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SYMPLECTIC RIGIDITY OF O'GRADY'S TENFOLDS

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ABSTRACT. We prove that any symplectic automorphism of finite order of an irreducible holomorphic symplectic manifold of O'Grady's 10-dimensional deformation type is trivial.

1. INTRODUCTION

O'Grady [16] constructs an irreducible holomorphic symplectic manifold $\widetilde{M}$ of dimension 10 and second Betti number 24 as a crepant resolution of a certain moduli space M of coherent sheaves on a K3 surface. Any complex manifold X which is deformation equivalent to $\widetilde{M}$ is said to be *of type* OG10.

In this short note, we prove the following theorem.

Theorem 1.1. *If X is a complex manifold of type* OG10, *and $f \in \mathrm{Aut}(X)$ is a symplectic automorphism of finite order, then f is the identity.*

Type OG10 was the only remaining known deformation type of irreducible holomorphic symplectic manifolds that lacked a systematic treatment of symplectic automorphisms in the literature. Theorem 1.1 fills this gap. For references to the works on the other types, see the historical note in [6, §1.1].

A similar theorem holds for symplectic automorphisms of the other sporadic deformation type OG6 found by O'Grady in dimension 6 (see [6, Theorem 1.1]). On the other hand, Theorem 1.1 does not hold for birational transformations. Indeed, it is known for instance that manifolds of type OG10 can admit symplectic birational involutions (see Remark 2.3 for a simple lattice theoretical argument, [1, §7.3] for a geometrical example, [12] for a complete classification, and [4] for induced symplectic birational involutions).

By a result by Mongardi and Wandel [14], any automorphism of a manifold X of type OG10 acting trivially on the second integral cohomology lattice $\mathrm{H}^2(X, \mathbb{Z})$ is the identity. Thus, the proof of Theorem 1.1 consists in showing that there is no symplectic automorphism f of finite order acting non-trivially on $\mathrm{H}^2(X, \mathbb{Z})$. The proof is divided

into two parts, according to the action of f on the discriminant group of $\mathrm{H}^2(X, \mathbb{Z})$. In the case of trivial action, we use a trick involving the Leech lattice already employed by Gaberdiel, Hohenegger and Volpato [5], and the classification of invariant sublattices of the Leech lattice by Höhn and Mason [7]. In the case of non-trivial action, we take advantage of some bounds by Rogers [19] and Leech [10] related to the sphere packing problem.

The paper is divided into two sections. In §2, we introduce the notation and recall some known results. In §3, we prove Theorem 1.1.

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2. Preliminaries

In this section, we recall basic definitions and some known results which we will use in the sequel.

2.1. **Lattices.** In this paper, we use exactly the same conventions and notation for lattices, i.e., finitely generated free $\mathbb{Z}$-modules $L \cong \mathbb{Z}^r$ with a non-degenerate symmetric bilinear pairing $(v, w) \mapsto v \cdot w$, as in our previous paper [6, §2].

A lattice L is called *even* if $v^2 \in 2\mathbb{Z}$ for all $v \in L$, and *unimodular* if $|\det(L)| = 1$. The *dual* of L is defined as $L^\vee := \{x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$. The divisibility of an element $v \in L$ is defined by $(v, L) := \gcd\{v \cdot w \mid w \in L\}$.

The *discriminant group* of a lattice L is the abelian group $L^\sharp := L^\vee/L$, which has order $|L^\sharp| = |\det(L)|$ and inherits from L a finite quadratic form when L is even. The orthogonal groups of L and $L^\sharp$ are denoted by $\mathrm{O}(L)$ and $\mathrm{O}(L^\sharp)$, respectively. The natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(L^\sharp)$ is denoted $g \mapsto g^\sharp$.

Given a subgroup $G \subset \mathrm{O}(L)$, we denote the *invariant sublattice* by $L^G := \{v \in L \mid g(v) = v \text{ for all } g \in G\}$, and the *coinvariant sublattice* by $L_G := (L^G)^\perp$. If the subgroup G is generated by a single element g, then we write $L^g := L^G$ and $L_g := L_G$.

A *root* of an even negative definite lattice L is an element $v \in L$ such that $v^2 = -2$. The *length* of a finite group A is the minimal number of generators of A and it is denoted by $\ell(A)$; similarly, for a prime number p, the *p-length* of A is the minimal number of generators of a Sylow p-subgroup of A and it is denoted by $\ell_p(A)$.

We denote by $\mathbf{U}$ and $\mathbf{E}_8$ the unique even unimodular lattices of signature $(1, 1)$ and $(0, 8)$, respectively. We denote by $\mathbf{A}_2$ the unique even lattice of signature $(0, 2)$ and determinant 3.

2.2. **Torelli theorem.** We define

$$\mathbf{L} := 3\mathbf{U} \oplus 2\mathbf{E}_8 \oplus \mathbf{A}_2.$$

By a result of Rapagnetta [18], the second integral cohomology group $\mathrm{H}^2(X, \mathbb{Z})$ of any manifold X of type OG10, together with the Beauville–Bogomolov–Fujiki form, is isomorphic to the lattice $\mathbf{L}$. We define the following subsets of $\mathbf{L}$:

$$\mathcal{W}^{\mathrm{pex}}_{\mathrm{OG10}} := \{v \in \mathbf{L} \mid v^2 = -2\} \cup \{v \in \mathbf{L} \mid v^2 = -6, (v, \mathbf{L}) = 3\},$$

$$\mathcal{W}_{\mathrm{OG10}} := \mathcal{W}^{\mathrm{pex}}_{\mathrm{OG10}} \cup \{v \in \mathbf{L} \mid v^2 = -4\} \cup \{v \in \mathbf{L} \mid v^2 = -24, (v, \mathbf{L}) = 3\}.$$

Geometrically we have the following interpretation of the two sets above. Let X be a manifold of type OG10 and recall that the positive cone of X is the cone $\mathcal{C}_X$ of classes

$x \in \mathrm{H}^{1,1}(X, \mathbb{R})$ such that $(x, x) > 0$ and $(x, \omega) > 0$ for one (and hence all) Kähler class ω. Once we fix a marking $\eta \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathbf{L}$, the Kähler cone (resp. the birational Kähler cone) is the chamber in

$$\mathcal{C}_X \setminus_{D \in \mathrm{H}^{1,1}(X,\mathbb{Z}) \cap \eta^{-1}(\mathcal{W}_{\mathrm{OG10}})} D^{\perp} \quad \left(\text{resp.} \quad \mathcal{C}_X \setminus_{D \in \mathrm{H}^{1,1}(X,\mathbb{Z}) \cap \eta^{-1}(\mathcal{W}_{\mathrm{OG10}}^{\mathrm{pex}})} D^{\perp} \right)$$

containing a Kähler class.

We can now state a consequence of the Torelli theorem which will be used to prove Theorem 1.1. We say that an isometry $g \in \mathrm{O}(\mathbf{L})$ is *induced* by an automorphism of a manifold of type OG10 if there exists a manifold X of type OG10, a marking $\eta \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathbf{L}$, and an automorphism $f \in \mathrm{Aut}(X)$ such that

$$g = \eta \circ (f^{-1})^* \circ \eta^{-1},$$

where $f^* \in \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$ is the pullback of f.

Theorem 2.1. *A finite subgroup $G \subset \mathrm{O}(\mathbf{L})$ is induced by a group of symplectic automorphisms of a manifold of type* OG10 *if and only if $\mathbf{L}_G$ is negative definite and*

$$\tag{1} \mathbf{L}_G \cap \mathcal{W}_{\mathrm{OG10}} = \emptyset.$$

Proof. The proof is analogous to [6, Theorem 2.16]. Let us sketch it here for the reader's convenience.

First of all, let X be a manifold of type OG10, and let us fix a marking, i.e., an isometry $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$. If $G \subset \mathrm{O}(\mathbf{L})$ is induced by a group of symplectic automorphisms, then $\mathbf{L}_G$ is negative definite by [6, Lemma 2.12]. Moreover, any element of G must send the Kähler cone into itself. The Kähler cone is described lattice-theoretically in [13, Theorem 5.5]: it coincides with one of the chambers of the positive cone determined by the orthogonal hyperplanes to the elements in $\mathcal{W}_{\mathrm{OG10}}$. Condition (1) follows then from [6, Lemma 2.14].

For the reverse implication, it is enough to show the existence of one manifold of type OG10 on which G is induced by symplectic automorphisms.

Since $\mathbf{L}_G$ is negative definite, there exists by [6, Lemma 2.13] a Hodge structure $\mathbf{L} \otimes \mathbb{C} = \mathbf{L}^{2,0} \oplus \mathbf{L}^{1,1} \oplus \mathbf{L}^{0,2}$ with the following properties: the Hodge structure is preserved by G, $\mathbf{L}^{1,1} \cap \mathbf{L} = \mathbf{L}_G$, and G acts as the identity on $(\mathbf{L}^{2,0} \oplus \mathbf{L}^{0,2}) \cap \mathbf{L}$. By the surjectivity of the period map (see [8, Theorem 8.1]), there exists a manifold X of type OG10 with a marking $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$ such that the Hodge structure on $\mathrm{H}^2(X, \mathbb{C}) \cong \mathrm{H}^2(X, \mathbb{Z}) \otimes \mathbb{C}$ corresponds through the marking to the Hodge structure on $\mathbf{L} \otimes \mathbb{C}$.

Now, by the Hodge-theoretic Torelli theorem [11, Theorem 1.3], the group G is induced by a group of automorphisms of X if and only if it acts via monodromy operators, it preserves the Hodge structure, and it preserves the Kähler cone.

By [17, Theorem 5.4], the monodromy group of a manifold of type OG10 is the whole group $\mathrm{O}^+(\mathbf{L})$. Since $\mathbf{L}_G$ is negative definite, we have by [6, Lemma 2.3] that $G \subset \mathrm{O}^+(\mathbf{L})$, so that the first condition of the Hodge-theoretic Torelli theorem is satisfied. Moreover, G preserves the Hodge structure of X by construction, thus satisfying the second condition of the Hodge-theoretic Torelli theorem.

Again by [13, Theorem 5.5], the Kähler cone is cut out inside the positive cone by the hyperplanes which are orthogonal to the vectors in $\mathcal{W}_{\mathrm{OG10}}$. Since $\mathbf{L}^{1,1} \cap \mathcal{W}_{\mathrm{OG10}} = \mathbf{L}_G \cap \mathcal{W}_{\mathrm{OG10}} = \emptyset$, it follows that the Kähler cone coincides with the positive cone, which

is preserved by G because $G \subset \mathrm{O}^+(\mathbf{L})$. Therefore, also the third condition of the Hodge-theoretic Torelli theorem is satisfied. Since G acts as the identity on $(\mathbf{L}^{2,0} \oplus \mathbf{L}^{0,2}) \cap \mathbf{L}$ by construction, the automorphisms inducing G are symplectic. $\qquad\square$

For the sake of completeness, we also state the corresponding theorem for symplectic birational transformations. The proof is analogous to the proof of Theorem 2.1, with the difference that we consider the set $\mathcal{W}^{\mathrm{pex}}_{\mathrm{OG10}}$ instead of $\mathcal{W}_{\mathrm{OG10}}$ and the birational Kähler cone instead of the Kähler cone.

Theorem 2.2. *A finite subgroup $G \subset \mathrm{O}(\mathbf{L})$ is induced by a group of symplectic birational transformations of a manifold of type* OG10 *if and only if $\mathbf{L}_G$ is negative definite and*

$$\mathbf{L}_G \cap \mathcal{W}^{\mathrm{pex}}_{\mathrm{OG10}} = \emptyset.$$

Remark 2.3. Manifolds of type OG10 with non-trivial symplectic birational transformations do exist. Consider for instance the involution $g \in \mathrm{O}(\mathbf{L})$ which consists in exchanging the two copies of $\mathbf{E}_8$ and is the identity elsewhere. Then, $\mathbf{L}_g$ is isomorphic to $\mathbf{E}_8(2)$. In particular, any $v \in \mathbf{L}_g$ satisfies $4 \mid v^2$, so the group generated by g satisfies the conditions of Theorem 2.2.

2.3. **Leech lattice.** The *Leech lattice*, which we denote by $\mathbf{\Lambda}_{24}$, is the unique even, unimodular, negative definite lattice of rank 24 without roots.

Let $\mathbf{\Lambda}_{1,25}$ be the unique even unimodular lattice of signature $(1, 25)$. The following proposition was originally proved by Gaberdiel, Hohenegger and Volpato [5, §B.2]. The same argument was reproduced by Huybrechts [9, §2.2, p. 398]. We sketch it here for the sake of completeness.

Proposition 2.4 ([5],[9])**.** *Let L be a lattice, and let $G \subset \mathrm{O}(L)$ be a subgroup of isometries which acts trivially on the discriminant group $L^\sharp$, such that L_G is negative definite and does not contain roots. If there exists a primitive embedding $L_G \hookrightarrow \mathbf{\Lambda}_{1,25}$, then G is isomorphic to a subgroup $G \subset \mathrm{O}(\mathbf{\Lambda}_{24})$ such that $L_G \cong (\mathbf{\Lambda}_{24})_G$.*

Proof. First of all, since G acts as the identity on the discriminant group of L, it also acts as the identity on the discriminant group of L_G. In particular, thanks to the primitive embedding $L_G \hookrightarrow \mathbf{\Lambda}_{1,25}$, the action of G on L_G can be extended to $\mathbf{\Lambda}_{1,25}$ as the identity on $L_G^\perp \subset \mathbf{\Lambda}_{1,25}$. Note that $\mathbf{\Lambda}_{1,25}^G \cong L_G^\perp$ is a non-degenerate lattice of signature $(1, 25 - \mathrm{rk}(L_G))$.

Let us now fix an isometry $\mathbf{\Lambda}_{1,25} \cong \mathbf{\Lambda}_{24} \oplus \mathbf{U}$, where $\mathbf{U}$ is the unimodular hyperbolic lattice of rank 2. We also fix an isotropic generator $w \in \mathbf{U}$ and we see it as an isotropic element of $\mathbf{\Lambda}_{1,25}$. A *Leech root* of $\mathbf{\Lambda}_{1,25}$ is any root δ such that $(\delta, w) = 1$. The Weyl group $W \subset \mathrm{O}(\mathbf{\Lambda}_{1,25})$ is the group generated by reflections associated with Leech roots (see [3, Chapter 27]). We denote by $\mathcal{C}_0$ the subset of $\mathbf{\Lambda}_{1,25} \otimes \mathbb{R}$ of positive classes x such that $(x, \delta) > 0$ for any Leech root δ.

Since L_G contains no roots by hypothesis, it follows that there is no root $\delta \in \mathbf{\Lambda}_{1,25}$ such that $\mathbf{\Lambda}_{1,25}^G \subset \delta^\perp$. In particular, this applies to Leech roots. Therefore, up to changing the primitive embedding $L_G \hookrightarrow \mathbf{\Lambda}_{1,25}$ by an element of W, we can assume that $\mathcal{C}_0$ is fixed by (the $\mathbb{R}$-linear extension of) G, i.e., $G \subset \mathrm{O}(\mathbf{\Lambda}_{1,25}, \mathcal{C}_0)$. By [2], the group $\mathrm{O}(\mathbf{\Lambda}_{1,25}, \mathcal{C}_0)$ is known to fix the isotropic vector $w \in \mathbf{\Lambda}_{1,25}$, hence $w \in \mathbf{\Lambda}_{1,25}^G \cong L_G^\perp$. It follows that there

TABLE 1. Rogers' upper bounds on center density δ, as computed by Leech.

n	1	2	3	4	5	6	7	8
b_n	0.5	0.28868	0.1847	0.13127	0.09987	0.08112	0.06981	0.06326
n	9	10	11	12	13	14	15	16
b_n	0.06007	0.05953	0.06136	0.06559	0.07253	0.08278	0.09735	0.11774
n	17	18	19	20	21	22	23	24
b_n	0.14624	0.18629	0.24308	0.32454	0.44289	0.61722	0.87767	1.27241

exists a primitive embedding $L_G \hookrightarrow \mathbf{\Lambda}_{24}$ given by the composition

$$L_G \hookrightarrow w^\perp \twoheadrightarrow w^\perp/\mathbb{Z}w \cong \mathbf{\Lambda}_{24}.$$

Finally, since again G acts as the identity on the discriminant group of L_G, the action of G on L_G can be extended to an action of G on $\mathbf{\Lambda}_{24}$ such that $L_G \cong (\mathbf{\Lambda}_{24})_G$ as claimed. $\quad\square$

2.4. **Sphere packings.** Let L be a positive definite lattice of rank n. Its *minimal norm* μ and *packing radius* ρ are defined as follows:

$$\tag{2} \mu := \min\{v^2 \mid v \in L \setminus \{0\}\} \quad \text{and} \quad \rho := \frac{1}{2}\sqrt{\mu}.$$

The *center density* of L is defined as

$$\tag{3} \delta := \frac{\rho^n}{\sqrt{\det L}}.$$

For $n \leq 24$, there exist upper bounds b_n on the center density δ found by Rogers [19], explicitly computed by Leech [10], and reproduced by Conway and Sloane in [3, Table 1.2, p. 15]. For the reader's convenience, we copied these bounds in Table 1.

3. PROOF OF THEOREM 1.1

We fix an irreducible holomorphic symplectic manifold X of type OG10 and a symplectic automorphism $f \in \operatorname{Aut}(X)$ of finite order. We claim that f acts trivially on the second integral cohomology group $\operatorname{H}^2(X, \mathbb{Z})$, and is thus the identity, by a result of Mongardi and Wandel [14, Theorem 3.1].

Choose any marking for X, that is, an isometry $\eta \colon \operatorname{H}^2(X, \mathbb{Z}) \to \mathbf{L}$, and let $g := \eta \circ (f^{-1})^* \circ \eta^{-1} \in \operatorname{O}(\mathbf{L})$ be the isometry of $\mathbf{L}$ induced by f. Suppose that g is not the identity. We seek a contradiction on the coinvariant lattice $\mathbf{L}_g$. For the rest of this section, we will use the fact that both $\mathbf{L}^g$ and $\mathbf{L}_g$ are primitive sublattices of $\mathbf{L}$.

The discriminant group $\mathbf{L}^\sharp$ has order $|\det(\mathbf{L})| = 3$. The only non-trivial automorphism of $\mathbf{L}^\sharp$, which we denote by $-\operatorname{id}$, is the one exchanging the two non-trivial elements of $\mathbf{L}^\sharp$. Hence, we have $\operatorname{O}(\mathbf{L}^\sharp) = \{\operatorname{id}, -\operatorname{id}\}$. In the following, we consider the image $g^\sharp \in \operatorname{O}(\mathbf{L}^\sharp)$ and we treat the two cases $g^\sharp = \operatorname{id}$ and $g^\sharp = -\operatorname{id}$ in §3.1 and §3.2, respectively, seeking a contradiction.

3.1. **Trivial action on discriminant group.** Suppose that g acts trivially on the discriminant group $\mathbf{L}^\sharp$, i.e., $g^\sharp = \mathrm{id}$.

First of all, we claim that $\ell(\mathbf{L}_g^\sharp) \leq \ell((\mathbf{L}^g)^\sharp) + \ell(\mathbf{L}^\sharp)$. In fact, applying [15, Proposition 1.15.1] to the primitive embedding of $\mathbf{L}^g$ in $\mathbf{L}$, we see that $\mathbf{L}_g^\sharp$ is isometric to an abelian group of the form $\Gamma^\perp/\Gamma$, where $\Gamma \subset (\mathbf{L}^g)^\sharp(-1) \oplus \mathbf{L}^\sharp$ is an isotropic subgroup. Therefore, we have

$$\ell(\mathbf{L}_g^\sharp) = \ell(\Gamma^\perp/\Gamma) \leq \ell(\Gamma^\perp) \leq \ell((\mathbf{L}^g)^\sharp) + \ell(\mathbf{L}^\sharp),$$

as claimed. Now, since $\mathbf{L}^\sharp = \mathbb{Z}/3\mathbb{Z}$, and $\ell(L^\sharp) \leq \mathrm{rk}(L)$ for any lattice L, we get the following chain of inequalities:

$$\ell(\mathbf{L}_g^\sharp) \leq \ell((\mathbf{L}^g)^\sharp) + \ell(\mathbf{L}^\sharp) \leq \mathrm{rk}(\mathbf{L}^g) + 1 = (24 - \mathrm{rk}(\mathbf{L}_g)) + 1 < \mathrm{rk}(\mathbf{\Lambda}_{1,25}) - \mathrm{rk}(\mathbf{L}_g).$$

It follows from [15, Corollary 1.12.3] that there exists a primitive embedding $\mathbf{L}_g \hookrightarrow \mathbf{\Lambda}_{1,25}$.

Note, moreover, that $\mathbf{L}_g$ is negative definite and does not contain roots by Theorem 2.1. Hence, all hypotheses of Proposition 2.4 are satisfied, and we conclude that there exists an element $g \in \mathrm{O}(\mathbf{\Lambda}_{24})$ of order p such that $\mathbf{L}_g \cong (\mathbf{\Lambda}_{24})_g$.

Höhn and Mason observe that all coinvariant lattices of the (positive definite) Leech lattice have minimal norm 4 (see [7, p. 633]). Thus, $(\mathbf{\Lambda}_{24})_g$ contains an element of square -4, and so does $\mathbf{L}_g$. Condition (1) of Theorem 2.1 is therefore not satisfied, and we have a contradiction.

3.2. **Non-trivial action on discriminant group.** Suppose now that g acts non-trivially on the discriminant group $\mathbf{L}^\sharp$, i.e., $g^\sharp = -\mathrm{id}$. Since $-\mathrm{id}$ has order 2, the order of g is necessarily even. Up to taking powers, we can suppose that the order of g is 2^r, for some integer $r \geq 1$. If $r > 1$, then g^2 acts trivially on the discriminant group, and, therefore, it is trivial by §3.1. Hence, from now on we can further assume that the order of g is exactly 2. In particular, g acts as the identity on the invariant sublattice $\mathbf{L}^g$, and as multiplication by -1 on the coinvariant sublattice $\mathbf{L}_g$.

By work of Nikulin [15, Proposition 1.5.1], the primitive embedding $\mathbf{L}^g \hookrightarrow \mathbf{L}$ is given by a subgroup H of $(\mathbf{L}^g)^\sharp$, a subgroup H' of $\mathbf{L}_g^\sharp$, and an isometry $\gamma\colon H \to H'(-1)$, which is called 'gluing isometry' in [6, §2.2]. By Nikulin's construction, one identifies $\Xi := \mathbf{L}/(\mathbf{L}^g \oplus \mathbf{L}_g)$ with an isotropic subgroup $\Xi \subset (\mathbf{L}^g)^\sharp \oplus \mathbf{L}_g^\sharp$. By definition, the groups H and H' are the image of Ξ under the projections to $(\mathbf{L}^g)^\sharp$ and $\mathbf{L}_g^\sharp$, respectively. Therefore, the isometry γ is equivariant with respect to the action of $g^\sharp$. Hence, for every $\xi \in H$ we have

$$\xi = g^\sharp(\xi) = \gamma^{-1}(\gamma(g^\sharp(\xi))) = \gamma^{-1}(g^\sharp(\gamma(\xi))) = \gamma^{-1}(-(\gamma(\xi)) = -\xi,$$

so all elements of H have order 2, that is, H is a 2-elementary abelian group, say of length ℓ. In particular, $|H| = 2^\ell$. Since H and H' are isomorphic, also $|H'| = 2^\ell$.

By [6, eq. (5)], it holds $|H|^2 \cdot |\det(\mathbf{L})| = |\det(\mathbf{L}^g) \cdot \det(\mathbf{L}_g)|$, which can be written as

$$3 = |\det(\mathbf{L})| = [(\mathbf{L}^g)^\sharp : H] \cdot [\mathbf{L}_g^\sharp : H'].$$

In particular, we have $[\mathbf{L}_g^\sharp : H'] \leq 3$ and, therefore,

$$|\det(\mathbf{L}_g)| = |\mathbf{L}_g^\sharp| = [\mathbf{L}_g^\sharp : H'] \cdot |H'| \leq 3 \cdot 2^\ell.$$

Put $n := \mathrm{rk}(\mathbf{L}_g)$. Note that $\ell \le \ell_2((\mathbf{L}^g)^\sharp) \le \mathrm{rk}\,\mathbf{L}^g = 24 - n$ and $\ell \le \ell_2(\mathbf{L}_g^\sharp) \le \mathrm{rk}\,\mathbf{L}_g = n$. Thus, we obtain the following upper bound:

$$(4) \qquad |\det(\mathbf{L}_g)| \le 3 \cdot 2^{\min(n, 24-n)}.$$

We now look for a lower bound. By Condition (1) of Theorem 2.1, $\mathbf{L}_g$ does not contain elements of square -2 or -4. Since $\mathbf{L}_g(-1)$ is an even (positive definite) lattice, its minimal norm μ satisfies

$$(5) \qquad \mu \ge 6.$$

We let b_n be Rogers' bound on the center density in dimension n, as in Table 1. Then, denoting δ and ρ respectively the center density and the packing radius of $\mathbf{L}_g(-1)$, it follows from (2), (3) and (5) that

$$(6) \qquad |\det(\mathbf{L}_g)| = \frac{\rho^{2n}}{\delta^2} \ge \frac{\rho^{2n}}{b_n^2} = \frac{\mu^n}{2^{2n}b_n^2} \ge \frac{3^n}{2^n b_n^2}.$$

Since $\mathbf{L}_g$ is negative definite by Theorem 2.1, and $\mathbf{L}$ has signature $(3, 21)$, we have $n \le 21$. By plugging in the values for b_n given by Table 1, we see that

$$3 \cdot 2^{\min(n, 24-n)} < \frac{3^n}{2^n b_n^2} \quad \text{for all } n \in \{2, \ldots, 21\},$$

so the two bounds (4) and (6) contradict each other. Hence, $\mathbf{L}_g$ can only exist in rank $n = 1$, in which case the two bounds coincide and, necessarily, $\mathbf{L}_g \cong [-6]$. Let v be a generator of $\mathbf{L}_g$ and w be any other element in $\mathbf{L}$. Since g is an involution, $w + g(w) \in \mathbf{L}^g$ and $w - g(w) \in \mathbf{L}_g$. Therefore, given that $(v, \mathbf{L}_g) = 6$, we have

$$(v, w) = \frac{1}{2}(v, w + g(w)) + \frac{1}{2}(v, w - g(w)) = \frac{1}{2}(v, w - g(w)) \in 3\mathbb{Z},$$

i.e., $3 \mid (v, \mathbf{L})$. Since v is a primitive vector of $\mathbf{L}$, we have $(v, \mathbf{L}) \mid \det \mathbf{L}$, hence $(v, \mathbf{L}) = 3$. In particular, Condition (1) of Theorem 2.1 is not satisfied because $v \in \mathbf{L}_g \cap \mathcal{W}_{\mathrm{OG10}}$. Thus, case $n = 1$ cannot occur either, and this finishes the proof of Theorem 1.1. $\qquad \square$

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