ij}(\lambda))^k = \sum_i z^{ik}(\lambda) - \sum_j z^{kj}(\lambda)\pi_t^{kj}(\lambda),$$

so $0 < z^{ik}$ for some $i = 1, \ldots, d$, and therefore

$$0 < C_t^{ik}(\lambda) \leq \sum_j 1_{z^{ij}(\lambda)>0} C_t^{ij}(\lambda) \leq X^k_t(\lambda) - X^k_{t+1}(\lambda) - \sum_{i,j} z^{ij}(\lambda)(f_t^{ij}(\lambda))^k.$$

It follows that

$$\begin{aligned} 0 &< X_t(\lambda) \cdot S_t(\lambda) - X_{t+1}(\lambda) \cdot S_t(\lambda) - \sum_{i,j} z^{ij}(\lambda) f_t^{ij}(\lambda) \cdot S_t(\lambda) \\ &\leq X_t(\lambda) \cdot S_t(\lambda) - X_{t+1}(\lambda) \cdot S_t(\lambda). \end{aligned}$$

As a result,

$$0 \leq X_{t+1}(\lambda) \cdot S_t(\lambda) < X_t(\lambda) \cdot S_t(\lambda) \leq 0$$

since $0 \leq S_t$ and $X_t \leq \ldots \leq X_0 = 0$ on λ. This contradiction proves that $\tau = T+1$.

We can conclude that $X_{T+1} \leq \cdots \leq X_0 = 0$, so X cannot be an arbitrage strategy, completing the proof of the theorem. □

Acknowledgements The author would like to express his gratitude to the anonymous reviewers, whose comments and suggestions helped to improve the clarity of the paper.

Funding Not applicable.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

Bouchard, B.: No-arbitrage in discrete-time markets with proportional transaction costs and general information structure. Finance Stoch. **10**, 276–297 (2006)

Brown, M., Zastawniak, T.: Fundamental theorem of asset pricing under fixed and proportional transaction costs. Ann. Finance **16**, 423–433 (2020)

Cherny, A.: General arbitrage pricing model: II–transaction costs. Séminaire de Probabilités XL, Lectures Notes in Mathematics, vol. 1899, pp. 447–461. Springer, Berlin (2007)

Dalang, R.C., Morton, A., Willinger, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stoch. Stoch. Rep. **29**, 185–201 (1990)

Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. **312**, 215–250 (1994)

Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Berlin (2006)

Denis, E., Kabanov, Y.: Consistent price systems and arbitrage opportunities of the second kind in models with transaction costs. Finance Stoch. **16**, 135–154 (2012)

Dolinsky, Y., Soner, Y.M.: Robust hedging with proportional transaction costs. Finance Stoch. **18**, 327–347 (2014)

Grigoriev, P.: On low dimensional case in the fundamental asset pricing theorem under transaction costs. Stat. Decis. **23**, 33–48 (2005)

Guasoni, P.: No arbitrage under transaction costs, with fractional Brownian motion and beyond. Math. Finance **16**, 569–582 (2006)

Guasoni, P., Rásonyi, M., Schachermayer, W.: The fundamental theorem of asset pricing for continuous processes under small transaction costs. Ann. Finance **6**, 157–191 (2010)

Guasoni, P., Lépinette, E., Rásonyi, M.: The fundamental theorem of asset pricing under transaction costs. Finance Stoch. **16**, 741–777 (2012)

Harrison, J.M., Pliska, S.R.: Martingales and stochastic integrals in the theory of continuous trading. Stoch. Process. Appl. **11**, 215–260 (1981)

Jouini, E., Kallal, H.: Martingales and arbitrage in securities markets with transaction costs. J. Econ. Theory **66**, 178–197 (1995)

Jouini, E., Kallal, H., Napp, C.: Arbitrage and viability in securities markets with fixed trading costs. J. Math. Econ. **35**, 197–221 (2001)

Kabanov, Y.M.: Hedging and liquidation under transaction costs in currency markets. Finance Stoch. **3**, 237–248 (1999)

Kabanov, Y., Safarian, M.: Markets with Transaction Costs: Mathematical Theory. Springer, Berlin (2009)

Kabanov, Y.M., Stricker, C.: The Harrison–Pliska arbitrage pricing theorem under transaction costs. J. Math. Econ. **35**, 185–196 (2001)

Kabanov, Y.M., Rásonyi, M., Stricker, C.: No-arbitrage criteria for financial markets with efficient friction. Finance Stoch. **6**, 371–382 (2002)

Lepinette, E., Tran, T.: General financial market model defined by a liquidation value. Stochastics **88**, 437–459 (2016)

Lepinette, E., Tran, T.: Arbitrage theory for non convex financial market models. Stoch. Process. Appl. **127**, 3331–3353 (2017)

Ortu, F.: Arbitrage, linear programming and martingales in securities markets with bid-ask spreads. Decis. Econ. Finance **24**, 79–105 (2001)

Rola, P.: Arbitrage in markets with bid-ask spreads. The fundamental theorem of asset pricing in finite discrete time markets with bid-ask spreads and a money account. Ann. Finance **11**, 453–475 (2015)

Roux, A.: The fundamental theorem of asset pricing in the presence of bid-ask and interest rate spreads. J. Math. Econ. **47**, 159–163 (2011)

Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Math. Finance **14**, 19–48 (2004)

Zhang, S., Xu, C., Deng, X.: Dynamic arbitrage-free asset pricing with proportional transaction costs. Math. Finance **12**, 89–97 (2002)

Zhao, J., Lepinette, E.: The Dalang–Morton–Willinger version of the fundamental theorem of asset pricing for the bid-ask financial market model, preprint (2018). https://hal.archives-ouvertes.fr/hal-01666860v4