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Band gaps of thermoelastic waves in 1D phononic crystal with fractional order generalized thermoelasticity and dipolar gradient elasticity

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Abstract：A coupled thermoelastic model with fractional order derivative which incorporates the microstructural effects and thermoelastic coupling effects simultaneously at small scale is provided and is used to study wave dispersion and bandgap features of Bloch waves in one-dimensional phononic crystals. Dipolar gradient elasticity is used to account for the effects of microstructure while the non-Fourier heat conduction with fractional order derivatives is used to model thermal conduction at small scale. The interaction of thermo-elastic coupled waves with a periodic structure leads to Bloch waves, and the transfer matrix method is used to obtain the dispersion equation of the Bloch waves based on the Bloch theorem. A parameter study is performed in the numerical example to investigate the influence of the strain gradient parameter, the micro-inertial parameter, the relaxation time and the fractional order on the dispersion and bandgap of Bloch waves.

Keyword: Gradient elasticity, Generalized thermoelasticity, Fractional order

differential, Thermoelastic wave, Phononic crystal.

1. Introduction

Phononic crystals are composite materials with a designed artificial periodic structure. Due to the interaction between elastic waves and a periodic structure, propagation of the elastic waves is observed within the specific frequency range, while the elastic waves outside the specific frequency range cannot propagate through this structure. That is, phononic crystals exhibit frequency selectivity which is also known as the band gap feature of phononic crystals. Furthermore, phononic crystals have many other special physical properties such as negative refraction that can be utilize in developing a flat lens, perfect lens, etc.. Therefore, phononic crystals have attracted extensive attention [1-7] as soon as they were proposed. Apart from the periodicity of the lattice, the material properties of the component materials are also important for creating band gaps. Zhao and Wei [8,9] studied the influence of viscoelastic properties on the dispersive relation and the appearance of band gaps in 1-D and 2-D phononic crystals with viscoelastic host by using complex moduli. Zhan and Wei [10,11] studied band gaps of a 2-D phononic crystal with orthotropic cylindrical fillers and a 3-D phononic crystal with orthotropic spherical inclusions embedded in the isotropic host. Fomenko et al [12] investigated wave transmission and band-gaps by considering in-plane wave propagation in layered phononic crystals composed of functionally graded interlayers and proposed a classification of band-gaps in layered phononic crystals. Wang et al. [13,14] studied the influences of the mechanical and electrical coupling effects on the dispersive relation and band gaps in phononic

crystals consisting of piezoelectric or piezomagnetic elastic solids. Lan and Wei [15,16] further studied the dispersive characteristics of elastic waves propagating through a laminated piezoelectric phononic crystal with mechanically imperfect interfaces and the gradient interlayer.

These above-mentioned investigations are applicable to Bloch waves with long wavelength. Because the wavelength of Bloch waves is much larger than the characteristic length of the microstructure in an equivalent macroscopic homogeneous medium, the microstructure effects can be ignored. However, when the Bloch wave's wavelength is comparable to the characteristic length of material microstructure, the material size effect gradually appears, and the microstructure effect, i.e. the dispersion and attenuation of waves due to microstructure, cannot be ignored anymore. In order to take the microstructure effects into consideration, classical linear elasticity theory should be replaced by generalized elasticity theories, for example the couple stress theory [17,18], the non-local theory [19], or the micropolar and micromorphic theory [20,21]. In 1964, Mindlin [22] proposed a linear elastic theory with microstructure, namely strain gradient theory. Strain gradient theory is a generalized continuum theory developed to capture the size effect of materials at micro and nano scales. According to strain gradient theory, the stress of a point is not only related to the strain of the point but also to higher-order gradients of the strain gradient at that point. However, in the traditional strain gradient elasticity theory established by Mindlin [22], the second-order deformation gradient includes 18 independent constitutive constants, and even for isotropic materials there are still seven linear elastic constants

in the second-order deformation gradient. In contrast, much simpler generalized elasticity models were formulated by Eringen[23] and Aifantis and coworkers[24-26] where a linear isotropic medium can be described by the two Lamé constants and one additional microstructural parameter. Similarly, in order to reduce the number of additional material constants, Lazar and Maugin [27] and Yang et al. [28] proposed a simple model in which only one additional material characteristic length parameter was added. Georgiadis [29] proposed the theory of dipolar gradient elasticity when he studied the problem of cracks in microstructured solids. Georgiadis et al. [30] studied the propagation of Rayleigh waves and Gourgiotis et al. [31] studied the reflection of elastic waves on free surfaces based on the theory of dipolar gradient elasticity. Using the theory of dipolar gradient elasticity, Li et al. studied the reflection and transmission of elastic waves [32], the propagation of waves in sandwich structures[33], the distribution law of "band gap" in infinite periodic composites [34], and the distribution of incident wave energy between reflected and transmitted under different interface conditions [35, 36]. These investigations show that there are four types of waves in dipolar strain-gradient solids, namely P wave, SV wave, SP surface wave and SS surface wave. These waves are all the dispersive waves due to the material property, and the shorter the wavelength or the higher the frequency, the more significant the microstructural influences.

The thermo-elastic coupled problem is not only of theoretic significance but also of important application value [37]. As discussed above, the bandgap nature of phononic crystals is widely studied although, but the thermo-elastic coupled effects are taken

into account less frequently. The governing equation of heat conduction based on classical Fourier law is a diffusion equation, therefore thermal disturbance has an infinite velocity of propagation. In order to eliminate this paradox, Fourier's law is modified into non-Fourier heat conduction and generalized thermoelastic theory [38-42]. In the generalized thermoelastic theory, the diffusion equation in the context of classical heat conduction is replaced by the wave equation, so that the thermal disturbance has a finite velocity of propagation [43]. At present, three kinds of generalized thermoelastic theories exist, namely Lord-Shulman (L-S) theory with one relaxation time, Green-Lindsay (G-L) theory with two relaxation times and Green-Naghdi (G-N) theory with three relaxation times. Different from L-S and G-L theories, the concept of thermal displacement is introduced in the G-N theory. The thermal displacement is the mean free path of phonon at micro scale and its time derivative is defined as temperature at macro scale. Based on the L-S theory and G-L theory, Kumar [44] studied (i) the propagation of plane waves in a microstretch thermoelastic diffusion solid of infinite extent, and (ii) the reflection and transmission of plane waves at a plane interface between inviscid fluid half-space and micropolar thermoelastic diffusion solid half-space. It is found that for a two-dimensional model, there exist four coupled longitudinal waves, that is, longitudinal displacement wave, thermal wave, mass diffusion wave and longitudinal microstretch wave, as well as two coupled transverse waves. Considering the strain gradient theory and coupled stress theory, Li et al. [45-48] further derived the governing equations of thermoelastic waves corresponding to the L-S, G-L and G-N generalized thermoelastic theories, and

studied the dispersion and attenuation characteristics of thermoelastic waves, as well as the characteristics of reflection and transmission. It was found that the thermo-mechanical coupling effect only affects the longitudinal wave, not the shear wave. Wu et al. [49] studied the propagation of thermoelastic wave in the phononic crystals by using G-N theory. 2D and 3D thermoelastic wave band were derived by using the plane wave expansion method. Hosseini and Zhang [50] computed the dispersion relation of thermoelastic waves by using G-N theory and Bloch's theorem, and analyzed thermoelastic waves propagating by employing a meshless collocation method in a cylindrical phononic crystal. Based on generalized thermoelasticity, Li et al. [51-53] studied thermal shock problem of a transient heated thick viscoelastic plate, thermal diffusion problems with fractional order strain and transient thermo-electromechanical responses of multi-layered piezoelectric laminated composite structure. Besides, Bouazza et al studied the thermoelastic buckling behavior of functionally graded rectangular plates [54-58] and the Laminated Beams [59-60] based on hyperbolic shear deformation theory. And then, Bouazza et al studied hygro-thermo-mechanical buckling of laminated plates [61-62] and beam [63] and analyzed the impacts of micromechanical approaches on the wave propagation in a functionally graded plate [64].

Although, the microstructure effects and the thermoelastic effects have been studied in the existing literatures. The model which incorporates the microstructure effects and the thermoelastic effects simultaneously is still rare. Consider that not only the mechanical behavior but also the heat conduction behavior at small scale are both different from that at macro scale. In this paper, a new coupled model with the

fractional order derivative which incorporate the microstructure effects and thermo-elastic coupled effects at small scale simultaneously is provided. Strain gradient elasticity is used to take into account the microstructural effects while non-Fourier heat conduction with fractional order derivatives is used to model the thermal conduction at small scale. The interaction of the thermo-elastic coupled waves with the periodic structure leads to Bloch waves in one-dimensional phononic crystals. The dispersion equation of the Bloch waves are derived based on the transfer matrix method. The influence of the microscale parameters and the heat conduction parameter, including the strain gradient parameter, the micro-inertial parameter, the relaxation time and the fractional order on the dispersion and bandgap of Bloch waves are established via a parameter study.

2. Formulation of thermoelasticity with fractional order derivatives

Fractional differentials are a generalization of integer differentials. Of particular relevance to the present paper is the property that fractional differentials introduce non-local properties of functions. In the problem of elastic wave propagation, the fractional differential of $u(t)=e^{-i\omega_0 t}$ needs to be considered. Therefore, in Section 2.1, the Caputo-type differential operation results of $u(t)=e^{-i\omega_0 t}$ are derived by applying Laplace and Fourier transforms. For a thermoelastic solid, the small scale thermoelastic response exhibits size effects. The classic heat conduction, i.e. Fourier's law, is not applicable anymore and replaced by non-Fourier heat conduction. Fractional heat conduction can well reflect the thermodynamic behavior of thermoelastic waves at high frequency and in small size. Therefore, LS fractional heat

conduction equation is given in Section 2.2.

2.1 Fractional order derivative of the exponential function

Let $u(t)$ be defined in an infinite interval $(-\infty < t < \infty)$, and $n \in N$, then the left and right Caputo-type fractional differential and integral are defined as, respectively,

$$ {}_{-\infty}^{C}D_{t}^{\sigma}u(t) = \frac{1}{\Gamma(n-\sigma)}\int_{-\infty}^{t}(t-\xi)^{n-\sigma-1}u^{(n)}(\xi)d\xi, \tag{1a} $$

$$ {}_{-\infty}^{C}D_{t}^{-\sigma}u(t) = \frac{1}{\Gamma(\sigma)}\int_{-\infty}^{t}(t-\xi)^{\sigma-1}u(\xi)d\xi, \tag{1b} $$

$$ {}_{t}^{C}D_{+\infty}^{\sigma}u(t) = \frac{1}{\Gamma(n-\sigma)}\int_{t}^{+\infty}(\xi-t)^{n-\sigma-1}(-1)^{n}u^{(n)}(\xi)d\xi, \tag{2a} $$

$$ {}_{t}^{C}D_{+\infty}^{-\sigma}u(t) = \frac{1}{\Gamma(\sigma)}\int_{t}^{+\infty}(\xi-t)^{\sigma-1}u(\xi)d\xi, \tag{2b} $$

where $\sigma > 0$ and n is the smallest integer greater than σ. $\Gamma(z)$ is the gamma function, i.e. $\Gamma(z) = \int_{0}^{+\infty} e^{-t}t^{z-1}dt\ (z \in C, \mathrm{Re}(z) > 0)$. Caputo-type fractional differentials satisfy the following operational relationship

$$ {}_{-\infty}^{C}D_{t}^{\sigma}u(t) = {}_{-\infty}^{C}D_{t}^{n}\,{}_{-\infty}^{C}D_{t}^{-(n-\sigma)}u(t) = {}_{-\infty}^{C}D_{t}^{-(n-\sigma)}\,{}_{-\infty}^{C}D_{t}^{n}u(t), \tag{3a} $$

$$ {}_{t}^{C}D_{+\infty}^{\sigma}u(t) = {}_{t}^{C}D_{+\infty}^{n}\,{}_{t}^{C}D_{+\infty}^{-(n-\sigma)}u(t) = D_{+\infty}^{-(n-\sigma)}\,{}_{t}^{C}D_{+\infty}^{n}u(t). \tag{3b} $$

The Laplace transform is defined by

$$ L[u(t)] = \int_{0}^{\infty} e^{-st}u(t)dt = \bar{u}(s) \tag{4} $$

The Laplace transform of Caputo-type fractional differential is

$$ L\left[{}_{0}^{C}D_{+\infty}^{\sigma}u(t)\right] = s^{\sigma}\bar{u}(s) - \sum_{k=0}^{n-1}s^{\sigma-k-1}u^{(k)}(0),\ n-1 \le \sigma < n \tag{5a} $$

Especially,

$$L\left[{}_0^C D_{+\infty}^{\sigma} u(t)\right] = s^{\sigma}\bar{u}(s) - s^{\sigma-1}u(0), 0 \le \sigma < 1. \tag{5b}$$

The Fourier transform and its inverse are defined as, respectively.

$$F\left[u(t)\right] = \int_{-\infty}^{+\infty} u(t)e^{-i\omega t}dt = \bar{u}(\omega), \tag{6}$$

$$F^{-1}\left[\bar{u}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{u}(\omega)e^{i\omega t}d\omega = u(t), \tag{7}$$

The Fourier and Laplace transforms satisfy

$$F\left[u(t) * v(t)\right] = F\left[u(t)\right] \cdot F\left[v(t)\right] = \bar{u}(s) \cdot \bar{v}(s), \tag{8}$$

$$L\left[t^m\right] = \frac{\Gamma(m+1)}{s^{m+1}} \quad (m > -1), \tag{9}$$

$$\begin{aligned} F\left[t^m H(t)\right] &= \int_{-\infty}^{+\infty} t^m H(t) e^{-i\omega t}dt = \int_0^{+\infty} t^m e^{-i\omega t}dt \\ &= \int_0^{+\infty} t^m e^{-st}dt \ (s = i\omega) \\ &= L\left[t^m\right] = \frac{\Gamma(m+1)}{s^{m+1}} = \frac{\Gamma(m+1)}{(i\omega)^{m+1}}. \end{aligned} \tag{10}$$

where $*$ represents convolution operation, i.e. $u(t) * v(t) = \int_{-\infty}^{+\infty} u(\tau)v(t-\tau)d\tau$.

Next, let's turn to the Fourier transform of Caputo-type fractional differentials of time. Let $h_+(t) = \frac{t^{\sigma-1}}{\Gamma(\sigma)}H(t)$, where $H(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$ is the Heaviside function, Then, Fourier transform of ${}_{-\infty}^C D_t^{-\sigma} u(t)$ can be written as

$$\begin{aligned} F\left[{}_{-\infty}^C D_t^{-\sigma} u(t)\right] &= F\left[h_+(t) * u(t)\right] \\ &= F\left[h_+(t)\right]F\left[u(t)\right] = (i\omega)^{-\sigma} F\left[u(t)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} F\left[h_+(-t)\right] &= \frac{1}{\Gamma(\sigma)}\int_{-\infty}^{0} (-t)^{\sigma-1} e^{-i\omega t}dt = \frac{1}{\Gamma(\sigma)}\int_0^{+\infty} y^{\sigma-1}e^{-i(-\omega)y}dy \\ &= F(h_+)(-\omega) = (-i\omega)^{-\sigma} \ , \end{aligned} \tag{12}$$

According to Eqs. (11) and (12), the Fourier transform of ${}_t^C D_{+\infty}^{-\sigma} u(t)$ can be written as

$$F\left[{}_{t}^{C}D_{+\infty}^{-\sigma}u(t)\right]=(-i\omega)^{-\sigma}F\left[u(t)\right]. \tag{13}$$

Applying Eqs.(3) and (11), the Fourier transform ${}_{-\infty}^{C}D_{t}^{\sigma}u(t)$ can be written as

$$\begin{aligned}F\left[{}_{-\infty}^{C}D_{t}^{\sigma}u(t)\right]&=F\left[{}_{-\infty}^{C}D_{t}^{-(n-\sigma)}\,{}_{-\infty}^{C}D_{t}^{n}u(t)\right]\\&=(i\omega)^{n}F\left[{}_{-\infty}^{C}D_{t}^{-(n-\sigma)}u(t)\right]=(i\omega)^{n}(i\omega)^{-(n-\sigma)}F\left[u(t)\right]\\&=(i\omega)^{\sigma}F\left[u(t)\right].\end{aligned} \tag{14}$$

In particular, when $u(t)=e^{-i\omega_0 t}$,

$$F\left[u(t)\right]=F\left[\mathrm{e}^{-i\omega_0 t}\right]=2\pi\delta(\omega+\omega_0), \tag{15}$$

then,

$$\begin{aligned}{}_{-\infty}^{C}D_{t}^{\sigma}u(t)&=F^{-1}\left\{F\left[{}_{-\infty}^{C}D_{t}^{\sigma}u(t)\right]\right\}=F^{-1}\left[(i\omega)^{\sigma}2\pi\delta(\omega+\omega_0)\right]\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(i\omega)^{\sigma}2\pi\delta(\omega+\omega_0)e^{i\omega t}d\omega=(-i\omega_0)^{\sigma}e^{-i\omega_0 t}.\end{aligned} \tag{16}$$

In similar way, we can get the Fourier transform ${}_{t}^{C}D_{\infty}^{\sigma}u(t)$ as

$$F\left[{}_{t}^{C}D_{+\infty}^{\sigma}u(t)\right]=F\left[{}_{t}^{C}D_{+\infty}^{-(n-\sigma)}\,{}_{t}^{C}D_{+\infty}^{n}u(t)\right]=(-i\omega)^{\sigma}F\left[u(t)\right] \tag{17}$$

$${}_{t}^{C}D_{+\infty}^{\sigma}u(t)={}_{t}^{C}D_{+\infty}^{\sigma}(\mathrm{e}^{-i\omega_0 t})=(-i\omega_0)^{\sigma}e^{-i\omega_0 t}. \tag{18}$$

2.2 The heat transport and thermo-elastic equations with fractional-order derivatives

In a dipolar gradient elastic solid, the total kinetic energy G, the total deformation energy E, and the total work done by external forces W, can be expressed as, respectively [29-31],

$$G=\frac{1}{2}\rho\dot{u}_j\dot{u}_j+\frac{1}{6}\rho d^2\dot{u}_{k,j}\dot{u}_{k,j}, \tag{19a}$$

$$E=\frac{1}{2}\left(\lambda\varepsilon_{ii}\varepsilon_{jj}+2\mu\varepsilon_{ij}\varepsilon_{ij}\right)+\frac{1}{2}c\left(\lambda\varepsilon_{ii,k}\varepsilon_{jj,k}+2\mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right), \tag{19b}$$

$$W=\int_S P_k\dot{u}_k dS+\int_S R_k n_l\partial_l\dot{u}_k dS, \tag{19c}$$

where ρ is the mass density, λ and μ are the Lamé constants of classic elastic

material, and c is a microstructural constant with the dimension of m^2. u_k is displacement, $\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$ represents infinitesimal strain, $\varepsilon_{ij,k}$ represents strain gradient. P_k is monopolar traction tensor while R_k is the dipolar traction tensor. d is the micro inertia parameter with the dimensions of m. The dot on the letters represents the derivative of time.

When the total kinetic energy G, the total deformation energy E, and the total work done by external force W, are inserted into Hamilton variational principle

$$\delta\int_{t_0}^{t_1}\int_V (G-E)dVdt+\int_{t_0}^{t_1}\int_S \delta W dSdt=0, \tag{20}$$

the following governing equation and the boundary conditions can be obtained [50]

$$\left(\tau_{jk}-\mu_{ijk,i}\right)_{,j}=\rho\ddot{u}_k-\frac{\rho d^2}{3}\ddot{u}_{k,jj} \quad \text{in } V, \tag{21a}$$

$$P_k=n_j\left(\tau_{jk}-\mu_{ijk,i}\right)-D_j\left(n_i\mu_{ijk}\right)+\left(D_l n_l\right)n_i n_j\mu_{ijk}+\frac{\rho d^2}{3}n_j\ddot{u}_{k,j} \quad \text{on surface } S, \tag{21b}$$

$$R_k=n_i n_j\mu_{ijk} \quad \text{on surface } S, \tag{21c}$$

for a dipolar gradient elastic solid. Here, volume forces are ignored, τ_{jk} is Cauchy stress, μ_{ijk} is high order stress with the dimension of Nm^{-1}, and $D_j(\ )=(\ )_{,j}-n_j D(\ )$, $D(\ )=n_l(\ )_{,l}$.

If the thermal effect is considered in a dipolar gradient elastic solid, a phenomenological simplified version of the Helmholtz free energy density for a centrosymmetric and isotropic material can be written as [45,48],

$$\rho\psi=\frac{1}{2}\left(\lambda\varepsilon_{ii}\varepsilon_{jj}+2\mu\varepsilon_{ij}\varepsilon_{ij}\right)+\frac{1}{2}c\left(\lambda\varepsilon_{ii,k}\varepsilon_{jj,k}+2\mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right)-\frac{\rho C_r\theta^2}{2T_0}-\Re\theta\varepsilon_{pp}, \tag{22}$$

where $\Re=\left(3\lambda+2\mu\right)\zeta$ is the thermo-elastic coupling coefficient, ζ is the linear thermal expansion coefficient, ψ is the free energy per unit mass, T is the absolute

temperature, C_r is the specific heat capacity per unit mass, $\theta = T - T_0$ is the temperature change, and T_0 is the reference temperature.

The constitutive equations that are consistent with Eq.(22) are

$$\tau_{ij} = \frac{\partial(\rho\psi)}{\partial\varepsilon_{ij}} = \lambda\delta_{ij}\varepsilon_{pp} + 2\mu\varepsilon_{ij} - \Re\theta\delta_{ij}, \tag{23a}$$

$$\mu_{ijk} = \frac{\partial(\rho\psi)}{\partial\varepsilon_{jk,i}} = c\left(\lambda\delta_{jk}\varepsilon_{pp,i} + 2\mu\varepsilon_{jk,i}\right), \tag{23b}$$

$$\rho\varphi = -\frac{\partial(\rho\psi)}{\partial\theta} = \frac{\rho C_r\theta}{T_0} + \Re\varepsilon_{pp}, \tag{23c}$$

where φ is the entropy density. Base on L-S generalized thermoelasticity of the fractional order differential,

$$\left(1 + \tau_0^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)q_i = -\kappa\theta_{,i} \qquad 0 \le \alpha \le 1 \tag{24}$$

where the scalar κ is the thermal conductivity of solid, τ_0 is called the thermal relaxation time, α is a fractional order parameter, and q_i is the heat flux.

The thermoelastic equation of fractional order derivative can thus be derived as [50]

$$\left(1 + \tau_0^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\rho C_r\dot{\theta} + T_0\Re\nabla\cdot\dot{\mathbf{u}}\right) = \kappa\nabla^2\theta. \tag{25}$$

Eq. (25) reduces to the classical heat conduction equation when α=0 and, similarly, it reduces to L-S generalized heat thermoelastic equation when α=1. Therefore, Eq. (25) can describe the arbitrary intermediate situation between classical heat thermoelasticity and L-S generalized heat thermoelasticity via $0 < \alpha < 1$. Thus the thermoelastic equation with the fractional derivative is a more flexible model which can be used to describe the heat conduction feature at small scale.

3. Propagation thermo-elastic coupled waves

Inserting Eq. (23a,b) into Eq. (20a) leads to the equation of motion in terms of displacement in isotropic material of dipolar gradient thermo-elasticity

$$\left(1-c\nabla^2\right)\left[\left(\lambda+\mu\right)\nabla\nabla\cdot\mathbf{u}+\mu\nabla^2\mathbf{u}\right]-\Re\nabla\theta=\left(1-\frac{d^2}{3}\nabla^2\right)\rho\ddot{\mathbf{u}}\,. \tag{26}$$

The propagation of in-plane wave is only investigated because the thermal effect does not affect SH wave, in other words, the mechanical waves and thermal waves are uncoupled in the case of the anti-plane wave.

In the case of in-plane wave, the solution of Eq. (26) can be derived by the application of Helmholtz vector decomposition,

$$\mathbf{u}\left(x,y\right)=u_x\left(x,y\right)\mathbf{e}_x+u_y\left(x,y\right)\mathbf{e}_y=\nabla\varphi(x,y)+\nabla\times\psi(x,y)\mathbf{e}_z, \tag{27a}$$

$$\nabla\cdot\mathbf{u}\left(x,y\right)=\nabla^2\varphi(x,y)\,,\quad \nabla\times\mathbf{u}\left(x,y\right)=-\nabla^2\psi(x,y)\mathbf{e}_z\,, \tag{27b,c}$$

where $\varphi(x,y)$ and $\psi(x,y)$ are the potential functions of displacements. The wave propagation is assumed to occur in the oxy plane, so that, the displacement component $u_z=0$ while u_x and u_y are dependent only on $\left(x,y\right)$.

Inserting Eq. (27a) into Eq. (26), we obtain

$$c\left(\lambda+2\mu\right)\nabla^4\varphi-\left(\lambda+2\mu\right)\nabla^2\varphi+\Re\theta+\rho\ddot{\varphi}-\frac{\rho d^2}{3}\nabla^2\ddot{\varphi}=0\,, \tag{28a}$$

$$c\mu\nabla^4\psi-\mu\nabla^2\psi+\rho\ddot{\psi}-\frac{\rho d^2}{3}\nabla^2\ddot{\psi}=0\,. \tag{28b}$$

From Eq. (28), it is observed that the thermal effects influence the dilatational waves only while the distortional waves are insusceptible. This follows from Eq. (22) where the thermoelastic coupling is expressed in terms of volumetric strain only.

Simultaneous application of Eq. (28a) and Eq. (25) leads to the following equation

$$\nabla^6\varphi+b_1\nabla^4\varphi+b_2\nabla^2\varphi+b_3\varphi=0\,, \tag{29}$$

where $b_1 = -\frac{1}{c} + \frac{d^2\omega^2}{3cV_P^2} + \frac{i\omega X \rho C_r}{\kappa}$, $b_2 = -\frac{\omega^2}{cV_P^2} - i\omega X\left(\frac{\rho C_r}{\kappa c} - \frac{\rho d^2\omega^2 C_r}{3c\kappa V_P^2} + \frac{T_0\Re^2}{c\kappa\rho V_P^2}\right)$,

$b_3 = -i\omega\frac{\rho\omega^2 C_r}{c\kappa V_P^2}$, $X = 1+\left(-i\omega\tau_0\right)^{\alpha}$ and $V_P = \sqrt{(\lambda+2\mu)/\rho}$ is the dilatational phase velocity of classical elastic solids.

Eq. (29) can be factorized as

$$\left(\nabla^2 + \sigma_{MT1}^2\right)\left(\nabla^2 + \sigma_{MT2}^2\right)\left(\nabla^2 + \sigma_{MT3}^2\right)\varphi = 0, \tag{30}$$

where σ_{MT1}, σ_{MT2} and σ_{MT3} are the wave numbers of the thermo-elastic coupled waves. The coupled wave with wavenumber σ_{MT1}, σ_{MT2} and σ_{MT3} is called MT1 wave, MT2 wave and MT3 wave, respectively. It is noted that MT1 wave, MT2 wave, and MT3 wave are all dispersive due to microstructure effect as well as thermal effect. By applying Cardano's formula of roots of a cubic equation with one variable, σ_{MT1}^2 σ_{MT2}^2 and σ_{MT3}^2 can be obtained by

$$\sigma_{MT1}^2 = -\left(pm + p^2 n - \frac{b_1}{3}\right),\ \sigma_{MT2}^2 = -\left(p^2 m + pn - \frac{b_1}{3}\right),\ \sigma_{MT3}^2 = -\left(m + n - \frac{b_1}{3}\right),$$

where $m = \sqrt[3]{s+r}$, for $|s+r| \geq |s-r|$; $m = \sqrt[3]{s-r}$, for $|s+r| < |s-r|$; if $|m| \neq 0$,

$n = \frac{b_1^2 - 3b_2}{9m}$; if $|m| = 0$, $n = 0$. $p = \frac{-1+\sqrt{3}i}{2}$, $s = \frac{9b_1b_2 - 27b_3 - 2b_1^3}{54}$,

$r = \frac{\sqrt{3\left(4b_2^3 - b_1^2b_2^2 - 18b_1b_2b_3 + 27b_3^2 + 4b_1^3b_3\right)}}{18}$.

Let $\sigma_{MT1}^2 = \xi^2 + \beta_{MT1}^2$, $\sigma_{MT2}^2 = \xi^2 + \beta_{MT2}^2$, $\sigma_{MT3}^2 = \xi^2 + \beta_{MT3}^2$, where ξ is the apparent wavenumber. Accordingly, the potential function φ and the temperature field θ can be expressed as

$$\varphi = A_1 e^{i(\xi x + \beta_{MT1} y - \omega t)} + A_2 e^{i(\xi x - \beta_{MT1} y - \omega t)} + B_1 e^{i(\xi x + \beta_{MT2} y - \omega t)} + B_2 e^{i(\xi x - \beta_{MT2} y - \omega t)}$$
$$+ C_1 e^{i(\xi x + \beta_{MT3} y - \omega t)} + C_2 e^{i(\xi x - \beta_{MT3} y - \omega t)}, \tag{31a}$$

$$\theta=\mathrm{g}_1\left[A_1e^{i(\xi x+\beta_{MT1}y-\omega t)}+A_2e^{i(\xi x-\beta_{MT1}y-\omega t)}\right]+\mathrm{g}_2\left[B_1e^{i(\xi x+\beta_{MT2}y-\omega t)}+B_2e^{i(\xi x-\beta_{MT2}y-\omega t)}\right]$$
$$+g_3\left[C_1e^{i(\xi x+\beta_{MT3}y-\omega t)}+C_2e^{i(\xi x-\beta_{MT3}y-\omega t)}\right], \tag{31b}$$

where $g_1=\dfrac{XT_0\Re\sigma_{MT1}^2}{X\rho C_r+\kappa\sigma_{MT1}^2}$, $g_2=\dfrac{XT_0\Re\sigma_{MT2}^2}{X\rho C_r+\kappa\sigma_{MT2}^2}$, $g_3=\dfrac{XT_0\Re\sigma_{MT3}^2}{X\rho C_r+\kappa\sigma_{MT3}^2}$. A_i, B_i and C_i are the amplitudes of coupled MT1 wave, coupled MT2 wave and coupled MT3 waves, respectively. The time harmonic dependence is assumed as $e^{-i\omega t}$ throughout the paper.

From Eq. (28b), we can derive the wave numbers of the distortional waves. Eq. (28b) can be rewritten as

$$\nabla^4\psi-\left(\frac{1}{c}-\frac{d^2\omega^2}{3cV_S^2}\right)\nabla^2\psi-\frac{\omega^2}{cV_S^2}\psi=0, \tag{32a}$$

where $V_S=\sqrt{\mu/\rho}$ is the phase speeds of distortional waves in the classic elastic solids.

If Eq. (32a) is factorized as

$$\left(\nabla^2+\sigma_{SV}^2\right)\left(\nabla^2-\tau_{SS}^2\right)\psi=0, \tag{32b}$$

then,

$$\sigma_{SV}^2=\left\{\left[\left(3V_S^2-d^2\omega^2\right)^2+36\omega^2cV_S^2\right]^{1/2}-\left(3V_S^2-d^2\omega^2\right)\right\}\Big/\left(6cV_S^2\right),$$

$$\tau_{SS}^2=\left\{\left[\left(3V_S^2-d^2\omega^2\right)^2+36\omega^2cV_S^2\right]^{1/2}+\left(3V_S^2-d^2\omega^2\right)\right\}\Big/\left(6cV_S^2\right),$$

Let $\sigma_{SV}^2=\xi^2+\beta_{SV}^2$, $\gamma_{SS}^2=\tau_{SS}^2+\xi^2$, where σ_{SV} is the wave number of the SV wave with consideration of microstructure effect and τ_{SS} is a S type surface wave with imaginary wavenumber $i\tau_{SS}$ and is called SS wave. It is noted that SV wave and SS waves are both dispersive due to the microstructural effect. Accordingly, the potential

function ψ can be expressed as

$$\psi = F_1 e^{i(\xi x+\beta_{SV} y-\omega t)} + F_2 e^{i(\xi x-\beta_{SV} y-\omega t)} + D_1 e^{-\gamma_{SS} y+i(\xi x-\omega t)} + D_2 e^{\gamma_{SS} y+i(\xi x-\omega t)}, \tag{33}$$

where F_i and D_i are the amplitudes of SV wave and SS surface wave, respectively.

4. Dispersion relation of Bloch waves in oblique propagation situation

Consider a one-dimensional laminated structure that consists of dipolar gradient thermal elastic solids. The single cell is composed of two different dipolar gradient thermal elastic solids with thickness of a_1 and a_2, see Fig.1. The wave propagation plane is assumed to be in the oxy plane, where y axis is along the normal direction of the laminated structure and x axis is along the interface, see Fig.2.

The thermomechanical coupled physical fields in any layer in a typical single cell are formed by the forward and the backward coupled MT1, coupled MT2, coupled MT3 and SV bulk waves plus two S type surface waves propagating along interface.

Define the state vector

$$\{V\} = \{u_x\ u_y\ u_{x,y}\ u_{y,y}\ \theta\ P_x\ P_y\ R_x\ R_y\ q_y\}^T. \tag{34}$$

Inserting Eq. (23a,b) into Eq. (21b,c), we can obtain monopolar traction and the dipolar traction, namely,

$$P_x = 2\mu\left(1-c\nabla^2\right)\varepsilon_{yx} - c\left[\left(\lambda+2\mu\right)\varepsilon_{xx,xy} + \lambda\varepsilon_{yy,xy}\right] + \frac{\rho d^2}{3}\ddot{u}_{x,y},$$

$$P_y = \left(1-c\nabla^2\right)\left[\left(\lambda+2\mu\right)\varepsilon_{yy} + \lambda\varepsilon_{xx}\right] - 2\mu c\varepsilon_{xy,xy} + \frac{\rho d^2}{3}\ddot{u}_{y,y} - \Re\theta,$$

$$R_x = 2\mu c\varepsilon_{yx,y}, \quad R_y = c\left[\left(\lambda+2\mu\right)\varepsilon_{yy,y} + \lambda\varepsilon_{xx,y}\right], \quad \left(1+\tau_0^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)q_y = -\kappa\theta_{,y}.$$

Then, the state vector at the left and right boundaries of a layer can be expressed as

$$\left\{V^L\right\}=\left[P^L\right]\left\{A_1,A_2,B_1,B_2,C_1,C_2,F_1,F_2,D_1,D_2\right\}^T e^{i(\xi x-\omega t)}, \tag{35a}$$

$$\left\{V^R\right\}=\left[P^R\right]\left\{A_1,A_2,B_1,B_2,C_1,C_2,F_1,F_2,D_1,D_2\right\}^T e^{i(\xi x-\omega t)}, \tag{35b}$$

where

$$\left[P^L\right]=\left[P_0\right]\left[G^L\right],\ \left[P^R\right]=\left[P_0\right]\left[G^R\right],\ \left[G^R\right]=\left[G^L\right]\left[G\left(a_j\right)\right],$$

$$\left[G^L\right]=diag(e^{i\beta_{MT1}y},e^{-i\beta_{MT1}y},e^{i\beta_{MT2}y},e^{-i\beta_{MT2}y},e^{\beta_{MT3}y},e^{-i\beta_{MT3}y},e^{i\beta_{SV}y},e^{-i\beta_{SV}y},e^{-\gamma_{SS}y},e^{\gamma_{SS}y})$$

$$\left[G\left(a_j\right)\right]=diag(e^{i\beta_{MT1}a_j},e^{-i\beta_{MT1}a_j},e^{i\beta_{MT2}a_j},e^{-i\beta_{MT2}a_j},e^{\beta_{MT3}a_j},e^{-i\beta_{MT3}a_j},$$

$$e^{i\beta_{SV}a_j},e^{-i\beta_{SV}a_j},e^{-\gamma_{SS}a_j},e^{-\gamma_{SS}a_j}\Big),\qquad (j=1\text{or }2).$$

$\left[P_0\right]$ is listed in the appendix A.

It is noted that the state vectors at the left and right sides of a layer are related by

$$\left\{V^R\right\}=\left[P^R\right]\left[P^L\right]^{-1}\left\{V^L\right\}=[T]\left\{V^L\right\}, \tag{36}$$

where the transfer matrix of the j^{th} layer is defined by

$$\left[T_j\right]=\left[P_{0j}\right]\left[G(a_j)\right]\left[P_{0j}\right]^{-1}. \tag{37}$$

The transfer matrix $\left[T_j\right]$ is determined by the material coefficients, the thickness of the layer and the wave modes in the j-layer for a given frequency ω.

For a perfect interface (displacement, displacement gradient, monopolar traction, dipolar traction, temperature and heat flux are all continuous across the interface), the state vector is continuous across the interface between layer A and layer B, namely,

$$\left\{V_A^L\right\}=\left\{V_A^R\right\}. \tag{38}$$

This interface condition leads to

$$\left\{V_B^R\right\}=[T_B][T_A]\left\{V_A^L\right\}, \tag{39}$$

The Bloch theorem for the wave propagation in the periodic structure can be expressed as

$$\left\{V_B^R\right\} = e^{ika}\left\{V_A^L\right\}, \tag{40}$$

where $a = a_1 + a_2$ is the thickness of a typical single cell, and k is the wavenumber of the Bloch wave in the periodic laminated structure.

Inserting Eq. (40) into Eq. (39) leads to

$$\left(\left[T_B\right]\left[T_A\right] - e^{ika}\left[I\right]\right)\left\{V_A^L\right\} = \{0\}, \tag{41}$$

The existence of a non-trivial solution requires

$$\det\left(\left[T_B\right]\left[T_A\right] - e^{ika}\left[I\right]\right) = f\left(\omega, \xi, k\right) = 0, \tag{42}$$

which is the dispersion relation of Bloch wave. The transfer matrix method is used to derive the transfer matrix of single cell of the periodic structure in the present work. The advantage of the transfer matrix method is that the total transfer matrix of single cell can be obtained by the continued multiplication of the transfer matrix of single layer.

5. Dispersion relation of Bloch wave in normal propagation situation

In the normal propagation situation, the longitudinal waves and transverse waves are decoupled from each other. The dispersive equations of Bloch longitudinal waves and Bloch transverse waves have the same form, namely,

$$\det\left(\left[T_B\right]\left[T_A\right] - e^{ika}\left[I\right]\right) = f\left(\omega, k\right) = 0, \tag{43}$$

The displacement field of Bloch longitudinal waves and the corresponding monopolar traction and the dipolar traction are

$$u_y = A_1 i\sigma_{MT1} e^{i(\sigma_{MT1} y - \omega t)} - A_2 i\sigma_{MT1} e^{i(-\sigma_{MT1} y - \omega t)} + B_1 i\sigma_{MT2} e^{i(\sigma_{MT2} y - \omega t)} - B_2 i\sigma_{MT2} e^{i(-\sigma_{MT2} y - \omega t)}$$
$$+ C_1 i\sigma_{MT3} e^{i(\sigma_{MT3} y - \omega t)} - 2_2 i\sigma_{MT3} e^{i(-\sigma_{MT3} y - \omega t)}, \tag{44a}$$

$$u_{y,y} = -A_1\sigma_{MT1}^2 e^{i(\sigma_{MT1} y - \omega t)} - A_2\sigma_{MT1}^2 e^{i(-\sigma_{MT1} y - \omega t)} - B_1\sigma_{MT2}^2 e^{i(\sigma_{MT2} y - \omega t)} - B_2\sigma_{MT2}^2 e^{i(-\sigma_{MT2} y - \omega t)}$$

$$-C_1\sigma_{MT3}^2 e^{i(\sigma_{MT3}y-\omega t)} - C_2\sigma_{MT3}^2 e^{i(-\sigma_{MT3}y-\omega t)}, \tag{44b}$$

$$\theta = g_1\left[A_1 e^{i(\sigma_{MT1}y-\omega t)} + A_2 e^{i(-\sigma_{MT1}y-\omega t)}\right] + g_2\left[B_1 e^{i(\sigma_{MT2}y-\omega t)} + B_2 e^{i(-\sigma_{MT2}y-\omega t)}\right]$$
$$+ g_3\left[C_1 e^{i(\sigma_{MT3}y-\omega t)} + C_2 e^{i(-\sigma_{MT3}y-\omega t)}\right], \tag{44c}$$

$$P_y = \left(1-c\nabla^2\right)\left(\lambda+2\mu\right)u_{y,y} + \frac{\rho d^2}{3}\ddot{u}_{y,y} - \Re\theta, \tag{44d}$$

$$R_y = c\left(\lambda+2\mu\right)u_{y,yy}, \quad q_y = -\frac{\kappa\theta_{,y}}{1+\left(-i\omega\tau_0\right)^{\alpha}}, \tag{44e,f}$$

The displacement field of Bloch transverse waves and the corresponding monopolar traction and the dipolar traction are

$$u_x = F_1 i\sigma_{SV} e^{i(\sigma_{SV}y-\omega t)} - F_2 i\sigma_{SV} e^{i(-\sigma_{SV}y-\omega t)} - D_1\tau_{SS} e^{-\tau_{SS}y-i\omega t} + D_2\tau_{SS} e^{\tau_{SS}y-i\omega t}, \tag{45a}$$

$$u_{x,y} = -F_1\sigma_{SV}^2 e^{i(\sigma_{SV}y-\omega t)} - F_2\sigma_{SV}^2 e^{i(-\sigma_{SV}y-\omega t)} + D_1\tau_{SS}^2 e^{-\tau_{SS}y-i\omega t} + D_2\tau_{SS}^2 e^{\tau_{SS}y-i\omega t}, \tag{45b}$$

$$P_x = \mu\left(1-c\nabla^2\right)u_{x,y} + \frac{\rho d^2}{3}\ddot{u}_{x,y}, \quad R_x = \mu c u_{x,yy}. \tag{45c,d}$$

The explicit expressions of the transfer matrix **T** for longitudinal waves and transverse waves are given in Appendix B and C, respectively.

6. Numerical results and discussion

The dispersive relation (the dependence of wavenumber k upon the angular frequency ω) of Bloch waves in the periodical laminated structure is dependent upon (1) the thickness $\left(a_1, a_2\right)$ of two dipolar gradient thermoelastic solids; (2) the material constants $(\lambda,\mu,\rho,c,d,C_r,\Re,\kappa,T_0,\tau_0,\alpha)$ of two dipolar gradient thermoelastic solids; (3) the apparent wavenumber $\left(\xi\right)$ of Bloch waves (in the oblique propagation situation). In general, the dispersive equation can be written as

$$f\left(\lambda_1,\mu_1,\rho_1,c_1,d_1,C_{r1},\Re_1,\kappa_1,T_{01},\tau_{01},a_1,\alpha_1,\right.$$
$$\left.\lambda_2,\mu_2,\rho_2,c_2,d_2,C_{r2},\Re_2,\kappa_2,T_{02},\tau_{02},a_2,\alpha_2,k,\xi,\omega\right)=0, \tag{46}$$

where $(\lambda_1,\mu_1,\rho_1,c_1,d_1,C_{r1},\Re_1,\kappa_1,T_{01},\tau_{01},a_1,\alpha_1)$ are the material constants in layer A, while $(\lambda_2,\mu_2,\rho_2,c_2,d_2,C_{r2},\Re_2,\kappa_2,T_{02},\tau_{02},a_2,\alpha_2)$ are the material constants in layer B. we choose material A is lead and material B is brass in here. Choosing (a_1,ρ_1,T_{01}) and $\omega_0=2\pi\Big/\left(\frac{a_1}{\sqrt{\mu_1/\rho_1}}+\frac{a_2}{\sqrt{\mu_2/\rho_2}}\right)$ as the base quantities, the non-dimensional material coefficients are defined by

$$\bar{\lambda}_1=\frac{\lambda_1}{\rho_1 a_1^2\omega_0^2}\ ,\quad \bar{\mu}_1=\frac{\mu_1}{\rho_1 a_1^2\omega_0^2}\ ,\quad \bar{c}_1=\frac{c_1}{a_1^2}\ ,\quad \bar{d}_1=\frac{d_1}{a_1}\ ,\quad \bar{C}_{r1}=\frac{T_{01}C_{r1}}{a_1^2\omega_0^2}\ ,\quad \bar{\Re}_1=\frac{T_{01}\Re_1}{\rho_1 a_1^2\omega_0^2}\ ,$$

$$\bar{\kappa}_1=\frac{T_{01}\kappa_1}{\rho_1 a_1^4\omega_0^3}\ ,\quad \bar{\tau}_1=\tau_{01}\omega_0\ ,\quad \lambda_R=\frac{\bar{\lambda}_2}{\bar{\lambda}_1}\ ,\quad \mu_R=\frac{\bar{\mu}_2}{\bar{\mu}_1}\ ,\quad \rho_R=\frac{\rho_2}{\rho_1}\ ,\quad c_R=\frac{\bar{c}_2}{\bar{c}_1}\ ,\quad d_R=\frac{\bar{d}_2}{\bar{d}_1}\ ,$$

$$C_{rR}=\frac{\bar{C}_{r2}}{\bar{C}_{r1}},\ \Re_R=\frac{\bar{\Re}_2}{\bar{\Re}_1},\ \kappa_R=\frac{\bar{\kappa}_2}{\bar{\kappa}_1},\ T_{0R}=\frac{T_{02}}{T_{01}},\ \tau_R=\frac{\bar{\tau}_{02}}{\bar{\tau}_{01}},\ a_R=\frac{a_2}{a_1},\ \bar{k}=\frac{ka_1}{\pi},\ \bar{\xi}=\frac{\xi a_1}{\pi},$$

$$\bar{\omega}=\frac{\omega}{\omega_0},\alpha_R=\frac{\alpha_2}{\alpha_1}.$$

Therefore, the dispersion equation can be rewritten as

$$f\Big(\bar{\lambda}_1,\bar{\mu}_1,1,\bar{c}_1,\bar{d}_1,\bar{C}_{r1},\bar{\Re}_1,\bar{\kappa}_1,1,\bar{\tau}_1,1,\alpha_1,$$
$$\lambda_R,\mu_R,\rho_R,c_R,d_R,C_{rR},\Re_R,\kappa_R,T_{0R},\tau_R,a_R,\alpha_R,\bar{k},\bar{\xi},\bar{\omega}\Big)=0,\qquad(35)$$

In this numerical example, we mainly focus on the influences of the following parameters: the fractional parameters α_R, the thermal relaxation time $\bar{\tau}_1,\tau_R$ as well as the microstructure parameters $\bar{c}_1,c_R$, $\bar{d}_1$ and d_R. The remaining parameters are given as follows:

$a_1=10^{-5}m$, ρ_1=$7.5\times10^3\,kg/m^3$, μ_1=2.3×10^{10}Pa , $T_{01}=300K$, ω_0=4.1×10^8Hz ,

$\bar{\lambda}_1=0.928$, $\bar{\mu}_1=0.182$, $\bar{C}_{r1}=0.069$, $\bar{\Re}_1=0.0167$, $\bar{\kappa}_1=2.3\times10^{-5}$, $\lambda_R=0.047$,

$\mu_R=0.056$, $\rho_R=0.157$, $C_{rR}=2.257$, $\Re_R=0.057$, $\kappa_R=0.407$, $T_{0R}=1$,

$a_R = 1(a_1 = a_2)$.

Fig. 3 shows the comparison of bandgaps for the three cases of classical elasticity, gradient elasticity and gradient thermo-elasticity. It is observed that strain gradient effects and thermal effects each have an evident influence on the dispersion curves and thus on the bandgaps of laminated structures. For example, the first bandgap becomes evidently wider when the strain gradient effects are taken into account but becomes narrower when the thermal effects are taken into account.

In order to ensure the reliability of the numerical results, the comparison with the existing literature is performed. Fig.4 shows the comparison of the dispersion and bandgap obtained by the present model with that reported in the existing literature. Fig.4(a) and Fig 4(b) show the dispersion curves and the bandgap obtained from the present model when the thermoelastic effects and the microstructure effects are both ignored and that reported in literature [65] and [34] for the classical elastic solids. It is founded that there is a good consistence. The deviation at the higher frequency is more bigger. Fig.4(c) shows the dispersion and bandgap for the dipolar gradient elastic solids but ignoring the thermoelastic coupling in the present model. It is noted that there is still a good consistence between our results and that reported in literature [34]. these comparisons provide a verification of the reliability of the present model to some extent.

Fig. 5 shows the influence of fractional parameter α of non-Fourier heat conduction on the dispersion curves and the band gaps of Bloch waves. It is noted that the dispersion curves shift toward the higher frequency region with the increasing α. As

the result of this shift, the bandgap widens. Figs. 6 and 7 show the influence of the thermal relaxation τ and the thermal relaxation time ratio τ_R on the dispersion curves and the band gaps of Bloch waves. Obviously, the relaxation time have evident influences on the dispersion curves and the dispersion curves seem become more flat with the increasing relaxation time. Fig.7 show the influences of the thermal relaxation time ratio of layer A and layer B in single unit. It is noted that the larger difference of relaxation times makes the dispersion curves shift towards the low frequency region and become flatter. This suppresses the formation of a bandgap.

Fig. 8 shows the influence of micro-stiffness length scale parameter ratio c_R on the dispersion curves and the band gaps of Bloch waves. Fig. 9 shows the influence of micro-inertial length scale parameter ratio d_R on the dispersion curves and the band gaps of Bloch waves. Because the influences of strain gradient effects are notable only at the higher frequency region, the dispersion curves at lower frequency region are not shown. By comparison of Figs. 8 and 9, it is observed that an increase of micro-stiffness length scale parameter ratio c_R decrease the number of curves within the specified frequency region while an increase of the micro-inertial length scale parameter ratio d_R increase the number of curves. The increase of the number of curves means an increase of the state density. Therefore, the micro-structure parameter c and d mainly have an influence on the state density and thus also influence the bandgap distribution. In general, the increase of micro-stiffness length scale parameter ratio c_R helps the formation of a bandgap while the opposite effects is noted for the increase of the micro-inertial length scale parameter ratio d_R.

Figs. 10 and 11 show the influence of the fractional parameter ratio α_R and the thermal relaxation time $\overline{\tau}$ on the dispersion curves and the band gaps of Bloch waves in the case of oblique propagation. Similar in the case of vertical propagation, the increase of fractional parameter ratio α_R mainly shifts the curves towards the low frequency region, while the increase of relaxation time mainly flattens the curves.

7. Conclusions

The periodically laminated structure causes frequency selection properties for wave propagation, also known as the bandgap property. The present work mainly focuses on the influence of microstructure and thermal effects on this bandgap feature. The microstructure effects are modelled with strain gradient elasticity and the thermal effects are captured by thermoelasticity with non-Fourier heat conduction. Based on the analytical formulation and the numerical example, the following conclusions can be drawn.

1) The strain gradient effects only have evident influences on the dispersion curves for relative high frequencies. The increase of the micro-stiffness parameter makes the dispersion curves shift toward higher frequency region.
2) The micro-inertial parameter has opposite effects compared with the micro-stiffness parameter. In general, the increase of the micro-inertial parameter makes the state density increase while the increase of the micro-stiffness parameter makes the state density decrease.
3) The non-Fourier heat conduction and the thermo-elastic coupled effects create the "second sound" phenomenon which greatly increases the mode complexity of

Bloch waves. In general, the thermal effects make the dispersion curves shift towards the low frequency region.

4) The influence of the fraction order mainly changes the mode frequency of Bloch waves while the increase of the relaxation time mainly flattens the dispersion curves, which helps the formation of bandgaps.

The merits of present model is that the microstructure effects and the thermoelastic effects can be considered simultaneously. The transfer matrix method can also be extend to the single cell which consists of multiple layers and the study of reflection and transmission problem of the finite laminated structure. The present work only is limited to the perfect interface, however, the imperfect interface with jump physical fields is expected in the future work.

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Compliance with ethics guidelines

All authors (Yueqiu Li, Harm Askes, Inna M. Gitman, Anton Krynkin and PeijunWei) declare that they have no conflict of interest or financial conflicts to disclose.

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Appendix A

$p_{11}=p_{12}=p_{13}=p_{14}=p_{15}=p_{16}=i\xi$, $p_{17}=i\beta_{SV}$, $p_{18}=-i\beta_{SV}$, $p_{19}=-\gamma_{SS}$, $p_{1,10}=-\gamma_{SS}$;

$p_{21}=i\beta_{MT1}$, $p_{22}=-i\beta_{MT1}$, $p_{23}=i\beta_{MT2}$, $p_{24}=-i\beta_{MT2}$, $p_{27}=p_{28}=p_{29}=p_{2,10}=-i\xi$;

$p_{25}=i\beta_{MT3}$, $p_{26}=-i\beta_{MT3}$, $p_{31}=-\xi\beta_{MT1}$, $p_{32}=\xi\beta_{MT1}$, $p_{33}=-\xi\beta_{MT2}$, $p_{34}=\xi\beta_{MT2}$,

$p_{35}=-\xi\beta_{MT3}$, $p_{36}=\xi\beta_{MT3}$, $p_{37}=p_{38}=-\beta_{SV}^2$, $p_{39}=p_{3,10}=\gamma_{SS}^2$; $p_{41}=p_{42}=-\beta_{MT1}^2$,

$p_{43}=p_{44}=-\beta_{MT2}^2$, $p_{45}=p_{46}=-\beta_{MT3}^2$, $p_{47}=\xi\beta_{SV}$, $p_{48}=-\xi\beta_{SV}$, $p_{49}=i\xi\gamma_{SS}$,

$p_{4,10}=-i\xi\gamma_{SS}$; $p_{51}=p_{52}=g_1$, $p_{53}=p_{54}=g_2$, $p_{55}=p_{56}=g_3$, $p_{57}=p_{58}=p_{59}=p_{5,10}=0$;

$p_{61}=-\xi\beta_{MT1}\left\{\left(2\mu-D\right)+c\lambda\sigma_{MT1}^2+2\mu c\left(\sigma_{MT1}^2+\xi^2\right)\right\}$, $p_{62}=-p_{61}$,

$p_{63}=-\xi\beta_{MT2}\left\{\left(2\mu-D\right)+c\lambda\sigma_{MT2}^2+2\mu c\left(\sigma_{MT2}^2+\xi^2\right)\right\}$, $p_{64}=-p_{63}$,

$p_{65}=-\xi\beta_{MT3}\left\{\left(2\mu-D\right)+c\lambda\sigma_{MT3}^2+2\mu c\left(\sigma_{MT3}^2+\xi^2\right)\right\}$, $p_{66}=-p_{65}$,

$p_{67}=\mu\left(\xi^2-\beta_{SV}^2\right)+D\beta_{SV}^2-\mu c\left(\sigma_{SV}^4-2\xi^4\right)$, $p_{68}=p_{67}$,

$p_{69}=\mu\left(\xi^2+\gamma_{SS}^2\right)-D\gamma_{SS}^2-\mu c\left(\tau_{SS}^4-2\xi^4\right)$, $p_{6,10}=p_{69}$;

$p_{71}=-\lambda\sigma_{MT1}^2-\left(2\mu-D\right)\beta_{MT1}^2-\Re g_1-c\lambda\sigma_{MT1}^4-2\mu c\beta_{MT1}^2\left(\sigma_{MT1}^2+\xi^2\right)$, $p_{72}=p_{71}$,

$p_{73}=-\lambda\sigma_{MT2}^2-\left(2\mu-D\right)\beta_{MT2}^2-\Re g_2-c\lambda\sigma_{MT2}^4-2\mu c\beta_{MT2}^2\left(\sigma_{MT2}^2+\xi^2\right)$, $p_{74}=p_{73}$,

$p_{75}=-\lambda\sigma_{MT3}^2-\left(2\mu-D\right)\beta_{MT3}^2-\Re g_3-c\lambda\sigma_{MT3}^4-2\mu c\beta_{MT3}^2\left(\sigma_{MT3}^2+\xi^2\right)$, $p_{76}=p_{75}$,

$p_{77}=\left(2\mu-D\right)\xi\beta_{SV}+\mu c\xi\beta_{SV}\left(\sigma_{SV}^2+2\xi^2\right)$, $p_{78}=-p_{77}$,

$p_{79}=\left(2\mu-D\right)i\xi\gamma_{SS}-\mu ci\xi\gamma_{SS}\left(\tau_{SS}^2-2\xi^2\right)$, $p_{7,10}=-p_{79}$;

$p_{81}=-2\mu ci\xi\beta_{MT1}^2$, $p_{82}=p_{81}$; $p_{83}=-2\mu ci\xi\beta_{MT2}^2$, $p_{84}=p_{83}$; $p_{85}=-2\mu ci\xi\beta_{MT3}^2$, $p_{86}=p_{85}$,

$p_{87}=-\mu ci\beta_{SV}\left(\sigma_{SV}^2-2\xi^2\right)$ $p_{88}=-p_{87}$, $p_{89}=-\mu c\gamma_{SS}\left(\tau_{SS}^2+2\xi^2\right)$, $p_{8,10}=-p_{89}$;

$p_{91}=-c\lambda i\beta_{MT1}\sigma_{MT1}^2-2\mu ci\beta_{MT1}^3$, $p_{92}=-p_{91}$, $p_{93}=-c\lambda i\beta_{MT2}\sigma_{MT2}^2-2\mu ci\beta_{MT2}^3$,

$p_{94}=-p_{93}$, $p_{95}=-c\lambda i\beta_{MT3}\sigma_{MT3}^2-2\mu ci\beta_{MT3}^3$, $p_{96}=-p_{95}$, $p_{97}=2\mu ci\xi\beta_{SV}^2$, $p_{98}=p_{97}$,

$p_{99}=-2\mu ci\xi\gamma_{SS}^2$, $p_{9,10}=p_{99}$, $p_{10,1}=Mg_1i\beta_{MT1}$, $p_{10,2}=-p_{10,1}$, $p_{10,3}=Mg_2i\beta_{MT2}$,

$p_{10,4}=-p_{10,3}$, $p_{10,5}=Mg_3i\beta_{MT3}$, $p_{10,6}=-p_{10,5}$ $p_{10,7}=p_{10,8}=p_{10,9}=p_{10,10}=0$,

Where $D=\dfrac{\rho d^2\omega^2}{3}$, $M=-\dfrac{\kappa}{1+\left(-i\omega\tau_0\right)^\alpha}$.

Appendix B

$p_{11}=i\sigma_{MT1}$, $p_{12}=-i\sigma_{MT1}$, $p_{13}=i\sigma_{MT2}$, $p_{14}=-i\sigma_{MT2}$, $p_{15}=i\sigma_{MT3}$, $p_{16}=-i\sigma_{MT3}$;

$p_{21}=p_{22}=\sigma_{MT1}^2$, $p_{23}=p_{24}=\sigma_{MT2}^2$, $p_{25}=p_{26}=\sigma_{MT3}^2$; $p_{31}=p_{32}=g_1$, $p_{33}=p_{34}=g_2$,

$p_{35} = p_{36} = g_3$;

$p_{41} = p_{42} = -(\lambda + 2\mu - D)\sigma_{MT1}^2 - (\lambda + 2\mu)\sigma_{MT1}^4 - \Re g_1$,

$p_{43} = p_{44} = -(\lambda + 2\mu - D)\sigma_{MT2}^2 - (\lambda + 2\mu)\sigma_{MT2}^4 - \Re g_2$,

$p_{45} = p_{46} = -(\lambda + 2\mu - D)\sigma_{MT3}^2 - (\lambda + 2\mu)\sigma_{MT3}^4 - \Re g_3$;

$p_{51} = -c(\lambda + 2\mu)i\sigma_{CP}^3$, $p_{52} = c(\lambda + 2\mu)i\sigma_{CP}^3$,

$p_{53} = -c(\lambda + 2\mu)i\sigma_{CT}^3$, $p_{54} = c(\lambda + 2\mu)i\sigma_{CT}^3$,

$p_{55} = -c(\lambda + 2\mu)\tau_{SP}^3$, $p_{56} = c(\lambda + 2\mu)\tau_{SP}^3$;

$p_{61} = Mg_1 i\sigma_{CP}$, $p_{62} = -Mg_1 i\sigma_{CP}$, $p_{63} = Mg_2 i\sigma_{CT}$,

$p_{64} = -Mg_2 i\sigma_{CT}$, $p_{65} = -Mg_3\tau_{SP}$, $p_{66} = Mg_3\tau_{SP}$.

Appendix C

$p_{11} = i\sigma_{SV}$, $p_{12} = -i\sigma_{SV}$, $p_{13} = -\tau_{SS}$, $p_{14} = \tau_{SS}$; $p_{21} = p_{22} = \sigma_{SV}^2$, $p_{23} = p_{24} = \tau_{SS}^2$;

$p_{31} = p_{32} = -(\mu - D)\sigma_{SV}^2 - \mu c\sigma_{SV}^4$, $p_{33} = p_{34} = (\mu - D)\tau_{SS}^2 - \mu c\tau_{SS}^4$;

$p_{41} = -c\mu i\sigma_{SV}^3$, $p_{42} = c\mu i\sigma_{SV}^3$, $p_{43} = -c\mu\tau_{SS}^3$, $p_{44} = c\mu\tau_{SS}^3$.